Chapter 4: Basic continuum mechanics

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In this chapter,

- Basic continuum mechanics is summarized to generalize the problems in the previous chapters

- **Vector and tensor notations**
  - Used for stress, strain, elastic stiffness, …
  - **Subscript** $_2$ for 2nd order tensor
  - **Subscript** $_4$ for 4th order tensor

- Sections 4.1, 4.2 and 4.4: basis of finite element employed in chapter 5, where the total Lagrangian formulation is introduced

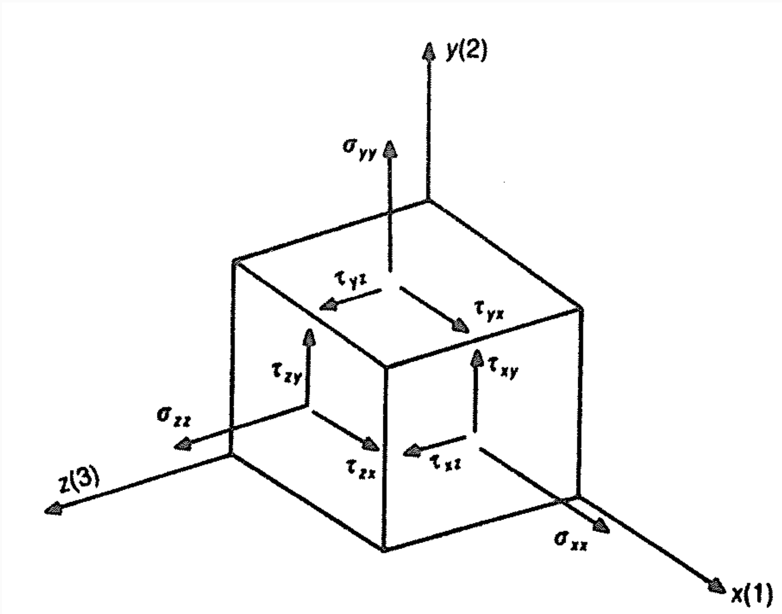
- Section 4.3: transformation between two coordinate systems

- Sections 4.5 and 4.6: updated Lagrangian formulations used in chapter 5.

- Section 4.7: relationship among various stress and strain measures

- Section 4.8: polar decomposition used in section 4.9 to relate the Green and Almansi strains to the principal stretches and in section 4.10 to give a simple explanation of the second Piola-Kirchhoff stress

- Section 4.11: brief overview of **constitutive models** with the aim that the finite element work of chapter 5 should make sense in relations to concepts such as **plasticity**



[Fig 4.1 Stress tensor]

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} \quad \text{or} \quad \boldsymbol{\sigma}_2 = \underbrace{\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{symmetric}$$

[eq. 4.1] [eq. 4.2]

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} \quad \text{or} \quad \boldsymbol{\varepsilon}_2 = \underbrace{\begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}}_{symmetric}$$

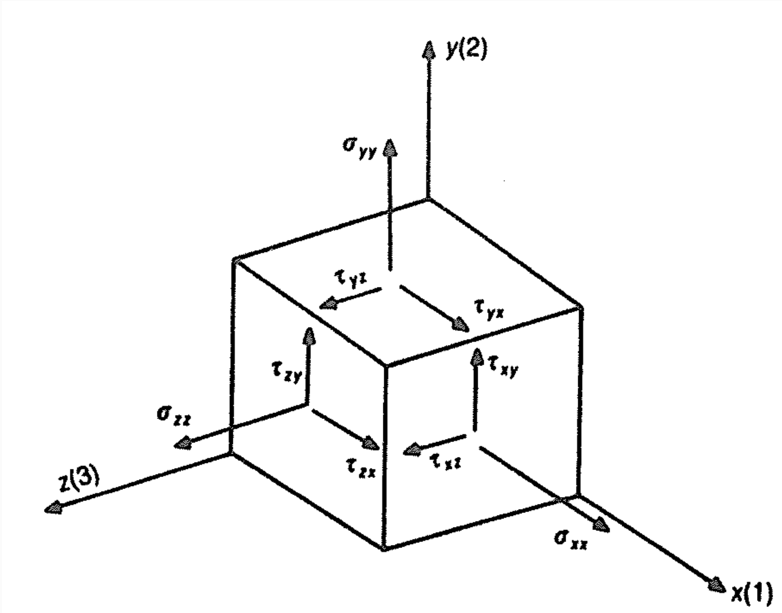
[eq. 4.3] [eq. 4.4]

For small strains,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

[eq. 4.5]



$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} \quad \text{or} \quad \boldsymbol{\sigma}_2 = \underbrace{\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{symmetric}$$

[eq. 4.1] [eq. 4.2]

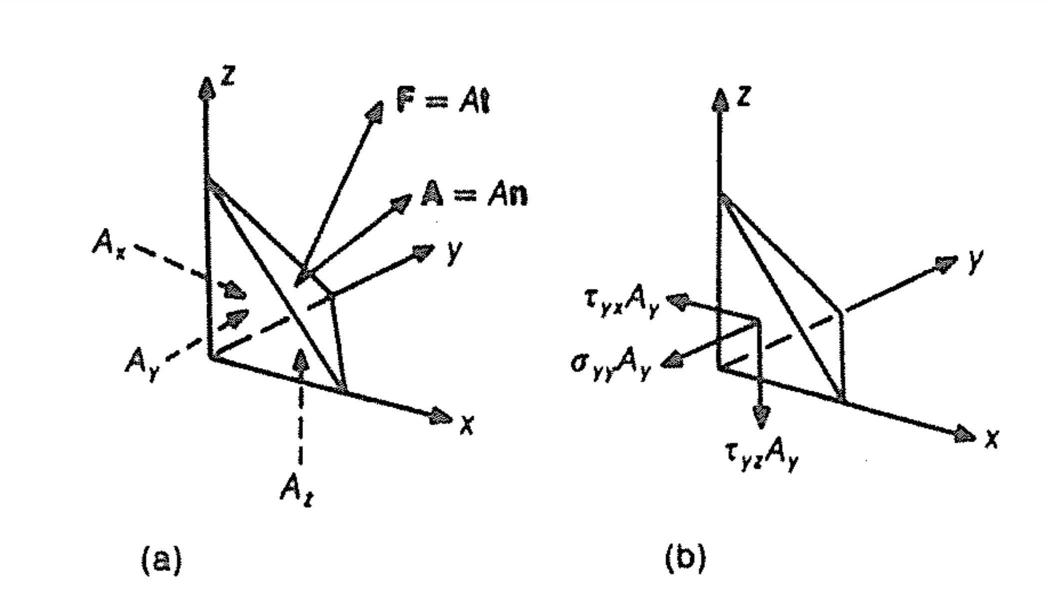
$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} \quad \text{or} \quad \boldsymbol{\varepsilon}_2 = \underbrace{\begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}}_{symmetric}$$

[eq. 4.3] [eq. 4.4]

$\frac{1}{2}$ ensures simple consistent linear strain energy representation

$$\varphi = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\sigma}_2 : \boldsymbol{\varepsilon}_2$$ [eq. 4.6]

$:$ double dot product, or contraction symbol



$$\boldsymbol{\sigma}_2 = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

symmetric

[eq. 4.2]

[Fig 4.2 Relationship between the external forces and the stresses]

Stress tensor notation allows the stresses to be very simply related to external forces

$$\boldsymbol{\sigma}_2 \mathbf{A} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = A \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$ [eq. 4.8]

➡ $\boldsymbol{\sigma}_2 \mathbf{n} = \mathbf{t}$ [eq. 4.9] "Traction"

- For a **linear elastic, isotropic** material the stresses and strains are related by:

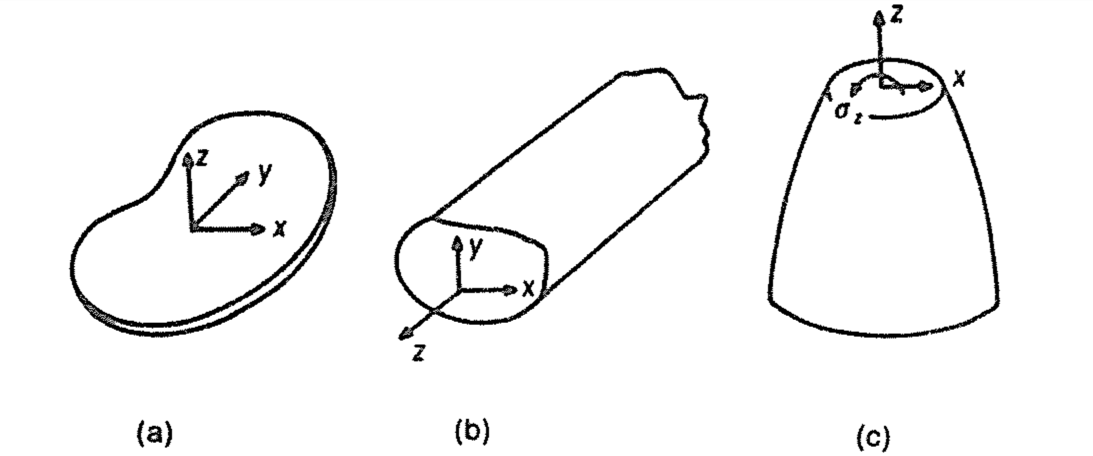
$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2v) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2v) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2v) \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix}$$ [eq. 4.10]

"Generalized Hooke's law"

$$\Leftrightarrow \quad \boldsymbol{\sigma} = \mathbf{C}_2 \boldsymbol{\varepsilon}$$ [eq. 4.11] Voigt notation

$$\Leftrightarrow \quad \boldsymbol{\sigma}_2 = \mathbf{C}_4 : \boldsymbol{\varepsilon}_2$$ [eq. 4.12] Indicial notation

4.2.1 Plane strain, axial symmetry (Axi-symmetric) and plane stress



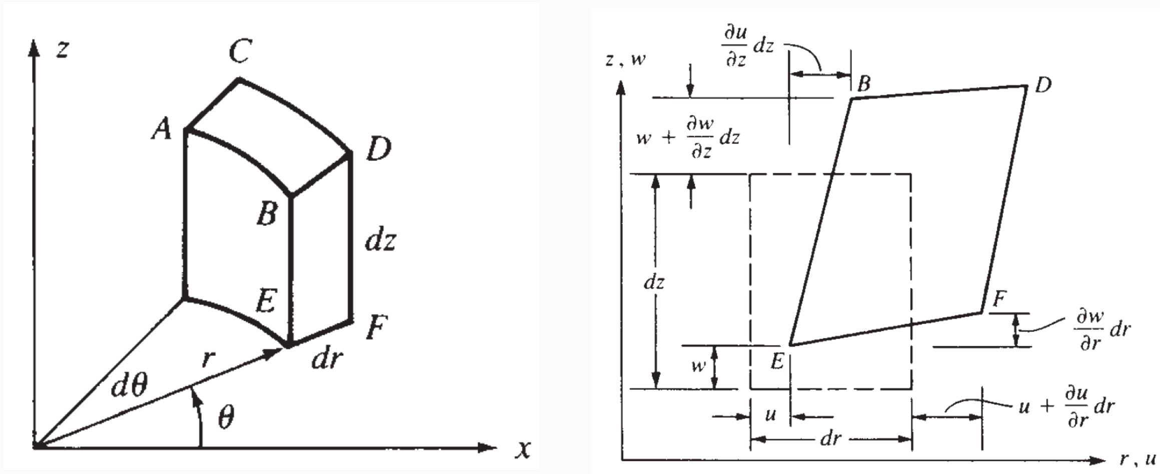
[Fig 4.3 (a) plane stress (b) plane strain (c) axi-symmetric]

- For plane stress, $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$

$$\Rightarrow \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

- For plane strain, $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$

$$\Rightarrow \begin{pmatrix} \sigma_{x} \\ \sigma_{y} \\ \tau_{xy} \end{pmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & (1-2v)/2 \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$



[axisymmetric coordinates]

- For axisymmetric case, $\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{u}{r}, \varepsilon_z = \frac{\partial w}{\partial z}, \gamma_{rz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$

$$\Rightarrow \begin{pmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{pmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 \\ v & (1-v) & v & 0 \\ v & v & (1-v) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2v) \end{bmatrix} \begin{pmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{pmatrix}$$

4.2.2 Decomposition into volumetric and deviatoric components

$$\boldsymbol{\sigma}_2 = \sigma_m \mathbf{I} + \boldsymbol{\sigma}_{2d} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_{xx} - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_m \end{bmatrix}$$ [eq. 4.18]

$$\boldsymbol{\varepsilon}_2 = \varepsilon_m \mathbf{I} + \boldsymbol{\varepsilon}_{2d} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_{xx} - \varepsilon_m & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{yy} - \varepsilon_m & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{zz} - \varepsilon_m \end{bmatrix}$$ [eq. 4.19]

where $\sigma_m = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$ $\quad \varepsilon_m = \frac{1}{3}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$ [eq. 4.20]

➡ $\boldsymbol{\sigma}_{2d} = \mathbf{s}_2 = 2\mu\boldsymbol{\varepsilon}_{2d} = 2\mu\mathbf{e}_2$ where $\mu = \frac{E}{2(1+\nu)}$ shear modulus, $k = \frac{E}{3(1-2\nu)}$ bulk modulus [eq. 4.22]

$\sigma_m = 3k\varepsilon_m$ [eq. 4.21]

4.2.3 An alternative expression using the Lame's constants

$$\begin{aligned}\boldsymbol{\sigma}_2 &= (\sigma_m - 2\mu\varepsilon_m)\mathbf{I} + 2\mu\boldsymbol{\varepsilon}_2 \\ &= (3k - 2\mu)\varepsilon_m\mathbf{I} + 2\mu\boldsymbol{\varepsilon}_2 \\ &= \left(\frac{3k-2\mu}{3}\right) tr(\boldsymbol{\varepsilon}_2)\mathbf{I} + 2\mu\boldsymbol{\varepsilon}_2 \\ &= \lambda\, tr(\boldsymbol{\varepsilon}_2)\mathbf{I} + 2\mu\boldsymbol{\varepsilon}_2\end{aligned}$$

[eq. 4.24-4.27]

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$ [eq. 4.28]

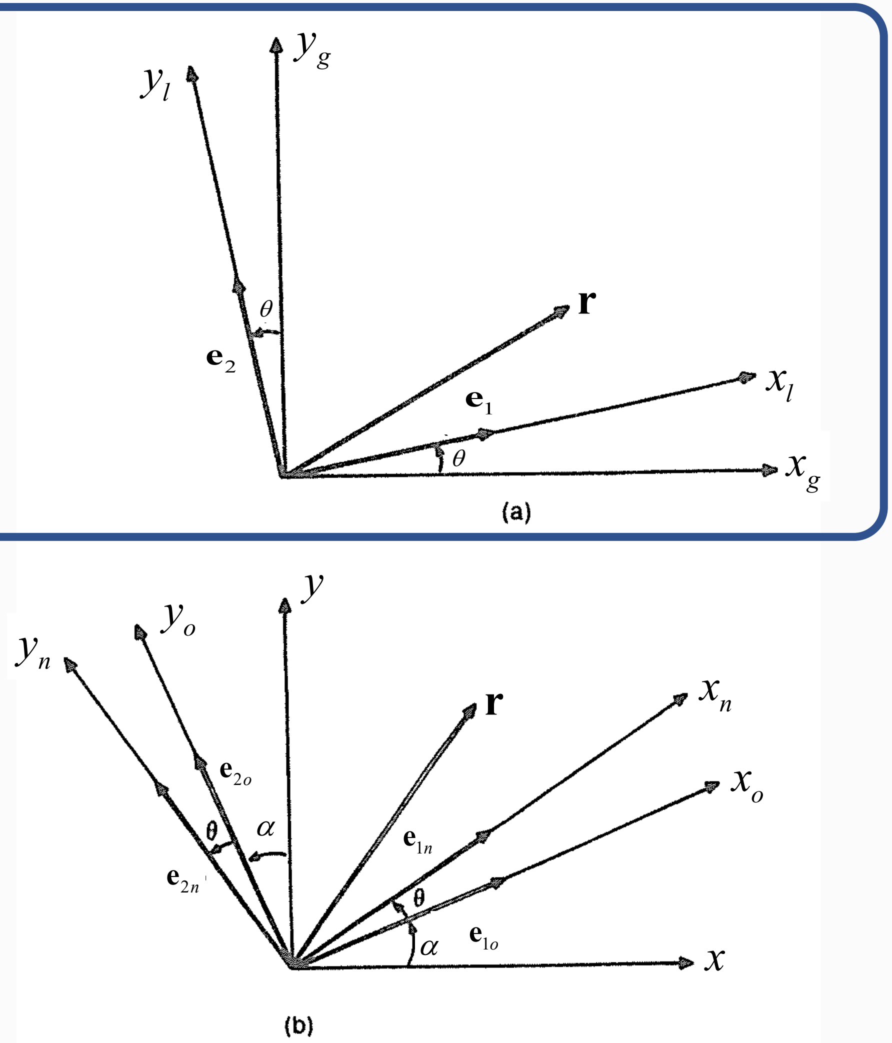
can be expressed as: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ [eq. 4.29]

