Chapter 3: Truss elements and solutions for different strain measures

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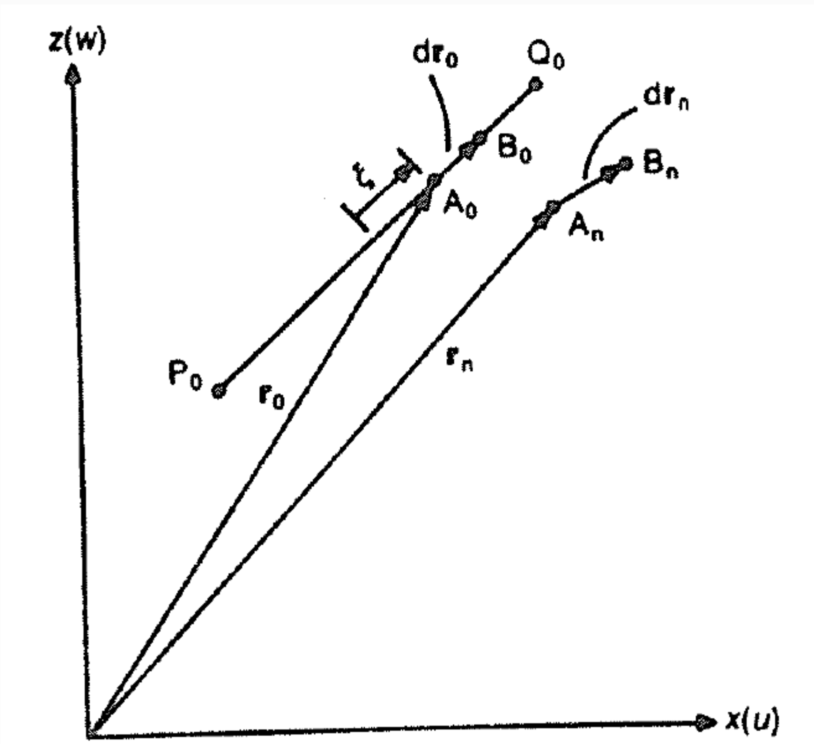
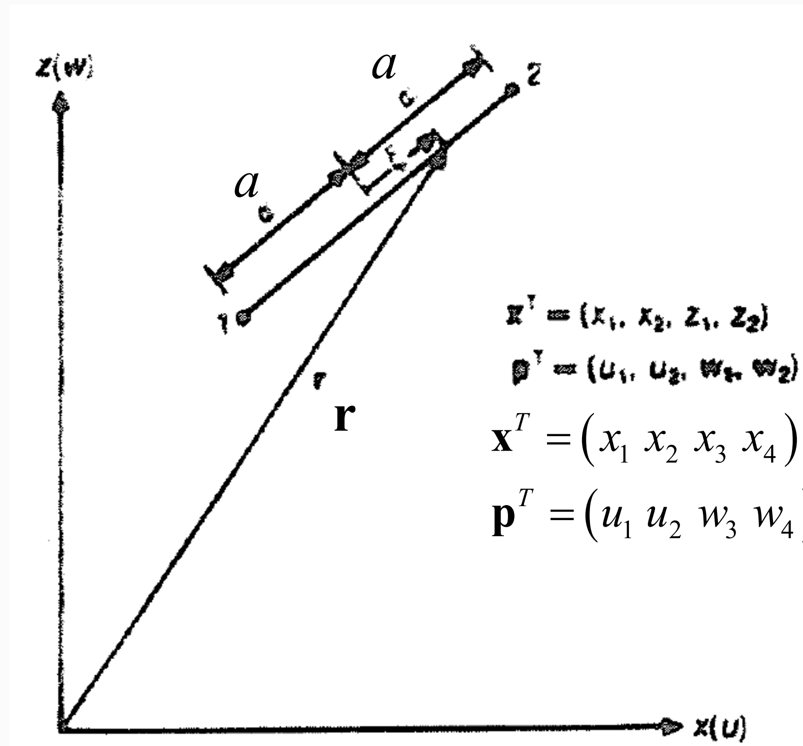
In this sub-chapter 3.3

- A truss (finite) element based on Green's strain will be introduced.
- Introduce the concepts that can be readily extended to continua (solid element), beams and shells.
- A standard finite element procedure based on shape functions will be utilized, though we keep the simple element formulation.
- Only two dimensional planar truss element will be given, but it can be readily extended to three dimensional space truss element (in chapter 3.7)

- **The two-dimensional planar truss element**



[Fig 3.5 Deformation of general truss element]



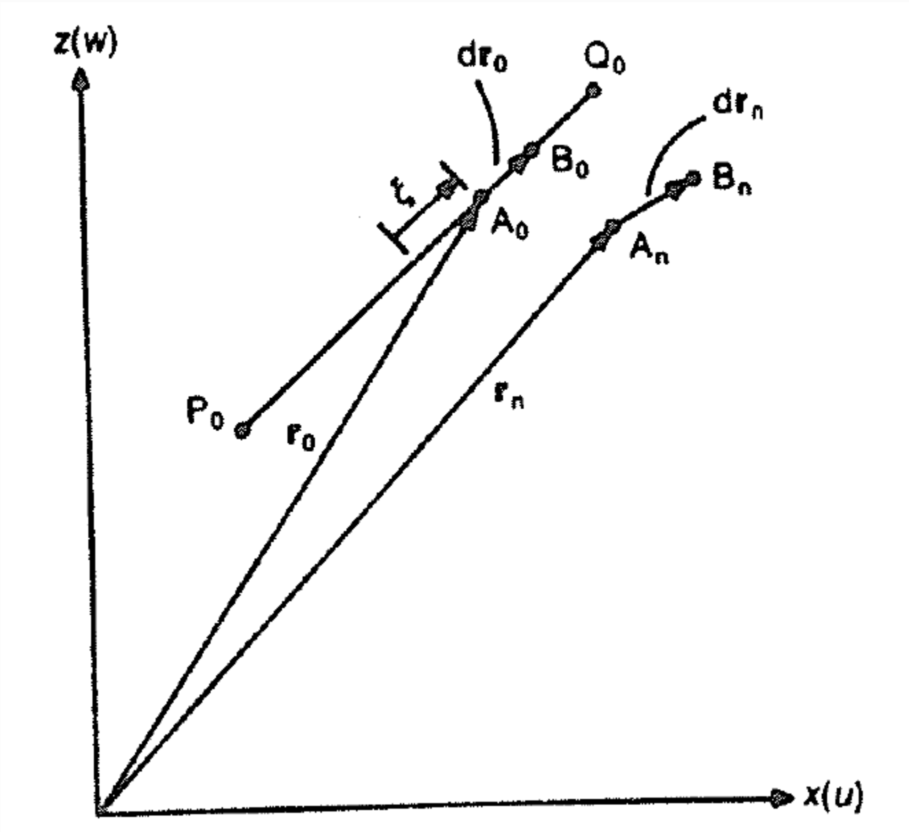
[Fig 3.6 Geometry and modes for general truss element]

