Motion Control of Robotic Manipulators

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Computed Torque Control

- Robot open-loop dynamics:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + f$$

where $\tau \in \Re^n$ is the control torque and $f \in \Re^n$ external force.

- Trajectory tracking control: design τ s.t., with $f \approx 0$,

$$(q(t), \dot{q}(t)) \rightarrow (q_d(t), \dot{q}_d(t))$$

where $q_d(t) \in \Re^n$ is smooth joint trajectory (e.g., from IK: for WS, later).

- **Computed torque control:** design τ s.t.,

$$\tau = M(q)[\ddot{q}_d - B(\dot{q} - \dot{q}_d) - K(q - q_d)] + C(q,\dot{q})\dot{q} + g(q)$$

exact dynamics cancelation

so that the closed-loop dynamics becomes: with $e := q_d - q$,

$$\ddot{e} + B\dot{e} + Ke = 0$$

implying that $(\dot{e}, e) \rightarrow 0$ exponentially, if $B, K \in \Re^{n \times n}$ are positive-definite and symmetric (even for non-diagonal B, K)?.

Stabiliy of Second-Order LTI System

- Closed-loop error dynamics:

$$\ddot{e} + B\dot{e} + Ke = 0$$

- Define $x := (\dot{e}, e) \in \Re^{2n}$. Then, the state-space representation:

$$\dot{x} = Ax = \begin{bmatrix} -B & -K \\ I & 0 \end{bmatrix} x = Ax$$

with all the **eigenvalues** $\lambda(A)$ in LHP, i.e., $x \to 0$ exponentially.

- Define $V = \frac{1}{2}\dot{e}^T\dot{e} + \frac{1}{2}e^T e = \frac{1}{2}x^T x$. Then,

$$\dot{V} = \dot{e}^T[-B\dot{e} - Ke] + \dot{e}^T Ke = -\dot{e}^T B\dot{e}$$

implying $\dot{e} \to 0$ likely (since $V \geq 0$); if so, $\ddot{e} \to 0$ likely and $e \to 0$ as well.

- This yet still implies that

$$\frac{1}{2}||x(t)||_2^2 = \frac{1}{2}\dot{e}^T\dot{e} + \frac{1}{2}e^T e = V(t) \leq V(0) = \frac{1}{2}||x(0)||_2^2$$

i.e., $||x(t)|| \leq ||x(0)|| \Rightarrow$ starts close, stay close (stability?).

Cross-Coupling Term

- Define $V = \frac{1}{2}\dot{e}^T\dot{e} + \epsilon e^T\dot{e} + \frac{1}{2}e^T[K + \epsilon B]e$. Then,

cross-coupling term

$$\dot{V} = -\dot{e}^T[B - \epsilon I]\dot{e} - \epsilon e^T Ke$$

where $V \geq 0$ and $\dot{V} \leq 0$ with small-enough $\epsilon > 0$: likely $\dot{V} \to 0$, $e \to 0$.

- With cross-coupling $\epsilon > 0$, we have

$$V = \frac{1}{2}\begin{pmatrix} \dot{e} \\ e \end{pmatrix}^T \begin{bmatrix} I & \epsilon I \\ \epsilon I & K + \epsilon B \end{bmatrix} \begin{pmatrix} \dot{e} \\ e \end{pmatrix}, \quad \dot{V} = -\begin{pmatrix} \dot{e} \\ e \end{pmatrix}^T \begin{bmatrix} B - \epsilon I & 0 \\ 0 & \epsilon K \end{bmatrix} \begin{pmatrix} \dot{e} \\ e \end{pmatrix}$$

with $P, Q \in \Re^{2n \times 2n}$ are positive-definite with small enough $\epsilon > 0$, from

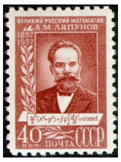
$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} > 0, \quad \text{if } A = A^* = \bar{A}^T > 0,\ C > B^* A^{-1} B$$

- Thus, with small-enough $\epsilon > 0$,

$$\dot{V} = -x^T Qx \leq -\lambda_{\min}[Q]||x||^2 \leq -\frac{\lambda_{\min}[Q]}{\lambda_{\max}[P]}V = -\gamma V$$

implying that $V(t) \leq V(0)e^{-\gamma t}$, i.e., $(\dot{e}, e) \to 0$ exponentially.

Lyapunov Stability - Definition

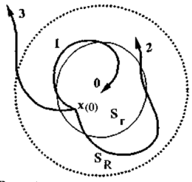


(1857-1918)

Def. 2.1: Consider an **autonomous** system

$$\dot{x} = f(x), \quad f(0) = 0$$

$f : \mathcal{D} \to \Re^n$ locally Lipschitz on $\mathcal{D}$ and $0 \in \mathcal{D}$. Then, equilibrium $x = 0$ is



- **Lyapunov stable**, if, $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ s.t.,

$$||x(0)|| < \delta \Longrightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0$$

- **unstable**, if it is not stable
- **asymptotically stable**, if it is stable and we can find $\delta' > 0$ s.t.