where $C_{ijkl} = 2\mu\left(\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right) + \lambda\delta_{ij}\delta_{kl}$ [eq. 4.30]

$$\mathbf{C} = \mathbf{C}_4 = 2\mu\mathbf{I} + \lambda(\mathbf{1}\otimes\mathbf{1}) = 2\mu\mathbf{I}_4 + \lambda(\mathbf{1}_2\otimes\mathbf{1}_2)$$ [eq. 4.31]

$\otimes$: dyadic product

4.3.1 Transformation to a new set of axes



[Fig 4.4 Different axis for transformations (a) 'local' and 'global' (b) 'old' and 'new']

- Local and global coordinates are related by:

$$\mathbf{r}_l = \mathbf{T}\mathbf{r}_g \quad \text{where} \quad \mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{bmatrix}$$

[eq. 4.32] [eq. 4.33]

- In three-dimensional case,

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{bmatrix}$$

[eq. 4.35]

- **T** is an orthonormal matrix,

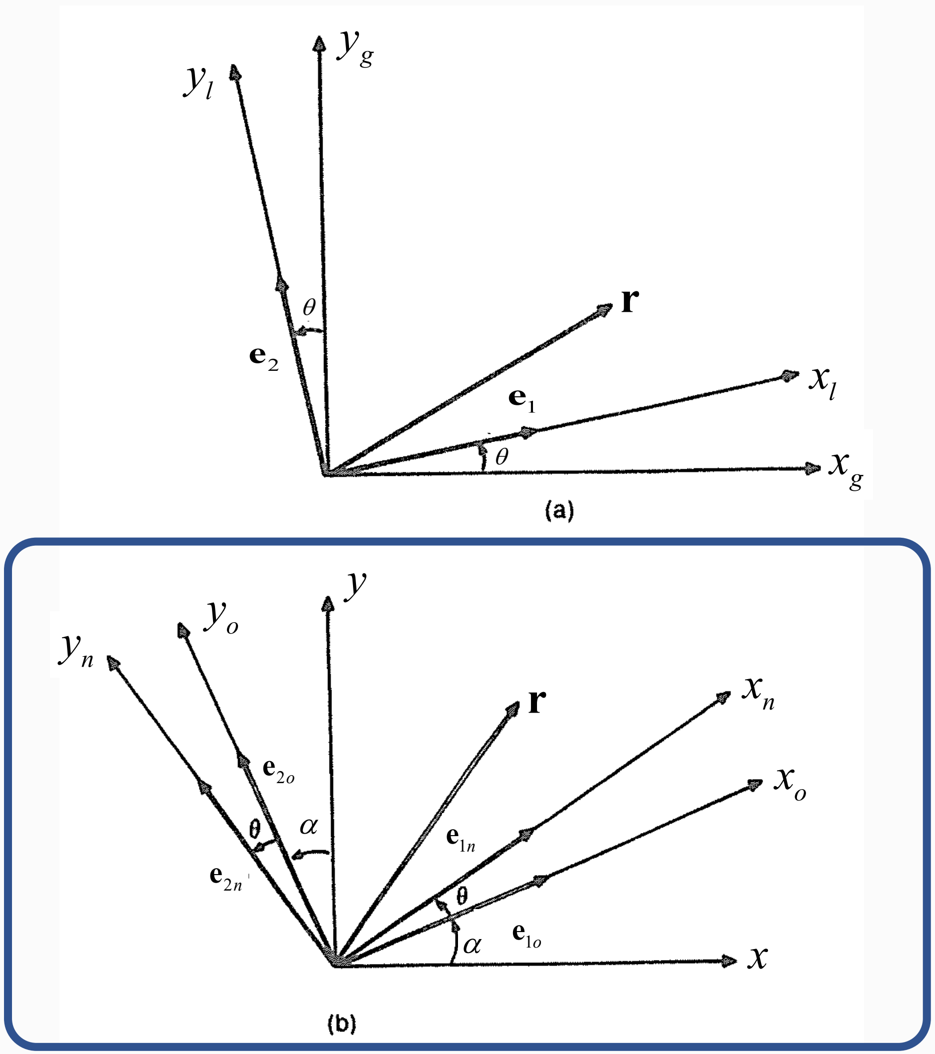
$$\mathbf{T}^{-1} = \mathbf{T}^T$$

[eq. 4.38]

- Stress is transformed by:

$$\boldsymbol{\sigma}_l = \mathbf{T}\boldsymbol{\sigma}_g\mathbf{T}^T \quad \text{or} \quad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}_l = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & \left(c^2 - s^2\right) \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}_g$$

[eq. 4.36] [eq. 4.37]



[Fig 4.4 Different axis for transformations (a) 'local' and 'global' (b) 'old' and 'new']

- New and old coordinates are related by:

$$\mathbf{r}_n = \mathbf{T}_n \mathbf{r}_g \qquad \mathbf{r}_o = \mathbf{T}_o \mathbf{r}_g$$

$$\mathbf{r}_n = \mathbf{T}_n \mathbf{T}_o^T \mathbf{r}_o = \mathbf{T}\mathbf{r}_o$$ [eq. 4.39-4.41]

where $\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{1n}^T \mathbf{e}_{1o} & \mathbf{e}_{1n}^T \mathbf{e}_{2o} \\ \mathbf{e}_{2n}^T \mathbf{e}_{1o} & \mathbf{e}_{2n}^T \mathbf{e}_{2o} \end{bmatrix} = \mathbf{T}_n \mathbf{T}_o^T$ [eq. 4.43]

$$\mathbf{e}_{1o} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \mathbf{e}_{2o} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$$ [eq. 4.44]

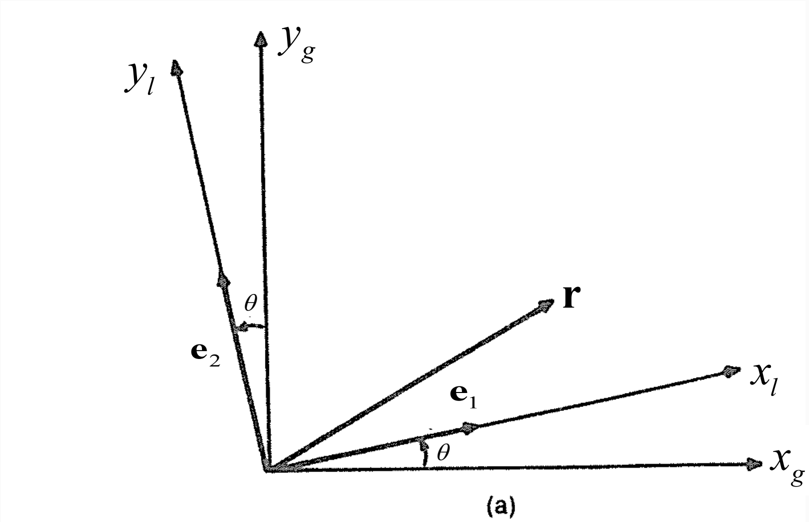
$$\mathbf{e}_{1n} = \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}, \mathbf{e}_{2n} = \begin{bmatrix} -\sin(\alpha+\theta) \\ \cos(\alpha+\theta) \end{bmatrix}$$ [eq. 4.45]

