# Gas Discharge Fundamentals

**Fall, 2017**


**Kyoung-Jae Chung**

***Department of Nuclear Engineering***

**Seoul National University**

# Thermionic emission

- Theoretically, the current density, $J_e$, of electrons emitted from a cathode at an absolute temperature, $T\,(\mathrm{K})$, is given by:

$$J_e = AT^2 e^{-e\varphi/kT}$$

Richardson-Dushman equation

Work function (eV)

- Schottky effect (field enhanced thermionic emission): For a constant temperature, the current still slowly increases with the applied extraction potential by lowering the surface barrier.

$$J_e = AT^2 \exp\left\{-\frac{e}{kT}\left(\varphi - \sqrt{\frac{eE}{4\pi\varepsilon_0}}\right)\right\}$$

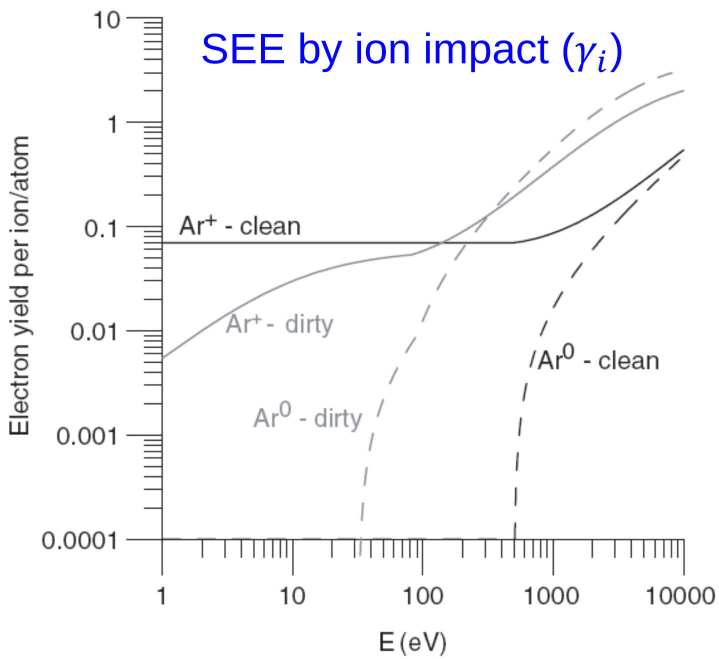
- Space charge limited current: When the external electric field is not sufficiently high, the emitted electron current density is limited by space charge. The space charge limited current is determined by the extraction voltage and independent of the cathode current.
- Emission (source) limited current: When the external electric field is sufficiently high, the saturation emission electron current is determined by the cathode temperature.

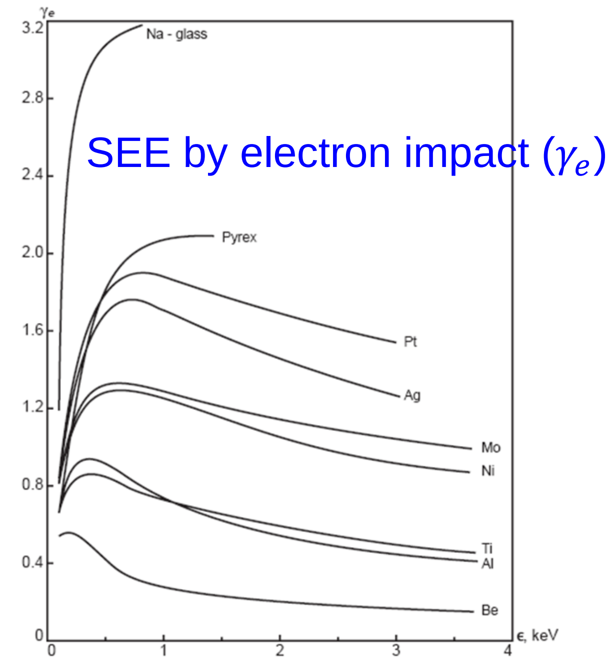
# Secondary electron emission (SEE)

- Electrons, ions or neutral particles cause secondary electron emission through different physical processes.
- The secondary electron emission is described by the secondary emission coefficient, $\gamma$, which is defined as the number of secondaries produced per an incident primary for normal incidence.

$$\gamma = \frac{n_e}{n_p}$$

- Since $\gamma$ depends on the state of the surface, e.g. depending on the surface film, there is an extremely wide variation in the measured values.



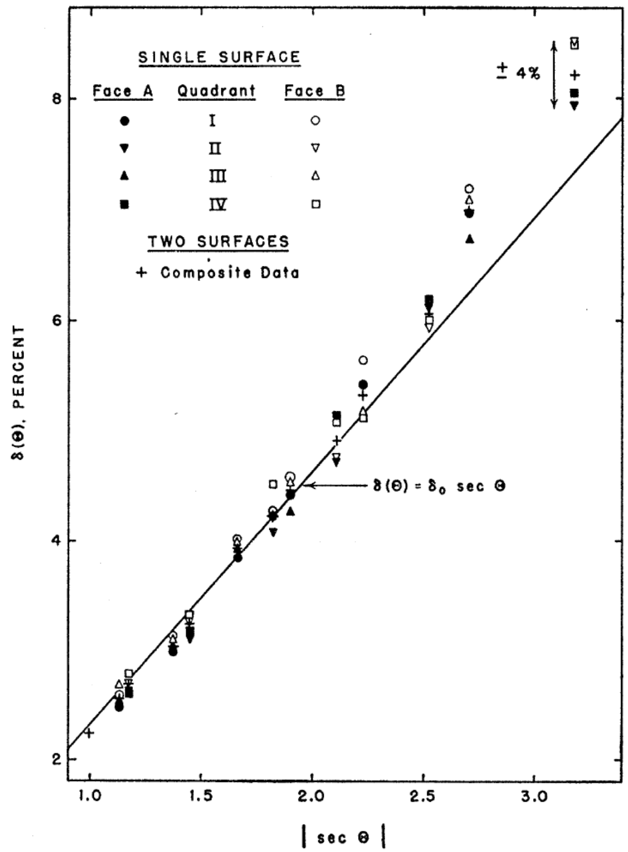
# Secondary electron emission (SEE)

- The coefficient of the secondary emission for a proton incident on a gold-plated tungsten filament with an energy of 1.6-15 keV is given by

$$\gamma_i = (1.06 \pm 0.01) E_p^{1/2}$$

$E_p$ is the proton energy in keV

- The dependence of SEE on the incidence angle is given approximately by a $\sec\theta$ law.
- SEE for an ion of a diatomic molecule is about double of that for an atomic ion with half the energy of the molecular ion.
- The energy distribution of the secondary electrons is almost Maxwellian. The mean energy of secondary electrons is about 5 eV. The majority of electrons have energies less than 20 eV.
- The angular distribution of the secondary electrons approximates a cosine law.