$$||x(0)|| < \delta' \Longrightarrow ||x(t)|| \to 0$$

- **exponentially stable** if $\exists \alpha, \gamma, \delta' > 0$ s.t.,

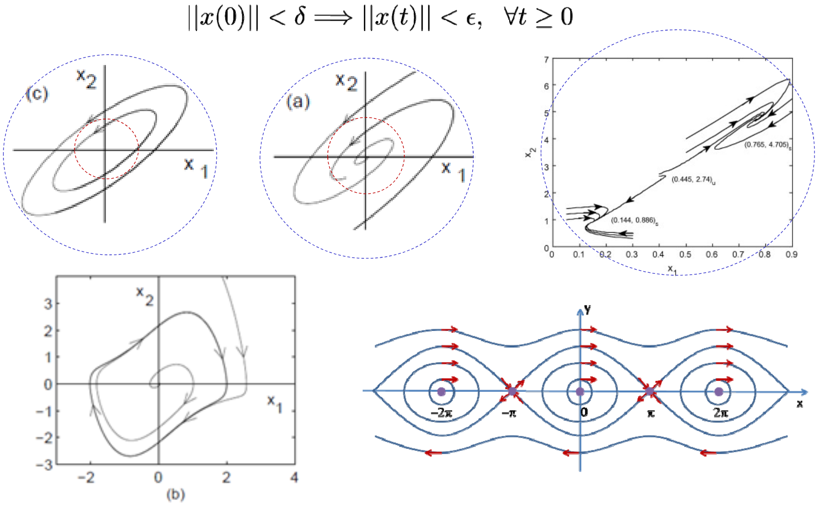
$$||x(0)|| < \delta' \Longrightarrow ||x(t)|| \leq \alpha ||x(0)|| e^{-\gamma t}$$

- globally asymptotically stable, if asymptotically stable for any $\forall x(0) \in \Re^n$.

Lyapunov Stability - Examples

Lyapunov stable, if, for any $\epsilon > 0$, there exits $\delta(\epsilon) > 0$ s.t.,

$$||x(0)|| < \delta \Longrightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0$$



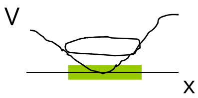
satisfy defintion: 1) for some ϵ or 2) $\forall \delta$, $\exists \epsilon$

Lyapunov Direct Theorem

Th. 2.1. (Lyapunov Direct): If we can find a Lyapunov function $V : \mathcal{D} \to \Re$, which is continuously differentiable and satisfies

$$V(x) \geq 0 \ \text{ in } \mathcal{D}, \text{ with } V(x) = 0 \ \text{ iff } \ x = 0$$

negative semi-definite ← $$\dot{V}(x) \leq 0 \ \text{ in } \mathcal{D} \text{ along the solution of } \dot{x} = f(x)$$



then, $x = 0$ is **Lyapunov stable** (ex. $V = \frac{1}{2}x^T x$). Moreover, if

$$\dot{V}(x) \leq 0 \ \text{ in } \mathcal{D}, \text{ with } \dot{V}(x) = 0 \ \text{ iff } \ x = 0$$ → negative-definite

then, $x = 0$ is **asymptotically stable** (ex. $V = \frac{1}{2}x^T P x$). Furthermore, if

$$k_1||x||^2 \leq V(x) \leq k_2||x||^2, \quad \text{with} \quad \dot{V} \leq -k_3||x||^2 \leq -\frac{k_3}{k_2}V$$

then, $x = 0$ is **exponentially stable** with $||x(t)|| \leq \sqrt{\frac{V(0)}{k_1}} e^{-\frac{k_3}{2k_2}t} \to 0$.

- Here, $\dot{V}(x)$ is time differentiation of V along the solution $\dot{x} = f(x)$, i.e.,

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) = \mathcal{L}_f V$$

Lyapunov Stability – Non-autonomous System

Consider non-autonomous system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times \mathcal{D} \to \Re^n$ piecewise continuous in t and locally Lipschitz in x, and $x = 0$ is an equilibrium at $t = t_o$, i.e.,

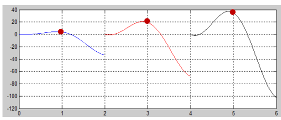
$$f(t, 0) = 0, \quad \forall t \geq t_o \geq 0$$

Def. 2.2: The equilibrium $x = 0$ of the non-autonomous system is

- stable, if, $\forall \epsilon > 0$, $\exists \delta(\epsilon, t_o) > 0$ s.t.

$$||x(t_o)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \quad \forall t \geq t_o \geq 0$$

- uniformly stable, if $\delta(\epsilon) > 0$ is independent of t_o.
- unstable, if not stable.



Ex) $\dot{x} = (6t \sin t - 2t)x$ with $t_o = 2n\pi$: $x(t) = e^{\int_{t_o}^{t}(6\tau \sin\tau - 2\tau)d\tau} x(t_o)$

$x(t) = e^{6\sin t - 6t\cos t - t^2 - 6\sin t_o + 6t_o\cos t_o + t_o^2} x(t_o) \to 0$ as $t \to \infty$

when evaluated at $t = t_o + \pi$: $|x(t_o + \pi)| = |x(t_o)e^{(4n+1)(6-\pi)\pi}| \leq \epsilon$

$\delta(\epsilon, n) = \epsilon / e^{(4n+1)(6-\pi)\pi} \to 0$ as $n \to \infty$

Lyapunov Direct Theorem

Th. 2.6: Consider non-autonomous system with equilibrium at $x = 0 \in \mathcal{D}$. Suppose we can find a continuously differentiable function $V(t,x)$ s.t.

$$\alpha_1(||x||) \le V(t,x) \boxed{\le \alpha_2(||x||)}$$

decrescent condition

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$$

$\forall t \ge 0$ and $\forall x \in \mathcal{D}$, where $\alpha_i \in \mathcal{K}$ on $\mathcal{D}$. Then, $x = 0$ is **uniformly stable**. Moreover, if

strictly-increasing with α(o)=o

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3(||x||)$$

$\forall t \ge 0$ and $\forall x \in \mathcal{D}$, where $\alpha_3 \in \mathcal{K}$ on $\mathcal{D}$, $x = 0$ is **uniformly A.S.** Further, if

$$k_1||x||^2 \le V(t,x) \le k_2||x||^2, \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -k_3||x||^2$$

$\forall t \ge 0$ and $\forall x \in \mathcal{D}$, where $k_i, a > 0$ are constants. Then, $x = 0$ is **U.E.S.**, i.e.,

$$||x(t)|| \le k||x(t_o)||e^{-\lambda(t-t_o)}$$

- Quadratic LF $V = x^T P x$ with $\dot{V} = -x^T Q x$ satisfies above conditions.

