Ch. 4 Mean-flow equations

⊙ Reynolds eqs.

- Reynolds decomposition

$$\underset{\text{instantaneous}}{\underline{U}(\underline{x},t)} = \underset{\text{mean}}{\langle \underline{U}(\underline{x},t)\rangle} + \underset{\text{fluctuation}}{\underline{u}(\underline{x},t)}$$

- continuity eq.

$$\boxed{\nabla\cdot\underline{u}=0} \leftarrow \boxed{\nabla\cdot\langle\underline{U}\rangle=0}$$

$$\nabla\cdot\underline{U}=0 \rightarrow \nabla\cdot(\langle\underline{U}\rangle+\underline{u})=0$$

$$\rightarrow \langle \nabla\cdot\langle\underline{U}\rangle + \nabla\cdot\underline{u} = 0\rangle \rightarrow \nabla\cdot\langle\langle\underline{U}\rangle\rangle + \nabla\cdot\langle\cancel{\underline{u}}\rangle = 0$$

(1)

0

$$\langle\cancel{\underline{U}}\rangle = \langle\langle\underline{U}\rangle\rangle + \langle\underline{u}\rangle = \langle\cancel{\underline{U}}\rangle + \langle\underline{u}\rangle \rightarrow \langle\underline{u}\rangle = 0$$

- momentum eqs. (N-S eqs.)

$$\left\langle \frac{DU_j}{Dt} = \frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_i}(U_i U_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu\nabla^2 U_j \right\rangle$$

$\langle \ \rangle$

$$\rightarrow \frac{\partial \langle U_j \rangle}{\partial t} + \underbrace{\frac{\partial}{\partial x_i} \langle U_i U_j \rangle} = -\frac{1}{\rho} \frac{\partial \langle P \rangle}{\partial x_j} + \nu \nabla^2 \langle U_j \rangle$$

$$= \frac{\partial}{\partial x_i} \langle (\langle U_i \rangle + u_i)(\langle U_j \rangle + u_j) \rangle$$

$\langle u_j \rangle = 0$

$\langle u_i \rangle = 0$

$$= \frac{\partial}{\partial x_i} \langle \langle U_i \rangle \langle U_j \rangle + \langle U_i \rangle u_j + u_i \langle U_j \rangle + u_i u_j \rangle$$

$$= \frac{\partial}{\partial x_i} \left(\langle U_i \rangle \langle U_j \rangle + \langle u_i u_j \rangle \right)$$

Reynolds stresses

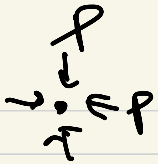
$$\rightarrow \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial}{\partial x_i} \langle U_i \rangle \langle U_j \rangle = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} + \nu \nabla^2 \langle U_j \rangle - \frac{\partial}{\partial x_i} \langle u_i u_j \rangle$$

Reynolds-averaged Navier-Stokes eq.
(RANS)

$$\rightarrow \rho \frac{\partial \langle U_j \rangle}{\partial t} + \rho \frac{\partial}{\partial x_i} \langle U_i \rangle \langle U_j \rangle$$

$$= \frac{\partial}{\partial x_i} \left[-\langle p \rangle \delta_{ij} + \mu \left(\frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\partial \langle U_i \rangle}{\partial x_j} \right) - \rho \langle u_i u_j \rangle \right]$$

2



$\uparrow$ isotropic stress

$\uparrow$ viscous stress

$\uparrow$ Reynolds stress from fluctuating velocities

eq. for $\langle U_j \rangle \longrightarrow \langle u_i u_j \rangle$

unknowns : $\langle U_j \rangle$, $\langle p \rangle$, and $\langle u_i u_j \rangle$ $\rightarrow$ 10 unknowns

$\qquad\qquad 3 \qquad\quad 1 \qquad\qquad 6 \quad i = 1, 2, 3$

# of eqs. $\rightarrow$ 4 — impossible to solve!

$$\frac{\partial}{\partial t}\underline{\langle u_i u_j \rangle} + \cdots = \cdots \quad \underline{\langle u_i u_j u_k \rangle} + \cdots$$

$\Rightarrow$ closure problem $\rightarrow$ turbulence modeling

- $\langle u_i u_j \rangle = \langle u_j u_i \rangle$: symmetric tensor

  $\langle u_1^2 \rangle, \langle u_2^2 \rangle, \langle u_3^2 \rangle$: normal stresses

  $\langle u_1 u_2 \rangle, \langle u_1 u_3 \rangle, \langle u_2 u_3 \rangle$: shear stresses

$$k = \frac{1}{2}\langle u_i u_i\rangle = \frac{1}{2}\left(\langle u_1^2\rangle + \langle u_2^2\rangle + \langle u_3^2\rangle\right)$$ turbulent kinetic energy.