# Surface ionization (thermal ionization)

- When an atom or molecule is adsorbed on a metal surface for a time long enough to reach thermodynamic equilibrium, the valence electron level of the adsorbed atom is sufficiently broadened and surface work function is modified, resulting in the valence electron moving between the adsorbed atom and the surface.
- When the surface temperature is hot enough to desorb the particles, the evaporated particles can be in the state of ions (positive of negative) or neutrals. This ionization phenomenon is called the positive or negative surface ionization.
- For a thermodynamic equilibrium process, the degree of positive surface ionization ($\alpha$) as a function of surface temperature is given by the Saha-Langmuir equation:

$$\alpha = \frac{n_i}{n_A} = G \exp\left\{-\frac{e(U_i - \varphi)}{kT}\right\} = G \exp\left\{\frac{W - E^i}{kT}\right\}$$

- Lowering of ionization potential by external electric field similar to Schottky effect:

$$\alpha = G \exp\left\{-\frac{e}{kT}\left(U_i - \varphi - \sqrt{\frac{eE}{4\pi\varepsilon_0}}\right)\right\}$$

# Surface ionization

- The surface ionization efficiency ($\beta$) is given by:

$$\beta = \frac{n_i}{n} = \frac{n_i}{n_i + n_A} = \frac{1}{1+\alpha^{-1}} = \frac{1}{1+\frac{1}{G}\exp\left\{\frac{e(U_i-\varphi)}{kT}\right\}}$$

- For $\varphi < U_i$ and $\alpha \ll 1$,

$$n_i = \beta n \approx \alpha n = nG\exp\left\{-\frac{e(U_i-\varphi)}{kT}\right\}$$

For $\alpha \ll 1$

- For $\varphi > U_i$ and $e(\varphi - U_i) \gg kT$, the majority of evaporated atoms will be ionized ($\beta \approx 1$)

$$n_i \approx n$$

For $\alpha \gg 1$

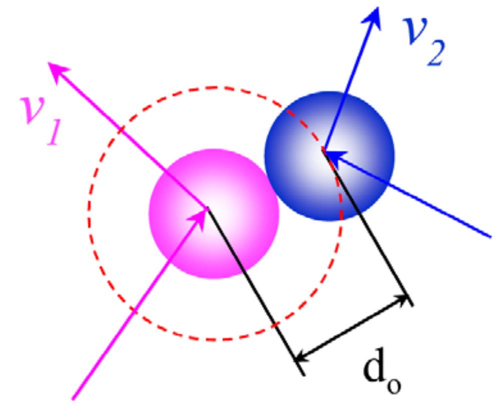
**Tab. 2.1.** Calculated values of $\alpha$ for some alkali-metals on a W surface.

| Adsorbate | $E^i$ (eV) | $W - E^i$ (eV) | $T$(K) | | | |
|---|---|---|---|---|---|---|
| | | | 1000 | 1500 | 2000 | 2500 |
| Cs | 3.88 | 0.64 | 790 | 72 | 19.9 | 9.8 |
| K | 4.32 | 0.20 | 6.3 | 2.16 | 1.60 | 1.27 |
| Na | 5.12 | −0.60 | $5 \times 10^{-4}$ | $5 \times 10^{-3}$ | $1.58 \times 10^{-2}$ | $3.16 \times 10^{-2}$ |
| Li | 5.40 | −0.88 | $1.8 \times 10^{-5}$ | $5.5 \times 10^{-4}$ | $3.0 \times 10^{-3}$ | $8.4 \times 10^{-3}$ |

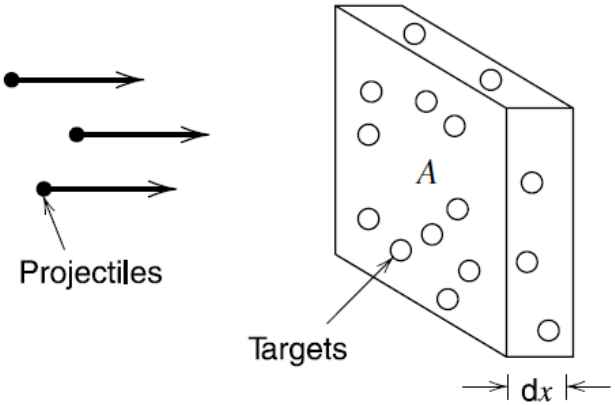
# Collision parameters

- Let $dn$ be the number of incident particles per unit volume at $x$ that undergo an "interaction" with the target particles within a differential distance $dx$, removing them from the incident beam. Clearly, $dn$ is proportional to $n$, $n_g$, and $dx$ for infrequent collisions within $dx$.

$$dn = -\sigma n n_g dx \qquad d\boldsymbol{\Gamma} = -\sigma\boldsymbol{\Gamma} n_g dx$$




- The collided flux: $\Gamma(\mathrm{x}) = \Gamma_0\left(1 - e^{-x/\lambda}\right)$
- Mean free path: $\lambda = \dfrac{1}{n_g\sigma}$
- Mean collision time: $\tau = \dfrac{\lambda}{v}$
- Collision frequency: $\nu = n_g\sigma v = n_g K$
- Rate constant: $K = \sigma v$

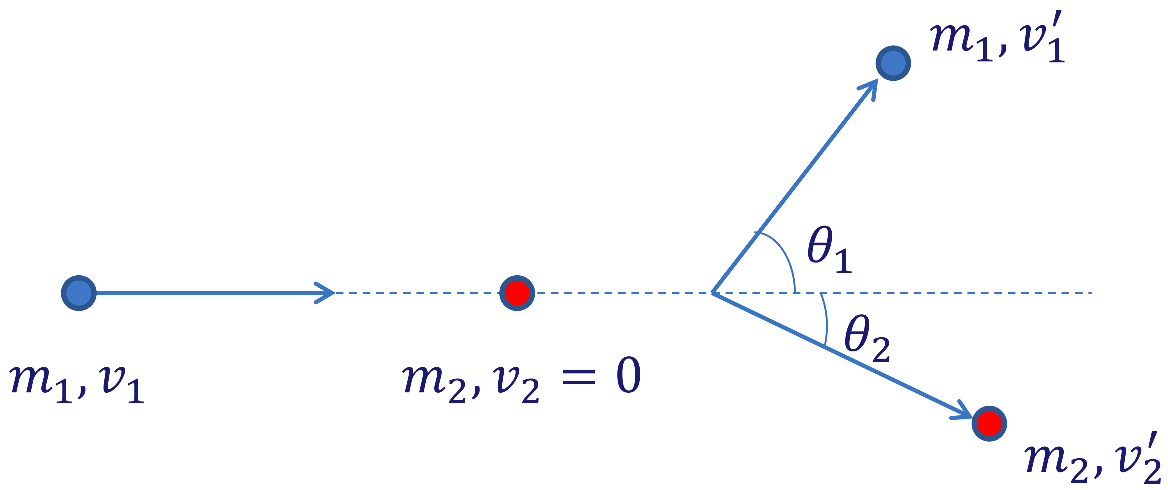


# Elastic and inelastic collisions

- Collisions conserve momentum and energy: the total momentum and energy of the colliding particles after collision are equal to that before collision.
- Electrons and fully stripped ions possess only kinetic energy. Atoms and partially stripped ions have internal energy level structures and can be excited, de-excited, or ionized, corresponding to changes in potential energy.
- It is the total energy, which is the sum of the kinetic and potential energy, that is conserved in a collision.

- Elastic: the sum of kinetic energies of the collision partners are conserved.
- Inelastic: the sum of kinetic energies are not conserved. ionization and excitation. the sum of kinetic energies after collision is less than that before collision.
- Super-elastic: the sum of kinetic energies are increased after collision. de-excitation.