Passivity-Based Control

- Robot open-loop dynamics:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + f$$

which is passive, i.e., $\dot{M} - 2C$ is skew-symmetric, or, equivalently, $\forall T \ge 0$,

$$\int_0^T [\tau + f]^T \dot{q} dt = E(T) - E(0)$$

- Computed torque control:

$$\tau = M(q)[\ddot{q}_d - B(\dot{q} - \dot{q}_d) - K(q - q_d)] + C(q,\dot{q})\dot{q} + g(q)$$

which is not so robust and also cancels out nonlinear dynamics rather than utilizes it (as typical for any **feedback linearization**).

- **Passivity-based control:**

$$\boxed{\tau = \hat{M}(q)\ddot{q}_d + \hat{C}(q,\dot{q})\dot{q}_d + \hat{g}(q) - K_d(\dot{q} - \dot{q}_d) - K_p(q - q_d)}$$

feedforward (inverse dynamics) kinematic feedback

with $\hat{\star}$ are estimates, $K_d, K_p \in \Re^{n \times n}$ are symmetric and PD gain matrices.

Passivity-Based Tracking Control

- Robot open-loop dynamics:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + f$$

with $f \approx 0$ and skew-symmetric $\dot{M} - 2C$.

- **Passivity-based control:**

$$\tau = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d + g(q) - K_d(\dot{q} - \dot{q}_d) - K_p(q - q_d)$$

- Closed-loop error dynamics: with $e = q - q_d$,

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + K_d\dot{e} + K_p e = 0$$ → dynamics utilized

- Define energy-like Lyapunov function $V = \frac{1}{2}\dot{e}^T M\dot{e} + \frac{1}{2}e^T K_p e$. Then, from the skew-symmetricity,

$$\dot{V} = \dot{e}^T M\ddot{e} + \frac{1}{2}\dot{e}^T\dot{M}\dot{e} + \dot{e}^T K_p e = -\dot{e}^T K_d \dot{e}$$

implying that $(\dot{e}, e) = 0$ is only Lyapunov stable. However, it is observed that $(\dot{e}, e) \to 0$ exponentially.

Passivity-Based Stabilization

- Consider passivity-based stabilization control:

$$\tau = g(q) - K_d\dot{q} - K_p(q - q_d)$$

to achieve $(\dot{q}, q) \to (0, q_d)$.

- Then, the closed-loop dynamics becomes:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + K_d\dot{q} + K_p(q - q_d) = 0$$

which should satisfy $(\dot{q}, q) \to (0, q_d)$ as it's mass-spring-damper system.

- Yet, Lyapunov analysis is inconclusive even for this simple system, i.e, if we use total energy as Lyapnuov function

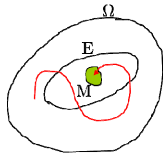
$$V = \frac{1}{2}\dot{q}^T M\dot{q} + \frac{1}{2}e^T K_p e$$

we have

$$\dot{V} = -\dot{q}^T K_d\dot{q} \leq 0$$

- There may be better Lyapunov function, which, yet, in general, is difficult to find (i.e., not constructive approach) ⇒ invariance principle.

Invariance Principle



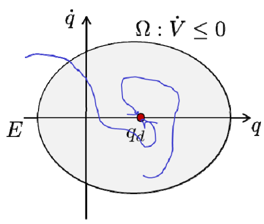
• **LaSalle's Invariance Principle:** Consider autonomous system

$$\dot{x} = f(x), \quad x \in \Re^n, \; f(0) = 0$$

Suppose $\exists$ compact (i.e., bounded and closed) set $\Omega \in \Re^n$ s.t., if $x(t) \in \Omega$, $x(t') \in \Omega \; \forall t' \geq t$ (i.e., Ω positive invariant set). Suppose further $\exists$ a continuously differentiable $V : \Re^n \to \Re$ s.t., $\dot{V}(x) \leq 0 \; \forall x \in \Omega$. Define

$$E := \{x \in \Omega \mid \dot{V}(x) = 0\}$$

and $M \subset E \subset \Omega$ be the largest invariant set in E. Then, if $x(0) \in \Omega$, $x(t) \to M$.



1. Compact and positive-invariant set Ω:

$$\frac{1}{2}\lambda_{\min}[K_p]||q - q_d||^2 + \frac{1}{2}\lambda_{\min}[M]||\dot{q}||^2 \leq V(t) \leq V(0)$$

2. The set $E = \{(q, \dot{q}) \mid \dot{V} = 0\} = \{(q, \dot{q}) \mid \dot{q} = 0\}$.
3. If $\dot{q} = 0$ but $q \neq q_d$, $\ddot{q} \neq 0 \Rightarrow \dot{q} \neq 0$. Thus,

$$M = \{(q, \dot{q}) = (q_d, 0)\} \quad \text{i.e.,} \quad (q, \dot{q}) \to (q_d, 0)$$

Passivity-Based Tracking Control

• Consider passivity-based trajectory tracking control:

$$\tau = M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + g(q) - K_d(\dot{q} - \dot{q}_d) - K_p(q - q_d)$$

• Then, the closed-loop error dynamics becomes: with $e = q - q_d$,

$$M(q)\ddot{e} + C(q, \dot{q})\dot{e} + K_d\dot{e} + K_p e = 0$$

• This closed-loop dynamics is **non-autonomous** system. Thus, LaSalle's invariance principle not applicable.