3.3.1 Geometry and the strain-displacement relationships

- Kinematics

$$\mathbf{r}_n = \mathbf{r}_0 + \mathbf{u}$$ [eq. 3.45] Position vector

$$\mathbf{r} = \begin{pmatrix} x \\ z \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} u \\ w \end{pmatrix}$$ [eq. 3.46]



Nodal coordinates

$$\mathbf{x}_n = \mathbf{x}' = \mathbf{x}_0 + \mathbf{p} = \mathbf{x} + \mathbf{p}$$ [eq. 3.47]

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} u_1 \\ u_2 \\ w_1 \\ w_2 \end{pmatrix}$$ [eq. 3.48, 3.49]

Without introducing the concept of shape functions

$$l_0^2 = 4\alpha_0^2 = \left(x_{21}^2 + z_{21}^2\right) = \mathbf{x}_{21}^T \mathbf{x}_{21}$$ [eq. 3.50]

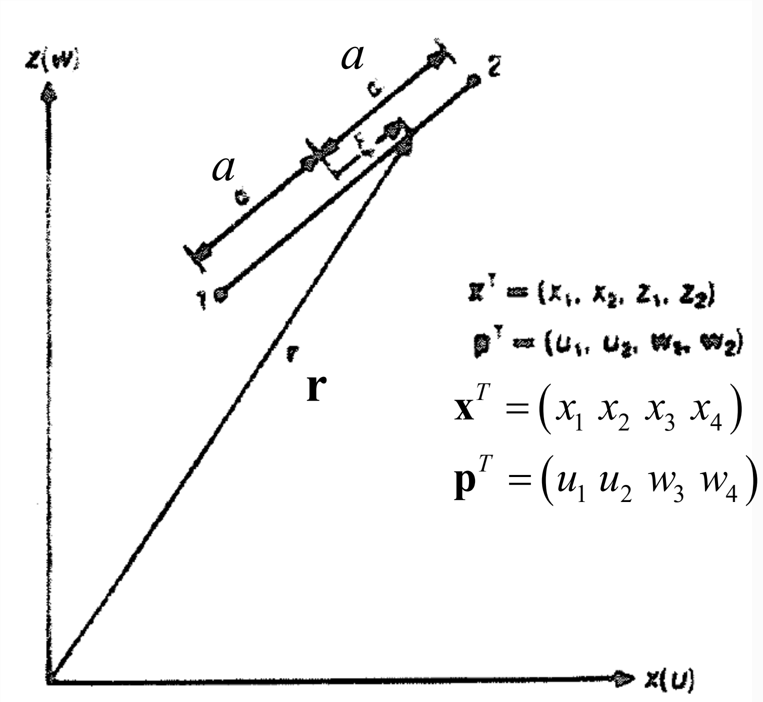
where $x_{21} = x_2 - x_1, \;\; z_{21} = z_2 - z_1, \;\; \mathbf{x}_{21} = \begin{pmatrix} x_{21} \\ z_{21} \end{pmatrix}$

[eq. 3.51, 3.52]

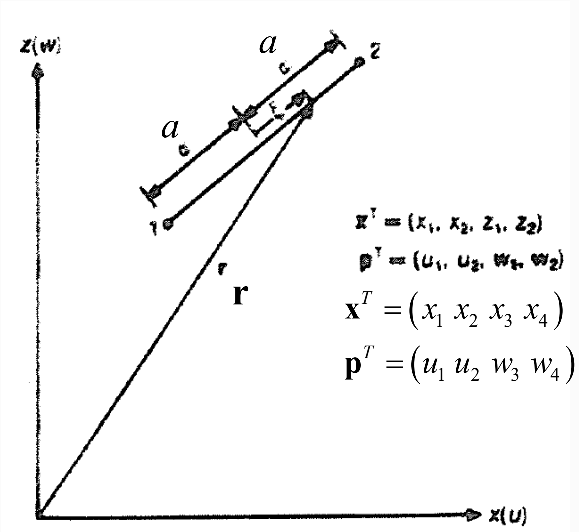
$$l_n^2 = 4\alpha_n^2 = \left(x_{21} + u_{21}\right)^2 + \left(z_{21} + w_{21}\right)^2 = \left(\mathbf{x}_{21} + \mathbf{p}_{21}\right)^T \left(\mathbf{x}_{21} + \mathbf{p}_{21}\right)$$

[eq. 3.53]

where $\mathbf{p}_{21} = \begin{pmatrix} u_{21} \\ w_{21} \end{pmatrix}$



Without introducing the concept of shape functions



- From the definition of Green strain

$$\varepsilon = \frac{l_n^2 - l_0^2}{2l_0^2} = \frac{\left(\mathbf{x}_{21} + \mathbf{p}_{21}\right)^T \left(\mathbf{x}_{21} + \mathbf{p}_{21}\right) - \mathbf{x}_{21}^T \mathbf{x}_{21}}{2\mathbf{x}_{21}^T \mathbf{x}_{21}} = \frac{1}{4\alpha_0^2}\left(\mathbf{x}_{21}^T \mathbf{p}_{21} + \frac{1}{2}\mathbf{p}_{21}^T \mathbf{p}_{21}\right)$$

$$= \mathbf{b}_1^T \mathbf{p} + \frac{1}{2\alpha_0^2}\mathbf{p}^T \mathbf{A} \mathbf{p}$$

where

$$\mathbf{b}_1 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -x_{21} \\ x_{21} \\ -z_{21} \\ z_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{x}), \qquad \mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

[eq. 3.54-3.57]