# Elastic collision: energy transfer




- The conservation of momentum and energy gives

$$m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2 \qquad 0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2$$

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

- Eliminate $v_1'$ and $\theta_1$, then

$$\frac{1}{2} m_2 v_2'^2 = \frac{1}{2} m_1 v_1^2 \frac{4 m_1 m_2}{(m_1 + m_2)^2} \cos^2\theta_2$$

- The fraction of energy lost by the projectile:

$$\zeta_L = \frac{4 m_1 m_2}{(m_1 + m_2)^2} \cos^2\theta_2$$

# Energy transfer by elastic collision

- Energy transfer rate (averaging over scattering angle)

$$\langle \zeta_L \rangle = \frac{4m_1 m_2}{(m_1+m_2)^2}\langle \cos^2\theta_2 \rangle = \frac{2m_1 m_2}{(m_1+m_2)^2}$$

- Electron-electron collision ($m_1 = m_2$) : effective energy transfer

$$\langle \zeta_L \rangle = \frac{2m_1 m_2}{(m_1+m_2)^2} = \frac{1}{2}$$

**: Quickly thermalized**

- Electron-ion or electron-neutral collision ($m_1 \ll m_2$)

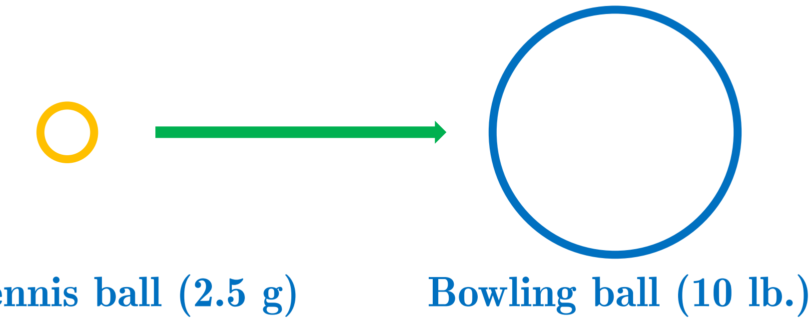
$$\langle \zeta_L \rangle = \frac{2m_1 m_2}{(m_1+m_2)^2} \approx \frac{2m_1}{m_2} \approx 10^{-4}$$

**: Hard to be thermalized**

- Therefore, in weakly ionized plasma

$$T_e \gg T_i \approx T_n$$

**: Non-equilibrium**



# Actual velocity dependence of elastic e-n collision

- Probability of collision: the average number of collisions in 1 cm of path through a gas at 1 Torr at 273K

$$\nu_{el} = v p_0 P_c$$

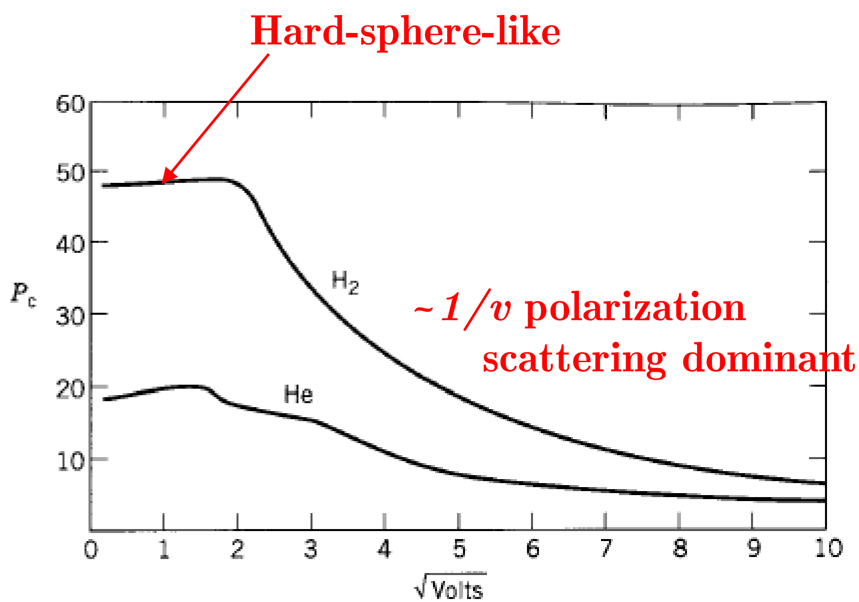
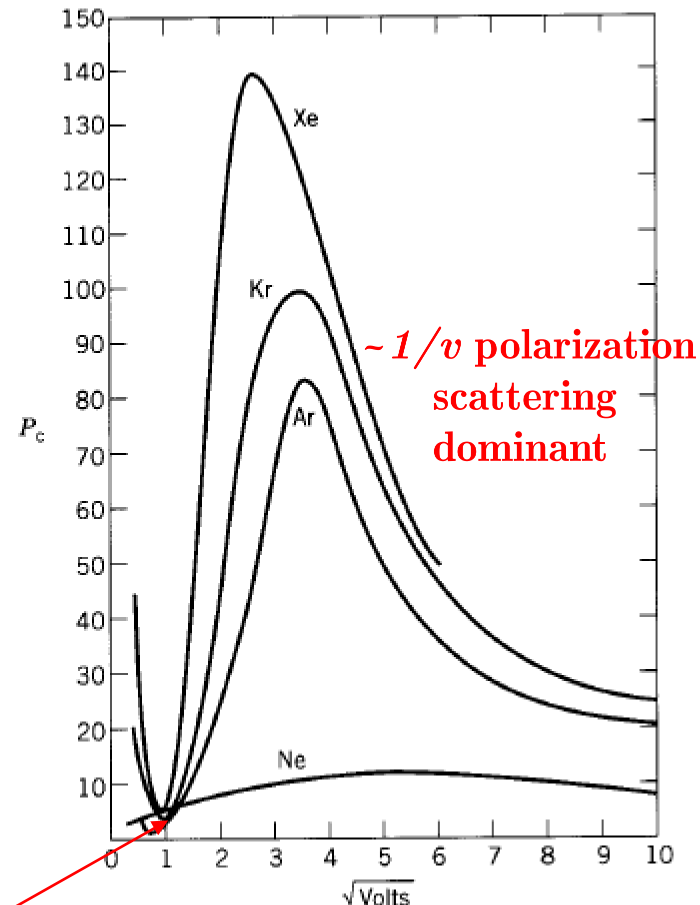


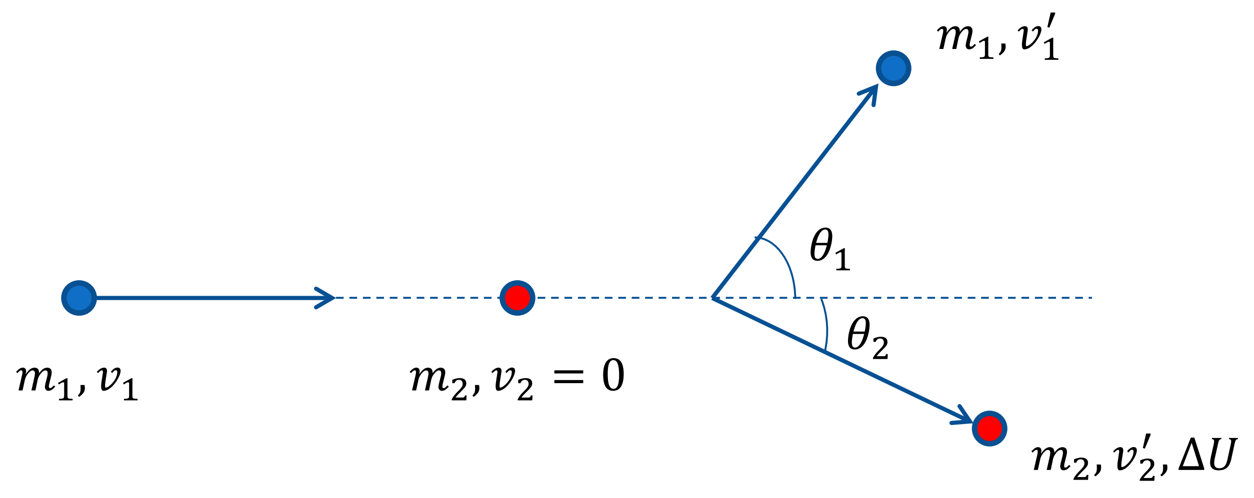

**FIGURE 3.9.** Probability of collision $P_c$ for electrons in $H_2$ and He; the cross section is $\sigma \approx 2.87 \times 10^{-17} P_c$ cm$^2$ (after Brown, 1959).