• Yet, with skew-symmetricity of original dynamics, the closed-loop dynamics still behaves like mass-spring-damper system, i.e.,

$$V = \frac{1}{2}\dot{e}^T M \dot{e} + \frac{1}{2}e^T K_p e \quad \Rightarrow \quad \dot{V} = -\dot{e}^T K_d \dot{e}$$

which looks like damping dissipation.

• This then would likely imply that $\dot{e} \to 0$ as V is lower-bounded by 0. Then, $e \to 0$? $\Rightarrow$ **Barbalat's lemma.**

Barbalat's Lemma

Lem. (Barbalat's) Suppose $f(t) \to c$. Then, $\dot{f}(t) \to 0$, if $\dot{f}(t)$ is uniformly continuous, i.e., $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ s.t.,

$$|t_1 - t_2| \leq \delta(\epsilon) \Longrightarrow |\dot{f}(t_1) - \dot{f}(t_2)| \leq \epsilon, \quad \forall t_1, t_2 \geq 0.$$

- If $\ddot{f}(t)$ is uniformly bounded, $\dot{f}(t)$ is uniformly continuous (MVT)
- Applicable both to autonomous & non-autonomous systems
- $f \to c$, yet, not $\dot{f} \to 0$: $f(t) = e^{-t} \sin e^{2t}$, $\dot{f}(t) = e^{-t} \sin e^{2t} + 2e^{t} \cos e^{2t}$
- $\dot{f} \to 0$, yet, not $f \to c$: $f(t) = \sin(\ln t)$, $\dot{f}(t) = \frac{1}{t} \cos(\ln t)$

Corollary: Suppose $f(t)$ is square-integrable (i.e., $\int_0^\infty f^2(\tau) d\tau \to c$) and $\dot{f}(t)$ is bounded. Then, $f(t) \to 0$.

(Proof) Define $g(t) = \int_0^t f^2(\sigma) d\sigma$. From the assumption, $\lim_{t \to \infty} g(t) = c$. Then, from Barbalat's, $\dot{g}(t) = f^2(t) \to 0$ if $\dot{g}$ is UC or $\ddot{g} = 2\dot{f}f$ is bounded.
- Here, 1) $\dot{f}$ is assumed to be bounded; and 2) $f(t)$ cannot be unbounded, since, if so, $\int_0^t f^2(\sigma) d\sigma$ will also be unbounded (with $\dot{f}$ bounded).
- Thus, $\ddot{g}$ is bounded $\Rightarrow$ $\dot{g}$ is UC $\Rightarrow$ $f \to 0$.

Trajectory Tracking Convergence Proof

- The closed-loop error dynamics:

$$M(q)\ddot{e} + C(q, \dot{q})\dot{e} + K_d \dot{e} + K_p e = 0$$

- Using $V = \frac{1}{2}\dot{e}^T M \dot{e} + \frac{1}{2} e^T K_p e$, $\dot{V} = -\dot{e}^T K_d \dot{e}$, i.e., $V(t) - V(0) = -\int_0^t \dot{e}^T K_d \dot{e} d\tau$, or,

$$0 \leq V(t) = V(0) - \int_0^t \dot{e}^T K_d \dot{e} d\tau \leq V(0)$$

  implying that $(\dot{e}, e)$ is bounded.

- From $\int_0^t \dot{e}^T K_d \dot{e} d\tau \leq V(0)$, $\dot{e}$ is square integrable. Also, from the dynamics with bounded $(e, \dot{e}, q_d, \dot{q}_d)$, if $\frac{\partial m_{ij}}{\partial q_k}$ and $M \geq \lambda_{\min} I$, $\ddot{e}$ will also be bounded, and $\dot{e} \to 0$ from the Corollary.

- Invoking BL again, with $\dot{e} \to 0$, $\ddot{e} \to 0$ if $\ddot{e}$ is UC or $\dddot{e}$ is bounded. This is true if $\frac{\partial^2 m_{ij}}{\partial q_p \partial q_r}$ is also bounded, since

$$M(q)\dddot{e} + \frac{dM(q)}{dt}\ddot{e} + C\ddot{e} + \frac{dC(q, \dot{q})}{dt}\dot{e} + K_d \ddot{e} + K_p \dot{e} = 0$$

  with $c_{kj} = \sum_i [\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}]\dot{q}_i$. This then establish $e \to 0$ too.

Trajectory Tracking Control - Theorem

Theorem: Consider robot dynamics with passivity-based tracking control as defined above. Then, if $q_d(t)$ is $\mathcal{C}^\infty$-smooth, $\frac{\partial m_{ij}}{\partial q_k}$ and $\frac{\partial^2 m_{ij}}{\partial q_p \partial q_r}$ are bounded, and $\exists\ \lambda_{\min} > 0$ s.t., $\lambda_{\min} I \leq M(q)$, $(e, \dot{e}) \to 0$.

- The assumptions are always guaranteed for revolute joint robots.
- $(\dot{e}, e) \to 0$, yet, how fast is the convergence is not specified (may be extremely slow).
- Asymptotic stability may be fragile, that is, it may become divergent with a bit of disturbance, noise and/or uncertainty.
- Exponential convergence is always preferred to asymptotic convergence, as it automatically guarantees a level of **robustness** against external disturbances and parametric uncertainty (ultimately bounded).
- We can establish exponential convergence of the passivity-based tracking control with Lyapunov function with **cross-coupling** term as before:

$$V = \frac{1}{2}\dot{e}^T M \dot{e} + \epsilon \dot{e}^T M e + \frac{1}{2} e^T [K_p + \epsilon K_d] e$$

Alternative Passivity-Based Control with r

- Consider again the robot dynamics:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + f, \quad f \approx 0$$

- Alternative passivity-based control with r-variable:

$$\tau = M[\ddot{q}_d - \Lambda \dot{e}] + C[\dot{q}_d - \Lambda e] + g(q) - K\dot{e} - K\Lambda e + \tau'$$

  with $K, \Lambda \in \Re^{n \times n}$ diagonal and positive-definite (i.e., $K\Lambda = \Lambda K$).