- Anisotropy

$$a_{ij} \equiv \langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}$$: anisotropic part of Reynolds stresses.

$$a_{ii} = \langle u_i u_i\rangle - \frac{2}{3}k\underbrace{\delta_{ii}}_{=3} = \underbrace{\langle u_i u_i\rangle - 2k}_{=0}$$

$$b_{ij} \equiv \frac{a_{ij}}{2k} = \frac{\langle u_i u_j\rangle}{\langle u_i u_i\rangle} - \frac{1}{3}\delta_{ij} \quad : \quad b_{ii} = 1 - \frac{1}{3}\cdot 3 = 0$$

$$\Rightarrow \rho\frac{\partial\langle U_j\rangle}{\partial t} + \rho\frac{\partial\langle U_i\rangle\langle U_j\rangle}{\partial x_i} = \frac{\partial}{\partial x_i}\left[-\langle p\rangle\delta_{ij} + \mu\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right) - \rho\langle u_i u_j\rangle\right]$$

$$\rho\frac{\partial}{\partial x_i}\langle u_i u_j\rangle + \frac{\partial\langle p\rangle}{\partial x_j} = \rho\frac{\partial}{\partial x_i}\left[a_{ij} + \frac{2}{3}k\delta_{ij}\right] + \frac{\partial\langle p\rangle}{\partial x_j}$$

$$= \rho\frac{\partial a_{ij}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\langle p\rangle + \frac{2}{3}\rho k\right]$$

$$\equiv \langle p'\rangle$$: modified press.

only the anisotropic part of the Reynolds stresses is effective in transporting mtm because the isotropic part is absorbed in a modified press.

- Irrotational motion $\rightarrow$ $\underline{\omega} = 0 \rightarrow \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 0$

$$\underline{\left\langle u_i \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right\rangle} = 0 \quad \because \underline{\omega} = 0$$

$\nabla \cdot \underline{u} = 0$

$$= \left\langle \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right) - \frac{\partial}{\partial x_i} u_i u_j \right\rangle = \frac{1}{2} \frac{\partial}{\partial x_j} \langle u_i u_i \rangle - \frac{\partial}{\partial x_i} \langle u_i u_j \rangle = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x_i} \langle u_i u_j \rangle = \frac{\partial k}{\partial x_j} \quad \text{for irrotational flow}}$$

Reynolds stress term is absorbed in a modified pressure. In other words, the Reynolds stresses arising from an irrotational field have no effect on the mean velocity field.

$$\Rightarrow \rho \frac{\partial \langle U_j \rangle}{\partial t} + \rho \frac{\partial}{\partial x_i} \langle U_i \rangle \langle U_j \rangle = - \frac{\partial \langle p'' \rangle}{\partial x_j} + \mu \nabla^2 \langle U_j \rangle$$

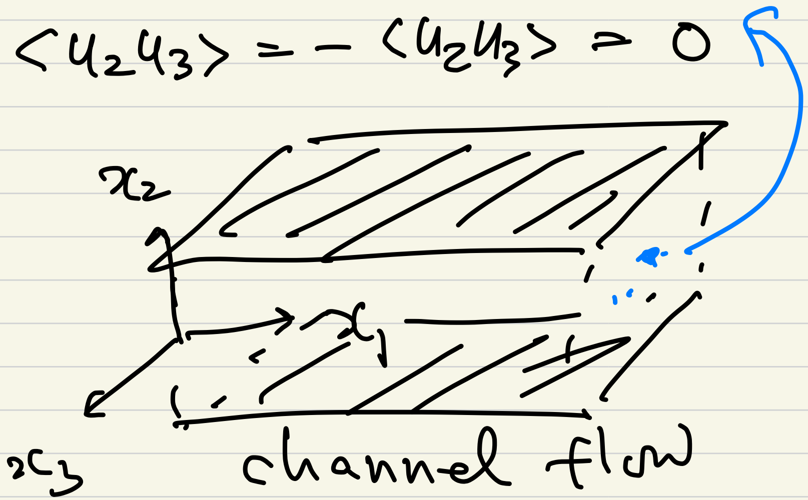
- Symmetries

Consider a statistically 2D flow (x_1, x_2) in which statistics are indep. of x_3, and which is statistically invariant under reflections of x_3 coord.

$$\rightarrow \langle U_3 \rangle = - \langle U_3 \rangle \Rightarrow \langle U_3 \rangle = 0$$

$$\langle u_1 u_3 \rangle = - \langle u_1 u_3 \rangle = 0 \,, \quad \langle u_2 u_3 \rangle = - \langle u_2 u_3 \rangle = 0$$

$$\begin{bmatrix} \langle u_1^2 \rangle & \langle u_1 u_2 \rangle & 0 \\ \langle u_1 u_2 \rangle & \langle u_2^2 \rangle & 0 \\ 0 & 0 & \langle u_3^2 \rangle \end{bmatrix}$$



channel flow

- Mean scalar eq.