Ramsauer effect: quantum mechanical wave diffraction of the electron around the atom at low energy

# Inelastic collision

- Inelastic collision: the sum of kinetic energies are not conserved.
  Where is the lost energy?
  → transferred to internal energy (ionization, excitation, dissociation)
  - Atomic gases : electronic transition
  - Molecules : excitation of rotational and vibrational states




- Find $\Delta U_{max}$ , Hint: $\dfrac{\Delta U}{dv_2'} = 0 \;\; at \;\; \Delta U_{max}$

# Inelastic collision

- The maximum obtainable internal energy

$$\zeta_L = \frac{\Delta U_{max}}{\frac{1}{2} m_1 v_1^2} = \frac{m_2}{m_1 + m_2} \cos^2 \theta_2$$

- $m_1 << m_2$ $$\zeta = \frac{m_2}{m_1 + m_2} \approx 1$$

→ Almost all electron energy may be transferred to the internal energy of the heavy particle

- $m_1 >> m_2$ $$\zeta = \frac{m_2}{m_1 + m_2} \approx 0$$

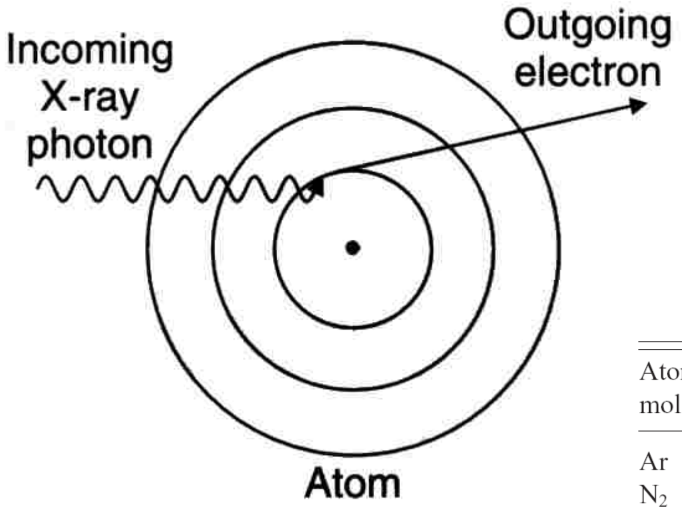
- $m_1 = m_2$ $$\zeta = \frac{m_2}{m_1 + m_2} \approx \frac{1}{2}$$

→ When using $H^+$ ions to ionize hydrogen atoms, the minimum required $H^+$ energy is double the ionization energy for atomic hydrogen.

# Photon induced ionization

- Photo-ionization $h\nu + A \rightarrow e + A^+$

  The physical process in which an incident photon ejects one or more electrons from an atom, ion or molecule. This is essentially the same process that occurs with the photoelectric effect with metals. To provide the ionization, the photo wavelength should be usually less than 1000 Å, which is ultraviolet radiation.



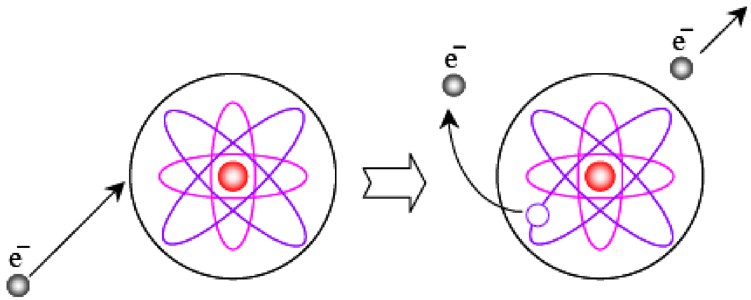

$$\lambda < \frac{12{,}400}{I(eV)} \text{Å}$$

$$K.E = h\nu - I$$

| Atoms or molecules | Wavelength $\lambda$, Å | Cross sections, $cm^2$ | Atoms or molecules | Wavelength $\lambda$, Å | Cross sections, $cm^2$ |
|---|---|---|---|---|---|
| Ar | 787 | $3.5 \cdot 10^{-17}$ | Ne | 575 | $0.4 \cdot 10^{-17}$ |
| $N_2$ | 798 | $2.6 \cdot 10^{-17}$ | O | 910 | $0.3 \cdot 10^{-17}$ |
| N | 482 | $0.9 \cdot 10^{-17}$ | $O_2$ | 1020 | $0.1 \cdot 10^{-17}$ |
| He | 504 | $0.7 \cdot 10^{-17}$ | Cs | 3185 | $2.2 \cdot 10^{-19}$ |
| $H_2$ | 805 | $0.7 \cdot 10^{-17}$ | Na | 2412 | $1.2 \cdot 10^{-19}$ |
| H | 912 | $0.6 \cdot 10^{-17}$ | K | 2860 | $1.2 \cdot 10^{-20}$ |