- The closed-loop dynamics becomes:

$$M(q)\dot{r} + C(q,\dot{q})r + Kr = \tau'$$

  where $r := \dot{e} + \Lambda e$ serves now as velocity, defining new passivity input-output pair (τ', r): with $V := \frac{1}{2} r^T M r$,

$$\dot{V} = r^T M \dot{r} + \frac{1}{2} r^T \dot{M} r = \tau'^T r - r^T K r$$

- With $\tau' = 0$, we have $\dot{V} = -r^T K r$ with $V = \frac{1}{2} r^T M r$, implying that $r = \dot{e} + \Lambda e \to 0$ exponentially (i.e., $(\dot{e}, e) \to 0$ exponentially too).

Effect of Uncertainty - I

- Consider the robot dynamics:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + f, \quad f \approx 0$$

under passivity-based control with **parametric uncertainty**:

$$\tau_o = \hat{M}(q)[\ddot{q}_d - \Lambda\dot{e}] + \hat{C}(q,\dot{q})[\dot{q}_d - \Lambda e] + \hat{g}(q) - K\dot{e} - K\Lambda e$$

where $\hat{\star}$ is estimate of $\star$ and $e = q - q_d$.

- From linearity in inertial parameters, $\forall y_1, y_2 \in \Re^n$,

$$M(q)y_1 + C(q,\dot{q})y_2 + g(q) = Y(q,\dot{q},y_1,y_2)\theta$$

where $Y \in \Re^{n\times l}$ (known) regressor; $\theta \in \Re^l$ (uncertain) inertia parameters.

- Since the estimated dynamics has the same structure,

$$\tau_o = Y(q,\dot{q},\dot{\nu},\nu)\hat{\theta} - K\dot{e} - K\Lambda e, \qquad \nu = \dot{q}_d - \Lambda e$$

- Augment nominal control τ_o with $\delta\tau$ to address uncertainty:

$$\tau = \tau_o + \delta\tau$$

Effect of Uncertainty - II

- Closed-loop dynamics:

$$M\ddot{q} + C\dot{q} + g = \hat{M}[\ddot{q}_d - \Lambda\dot{e}] + \hat{C}[\dot{q}_d - \Lambda e] + \hat{g} - K[\dot{e} + \Lambda e] + \delta\tau$$

- Substracting $M[\ddot{q}_d - \Lambda\dot{e}] + C[\dot{q}_d - \Lambda e] + g$ from both sides, we obtain:

$$M[\ddot{e} + \Lambda\dot{e}] + C[\dot{e} + \Lambda e] + K[\dot{e} + \Lambda e] = Y(q,\dot{q},\nu,\dot{\nu})\hat{\theta} - Y(q,\dot{q},\nu,\dot{\nu})\theta + \delta\tau$$
$$= Y(q,\dot{q},\nu,\dot{\nu})[\hat{\theta} - \theta] + \delta\tau$$

uncertainty

or, using $r := \dot{e} + \Lambda e$,

$$M\dot{r} + Cr + Kr = Y[\hat{\theta} - \theta] + \delta\tau$$

→ matching uncertainty: appears in same channel with control

- Define Lyapunov function as before: $V = \frac{1}{2}r^T M r$ (or $V = \frac{1}{2}r^T M r + e^T K\Lambda e$),

$$\frac{dV}{dt} = -r^T K r + r^T Y[\theta - \hat{\theta}] + r^T \delta\tau$$

negative-definite as before; want to make still negative even with uncertainty

- **Robust control:** large-enough $\delta\tau$ to **absorb** the uncertainty effect
- **Adaptive control:** adaptively change estimate $\hat{\theta}$ by seeing error r.

Passivity-Based Robust Control

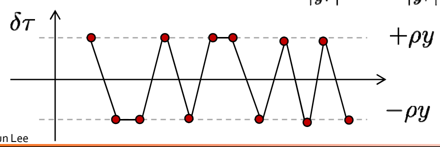
- Closed-loop dynamics: $M\dot{r} + Cr + Kr = Y[\hat{\theta} - \theta] + \delta\tau$.
- Using $V = \frac{1}{2}r^T M r$, $\frac{dV}{dt} = -r^T K r + r^T Y[\theta - \hat{\theta}] + r^T \delta\tau$. → matched uncertainty
- Suppose $\exists \rho \geq 0$ s.t., $||\hat{\theta} - \theta|| \leq \rho$ (i.e., θ estimation error bounded). Then,

$$r^T Y(\theta - \hat{\theta}) + r^T \delta\tau \leq ||Y^T r|| \cdot ||\hat{\theta} - \theta|| + r^T \delta\tau \leq ||Y^T r||\rho + r^T \delta\tau$$

- Thus, if we choose the robust control term $\delta\tau := -\rho \frac{YY^T r}{||Y^T r||}$, we have

$$\frac{dV}{dt} \leq -r^T K r + ||Y^T r||\rho + r^T \left[-\rho \frac{YY^T r}{||Y^T r||}\right] \leq -r^T K r$$

  implying that $(e, \dot{e}) \to 0$ exponentially even under certainty.
- **Chattering:** control pushes $r \to 0$, yet, as $r \to 0$, $\delta\tau$ becomes un-defined. For scalar case, $\delta\tau = -\rho \frac{y^2 r}{|yr|} = \rho y \frac{yr}{|yr|} = -\rho y \, \mathrm{sgn}(yr)$

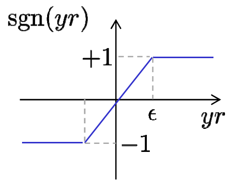


- as $r \to 0$ with nosy measurement.
- high-frequency (sampling rate) chatters.