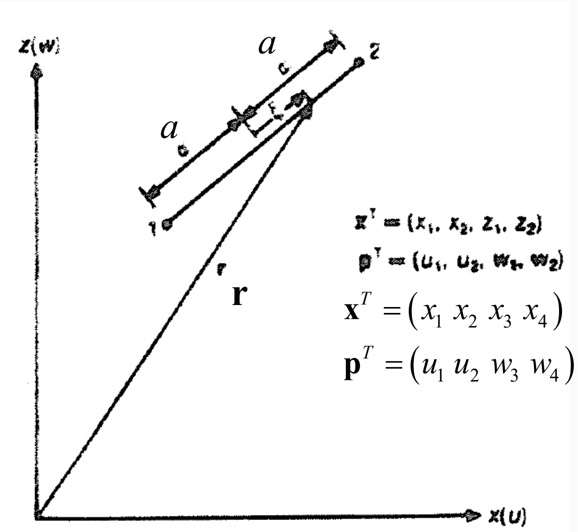
$$\varepsilon = \frac{l_n^2 - l_0^2}{2l_0^2} = \frac{\left(\mathbf{x}_{21} + \mathbf{p}_{21}\right)^T \left(\mathbf{x}_{21} + \mathbf{p}_{21}\right) - \mathbf{x}_{21}^T \mathbf{x}_{21}}{2\mathbf{x}_{21}^T \mathbf{x}_{21}} = \frac{1}{4\alpha_0^2}\left(\mathbf{x}_{21}^T \mathbf{p}_{21} + \frac{1}{2}\mathbf{p}_{21}^T \mathbf{p}_{21}\right)$$

$$= \mathbf{b}_1^T \mathbf{p} + \frac{1}{2\alpha_0^2}\mathbf{p}^T \mathbf{A} \mathbf{p}$$

- Increment of Green strain

$$\Delta\varepsilon = \left(\mathbf{b}_1 + \mathbf{b}_2\left(\mathbf{p}\right)\right)^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A} \Delta\mathbf{p} = \mathbf{b}(\mathbf{p})^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A}\Delta\mathbf{p}$$ [eq. 3. 58]

where $$\mathbf{b}_2 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -u_{21} \\ u_{21} \\ -w_{21} \\ w_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{p}) = \frac{1}{\alpha_0^2}\mathbf{A}\mathbf{p}$$ [eq. 3. 59]



- For a small virtual displacement,

$$\delta\varepsilon_v = \left(\mathbf{b}_1 + \mathbf{b}_2\left(\mathbf{p}\right)\right)^T \delta\mathbf{p}_v = \mathbf{b}(\mathbf{p})^T \delta\mathbf{p}_v$$ [eq. 3. 61]

3.3.2 Equilibrium and the internal force vector

"summation over all elements"

- Total internal virtual work should be summed over all elements: $\sum_e$

$$\sum_e \delta \mathbf{p}_v^T \mathbf{q}_i = \sum_e \int \sigma_G \delta \varepsilon_v dV_0 = \sum_e \mathbf{p}_v^T \int \sigma_G \mathbf{b} dV_0 \quad \text{[eq. 3.62]}$$

- The virtual internal work can be treated on each element individually and then assembled in regular FE scheme.

$$\mathbf{q}_i = \int \sigma_G \mathbf{b} dV_0 = 2\alpha_0 A_0 \sigma_G \mathbf{b} = 2\alpha_0 A_0 \sigma_G \left(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\right) = \frac{\sigma_G A_0}{2\alpha_0}\left(\mathbf{c}(\mathbf{x}) + \mathbf{c}(\mathbf{p})\right) = \mathbf{q}_{i1} + \mathbf{q}_{i2} \quad \text{[eq. 3.63]}$$

- The procedure for computing the internal forces, $\mathbf{q}_i$, from a set of nodal displacements, $\mathbf{p}$, is as follows:

1. Compute the **strain** from:

$$\varepsilon = \frac{l_n^2 - l_0^2}{2 l_0^2} = \mathbf{b}_1^T \mathbf{p} + \frac{1}{2\alpha_0^2} \mathbf{p}^T \mathbf{A} \mathbf{p}$$

2. Compute the **stress**, σ_G (here, constant over the element), for now assuming a linear material response from: $\sigma_G = E\varepsilon$

3. Compute the **internal forces**, $\mathbf{q}_i$, from [eq. 3.63] above.

3.3.3 The tangent stiffness matrix

- Tangent stiffness matrix

$$\mathbf{q}_i = \int \sigma_G \mathbf{b} dV_0 = 2\alpha_0 A_0 \sigma_G \mathbf{b} = 2\alpha_0 A_0 \sigma_G \left(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\right) = \frac{\sigma_G A_0}{2\alpha_0}\left(\mathbf{c}(\mathbf{x}) + \mathbf{c}(\mathbf{p})\right) = \mathbf{q}_{i1} + \mathbf{q}_{i2}$$

$$\mathbf{K}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{q}_i}{\partial \mathbf{p}} = 2\alpha_0 A_0 \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{p}} + 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \sigma_G$$ [eq. 3.64]

- First term:

$$\Delta\varepsilon = \left(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\right)^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A} \Delta\mathbf{p} = \mathbf{b}(\mathbf{p})^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A}\Delta\mathbf{p}$$

$$\frac{\partial \sigma_G}{\partial \mathbf{p}} = E\frac{\partial \varepsilon}{\partial \mathbf{p}} = E\left(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\right)^T = E\mathbf{b}(\mathbf{p})^T$$ [eq. 3.65]

$$2\alpha_0 A_0 \mathbf{b}\frac{\partial \sigma_G}{\partial \mathbf{p}} = 2E\alpha_0 A_0 \mathbf{b}\mathbf{b}^T = \mathbf{K}_{t1} + \mathbf{K}_{t2}$$ [eq. 3.66]

$$\mathbf{b}_1 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -x_{21} \\ x_{21} \\ -z_{21} \\ z_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{x}),$$

where

$$\mathbf{K}_{t1} = 2EA_0\alpha_0 \mathbf{b}_1 \mathbf{b}_1^T = \frac{EA_0}{8\alpha_0^3}\mathbf{c}(\mathbf{x})\mathbf{c}(\mathbf{x})^T$$ [eq. 3.67]

$$\mathbf{b}_2 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -u_{21} \\ u_{21} \\ -w_{21} \\ w_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{p}) = \frac{1}{\alpha_0^2}\mathbf{A}\mathbf{p}$$

$$\mathbf{K}_{t2} = 2EA_0\alpha_0\left(\mathbf{b}_1\mathbf{b}_2^T + \mathbf{b}_2\mathbf{b}_1^T + \mathbf{b}_2\mathbf{b}_2^T\right) = \mathbf{K}_{t2a} + \mathbf{K}_{t2a}^T + \mathbf{K}_{t2b}$$ [eq. 3.68]