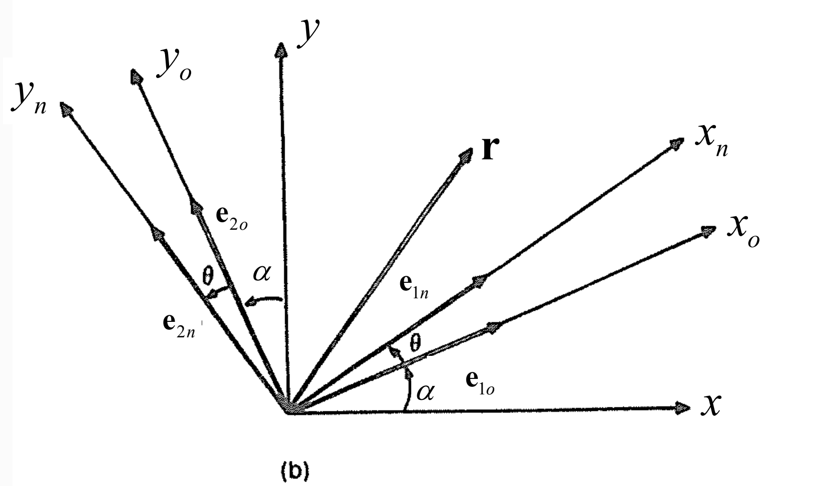
- Stress is transformed by:

$$\boldsymbol{\sigma}_n = \mathbf{T}\boldsymbol{\sigma}_o \mathbf{T}^T$$ [eq. 4.42]

- In three-dimensional case:

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_{1n}^T \mathbf{e}_{1o} & \mathbf{e}_{1n}^T \mathbf{e}_{2o} & \mathbf{e}_{1n}^T \mathbf{e}_{3o} \\ \mathbf{e}_{2n}^T \mathbf{e}_{1o} & \mathbf{e}_{2n}^T \mathbf{e}_{2o} & \mathbf{e}_{2n}^T \mathbf{e}_{3o} \\ \mathbf{e}_{3n}^T \mathbf{e}_{1o} & \mathbf{e}_{3n}^T \mathbf{e}_{2o} & \mathbf{e}_{3n}^T \mathbf{e}_{3o} \end{bmatrix}$$ [eq. 4.49] or $T_{ij} = \mathbf{e}_{in}^T \mathbf{e}_{jo}$ [eq. 4.50]





[Fig 4.4 Different axis for transformations (a) 'local' and 'global' (b) 'old' and 'new']

- Stress is transformed by:

$$\boldsymbol{\sigma}_n = \mathbf{T}\boldsymbol{\sigma}_o\mathbf{T}^T$$ [eq. 4.42]

or $$\sigma_{ij}^n = T_{ia}\sigma_{ab}^o T_{bj}^T = T_{ia}\sigma_{ab}^o T_{jb} = T_{ia}T_{jb}\sigma_{ab}^o$$ [eq. 4.51]

- In similar fashion, strain is transformed by:

$$\varepsilon_{ij}^n = T_{ia}T_{jb}\varepsilon_{ab}^o \underset{\mathbf{T}^{-1}=\mathbf{T}^T}{\Leftrightarrow} \varepsilon_{ij}^o = T_{ai}T_{bj}\varepsilon_{ab}^n$$ [eq. 4.52]

- Consequently, a constitutive relationship,

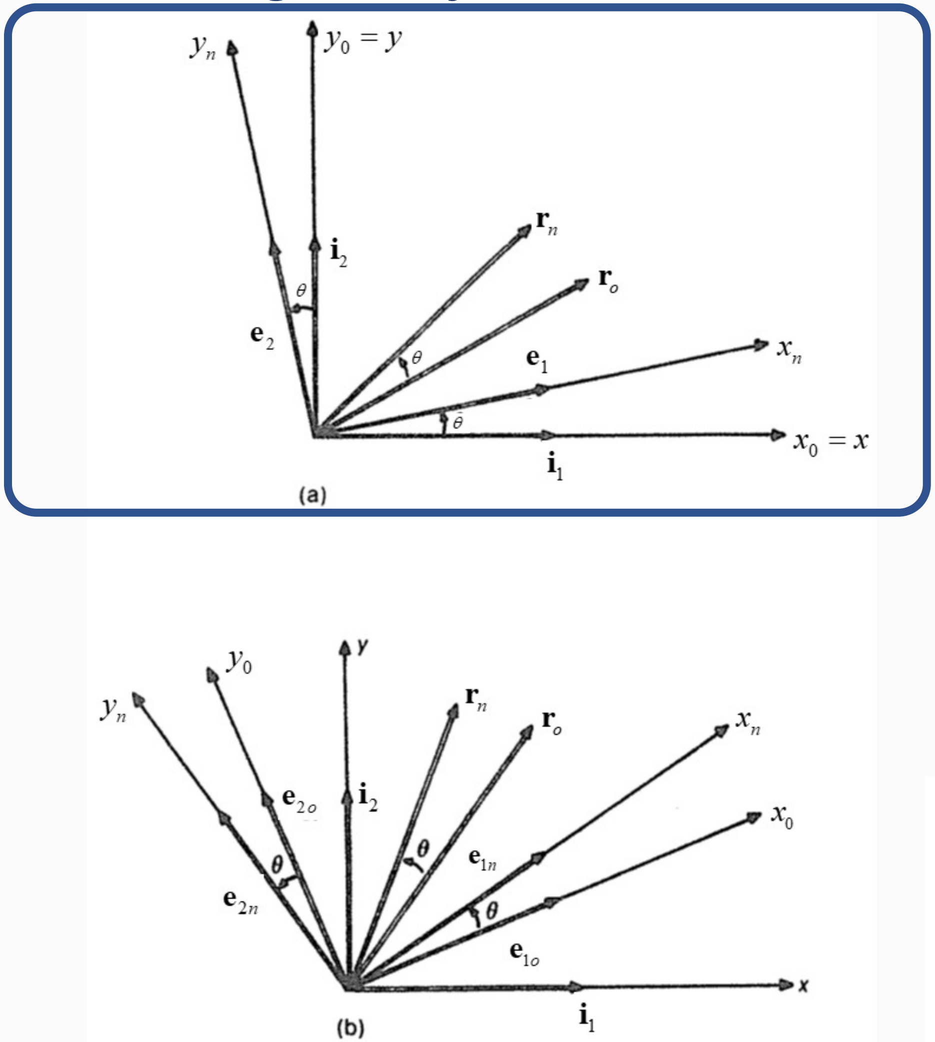
$$\sigma_{ab}^o = C_{abcd}^o\varepsilon_{cd}^o$$ [eq. 4.53]

would transform to: $$\sigma_{ij}^n = C_{ijkl}^n\varepsilon_{kl}^n$$ [eq. 4.54]

where $$C_{ijkl}^n = T_{ia}T_{jb}T_{kc}T_{ld}C_{abcd}^o$$

[eq. 4.55]

4.3.2 A rigid-body rotation



[Fig 4.6 Applying a rotation
(a) with $x = x_o$, $y = y_o$ (b) $x \neq x_o$, $y \neq y_o$]

- In the previous section, the line element remained fixed but was related to different sets of axes: in the present, the line element, **r**, will be physically rotated from $\mathbf{r}_o$ to $\mathbf{r}_n$.

$$\mathbf{r}_n = \mathbf{R}\mathbf{r}_o \quad \text{[eq. 4.56]}$$

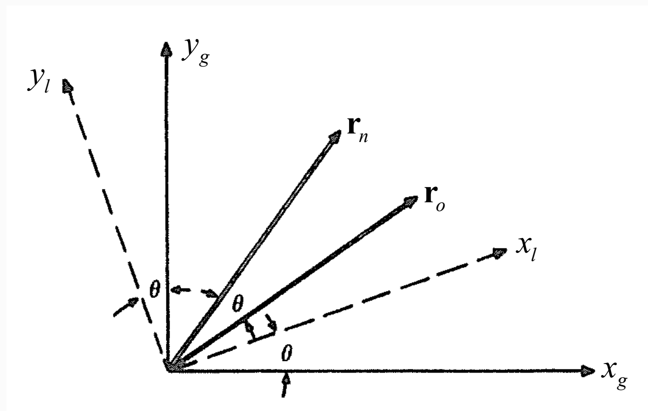
$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \quad \text{[eq. 4.57]}$$