Boundary-Layer Approximation

- To avoid chattering problem, instead of discontinuous control $\delta\tau = -\rho \frac{YY^T r}{||Y^T r||}$, we use its **boundary-layer approximation:**

$$\delta\tau = \begin{cases} -\rho Y \frac{Y^T r}{||Y^T r||} & \text{if } ||Y^T r|| > \epsilon \\ -\rho Y \frac{Y^T r}{\epsilon} & \text{if } ||Y^T r|| \leq \epsilon \end{cases}$$



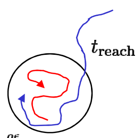
- Then, if $||Y^T r|| > \epsilon$, $\frac{dV}{dt} \leq -r^T K r$. Also, if $||Y^T r|| \leq \epsilon$,

$$\begin{aligned} \frac{dV}{dt} &\leq -r^T K r + \rho ||Y^T r|| - \rho \frac{r^T Y Y^T r}{\epsilon} \\ &= -r^T K r + \rho ||Y^T r|| - \rho \frac{||Y^T r||^2}{\epsilon} \\ &= -r^T K r - \frac{\rho}{\epsilon}\left[||Y^T r|| - \frac{\epsilon}{2}\right]^2 + \frac{\rho\epsilon}{4} \leq -r^T K r + \frac{\rho\epsilon}{4} \end{aligned}$$

→ hold for all t

- Here, if $||r||$ increases, $\dot{V}$ will become negative $\to$ $||r||$ starts to decrease $\to$ V will be bounded. In other words,

$$\frac{dV}{dt} \leq -\lambda_{\min}[K]||r||^2 + \frac{\rho\epsilon}{4} \leq -\frac{\lambda_{\min}[K]}{\lambda_{\max}[M]} V + \frac{\rho\epsilon}{4}$$

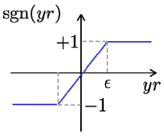


i.e., $V(t)$ is **ultimately bounded** by $\frac{\lambda_{\max}[M]}{\lambda_{\min}[K]} \frac{\rho\epsilon}{4}$.

$V \leq \frac{\lambda_{\max}[M]}{\lambda_{\min}[K]} \frac{\rho\epsilon}{4}$

Ultimate Boundedness

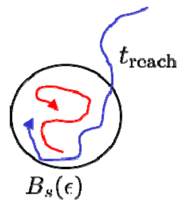
- To avoid chattering problem, instead of discontinuous control $\delta\tau = -\rho \frac{YY^T r}{||Y^T r||}$, we use its **boundary-layer approximation:**



$$\delta\tau = \begin{cases} -\rho Y \frac{Y^T r}{||Y^T r||} & \text{if } ||Y^T r|| > \epsilon \\ -\rho Y \frac{Y^T r}{\epsilon} & \text{if } ||Y^T r|| \leq \epsilon \end{cases}$$

- $V(t)$ is **ultimately bounded** by $\frac{\lambda_{\max}[M]}{\lambda_{\min}[K]} \frac{\rho\epsilon}{4}$, since

$$\frac{dV}{dt} \leq -\lambda_{\min}[K]||r||^2 + \frac{\rho\epsilon}{4} \leq -\frac{\lambda_{\min}[K]}{\lambda_{\max}[M]} V + \frac{\rho\epsilon}{4}$$



- $r(t)$ eventually enters into the bounded set

$$B_s := \{r \mid ||r|| \leq \sqrt{\frac{\lambda_{\max}[M]}{\lambda_{\min}[M]\lambda_{\min}[K]} \frac{\rho\epsilon}{4}}\}$$

- U.B. set B_s gets smaller if ρ is small (i.e., good estimtion of θ) or ϵ is small (i.e., tigher approximatoin) or $\lambda_{\min}[K]$ is large (i.e., large feedback gain).
- Note that B_s is positive-invariant, i.e., $r(t)$ may start outside, yet, eventually enters into it, and, once in it, $r(t)$ will stay in B_s afterward.

Passivity-Based Adaptive Control

- Robot dynamics: $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + f$, $f \approx 0$.
- Tracking control: $\tau = \hat{M}(q)[\ddot{q}_d - \Lambda\dot{e}] + \hat{C}(q,\dot{q})[\dot{q}_d - \Lambda e] + \hat{g}(q) - K\dot{e} - K\Lambda e$. Here, we **adaptively change** $\hat{\theta}(t)$ instead of large-action $\delta\tau$.
- Closed-loop dynamics:

$$M\dot{r} + Cr + Kr = Y[\hat{\theta} - \theta] = Y\tilde{\theta}$$

where $\tilde{\theta} = \hat{\theta}(t) - \theta$ is parameter estimation error for constant θ. Note that $\hat{\theta}$ consitutues another state vecotr, since it has its own dynamics.

- Define $V = \frac{1}{2} r^T M r + \frac{1}{2}\tilde{\theta}^T \Gamma \tilde{\theta}$ with symmetric and pd $\Gamma \in \Re^{l \times l}$. Then,

$$\frac{dV}{dt} = -r^T K r + r^T Y \tilde{\theta} + \dot{\hat{\theta}}^T \Gamma \tilde{\theta} = -r^T K r + \tilde{\theta}^T [Y^T r + \Gamma \dot{\hat{\theta}}]$$

passivity (cf. CLF)

- This then suggests the following adaptation law

$$\dot{\hat{\theta}} = \frac{d\hat{\theta}}{dt} = \Gamma^{-1} Y^T (q, \dot{q}, \nu, \dot{\nu}) r(t), \quad \nu = \dot{q}_d(t) - \Lambda e(t)$$

with which we have $\frac{dV}{dt} = -r^T K r$, i.e., $(r, \tilde{\theta})$ is Lyapunov stable.