- **Tangent stiffness matrix**

$$\mathbf{K}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{q}_i}{\partial \mathbf{p}} = 2\alpha_0 A_0 \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{p}} + 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \sigma_G$$

[eq. 3.64]

$$\mathbf{b}_2 = \frac{1}{4\alpha_0^2} \begin{pmatrix} -u_{21} \\ u_{21} \\ -w_{21} \\ w_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2} \mathbf{c}(\mathbf{p}) = \frac{1}{\alpha_0^2} \mathbf{A}\mathbf{p}$$

- Second term:

$$\mathbf{K}_{t\sigma} = 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \sigma_G = 2\alpha_0 A_0 \frac{\partial \mathbf{b}_2}{\partial \mathbf{p}} \sigma_G = \frac{2\alpha_0 A_0 \sigma_G}{\alpha_0} \mathbf{A} = \frac{\alpha_0 A_0 \sigma_G}{2\alpha_0} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

[eq. 3.69]

$$\mathbf{A} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{K}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{q}_i}{\partial \mathbf{p}} = 2\alpha_0 A_0 \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{p}} + 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \sigma_G$$

[eq. 3.64]

- Each stiffness term can be expanded as:

$$\mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t2} + \mathbf{K}_{t\sigma} = \mathbf{K}_{t1} + \mathbf{K}_{t2a} + \mathbf{K}_{t2b} + \mathbf{K}_{t\sigma}$$

[eq. 3.73]

$$\mathbf{K}_{t1} = \frac{EA_0}{8\alpha_0^3} \begin{bmatrix} x_{21}^2 & -x_{21}^2 & x_{21}z_{21} & -x_{21}z_{21} \\ -x_{21}^2 & x_{21}^2 & -x_{21}z_{21} & x_{21}z_{21} \\ x_{21}z_{21} & -x_{21}z_{21} & z_{21}^2 & -z_{21}^2 \\ -x_{21}z_{21} & x_{21}z_{21} & -z_{21}^2 & z_{21}^2 \end{bmatrix} = \frac{EA_0}{8\alpha_0^3} \mathbf{c}(\mathbf{x})\mathbf{c}(\mathbf{x})^T$$

[eq. 3.70]

"Linear stiffness matrix – for linear material"

$$\mathbf{K}_{t2a} = 2EA_0\alpha_0\mathbf{b}_1\mathbf{b}_2^T = \frac{EA_0}{8\alpha_0^3}\begin{bmatrix} x_{21}u_{21} & -x_{21}u_{21} & z_{21}u_{21} & -z_{21}u_{21} \\ -x_{21}u_{21} & x_{21}u_{21} & -z_{21}u_{21} & z_{21}u_{21} \\ z_{21}u_{21} & -z_{21}u_{21} & z_{21}w_{21} & -z_{21}w_{21} \\ -z_{21}u_{21} & z_{21}u_{21} & -z_{21}w_{21} & z_{21}w_{21} \end{bmatrix}$$

[eq. 3.71]

$$\mathbf{K}_{t2b} = \frac{EA_0}{8\alpha_0^3}\begin{bmatrix} u_{21}^2 & -u_{21}^2 & u_{21}w_{21} & -u_{21}w_{21} \\ -u_{21}^2 & u_{21}^2 & -u_{21}w_{21} & u_{21}w_{21} \\ u_{21}w_{21} & -u_{21}w_{21} & w_{21}^2 & -w_{21}^2 \\ -u_{21}w_{21} & u_{21}w_{21} & -w_{21}^2 & w_{21}^2 \end{bmatrix}$$

"Initial displacement matrix"

[eq. 3.72]

$$\mathbf{K}_{t\sigma} = 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}}\sigma_G = 2\alpha_0 A_0 \frac{\partial \mathbf{b}_2}{\partial \mathbf{p}}\sigma_G = \frac{2\alpha_0 A_0 \sigma_G}{\alpha_0}\mathbf{A} = \frac{\alpha_0 A_0 \sigma_G}{2\alpha_0}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

"Geometric or initial stress matrix"

[eq. 3.69]

The internal force vector, tangent stiffness matrics can be incorporated into the computer program adopted in the shallow truss theory (in Ch. 2). We call this procedure as "element development".
The formulation here is known as "total Lagrangian" because all measures are related to the "initial configuration".

3.3.4 Using shape functions

- While shape functions are unnecessary for the current simple elements, with a view to more complex elements, it is useful to apply them.

- The incremental vector along A_0B_0 and A_nB_n:

"with non-dimensional coordinates"

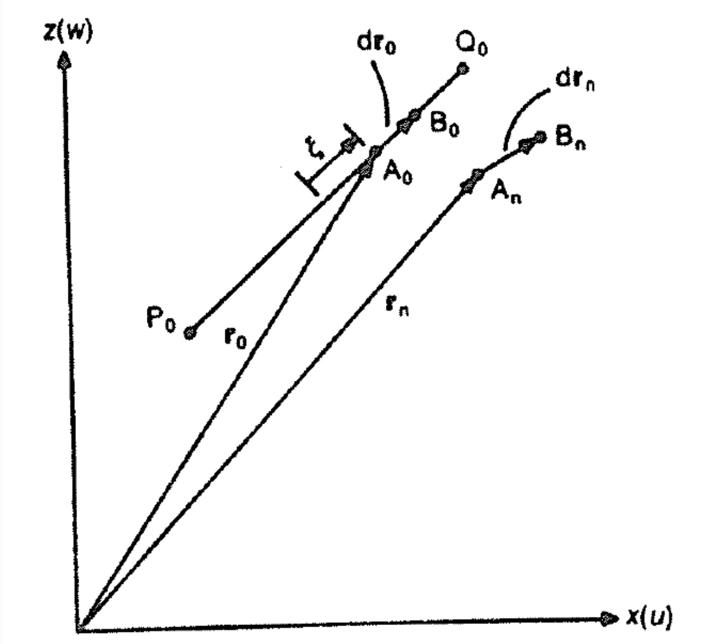
$$d\mathbf{r}_0 = \frac{d\mathbf{r}_0}{d\xi} d\xi$$ [eq. 3.74]

$$d\mathbf{r}_n = \frac{d(\mathbf{r}_0 + \mathbf{u})}{d\xi} d\xi$$ [eq. 3.75]

$$dr_n = \| \mathrm{d}\,\mathbf{r}_n \| = \left(\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{r}_0}{d\xi} + 2\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{u}}{d\xi} + \frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi} \right)^{1/2} d\xi = \alpha_n d\xi$$