- In three dimension,

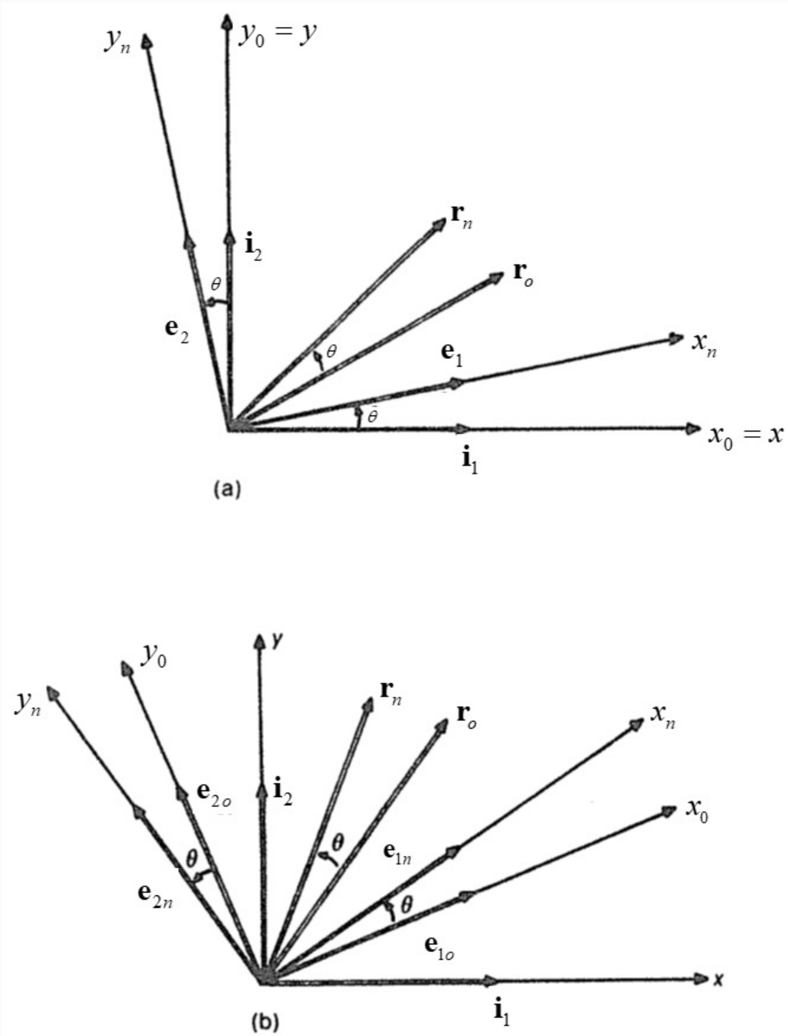
$$\mathbf{e}_i = \mathbf{R}\mathbf{i}_i, \quad (i = 1,2,3) \quad \text{[eq. 4.58]}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \quad \text{[eq. 4.59]}$$

- Relationship between **R** and **T** is: $\mathbf{R} = \mathbf{T}^T$ [eq. 4.60]



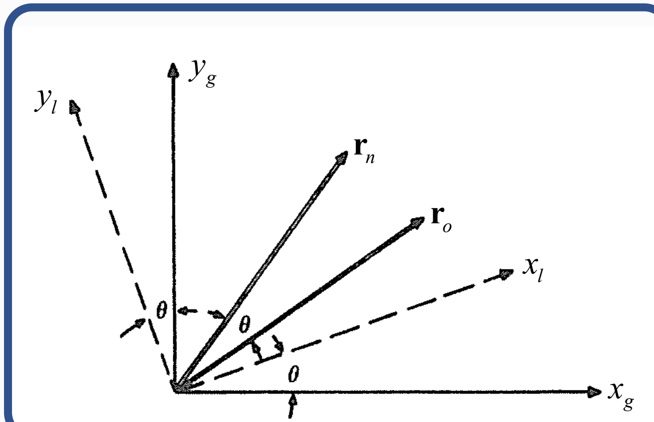
[Fig 4.7 Illustrating rotations and transformations]



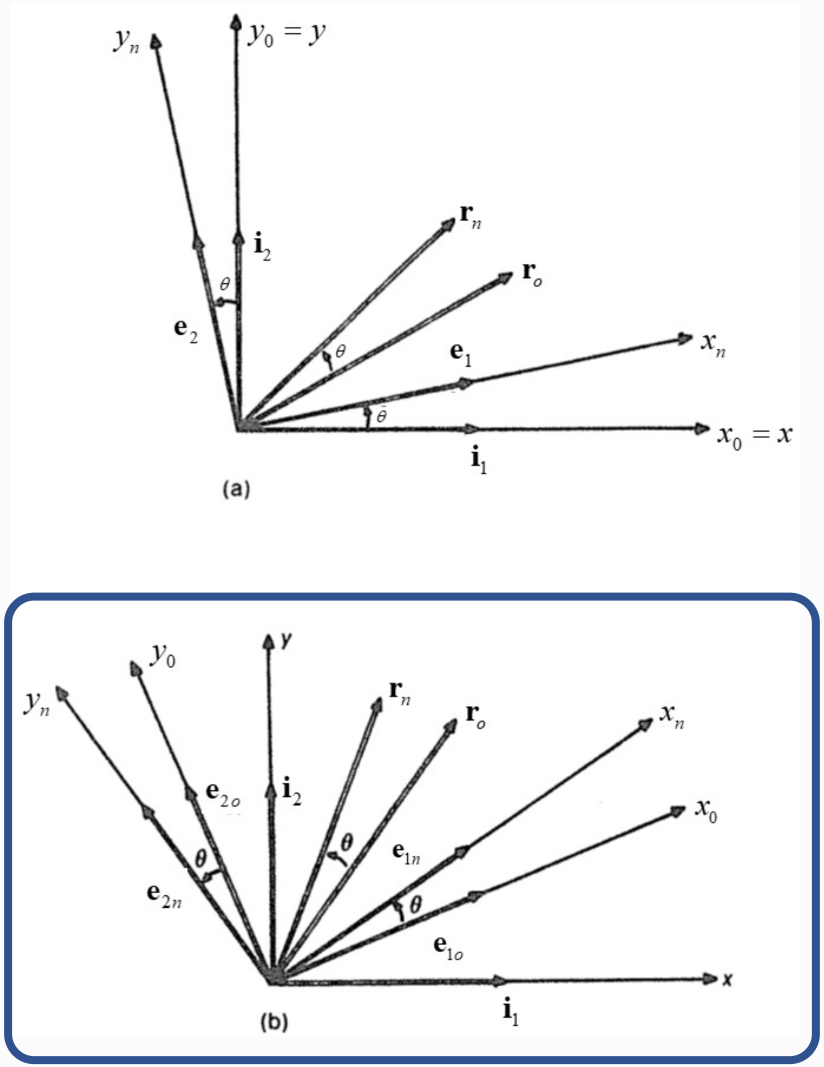
[Fig 4.6 Applying a rotation
(a) with x = x_o, y = y_o (b) x ≠ x_o, y ≠ y_o]

- Local and global coordinates are related by:

$$\mathbf{r}_{n,g} = \mathbf{R}\mathbf{r}_{o,g} = \mathbf{R}\mathbf{r}_{\mathrm{n,1}} = \mathbf{T}^T\mathbf{r}_{n,l}$$ [eq. 4.61, 4.62]



[Fig 4.7 Illustrating rotations and transformations]



[Fig 4.6 Applying a rotation
(a) with $x = x_o$, $y = y_o$ (b) $x \neq x_o$, $y \neq y_o$]

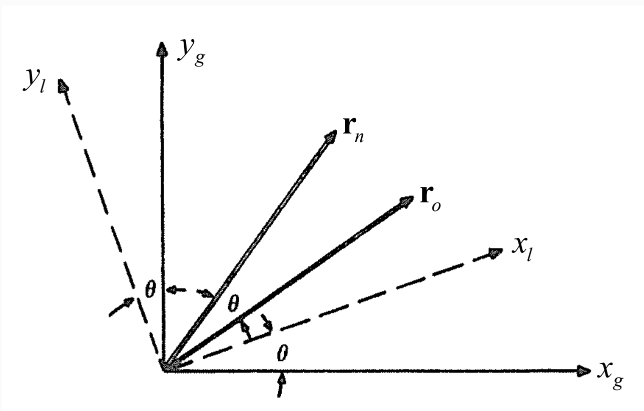
- Position vectors and basis are given by:

$$\mathbf{e}_{in} = \mathbf{R}_n \mathbf{i}_i, \quad \mathbf{e}_{io} = \mathbf{R}_o \mathbf{i}_i, \qquad i = 1, 2, 3 \qquad \text{[eq. 4.64]}$$

$$\mathbf{r}_n = \mathbf{R}_n \mathbf{r}, \quad \mathbf{r}_o = \mathbf{R}_o \mathbf{r} \qquad \text{[eq. 4.65]}$$