Convergence Proof

- Closed-loop dynamics: $M\dot{r} + Cr + Kr = Y[\hat{\theta} - \theta] = Y\tilde{\theta}$.
- Define $V = \frac{1}{2}r^T M r + \frac{1}{2}\tilde{\theta}^T \Gamma \tilde{\theta}$. Then, with adaptation law $\frac{d\hat{\theta}}{dt} = \Gamma^{-1} Y^T r$, we have $\frac{dV}{dt} = -r^T K r$, i.e., $(r, \tilde{\theta}) = 0$ is only Lyapnuv stable.
- Integrating this, we have, $\forall T \geq 0$,

$$V(T) = V(0) - \int_0^T r^T K r dt \leq V(0)$$

implying that $r \in \mathcal{L}_2$. Also, if $|\frac{\partial m_{ij}}{\partial q_k}|$ is bounded, $\dot{r} \in \mathcal{L}_\infty$. Then, from the Corollary of Barbalat's lemma, $r \to 0$.

- We now have $(e, \dot{e}) \to 0$. Then, $\tilde{\theta}(t) = \hat{\theta}(t) - \theta \to 0$ as well? When this parameter convergence possible? $\Rightarrow$ **persistency of excitation.**
- With $r \to 0$, we have $\dot{\tilde{\theta}} \to 0$ from adaptation. We also have $Y^T\tilde{\theta} \to 0$ from C.L. dynamics. This then collectively defines LTV dynamics

$$\dot{\tilde{\theta}} = 0, \quad y = Y^T(q, \dot{q}, \nu, \dot{\nu})\tilde{\theta} = Y^T(q_d, \dot{q}_d, \ddot{q}_d)\tilde{\theta}$$

with the output $y \to 0$. Does this then also implies $\tilde{\theta} \to 0$?

Observability of LTV Systems

- LTV dynamics of parameter estimation error $\tilde{\theta}$:

$$\dot{\tilde{\theta}} = 0, \quad y = Y^T(q_d, \dot{q}_d, \ddot{q}_d)\tilde{\theta}$$

with the output $y \to 0$. If $Y \in \Re^{n \times l}$ is rich enough (e.g., non-singular square), $y \to 0$ would imply $\tilde{\theta} \to 0$. However, if Y is not rich enough (e.g., constant fat), $y \to 0$ wouldn't imply $\tilde{\theta} \to 0$.

- **Definition:** A LTV system $\dot{x} = A(t)x$, $y = C(t)x$ is **observable** if, $\forall t' > 0$, the initial state $x(0)$ is uniquely determined by $y(t)$, $t = [0, t']$.

- **Theorem:** LTV system $\dot{x} = A(t)x$, $y = C(t)x$ is **observable** on $[t_o, t_f]$ iff

$$W_o(t_o, t_f) = \int_{t_o}^{t_f} \Phi(t, t_o) C^T(t) C(t) \Phi(t, t_o) dt \geq \sigma_o I$$

where $\sigma_o > 0$, $\Phi(t, t_o)$ is the transition matrix with $x(t) = \Phi(t, t_o)x(0)$, and W_o is **observability grammian**.

Observability Grammian

- **Theorem:** LTV system $\dot{x} = A(t)x$, $y = C(t)x$ is **observable** on $[t_o, t_f]$ iff

$$W_o(t_o, t_f) = \int_{t_o}^{t_f} \Phi(t, t_o) C^T(t) C(t) \Phi(t, t_o) dt \geq \sigma_o I$$

where $\sigma_o > 0$, $\Phi(t, t_o)$ is the transition matrix with $x(t) = \Phi(t, t_o)x(0)$, and W_o is **observability grammian.**

(Proof) From $y = C(t)x = C(t)\Phi(t, t_o)x(0)$, we have

$$\int_{t_o}^{t_f} \Phi^T(t, t_o) C^T(t) y(t) dt = \int_{t_o}^{t_f} \Phi^T(t, t_o) C^T(t) C(t) \Phi(t, t_o) dt \cdot x(0)$$

thus, if the above condition holds, $x(0)$ is uniquely determined by

$$x(0) = W_o^{-1}(t_o, t_f) \int_{t_o}^{t_f} \Phi^T(t, t_o) C^T(t) y(t) dt$$

Persistency of Excitation

- **Theorem:** LTV system $\dot{x} = A(t)x$, $y = C(t)x$ is **observable** on $[t_o, t_f]$ iff

$$W_o(t_o, t_f) = \int_{t_o}^{t_f} \Phi(t, t_o) C^T(t) C(t) \Phi(t, t_o) dt \geq \sigma_o I$$

with

$$x(0) = W_o^{-1}(t_o, t_f) \int_{t_o}^{t_f} \Phi^T(t, t_o) C^T(t) y(t) dt$$

- Consider the LTV dynamics of parameter estimation error $\tilde{\theta}$:

$$\dot{\tilde{\theta}} = 0, \quad y = Y^T(q_d, \dot{q}_d, \ddot{q}_d)\tilde{\theta}$$

with $\Phi(t_o, t_f) = I$. Then, with the output $y \to 0$, we will have $\tilde{\theta} \to 0$, if

$$W_o(t+T, t) = \int_t^{t+T} Y^T(q_d, \dot{q}_d, \ddot{q}_d) Y(q_d, \dot{q}_d, \ddot{q}_d) dt \geq \sigma_o I, \quad \forall t \geq 0$$

for some positive constants $T > 0$ and $\sigma_o > 0$.