[eq. 3.76]

$$dr_0 = \| \mathrm{d}\,\mathbf{r}_0 \| = \left(\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{r}_0}{d\xi} \right)^{1/2} d\xi = \alpha_0 d\xi$$ [eq. 3.77]



$$\Rightarrow \quad \varepsilon = \frac{dr_n^2 - dr_0^2}{2dr_0^2} = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi}$$ [eq. 3.78, 3.79]

"for Green strain"

$$\varepsilon = \frac{dr_n^2 - dr_0^2}{2dr_0^2} = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi}$$

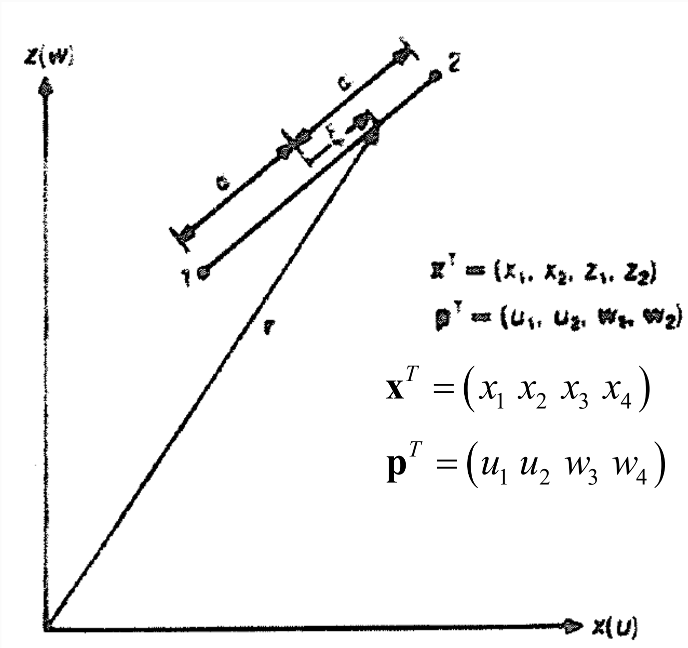
[eq. 3.78, 3.79]

An increment related to a displacement increment

$$\Delta\varepsilon = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\Delta\mathbf{u}}{d\xi} + \frac{1}{\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\Delta\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\Delta\mathbf{u}^T}{d\xi}\frac{d\Delta\mathbf{u}}{d\xi}$$

[eq. 3.80]

- For the displacement-based finite element method, shape functions are used to relate both the geometry and the displacements to nodal values:



$$\mathbf{r}_0 = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{bmatrix} \mathbf{h}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{h}^T \end{bmatrix} \mathbf{x} = \mathbf{H}\mathbf{x}$$

[eq. 3.81]

$$\mathbf{u} = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} \mathbf{h}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{h}^T \end{bmatrix} \mathbf{p} = \mathbf{H}\mathbf{p}$$

[eq. 3.82]

"Linear relation, or linear shape function"

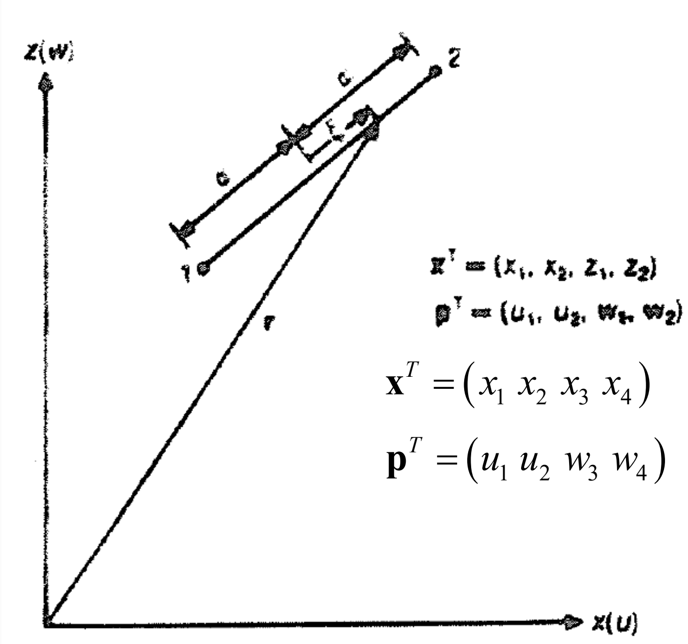
"Isoparametric element"

$$\varepsilon = \frac{dr_n^2 - dr_0^2}{2dr_0^2} = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi}$$

- Differentiations:

$$\frac{d\mathbf{r}_0}{d\xi} = \mathbf{r}_{0\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{h}_\xi^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{h}_\xi^T \end{bmatrix}\mathbf{x} = \mathbf{H}_\xi \mathbf{x}$$ [eq. 3.83]

$$\frac{d\mathbf{u}}{d\xi} = \mathbf{u}_\xi = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\mathbf{p} = \begin{bmatrix} \mathbf{h}_\xi^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{h}_\xi^T \end{bmatrix}\mathbf{p} = \mathbf{H}_\xi \mathbf{p}$$ [eq. 3.84]



$$\mathbf{x}^T = \left(x_1\ x_2\ x_3\ x_4\right)$$

$$\mathbf{p}^T = \left(u_1\ u_2\ w_3\ w_4\right)$$

➡ $$\varepsilon = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi}$$ [eq. 3.79]

$$= \left(\frac{1}{\alpha_0^2}\mathbf{r}_{0\xi}^T\mathbf{H}_\xi\right)\mathbf{p} + \left(\frac{1}{2\alpha_0^2}\mathbf{p}^T\mathbf{H}_\xi^T\mathbf{H}_\xi\mathbf{p}\right) = \mathbf{b}_1^T\mathbf{p} + \frac{1}{2\alpha_0^2}\mathbf{p}^T\mathbf{A}\mathbf{p}$$