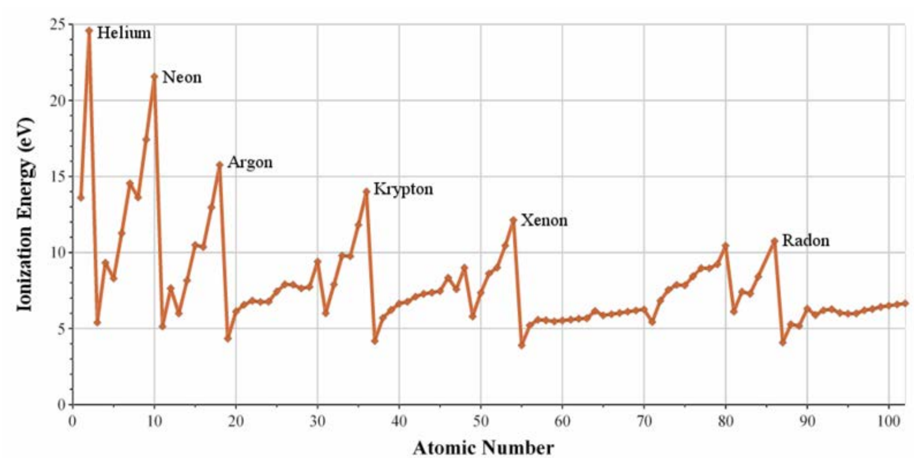
# Electron induced ionization

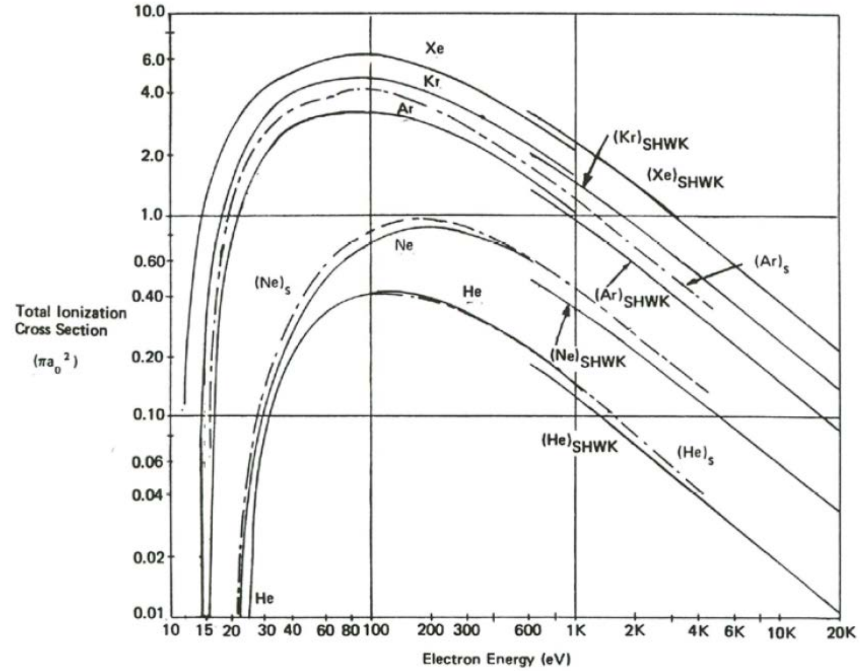
- Electron impact ionization $e + A \rightarrow e + e + A^+$
  Electrons with sufficient energy (> 10 eV) can remove an electron from an atom and produce one extra electron and an ion.




Thomson cross section:

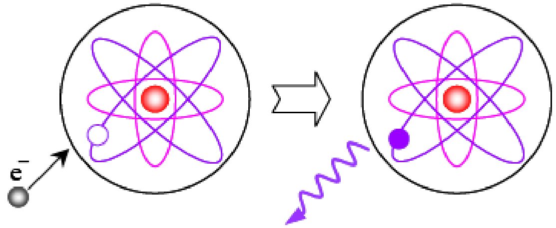
$$\sigma_{\rm iz} = \pi\left(\frac{e}{4\pi\epsilon_0}\right)^2 \frac{1}{\mathcal{E}}\left(\frac{1}{\mathcal{E}_{\rm iz}} - \frac{1}{\mathcal{E}}\right) \quad \mathcal{E} > \mathcal{E}_{\rm iz}$$



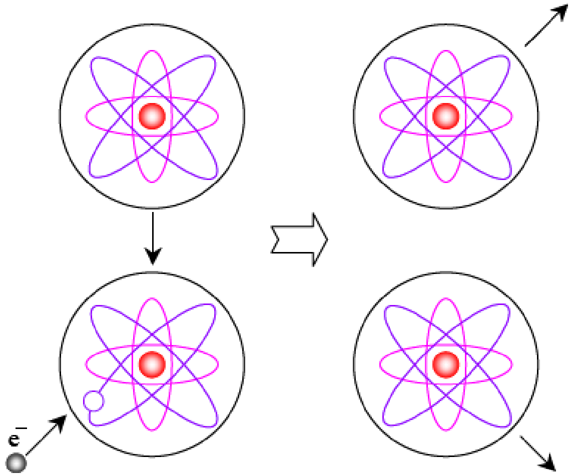
# Recombination of charged particles

- Radiative recombination $e + A^{+} \rightarrow A + h\nu$




- Electron-ion recombination $e + A^{+} + A \rightarrow A^{*} + A$

For electron-ion recombination, a third-body must be involved to conserve the energy and momentum conservation. Abundant neutral species or reactor walls are ideal third-bodies. This recombination process typically results in excited neutrals.



# Diffusion and mobility

- The fluid equation of motion including collisions

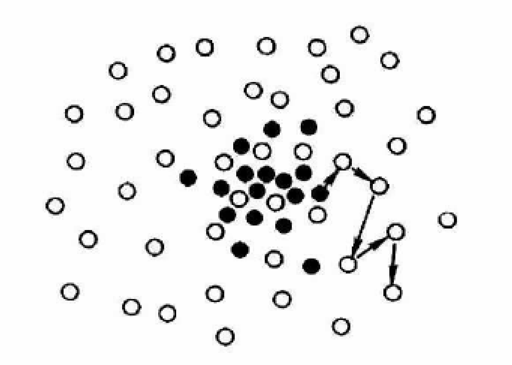
$$mn\frac{d\boldsymbol{u}}{dt} = mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = qn\boldsymbol{E} - \boldsymbol{\nabla}p - mn\nu_m\boldsymbol{u}$$

- In steady-state, for isothermal plasmas

$$\boldsymbol{u} = \frac{1}{mn\nu_m}(qn\boldsymbol{E} - \boldsymbol{\nabla}p) = \frac{1}{mn\nu_m}(qn\boldsymbol{E} - kT\boldsymbol{\nabla}n)$$

$$= \frac{q}{m\nu_m}\boldsymbol{E} - \frac{kT}{m\nu_m}\frac{\boldsymbol{\nabla}n}{n} = \pm\mu\boldsymbol{E} - D\frac{\boldsymbol{\nabla}n}{n}$$

Drift Diffusion



Diffusion is a random walk process.

- In terms of particle flux

$$\boldsymbol{\Gamma} = n\boldsymbol{u} = \pm n\mu\boldsymbol{E} - D\boldsymbol{\nabla}n$$

$\mu = \frac{|q|D}{kT}$ : Einstein relation

$\mu = \frac{|q|}{m\nu_m}$ : Mobility

$D = \frac{kT}{m\nu_m}$ : Diffusion coefficient

# Ambipolar diffusion

- The flux of electrons and ions out of any region must be equal such that charge does not build up. Since the electrons are lighter, and would tend to flow out faster in an unmagnetized plasma, an electric field must spring up to maintain the local flux balance.

$$\Gamma_i = +n\mu_i E - D_i \nabla n \qquad \Gamma_e = -n\mu_e E - D_e \nabla n$$

- Ambipolar electric field for $\Gamma_i = \Gamma_e$

$$E = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}$$

- The common particle flux

$$\Gamma = -\frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e}\nabla n = -D_a \nabla n$$

- The ambipolar diffusion coefficient for weakly ionized plasmas

$$D_a = \frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e} \approx D_i + \frac{\mu_i}{\mu_e} D_e \approx D_i\left(1 + \frac{\mathrm{T_e}}{\mathrm{T_i}}\right) \sim \mu_i \mathrm{T_e}$$

# Diffusion across a magnetic field

- The perpendicular component of the force equation for either species

$$0 = qn(\boldsymbol{E} + \boldsymbol{u}_\perp \times \boldsymbol{B}_0) - kT\boldsymbol{\nabla}n - mn\nu_m\boldsymbol{u}_\perp$$

- In terms of the rectangular components

$$mn\nu_m u_x = qnE_x - kT\frac{\partial n}{\partial x} + qnu_yB_0 \qquad u_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \frac{\omega_c}{\nu_m}u_y$$

$$mn\nu_m u_y = qnE_y - kT\frac{\partial n}{\partial y} - qnu_xB_0 \quad \Rightarrow \quad u_y = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \frac{\omega_c}{\nu_m}u_x$$