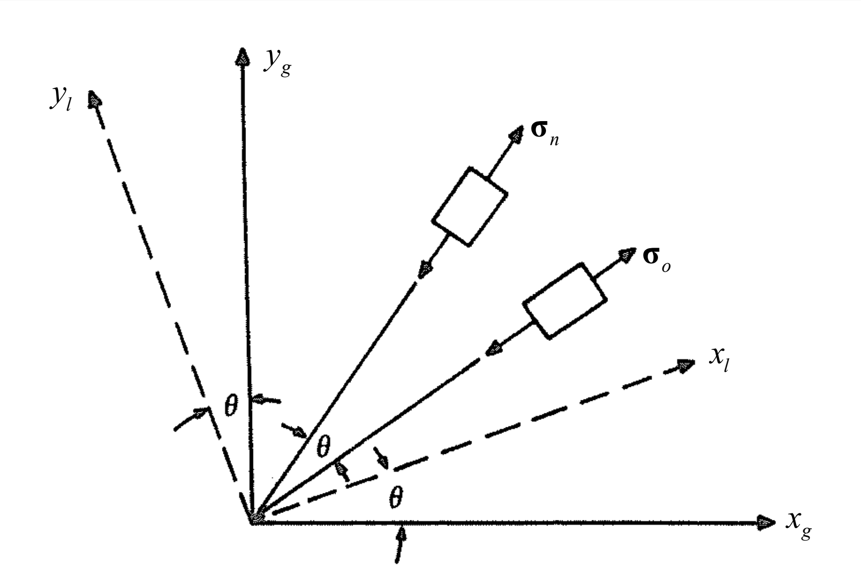
$$\mathbf{r}_n = \mathbf{R}_n \mathbf{R}_o^T \mathbf{r}_o = \mathbf{R} \mathbf{r}_o \qquad \text{[eq. 4.66]}$$

$$\mathbf{R} = \mathbf{e}_{1n}\mathbf{e}_{1o}^T + \mathbf{e}_{2n}\mathbf{e}_{2o}^T + \mathbf{e}_{3n}\mathbf{e}_{3o}^T \qquad \text{[eq. 4.67]}$$



[Fig 4.7 Illustrating rotations and transformations]

- If a set of **stress** in the global coordinates system is rotated, the global system can be assumed to have rotated by same angle to local coordinate system.



[Fig 4.8 Rotating a stress state]

$$\boldsymbol{\sigma}_{n,l} = \boldsymbol{\sigma}_{\mathrm{o,g}}$$

[eq. 4.63]

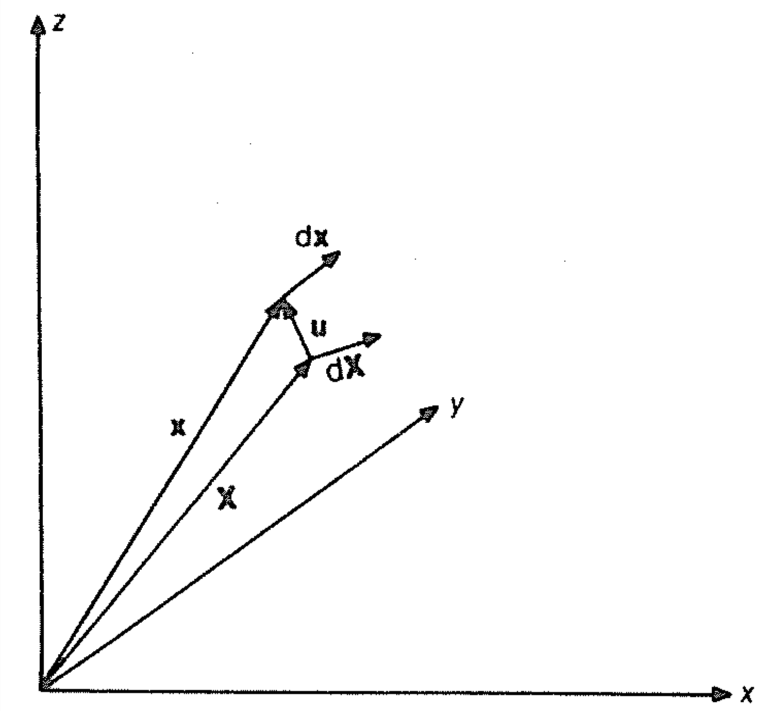
$$\boldsymbol{\sigma}_{n,g} = \mathbf{T}^T \boldsymbol{\sigma}_{n,l} \mathbf{T} = \mathbf{T}^T \boldsymbol{\sigma}_{\mathrm{o,g}} \mathbf{T} = \mathbf{R} \boldsymbol{\sigma}_{\mathrm{o,g}} \mathbf{R}^T$$

- For small strains,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

[eq. 4.5]



[Fig 4.9 position vectors and displacements]

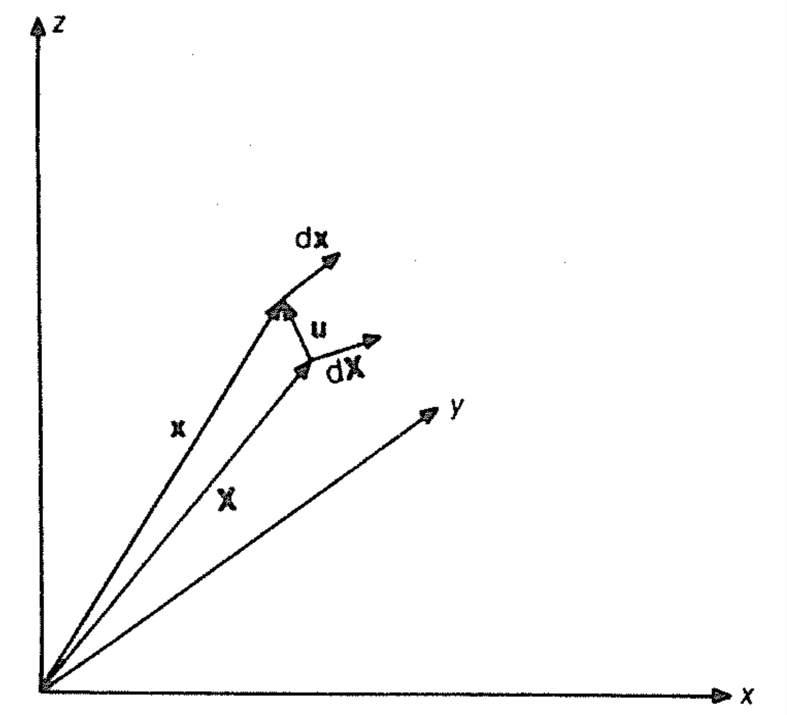
- For large strains, strains from sections 3.1.2 and 3.3.1 should be generalized.

$$d\mathbf{r}_n^2 - d\mathbf{r}_o^2 = d\mathbf{x}^2 - d\mathbf{X}^2 = 2d\mathbf{X}^T\mathbf{E}_2 d\mathbf{X}$$ [eq. 4.68]

$$\mathbf{E}_2 : Green(-Lagrange)\ strain$$

$$\mathbf{x} = \mathbf{X} + \mathbf{u}$$ [eq. 4.69]

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X} = \mathbf{F} d\mathbf{X} = \frac{\partial(\mathbf{X}+\mathbf{u})}{\partial \mathbf{X}} d\mathbf{X}$$ [eq. 4.70]



[Fig 4.9 position vectors and displacements]

$$d\mathbf{x} = \mathbf{F}\,d\mathbf{X}$$

$\mathbf{F}$: *deformation gradient*

$\mathbf{D}$: *displacement – derivative matrix*

$$= \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

$$= \begin{bmatrix} 1+\frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} & 1+\frac{\partial v}{\partial Y} & \frac{\partial v}{\partial Z} \\ \frac{\partial w}{\partial X} & \frac{\partial w}{\partial Y} & 1+\frac{\partial w}{\partial Z} \end{bmatrix} \begin{pmatrix} dX \\ dY \\ dZ \end{pmatrix}$$