[eq. 3.85]

$$\Delta\varepsilon = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\Delta\mathbf{u}}{d\xi} + \frac{1}{\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\Delta\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\Delta\mathbf{u}^T}{d\xi}\frac{d\Delta\mathbf{u}}{d\xi}$$ [eq. 3.80]

c.f

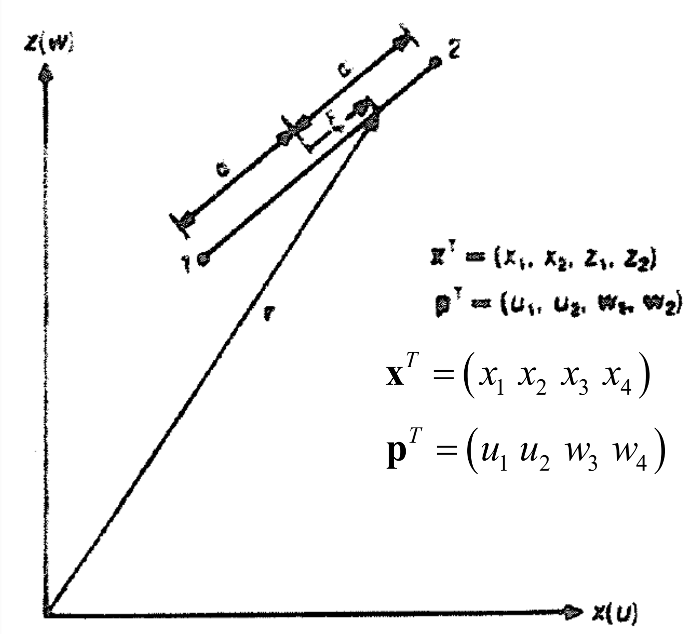
$$\Delta\varepsilon = \left(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\right)^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T\mathbf{A}\Delta\mathbf{p} = \mathbf{b}(\mathbf{p})^T\Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T\mathbf{A}\Delta\mathbf{p}$$

$\mathbf{r}_0 = \mathbf{H}\mathbf{x}$

$\mathbf{u} = \mathbf{H}\mathbf{p}$

$$\Rightarrow \Delta\varepsilon = \left(\frac{1}{\alpha_0^2}\mathbf{r}_{0\xi}^T\mathbf{H}_\xi\right)\Delta\mathbf{p} + \left(\frac{1}{\alpha_0^2}\mathbf{u}_\xi^T\mathbf{H}_\xi\right)\Delta\mathbf{p} + \left(\frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T\mathbf{H}_\xi^T\mathbf{H}_\xi\Delta\mathbf{p}\right)$$

$$= \mathbf{b}_1^T\Delta\mathbf{p} + \mathbf{b}_2^T\Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T\mathbf{A}\Delta\mathbf{p}$$ [eq. 3.86]



$\mathbf{x}^T = (x_1\ x_2\ x_3\ x_4)$

$\mathbf{p}^T = (u_1\ u_2\ w_3\ w_4)$

- Internal force

$$\mathbf{q}_i = \int(\mathbf{b}_1 + \mathbf{b}_2)\sigma_G dV_0 = 2A_0\alpha_0(\mathbf{b}_1 + \mathbf{b}_2)\sigma_G$$

- Tangent stiffness matrix

"Total Lagrangian"

$$\mathbf{K}_t = \frac{\partial\mathbf{q}_i}{\partial\mathbf{p}} = 2A_0\alpha_0\left(\mathbf{b}\frac{\partial\sigma_G}{\partial\mathbf{p}} + \frac{\partial\mathbf{b}}{\partial\mathbf{p}}\sigma_G\right)$$ [eq. 3.88]

where $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$

3.3.5 Alternative expressions involving updated coordinates

- The updated coordinates, $\mathbf{x}'$, can be expressed as:

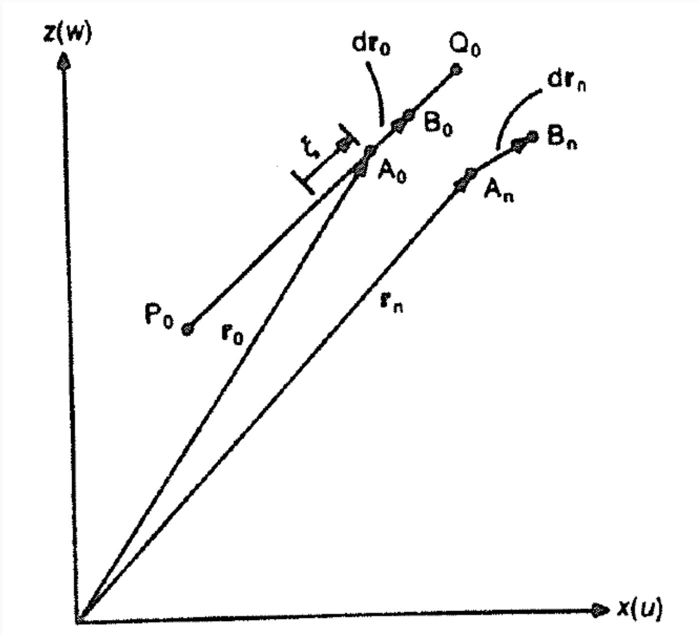
$$\mathbf{x}' = \mathbf{x} + \mathbf{p} \quad \text{or} \quad \mathbf{r}_n = \mathbf{r}_0 + \mathbf{u}$$ [eq 3.89]

"Nodal variables" *"General points"*

- Then, the strain increment of eq 3.58

$$\Delta\varepsilon = \left(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})\right)^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A}\Delta\mathbf{p} = \mathbf{b}(\mathbf{p})^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A}\Delta\mathbf{p}$$

[eq. 3. 58]



$$\Rightarrow \quad \Delta\varepsilon = \mathbf{b}(\mathbf{x}')^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A}\Delta\mathbf{p}$$

$$= \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{x}')^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T \mathbf{A}\Delta\mathbf{p}$$ [eq 3.90]

where

$$\mathbf{b}_1 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -x_{21} \\ x_{21} \\ -z_{21} \\ z_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{x})$$ [eq 3.56]

and

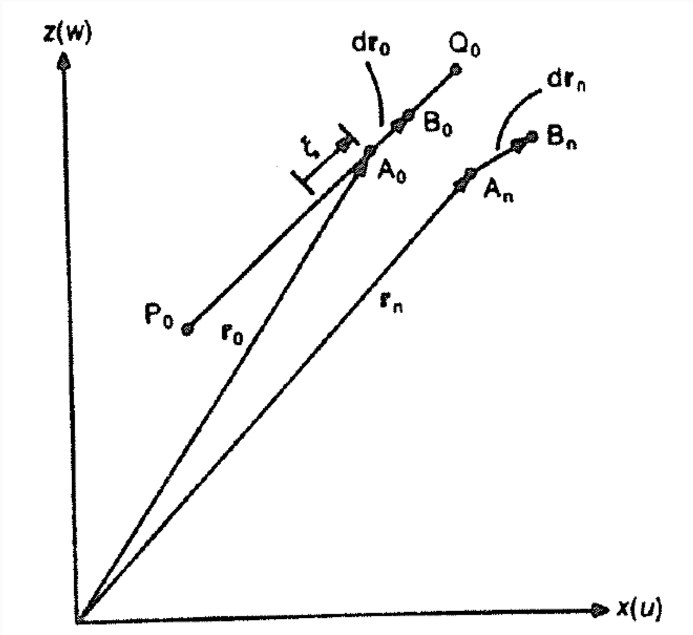
$$\mathbf{b}_2 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -u_{21} \\ u_{21} \\ -w_{21} \\ w_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{p}) = \frac{1}{\alpha_0^2}\mathbf{A}\mathbf{p}$$ [eq 3.59]