$$\left\langle \frac{\partial \phi}{\partial t} + \nabla\cdot(\underline{U}\phi) = \Gamma \nabla^2 \phi \right\rangle \qquad \phi = \langle\phi\rangle + \phi'$$

$$\rightarrow \frac{\partial\langle\phi\rangle}{\partial t} + \frac{\partial}{\partial x_j}\langle U_j\rangle\langle\phi\rangle = \frac{\partial}{\partial x_j}\left(\Gamma \frac{\partial\langle\phi\rangle}{\partial x_j} \underbrace{- \langle u_j \phi'\rangle}_{\text{scalar flux}}\right)$$

again closure prob.! fluid property

- Gradient-diffusion and turbulent-viscosity hypotheses

$$\boxed{-\langle \underline{u}\phi'\rangle = \Gamma_T \nabla\langle\phi\rangle}$$

Γ_T : turbulent diffusivity > 0

flow property

gradient-diffusion hypothesis

$$\rightarrow \frac{\partial\langle\phi\rangle}{\partial t} + \frac{\partial}{\partial x_j}\langle U_j\rangle\langle\phi\rangle = \frac{\partial}{\partial x_j}\left(\Gamma_{eff}\frac{\partial\langle\phi\rangle}{\partial x_j}\right)$$

$\Gamma_{eff} = \Gamma + \Gamma_T(\underline{x},t)$: effective diffusivity

$$-\rho a_{ij}$$

$$\boxed{-\rho\langle u_i u_j\rangle + \frac{2}{3}\rho k\delta_{ij} = \rho\nu_T\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right)} = 2\rho\nu_T\langle S_{ij}\rangle$$

$$\underline{\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right)} = 2\langle S_{ij}\rangle$$

turbulent-viscosity hypothesis

Boussinesq approximation (1877)

ν_T : turbulent (or eddy) viscosity

↑ flow property

$$\Rightarrow \frac{\partial\langle U_j\rangle}{\partial t} + \frac{\partial}{\partial x_i}\langle U_i\rangle\langle U_j\rangle = \frac{\partial}{\partial x_i}\left[\nu_{eff}\left(\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial\langle U_i\rangle}{\partial x_j}\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\langle p\rangle + \frac{2}{3}\rho k\right]$$

$$\langle p\rangle + \frac{2}{3}\rho k = \underline{\underline{\langle p'\rangle}}$$

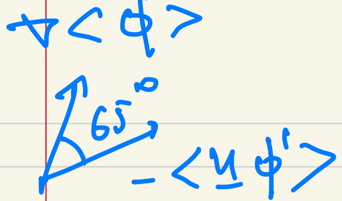
$\nu_{eff} = \nu + \nu_T(\underline{x}, t)$: effective viscosity

* Some important observations

- ~~$\langle \underline{u}\phi'\rangle$ is aligned w/ mean scalar gradient vector.~~ $\nabla\langle\phi\rangle$

$\langle \underline{u}\phi'\rangle$ $\nabla\langle\phi\rangle$

→ this is not true even in simple turbulent flow



e.g., angle betweeen $\nabla\langle\phi\rangle$ and $-\langle \underline{u}\,\phi' \rangle$ is 65° in homo. turb. shear flow

- $a_{ij} \sim \langle S_{ij} \rangle$ $\quad \underset{3\times3}{A} = \underset{1}{\nu_T}\, \underset{3\times3}{S} \Rightarrow \underset{3\times3}{A} = \underset{3\times3}{\overset{\text{matrix}}{\nu_T}}\, \underset{3\times3}{S}$

  proportional coeff is scalar for six different components.
  ν_T

  → not true even in simple shear flow

- In 2-D turb. bdry layer flow,
  important flow variables are $\langle u_1 u_2 \rangle$ and $\langle u_2 \phi' \rangle$.

$$
\rightarrow \left(\begin{aligned} \langle u_1 u_2 \rangle &= -\nu_T \frac{\partial \langle U_1 \rangle}{\partial x_2} \\ \underset{\text{scalar}}{\langle u_2 \phi' \rangle} &= -\underset{\text{scalar}}{\Gamma_T} \underset{\text{scalar}}{\frac{\partial \langle \phi \rangle}{\partial x_2}} \end{aligned} \right.
$$

Not bad!

- If ν_T & Γ_T are properly modeled,
  the mean eqs. can be solved accurately.

- At high Re and away from the wall,

$$\frac{\nu_T}{\nu} \text{ and } \frac{\Gamma_T}{\Gamma} \sim Re \gg 1$$

$\rightarrow$ molecular transport is negligible.

- $\sigma_T = \frac{\nu_T}{\Gamma_T}$: turbulent Prandtl number

in most simple turb. flows, $\sigma_T \sim O(1)$.