- This condition is referred to (uniform) **persistency of excitation**, i.e., the signal q_d is righ enough and persistently exciting the system so that $Y^T(q_d, \dot{q}_d, \ddot{q}_d)\tilde{\theta} \to 0$ should necessarily imply $\tilde{\theta} \to 0$ as well.

Workspace EF Control

Consider the robot dynamics

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + \tau_e$$

where $q \in \Re^n$ is joint configuration and $\tau_e \in \Re^n$ is external wrench effect.

- In most applications, tasks are specified for the end-effector $x \in f(q) \in \Re^m$, not for the joint configuration $q \in \Re^n$, thus, want to control x directly.
- Let $f(q) \in \Re^m$ be a minimal representation of EF motion for the task. Assume $n = m$ and $J(q) = \frac{\partial f}{\partial q} \in \Re^{n\times n}$ locally full-rank (i.e., f also locally invertible).
- For instance, $(x, y, z, \phi, \theta, \psi)$ for EF in SE(3) with $\tau_e = J^T(q) f_e$, where $f_e \in$ se$^*(3)$ is wrench, i.e., force and torque.
- We then have the following kinematics relations:

$$\dot{x} = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q} \quad \rightarrow \quad \dot{q} = J^{-1}(q)\dot{x}$$
$$\ddot{q} = \frac{d}{dt}[J^{-1}(q)\dot{x}] = \frac{dJ^{-1}(q)}{dt}\dot{x} + J^{-1}(q)\ddot{x}$$

where $\frac{d}{dt}[J^{-1}(q)] = -J^{-1}(q)\dot{J}(q)J^{-1}(q)$.

Workspace Dynamics

- The robot dynamics in joint-space

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + J^T(q) f_e$$

- Using $\dot{q} = J^{-1}(q)\dot{x}$ and $\ddot{q} = \frac{dJ^{-1}}{dt}\dot{x} + J^{-1}(q)\ddot{x}$, can rewirte joint-space dynamics:

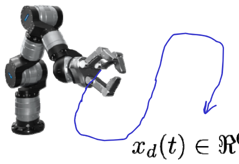
$$M(q)[J^{-1}\ddot{x} + \frac{dJ^{-1}}{dt}\dot{x}] + C(q,\dot{q})J^{-1}\dot{x} + g(q) = \tau + J^T f_e$$
$$\Rightarrow \; J^{-T}M(q)J^{-1}\ddot{x} + J^{-T}[M(q)\frac{dJ^{-1}}{dt} + C(q,\dot{q})J^{-1}]\dot{x} + J^{-T}g(q) = J^{-T}\tau + f_e$$

from which we can obtain the **robot workspace dynamics**:

$$D(x)\ddot{x} + Q(x,\dot{x})\dot{x} + g_x(x) = u + f_e$$

- $D(x) = J^{-T}M(q)J^{-1} \in \Re^{n\times n}$ is symmetric and positive-definite.
- $Q(x,\dot{x}) = J^{-T}[M(q)\frac{dJ^{-1}}{dt} + C(q,\dot{q})J^{-1}]$ is workspace Corilois matrix.
- $\dot{D}(x) - 2Q(x,\dot{x})$ is skew-symmetric.
- $g_x(x) = J^{-T}g(q)$ is workspace gravity.
- $u = J^{-T}\tau$ is workspace control.
- $J(q)$ is analytic Jacobian with representation singularity.

Workspace Control



- Consider the robot workspace dynamics:

$$D(x)\ddot{x} + Q(x,\dot{x})\dot{x} + g_x(x) = u + f_e$$

$x_d(t) \in \Re^6$

- Workspace EF trajectory tracking control objective:

$$(x(t), \dot{x}(t)) \to (x_d(t), \dot{x}_d(t))$$

- Passivity-based trajectory tracking control with r-variable:

$$u = D[\ddot{x}_d - \Lambda(\dot{x}-\dot{x}_d)] + Q[\dot{x}_d - \Lambda(x-x_d)] - K(\dot{x}-\dot{x}_d) - K\Lambda(x-x_d) + g_x - f_e$$

which results in exponentially-stable system

$$D(x)\dot{r} + Q(x,\dot{x})r + Kr = 0, \quad r := \dot{e} + \Lambda e, \; e = x - x_d$$

- Real joint torque control: with $e = x(q(t)) - x_d(t)$,

$$\tau = J^T u = J^T(q)\cdot[D(q)(\ddot{x}_d - \Lambda\dot{e}) + Q(q,\dot{q})(\dot{x}_d - \Lambda e) - Kr] + g(q) - J^T(q) f_e$$

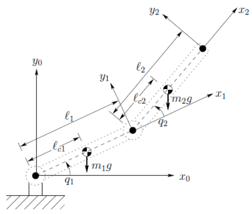
wrench sensing

Workspace Control: Example

- Robot workspace dynamics:

$$D(q)\ddot{x} + Q(q,\dot{q})\dot{x} + g_x(q) = u$$

where $q = (\theta_1, \theta_2)$.



- We want to control the EF position in plane. Then,

$$f(q) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{pmatrix} \in \Re^2$$

- Jacobian relation

$$\frac{df(q)}{dt} = J(q)\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

with $J(q) \in \Re^{2\times 2}$ is full-rank if $\theta_2 \neq n\pi$.

- Workspace trajectory tracking control: with $e = x(q) - x_d(t)$,

$$\tau = J^T(q)\cdot[D(q)(\ddot{x}_d - \Lambda\dot{e}) + Q(q,\dot{q})(\dot{x}_d - \Lambda e) - Kr] + g(q)$$