- The virtual strain can be expressed as:

$$\Delta\varepsilon = \mathbf{b}(\mathbf{x}')^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T\mathbf{A}\Delta\mathbf{p}$$
$$= \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{x}')^T \Delta\mathbf{p} + \frac{1}{2\alpha_0^2}\Delta\mathbf{p}^T\mathbf{A}\Delta\mathbf{p} \quad \text{[eq 3.90]}$$

$$\Rightarrow \quad \delta\varepsilon_v = \mathbf{b}_1(\mathbf{x}')^T \delta\mathbf{p}_v \quad \text{[eq 3.91]}$$

$$\mathbf{q}_i = \int \sigma_G \mathbf{b}_1(\mathbf{x}')\, dV_0 = 2A_0\alpha_0\sigma_G\mathbf{b}_1(\mathbf{x}')$$
$$= \frac{A_0\sigma_G}{2\alpha_0}\mathbf{c}(\mathbf{x}') = \frac{A_0\sigma_G}{2\alpha_0}\begin{pmatrix} -x'_{21} \\ x'_{21} \\ -z'_{21} \\ z'_{21} \end{pmatrix} = \mathbf{q}'_{i1} \quad \text{[eq 3.92]}$$



- There is no $\mathbf{q}'_{i2}$ term compared to:

$$\mathbf{q}_i = \int \sigma_G \mathbf{b}\, dV_0 = 2\alpha_0 A_0 \sigma_G \mathbf{b} = 2\alpha_0 A_0 \sigma_G(\mathbf{b}_1 + \mathbf{b}_2(\mathbf{p})) = \frac{\sigma_G A_0}{2\alpha_0}(\mathbf{c}(\mathbf{x}) + \mathbf{c}(\mathbf{p})) = \mathbf{q}_{i1} + \mathbf{q}_{i2} \quad \text{[eq. 3.63]}$$

- Again using updated coordinates, $\mathbf{K}'_{t1}$ as well as $\mathbf{K}'_{t2}$ can be combined to give:

$$\mathbf{K}_{t1} = 2EA_0\alpha_0\mathbf{b}_1\mathbf{b}_1^T = \frac{EA_0}{8\alpha_0^3}\mathbf{c}(\mathbf{x})\mathbf{c}(\mathbf{x})^T \quad \text{[eq. 3.67]}$$

$$\mathbf{K}_{t2} = 2EA_0\alpha_0\left(\mathbf{b}_1\mathbf{b}_2^T + \mathbf{b}_2\mathbf{b}_1^T + \mathbf{b}_2\mathbf{b}_2^T\right) = \mathbf{K}_{t2a} + \mathbf{K}_{t2a}^T + \mathbf{K}_{t2b} \quad \text{[eq. 3.68]}$$

$$\mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t2} + \mathbf{K}_{t\sigma} = \mathbf{K}_{t1} + \mathbf{K}_{t2a} + \mathbf{K}_{t2b} + \mathbf{K}_{t\sigma} \quad \text{[eq. 3.73]}$$

$$\Rightarrow \quad \mathbf{K}'_{t1} = 2EA_0\alpha_0\mathbf{b}_1(\mathbf{x}')\mathbf{b}_1(\mathbf{x}')^T = \frac{EA_0}{8\alpha_0^3}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T$$

$$= \mathbf{K}_{t1} + \mathbf{K}_{t2a} + \mathbf{K}_{t2a}^T + \mathbf{K}_{t2b} \quad \text{[eq. 3.93]}$$

- While the geometric stiffness matrix is unaltered.

$$\mathbf{K}_{t\sigma} = 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}}\sigma_G = 2\alpha_0 A_0 \frac{\partial \mathbf{b}_2}{\partial \mathbf{p}}\sigma_G = \frac{2A_0\sigma_G}{\alpha_0}\mathbf{A} = \frac{A_0\sigma_G}{2\alpha_0}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{[eq. 3.69]}$$

- The introduction of updated coordinates can simply be considered as an alternative way of expressing the 'Green-strain system' which avoids $\mathbf{b}_2(\mathbf{p})$ terms.

3.3.6 An updated Lagrangian formulation

- The procedure in section 3.3.5 is alternative way of writing the **total Lagrangian formulation**, but it is closely related to an **updated Lagrangian formulation**.

- Before proceeding to the next increment or iteration, the 2nd Piola-Kirchhoff stresses must be converted to 'true stresses' relating to new configuration so that σ_G,

$$'\sigma' = \frac{A_0 l_n}{A_n l_0}\sigma_G = \frac{A_0 \alpha_n}{A_n \alpha_0}\sigma_G \qquad \text{[eq. 3.30]}$$

- The internal force vector is given by:

$$\mathbf{q}_i = \int '\sigma' \mathbf{b}_1(\mathbf{x}')\, dV_n = \frac{A_n '\sigma'}{2\alpha_n}\mathbf{c}(\mathbf{x}') \qquad \text{[eq. 3. 96]}$$

$$\mathbf{K}_{t\sigma} = 2\alpha_0 A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \sigma_G = 2\alpha_0 A_0 \frac{\partial \mathbf{b}_2}{\partial \mathbf{p}} \sigma_G = \frac{2A_0\sigma_G}{\alpha_0}\mathbf{A} = \frac{A_0\sigma_G}{2\alpha_0}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

[eq. 3.69]

$$\mathbf{K}_{t1} = \frac{EA_0}{8\alpha_0^3}\begin{bmatrix} x_{21}^2 & -x_{21}^2 & x_{21}z_{21} & -x_{21}z_{21} \\ -x_{21}^2 & x_{21}^2 & -x_{21}z_{21} & x_{21}z_{21} \\ x_{21}z_{21} & -x_{21}z_{21} & z_{21}^2 & -z_{21}^2 \\ -x_{21}z_{21} & x_{21}z_{21} & -z_{21}^2 & z_{21}^2 \end{bmatrix} = \frac{EA_0}{8\alpha_0^3}\mathbf{c}(\mathbf{x})\mathbf{c}(\mathbf{x})^T$$