- In terms of the rectangular components

$$[1 + (\omega_c\tau_m)^2]u_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + (\omega_c\tau_m)^2\frac{E_y}{B_0} - (\omega_c\tau_m)^2\frac{kT}{qB_0}\frac{1}{n}\frac{\partial n}{\partial y}$$

$$[1 + (\omega_c\tau_m)^2]u_y = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} + (\omega_c\tau_m)^2\frac{E_x}{B_0} + (\omega_c\tau_m)^2\frac{kT}{qB_0}\frac{1}{n}\frac{\partial n}{\partial x}$$

where, $\tau_m \equiv 1/\nu_m$

# Perpendicular mobility and diffusion coefficient

- In vector form

ExB drift　　Diamagnetic drift

$$\boldsymbol{u}_\perp = \pm\mu_\perp \boldsymbol{E} - D_\perp \frac{\boldsymbol{\nabla} n}{n} + \frac{\boldsymbol{u}_E + \boldsymbol{u}_D}{1 + (\omega_c \tau_m)^{-2}}$$

$$\boldsymbol{u}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}_0}{B_0^2}$$

where,

$$\mu_\perp = \frac{\mu}{1 + (\omega_c \tau_m)^2} \qquad D_\perp = \frac{D}{1 + (\omega_c \tau_m)^2}$$

$$\boldsymbol{u}_D = -\frac{kT}{qB_0^2} \frac{\boldsymbol{\nabla} n \times \boldsymbol{B}_0}{n}$$

- The mobility and diffusion fluxes perpendicular to the field exist only in the presence of collisions, and are slowed by the presence of the magnetic field.

- Note that

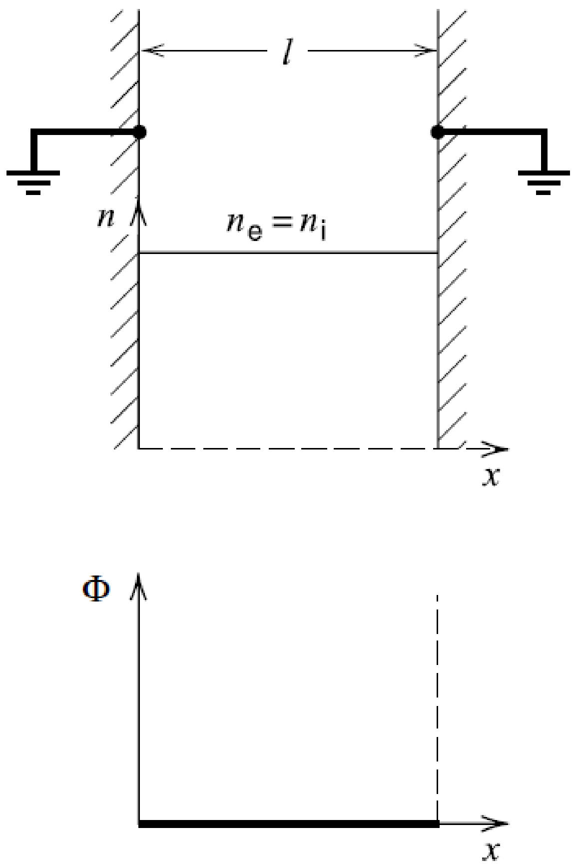
$$D_\perp \approx \frac{kT}{m\nu_m} \frac{1}{(\omega_c \tau_m)^2} = \frac{kT\nu_m}{m\omega_c^2} \qquad D_\parallel \approx \frac{kT}{m\nu_m}$$

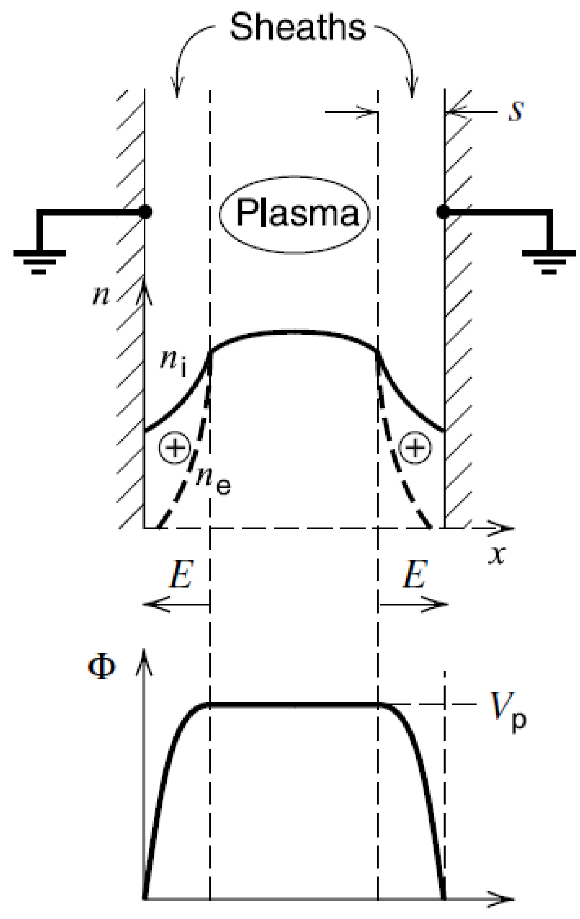
- Cross-field ambipolar diffusion coefficient (with assumption of no loss along B)

$$D_{\perp a} \approx \frac{\mu_{\perp i} D_{\perp e} + \mu_{\perp e} D_{\perp i}}{\mu_{\perp i} + \mu_{\perp e}} \approx D_{\perp e}\left(1 + \frac{T_i}{T_e}\right)$$