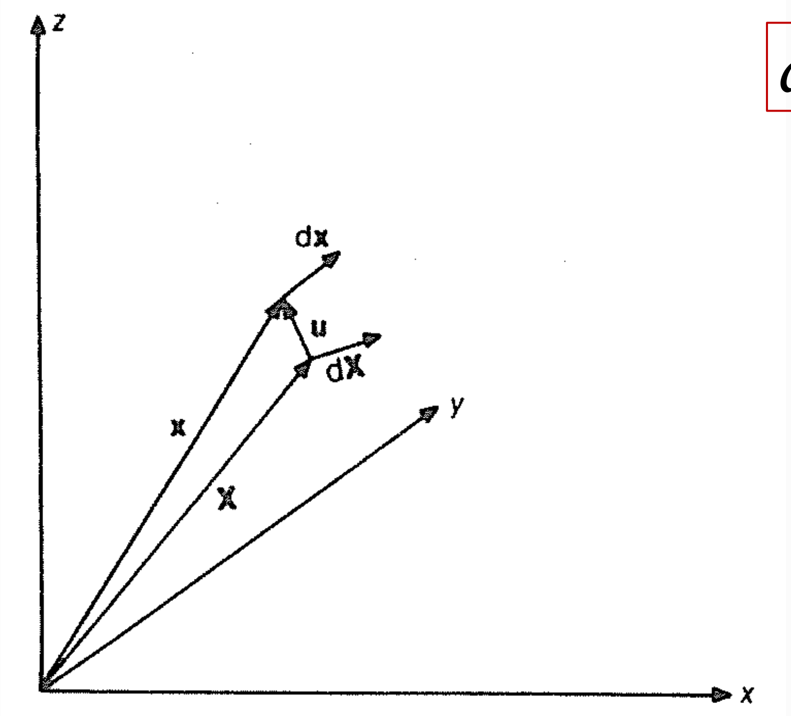
$$= [\mathbf{I}+\mathbf{D}]\,d\mathbf{X} \qquad \text{where} \qquad \mathbf{D} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$

[eq. 4.71] [eq. 4.72]

$$dx^2 - dX^2 = 2d\mathbf{X}^T\mathbf{E}_2 d\mathbf{X}$$ [eq. 4.68]

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}$$ [eq. 4.71]

$$\mathbf{D} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$ [eq. 4.72]



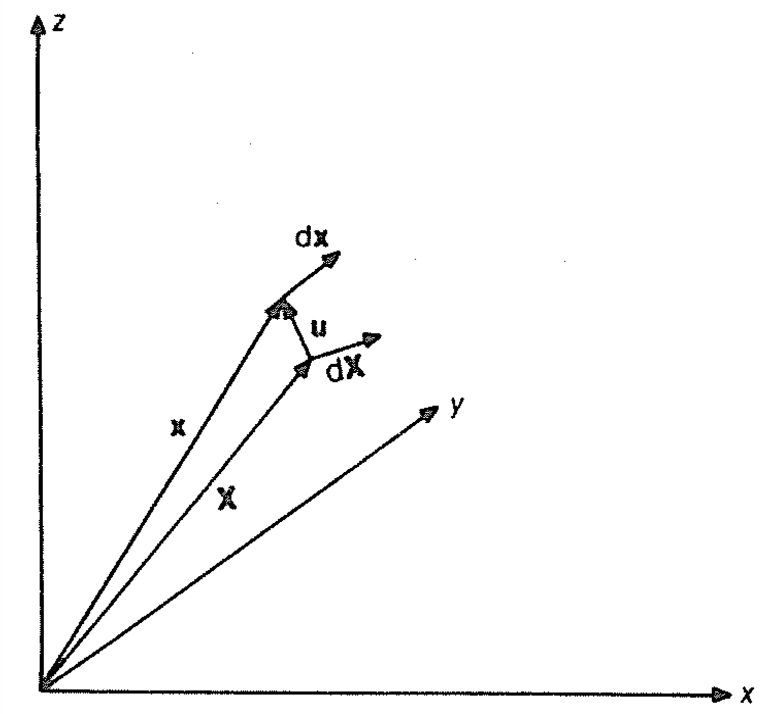
[Fig 4.9 position vectors and displacements]

$$\Rightarrow 2\mathbf{E}_2 = \mathbf{F}^T\mathbf{F} - \mathbf{I} = [\mathbf{I}+\mathbf{D}]^T[\mathbf{I}+\mathbf{D}] - \mathbf{I}$$ [eq. 4.73]

$$\mathbf{E}_2 = \frac{1}{2}\left[\mathbf{F}^T\mathbf{F} - \mathbf{I}\right] = \underbrace{\frac{1}{2}\left[\mathbf{D}+\mathbf{D}^T\right]}_{\boldsymbol{\varepsilon}_2} + \frac{1}{2}\mathbf{D}^T\mathbf{D}$$ [eq. 4.74]

$$\mathbf{E} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial v}{\partial Y} \\ \frac{\partial w}{\partial Z} \\ \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} \\ \frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X} \\ \frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\left(\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2\right) \\ \frac{1}{2}\left(\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right) \\ \frac{1}{2}\left(\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2 + \left(\frac{\partial w}{\partial Z}\right)^2\right) \\ \left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial u}{\partial Y}\right) + \left(\frac{\partial v}{\partial X}\right)\left(\frac{\partial v}{\partial Y}\right) + \left(\frac{\partial w}{\partial X}\right)\left(\frac{\partial w}{\partial Y}\right) \\ \left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial u}{\partial Z}\right) + \left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial v}{\partial Z}\right) + \left(\frac{\partial w}{\partial X}\right)\left(\frac{\partial w}{\partial Z}\right) \\ \left(\frac{\partial u}{\partial Y}\right)\left(\frac{\partial u}{\partial Z}\right) + \left(\frac{\partial v}{\partial Y}\right)\left(\frac{\partial v}{\partial Z}\right) + \left(\frac{\partial w}{\partial Y}\right)\left(\frac{\partial w}{\partial Z}\right) \end{pmatrix}$$

[eq. 4.75]



[Fig 4.9 position vectors and displacements]

- If only rotation happens,

$$\mathbf{x} = \mathbf{R}\mathbf{X}$$

$$d\mathbf{x} = \mathbf{R}d\mathbf{X} \quad \Rightarrow \quad \mathbf{F} = \mathbf{R}$$

$$\Rightarrow \quad \mathbf{E}_2 = \mathbf{0}$$

- However, if we neglect $\frac{1}{2}\mathbf{D}^T\mathbf{D}$

$$\boldsymbol{\varepsilon}_2 = \frac{1}{2}\left(\mathbf{R} + \mathbf{R}^T\right) - \mathbf{I}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

[eq. 4.5]

even though there is no deformation.

This is why we need exact definition of strain.

- Of course, when the rotating angle goes to zero,

$\frac{1}{2}\mathbf{D}^T\mathbf{D}$ goes to zero too.