[eq. 3.70]

$$\mathbf{K}_{t2a} = 2EA_0\alpha_0\mathbf{b}_1\mathbf{b}_2^T = \frac{EA_0}{8\alpha_0^3}\begin{bmatrix} x_{21}u_{21} & -x_{21}u_{21} & z_{21}u_{21} & -z_{21}u_{21} \\ -x_{21}u_{21} & x_{21}u_{21} & -z_{21}u_{21} & z_{21}u_{21} \\ z_{21}u_{21} & -z_{21}u_{21} & z_{21}w_{21} & -z_{21}w_{21} \\ -z_{21}u_{21} & z_{21}u_{21} & -z_{21}w_{21} & z_{21}w_{21} \end{bmatrix}$$

[eq. 3.71]

$$\mathbf{K}_{t2b} = \frac{EA_0}{8\alpha_0^3}\begin{bmatrix} u_{21}^2 & -u_{21}^2 & u_{21}w_{21} & -u_{21}w_{21} \\ -u_{21}^2 & u_{21}^2 & -u_{21}w_{21} & u_{21}w_{21} \\ u_{21}w_{21} & -u_{21}w_{21} & w_{21}^2 & -w_{21}^2 \\ -u_{21}w_{21} & u_{21}w_{21} & -w_{21}^2 & w_{21}^2 \end{bmatrix}$$

[eq. 3.72]

$$\Rightarrow \quad \mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t2} + \mathbf{K}_{t\sigma} = \mathbf{K}_{t1} + \mathbf{K}_{t2a} + \mathbf{K}_{t2b} + \mathbf{K}_{t\sigma}$$

[eq. 3.73]

- In the updated Lagrangian formulation,

$$\mathbf{p} = \begin{pmatrix} u_1 \\ u_2 \\ w_1 \\ w_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \mathbf{K}_{t2a} = \mathbf{K}_{t2b} = 0 \quad \text{and} \quad \mathbf{K}_t = \frac{E'A_n}{8\alpha_n^2}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T + \frac{2A_n'\sigma'}{\alpha_n}\mathbf{A}$$

[eq. 3. 97]

- The first term corresponds to (3.94) if,

$$\frac{E'A_n}{\alpha_n^3} = \frac{EA_0}{\alpha_0^3}$$

[eq. 3. 98]

$$\mathbf{K}'_{t1} = 2EA_0\alpha_0\mathbf{b}_1(\mathbf{x}')\mathbf{b}_1(\mathbf{x}')^T = \frac{EA_0}{8\alpha_0^3}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T$$
$$= \mathbf{K}_{t1} + \mathbf{K}_{t2a} + \mathbf{K}_{t2a}^T + \mathbf{K}_{t2b}$$

[eq. 3. 94]

$$\varepsilon = \frac{l_n^2 - l_0^2}{2l_0^2} = \frac{\left(\mathbf{x}_{21} + \mathbf{p}_{21}\right)^T \left(\mathbf{x}_{21} + \mathbf{p}_{21}\right) - \mathbf{x}_{21}^T \mathbf{x}_{21}}{2\mathbf{x}_{21}^T \mathbf{x}_{21}} = \frac{1}{4\alpha_0^2}\left(\mathbf{x}_{21}^T \mathbf{p}_{21} + \frac{1}{2}\mathbf{p}_{21}^T \mathbf{p}_{21}\right)$$

$$= \mathbf{b}_1^T \mathbf{p} + \frac{1}{2\alpha_0^2}\mathbf{p}^T \mathbf{A}\mathbf{p}$$ [eq. 3.54-3.57]

where

$$\mathbf{b}_1 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -x_{21} \\ x_{21} \\ -z_{21} \\ z_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{x}), \quad \mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{b}_2 = \frac{1}{4\alpha_0^2}\begin{pmatrix} -u_{21} \\ u_{21} \\ -w_{21} \\ w_{21} \end{pmatrix} = \frac{1}{4\alpha_0^2}\mathbf{c}(\mathbf{p}) = \frac{1}{\alpha_0^2}\mathbf{A}\mathbf{p}$$ [eq. 3. 59]

$\mathbf{x}' = \mathbf{x} + \mathbf{p}$ or $\mathbf{r}_n = \mathbf{r}_0 + \mathbf{u}$ [eq 3.89]

- In the updated Lagrangian formulation,

$$\varepsilon = \mathbf{b}_1\left(\mathbf{x}'\right)^T \mathbf{p} - \frac{1}{2\alpha_0^2}\mathbf{p}^T \mathbf{A}\mathbf{p} = \frac{1}{4\alpha_0^2}\mathbf{c}\left(\mathbf{x}'\right)^T \mathbf{p} - \frac{1}{2\alpha_0^2}\mathbf{p}^T \mathbf{A}\mathbf{p}$$ [eq. 3. 99]

$$\varepsilon = \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_0^T}{d\xi}\frac{d\mathbf{u}}{d\xi} + \frac{1}{2\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi}$$ [eq. 3.79]

$$= \frac{1}{\alpha_0^2}\frac{d\mathbf{r}_n^T}{d\xi}\frac{d\mathbf{u}}{d\xi} - \frac{1}{2\alpha_0^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi}$$ [eq. 3.100]

- Above equations are not fully related to the current configuration because of α_0

- However, we can generalize Almansi strain:

$$\varepsilon_A = \frac{1}{2}\left(1-\left(\frac{l_n}{l_0}\right)^{-2}\right) = \frac{l_n^2 - l_0^2}{2l_n^2} \quad \text{[eq. 3.41]}$$

$$\varepsilon_A = \frac{dr_n^2 - dr_0^2}{2dr_n^2} = \frac{\alpha_n^2 - \alpha_0^2}{2\alpha_n^2} \quad \text{[eq. 3.101]}$$

$$= \frac{1}{\alpha_n^2}\frac{d\mathbf{r}_n^T}{d\xi}\frac{d\mathbf{u}}{d\xi} - \frac{1}{2\alpha_n^2}\frac{d\mathbf{u}^T}{d\xi}\frac{d\mathbf{u}}{d\xi} \quad \text{[eq. 3.102]}$$

- Further discussion on the total and updated Lagrangian formulations will be given in a continuum context in Ch5.

- It is emphasized that section 3.3.5 involves **updated but unrotated coordinates**.

Thank you!