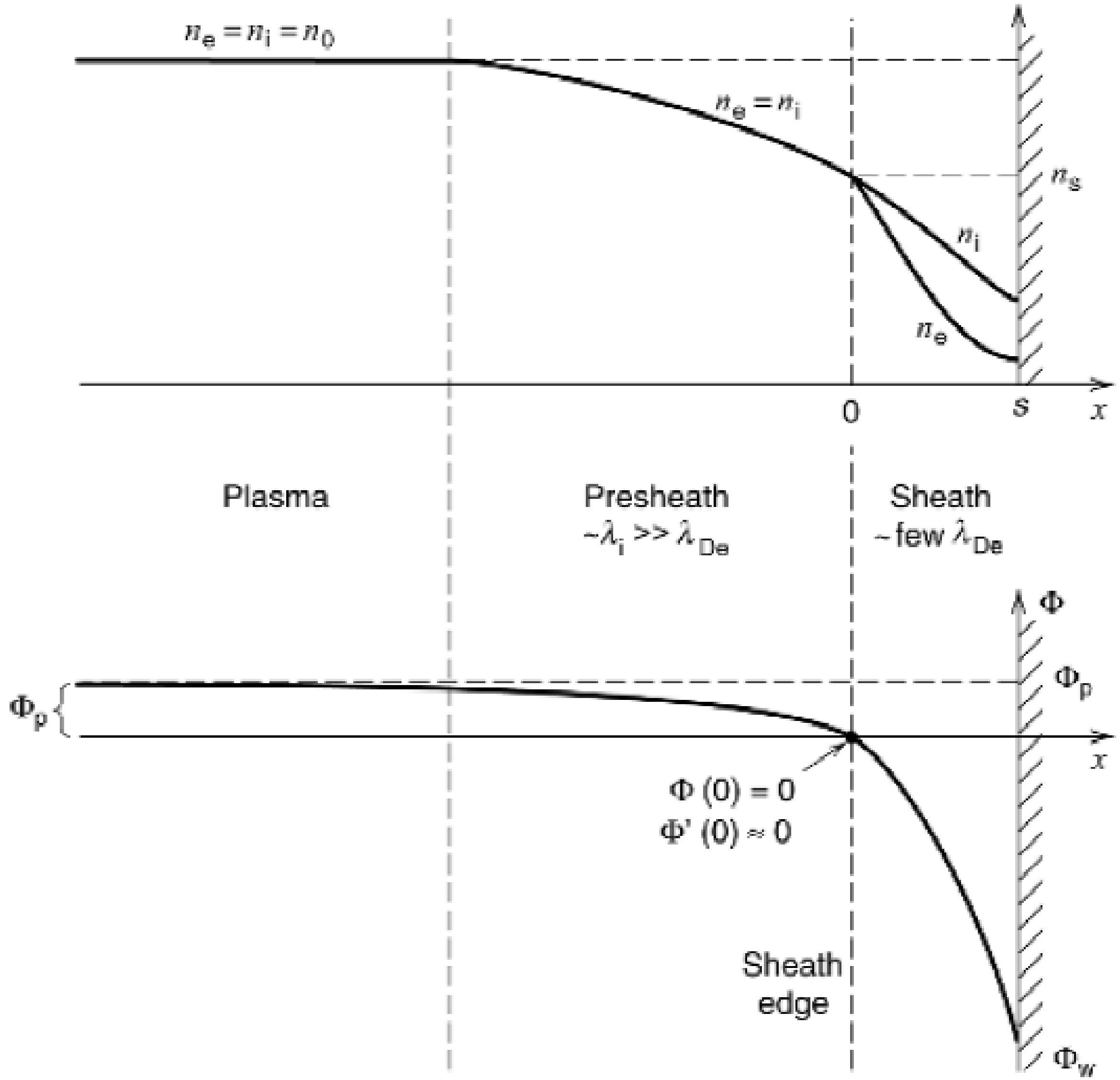
# Formation of plasma sheaths

- Plasma sheath: the non-neutral potential region between the plasma and the wall caused by the balanced flow of particles with different mobility such as electrons and ions.



# Plasma-sheath structure

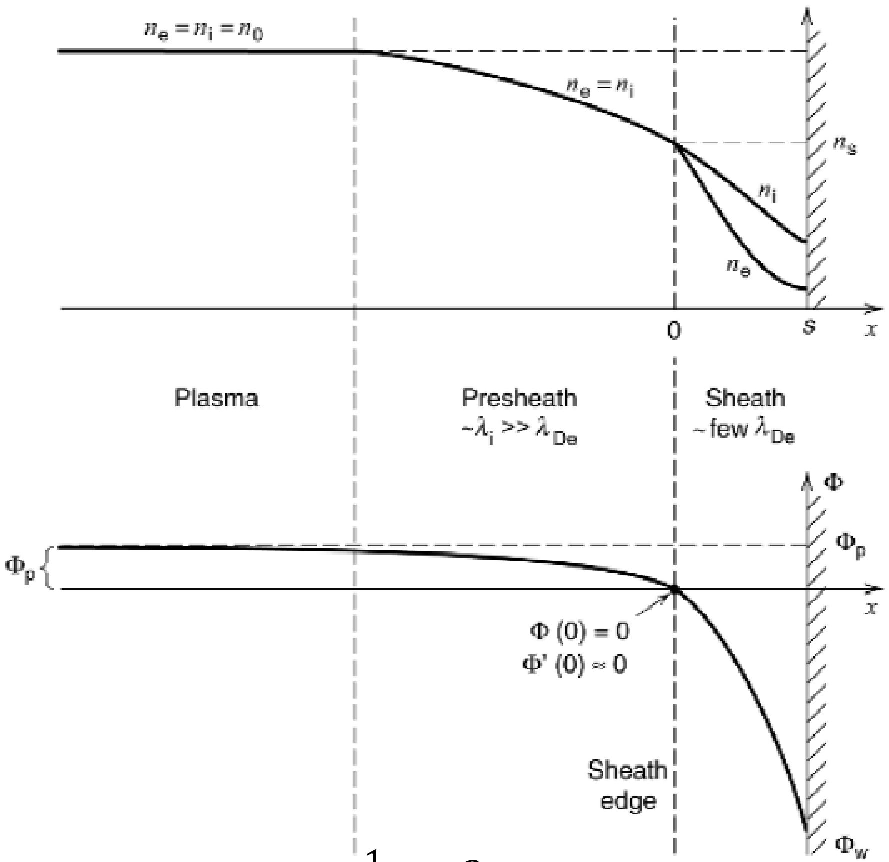


# Collisionless sheath

- Ion energy & flux conservations (no collision)

$$\frac{1}{2}Mu(x)^2 + e\Phi(x) = \frac{1}{2}Mu_s^2$$

$$n_i(x)u(x) = n_{is}u_s$$




- Ion density profile

$$n_i(x) = n_{is}\left(1 - \frac{2e\Phi(x)}{Mu_s^2}\right)^{-1/2}$$

- Electron density profile

$$n_e(x) = n_{es}\exp\left(\frac{\Phi(x)}{\mathrm{T_e}}\right)$$

- Setting $n_{es} = n_{is} \equiv n_s$

$$\frac{d^2\Phi}{dx^2} = \frac{en_s}{\epsilon_0}\left[\exp\left(\frac{\Phi}{\mathrm{T_e}}\right) - \left(1 - \frac{\Phi}{\mathcal{E}_s}\right)^{-1/2}\right]$$

where, $e\mathcal{E}_s \equiv \frac{1}{2}Mu_s^2$

# Bohm sheath criterion

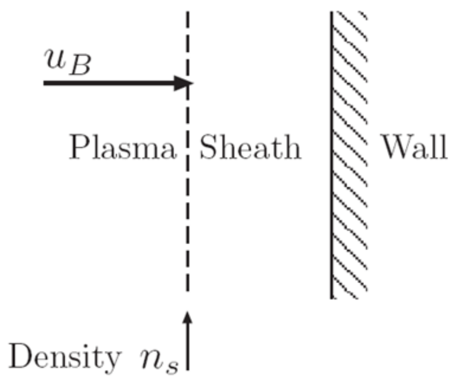
- Multiplying the sheath equation by $d\Phi/dx$ and integrating over $x$

$$\int_0^{\Phi} \frac{d\Phi}{dx}\frac{d}{dx}\left(\frac{d\Phi}{dx}\right)dx = \frac{en_s}{\epsilon_0}\int_0^{\Phi}\frac{d\Phi}{dx}\left[\exp\left(\frac{\Phi}{\mathrm{T_e}}\right)-\left(1-\frac{\Phi}{\mathcal{E}_s}\right)^{-1/2}\right]dx$$

- Cancelling $dx$'s and integrating with respect to $\Phi$

$$\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 = \frac{en_s}{\epsilon_0}\left[\mathrm{T_e}\exp\left(\frac{\Phi}{\mathrm{T_e}}\right)-\mathrm{T_e}+2\mathcal{E}_s\left(1-\frac{\Phi}{\mathcal{E}_s}\right)^{1/2}-2\mathcal{E}_s\right]\geq 0$$

$$\Rightarrow \quad \boxed{\mathcal{E}_s \equiv \frac{1}{2e}Mu_s^2 \geq \frac{\mathrm{T_e}}{2}}$$




- Bohm sheath criterion

$$\boxed{u_s \geq \left(\frac{e\mathrm{T_e}}{M}\right)^{1/2} \equiv \left(\frac{kT_e}{M}\right)^{1/2} \equiv u_B \quad \text{(Bohm speed)}}$$