4.4.1 Virtual work expressions using Green's strain

- The stress measure that is conjugate the Green's strain is the **2nd Piola-Kirchhoff** stress tensor.

$$\mathbf{S}(or\ \mathbf{S}_2) : 2^{nd}\ \ Piola-Kirchhoff\ \ stress\ tensor$$

- Virtual work expression is given by:

$$V = V_i - V_e = \int \mathbf{S}^T \delta \mathbf{E}_v dV_0 - V_e = \int \mathbf{S}_2 : \delta \mathbf{E}_{v2} dV_0 - V_e \qquad \text{[eq. 4.76]}$$

by $\mathbf{E}_2 = \frac{1}{2}\left[\mathbf{F}^T\mathbf{F} - \mathbf{I}\right] = \frac{1}{2}\left[\mathbf{D} + \mathbf{D}^T\right] + \frac{1}{2}\mathbf{D}^T\mathbf{D}$,

$$\delta \mathbf{E}_2 = \frac{1}{2}\left[\delta\mathbf{D} + \delta\mathbf{D}^T\right] + \underbrace{\frac{1}{2}\mathbf{D}^T\delta\mathbf{D} + \frac{1}{2}\delta\mathbf{D}^T\mathbf{D}}_{=\mathbf{D}^T\delta\mathbf{D}} + \frac{1}{2}\delta\mathbf{D}^T\delta\mathbf{D} \qquad \text{[eq. 4.74]}$$

$$\delta \mathbf{E}_{2v} = \frac{1}{2}\left[\delta\mathbf{D}_v + \delta\mathbf{D}_v^T\right] + \mathbf{D}^T\delta\mathbf{D}_v$$

or $\delta \mathbf{E}_2 = \frac{1}{2}\mathbf{F}^T\delta\mathbf{D} + \frac{1}{2}\delta\mathbf{D}^T\mathbf{F} + \frac{1}{2}\delta\mathbf{D}^T\delta\mathbf{D}$ [eq. 4.77]

$$\delta \mathbf{E}_{v2} = \frac{1}{2}\mathbf{F}^T\delta\mathbf{D}_v + \frac{1}{2}\delta\mathbf{D}_v^T\mathbf{F} \qquad \text{[eq. 4.79]}$$

$$\delta V = \int\left(\delta\mathbf{S}^T\delta\mathbf{E}_v + \mathbf{S}^T\delta\left(\delta(\mathbf{E}_v)\right)\right)dV_0 = \int\left(\delta\mathbf{S}_2 : \delta\mathbf{E}_{v2} + \mathbf{S}_2 : \delta\left(\delta(\mathbf{E}_{v2})\right)\right)dV_0 \qquad \text{[eq. 4.80]}$$

$$\delta V = \int \left(\delta \mathbf{S}^T \delta \mathbf{E}_v + \mathbf{S}^T \delta \left(\delta (\mathbf{E}_v) \right) \right) dV_0 = \int \left(\delta \mathbf{S}_2 : \delta \mathbf{E}_{v2} + \mathbf{S}_2 : \delta \left(\delta (\mathbf{E}_{v2}) \right) \right) dV_0$$ [eq. 4.80]

$$\delta \mathbf{E}_{2v} = \frac{1}{2} \left[\delta \mathbf{D}_v + \delta \mathbf{D}_v^T \right] + \mathbf{D}^T \delta \mathbf{D}_v$$

$$\delta \left(\delta (\mathbf{E}_{v2}) \right) = \frac{1}{2} \left[\delta \mathbf{D}_v^T \delta \mathbf{D} + \delta \mathbf{D}^T \delta \mathbf{D}_v \right] = \delta \mathbf{D}_v^T \delta \mathbf{D}$$ [eq. 4.81]

δ^2 is neglected for virtual change

$$\delta \mathbf{S} = \mathbf{C}_{t2} \delta \mathbf{E}, \quad \delta \mathbf{S}_2 = \mathbf{C}_{t4} : \delta \mathbf{E}_2$$ [eq. 4.82]

$$\Rightarrow \quad \delta V = \int \left(\delta \mathbf{E}_{v2} : \mathbf{C}_{t4} : \delta \mathbf{E}_2 + \mathbf{S} : \delta \mathbf{D}_v^T \delta \mathbf{D} \right) dV_0 = \int \left(\delta \mathbf{E}^T \mathbf{C}_{t2} \delta \mathbf{E} + \mathbf{S}^T \delta \left(\delta \mathbf{E}_{v2} \right) \right) dV_0$$

[eq. 4.83]

- This equation will be used in total Lagrangian finite element formulation for a continuum (sections 5.1 and 5.2)

4.4.2 Work expressions using von Karman's non-linear strain-displacement relationships for a plate

- If, for a plate in x-y plane,
- and if in-plane strain terms, $\left(\frac{\partial u}{\partial X}\right)^2, \left(\frac{\partial v}{\partial X}\right)^2$, etc are negligible,

$$\mathbf{E} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial v}{\partial Y} \\ \frac{\partial w}{\partial Z} \\ \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} \\ \frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X} \\ \frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\left(\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2\right) \\ \frac{1}{2}\left(\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right) \\ \frac{1}{2}\left(\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2 + \left(\frac{\partial w}{\partial Z}\right)^2\right) \\ \left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial u}{\partial Y}\right) + \left(\frac{\partial v}{\partial X}\right)\left(\frac{\partial v}{\partial Y}\right) + \left(\frac{\partial w}{\partial X}\right)\left(\frac{\partial w}{\partial Y}\right) \\ \left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial u}{\partial Z}\right) + \left(\frac{\partial u}{\partial X}\right)\left(\frac{\partial v}{\partial Z}\right) + \left(\frac{\partial w}{\partial X}\right)\left(\frac{\partial w}{\partial Z}\right) \\ \left(\frac{\partial u}{\partial Y}\right)\left(\frac{\partial u}{\partial Z}\right) + \left(\frac{\partial v}{\partial Y}\right)\left(\frac{\partial v}{\partial Z}\right) + \left(\frac{\partial w}{\partial Y}\right)\left(\frac{\partial w}{\partial Z}\right) \end{pmatrix}$$

[eq. 4.75]

$$\Rightarrow \quad \mathbf{E} = \begin{pmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial v}{\partial Y} \\ \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial X}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial Y}\right)^2 \\ \left(\frac{\partial w}{\partial X}\right)\left(\frac{\partial w}{\partial Y}\right) \end{pmatrix}$$

[eq. 4.84]

$$\mathbf{E}=\begin{pmatrix}\dfrac{\partial u}{\partial X}\\ \dfrac{\partial v}{\partial Y}\\ \dfrac{\partial u}{\partial Y}+\dfrac{\partial v}{\partial X}\end{pmatrix}+\begin{pmatrix}\dfrac{1}{2}\left(\dfrac{\partial w}{\partial X}\right)^2\\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial Y}\right)^2\\ \left(\dfrac{\partial w}{\partial X}\right)\left(\dfrac{\partial w}{\partial Y}\right)\end{pmatrix} \Rightarrow \delta\mathbf{E}=\begin{pmatrix}\dfrac{\partial \delta u}{\partial X}\\ \dfrac{\partial \delta v}{\partial Y}\\ \dfrac{\partial \delta u}{\partial Y}+\dfrac{\partial \delta v}{\partial X}\end{pmatrix}+\begin{bmatrix}\dfrac{\partial w}{\partial X} & 0\\ 0 & \dfrac{\partial w}{\partial Y}\\ \dfrac{\partial w}{\partial Y} & \dfrac{\partial w}{\partial X}\end{bmatrix}\begin{pmatrix}\dfrac{\partial \delta w}{\partial X}\\ \dfrac{\partial \delta w}{\partial Y}\end{pmatrix}+\begin{pmatrix}\dfrac{1}{2}\left(\dfrac{\partial \delta w}{\partial X}\right)^2\\ \dfrac{1}{2}\left(\dfrac{\partial \delta w}{\partial Y}\right)^2\\ \left(\dfrac{\partial \delta w}{\partial X}\right)\left(\dfrac{\partial \delta w}{\partial Y}\right)\end{pmatrix}$$

[eq. 4.85]

$$\Rightarrow \delta(\delta\mathbf{E})=\begin{pmatrix}\left(\dfrac{\partial \delta w}{\partial X}\right)\left(\dfrac{\partial \delta w}{\partial X}\right)_v\\ \left(\dfrac{\partial \delta w}{\partial Y}\right)\left(\dfrac{\partial \delta w}{\partial Y}\right)_v\\ \left(\dfrac{\partial \delta w}{\partial X}\right)\left(\dfrac{\partial \delta w}{\partial Y}\right)_v+\left(\dfrac{\partial \delta w}{\partial X}\right)_v\left(\dfrac{\partial \delta w}{\partial Y}\right)\end{pmatrix}$$

[eq. 4.86]

$$\Rightarrow dV=\int\left[\delta\mathbf{S}^T\delta\mathbf{E}_v+\begin{pmatrix}\dfrac{\partial \delta w}{\partial X} & \dfrac{\partial \delta w}{\partial Y}\end{pmatrix}\mathbf{S}_2\begin{pmatrix}\dfrac{\partial \delta w}{\partial X}\\ \dfrac{\partial \delta w}{\partial Y}\end{pmatrix}\right]dV_0$$

[eq. 4.87]

where $\mathbf{S}=\begin{pmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{pmatrix}$ [eq. 4.87] $\mathbf{S}_2=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}$ [eq. 4.88]


Thank you!