# Presheath

- To give the ions the directed velocity $u_B$, there must be a finite electric field in the plasma over some region, typically much wider than the sheath, called the presheath.
- At the sheath–presheath interface there is a transition from subsonic ($u_i < u_B$) to supersonic ($u_i > u_B$) ion flow, where the condition of charge neutrality must break down.
- The potential drop across a collisionless presheath, which accelerates the ions to the Bohm velocity, is given by

$$\frac{1}{2}Mu_B^2 = \frac{e\mathrm{T_e}}{2} = e\Phi_p$$

where, $\Phi_p$ is the plasma potential with respect to the potential at the sheath–presheath edge

- The spatial variation of the potential $\Phi_p(x)$ in a collisional presheath (Riemann)

$$\frac{1}{2} - \frac{1}{2}\exp\left(\frac{2\Phi_p}{\mathrm{T_e}}\right) - \frac{\Phi_p}{\mathrm{T_e}} = \frac{x}{\lambda_i}$$

- The ratio of the density at the sheath edge to that in the plasma

$$n_s = n_b e^{-\Phi_p/\mathrm{T_e}} \approx 0.61 n_b$$

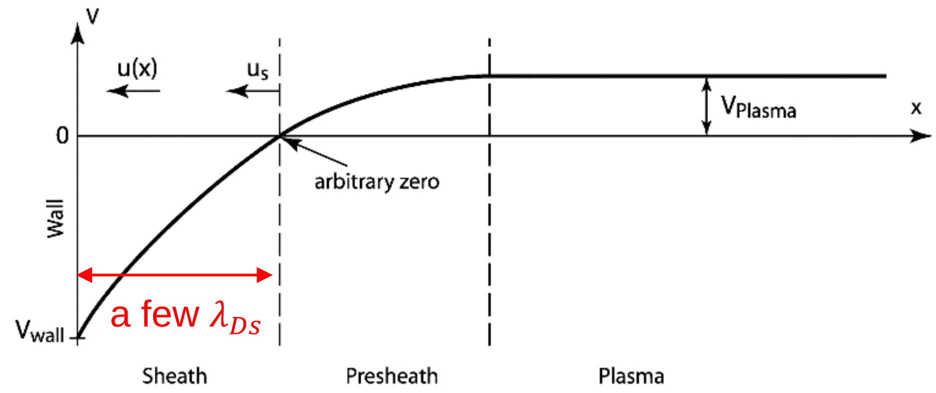
# Sheath potential at a floating Wall

- Ion flux

$$\Gamma_i = n_s u_B$$

- Electron flux

$$\Gamma_e = \frac{1}{4} n_{ew} \bar{v}_e = \frac{1}{4} n_s \bar{v}_e \exp\left(\frac{\Phi_w}{\mathrm{T_e}}\right)$$




- Ion flux = electron flux for a floating wall

$$n_s \left(\frac{e\mathrm{T_e}}{M}\right)^{1/2} = \frac{1}{4} n_s \left(\frac{8e\mathrm{T_e}}{\pi m}\right)^{1/2} \exp\left(\frac{\Phi_w}{\mathrm{T_e}}\right)$$

- Wall potential

$$\Phi_w = -\frac{\mathrm{T_e}}{2} \ln\left(\frac{M}{2\pi m}\right)$$

$\Phi_w \approx -2.8\mathrm{T_e}$ for hydrogen and $\Phi_w \approx -4.7\mathrm{T_e}$ for argon

- Ion bombarding energy

$$\mathcal{E}_{ion} = \frac{e\mathrm{T_e}}{2} + e|\Phi_w| = \frac{e\mathrm{T_e}}{2}\left(1 + \ln\left(\frac{M}{2\pi m}\right)\right)$$

# High-voltage sheath: matrix sheath

- The potential $\Phi$ in high-voltage sheaths is highly negative with respect to the plasma–sheath edge; hence $n_e \sim n_s e^{\Phi/\mathrm{T_e}} \rightarrow 0$ and only ions are present in the sheath.
- The simplest high-voltage sheath, with a uniform ion density, is known as a matrix sheath (not self-consistent in steady-state).
- Poisson's eq.

$$\frac{d^2\Phi}{dx^2} = -\frac{en_s}{\epsilon_0} \quad \Longrightarrow \quad \Phi = -\frac{en_s}{\epsilon_0}\frac{x^2}{2}$$

- Setting $\Phi = -V_0$ at $x = s$, we obtain the matrix sheath thickness

$$s = \left(\frac{2\epsilon_0 V_0}{en_s}\right)^{1/2}$$

- In terms of the electron Debye length at the sheath edge

$$s = \lambda_{Ds}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{1/2} \qquad \text{where, } \lambda_{Ds} = (\epsilon_0 \mathrm{T_e}/en_s)^{1/2}$$

# High-voltage sheath: space-charge-limited current

- In the limit that the initial ion energy $\mathcal{E}_s$ is small compared to the potential, the ion energy and flux conservation equations reduce to

$$\frac{1}{2}Mu^2(x) = -e\Phi(x)$$

$$en(x)u(x) = J_0$$

$$\Longrightarrow \quad n(x) = \frac{J_0}{e}\left(-\frac{2e\Phi}{M}\right)^{-1/2}$$

- Poisson's eq.

$$\frac{d^2\Phi}{dx^2} = -\frac{e}{\epsilon_0}(n_i - n_e) = -\frac{J_0}{\epsilon_0}\left(-\frac{2e\Phi}{M}\right)^{-1/2}$$

- Multiplying by $d\Phi/dx$ and integrating twice from 0 to $x$

$$-\Phi^{3/4} = \frac{3}{2}\left(\frac{J_0}{\epsilon_0}\right)^{1/2}\left(\frac{2e}{M}\right)^{-1/4} x$$

B.C. $\left.\frac{d\Phi}{dx}\right|_{x=0} = 0$

- Letting $\Phi = -V_0$ at $x = s$ and solving for $J_0$, we obtain

$$J_0 = \frac{4}{9}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{V_0^{3/2}}{s^2}$$

Child law:
Space-charge-limited current in a plane diode

# High-voltage sheath: Child law sheath

- For a plasma $J_0 = e n_s u_B$

$$e n_s u_B = \frac{4}{9}\epsilon_0 \left(\frac{2e}{M}\right)^{1/2} \frac{V_0^{3/2}}{s^2}$$

- Child law sheath

$$s = \frac{\sqrt{2}}{3}\left(\frac{\epsilon_0 \mathrm{T_e}}{e n_s}\right)^{1/2}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{3/4} = \frac{\sqrt{2}}{3}\lambda_{Ds}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{3/4}$$

- Potential, electric field and density within the sheath

$$\Phi = -V_0\left(\frac{x}{s}\right)^{4/3} \qquad E = \frac{4}{3}\frac{V_0}{s}\left(\frac{x}{s}\right)^{1/3} \qquad n = \frac{4}{9}\frac{\epsilon_0}{e}\frac{V_0}{s^2}\left(\frac{x}{s}\right)^{-2/3}$$

- Assuming that an ion enters the sheath with initial velocity $u(0) = 0$

$$\frac{dx}{dt} = v_0\left(\frac{x}{s}\right)^{2/3}$$

where, $v_0$ is the characteristic ion velocity in the sheath

$$v_0 = \left(\frac{2eV_0}{M}\right)^{1/2}$$

- Ion transit time across the sheath

$$\frac{x(t)}{s} = \left(\frac{v_0 t}{3s}\right)^3 \quad \Rightarrow \quad \tau_i = \frac{3s}{v_0}$$