Chapter 4: Basic continuum mechanics
Part 2

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Review - Almansi strain

- The strain measure : $$\varepsilon_A = \frac{1}{2}\left(1 - \left(\frac{l_n}{l_0}\right)^{-2}\right) = \frac{l_n^2 - l_0^2}{2l_n^2}$$ [eq. 3.41]
- Virtual strain increment:

$$\delta\varepsilon_A = \left(\frac{l_n}{l_0}\right)^{-3}\frac{\delta u}{l_0} = \frac{l_0^2}{l_n^3}\delta u$$ [eq. 3.42]

$$= b\delta u$$

- Internal virtual work and internal force:

$$\int \sigma_A \delta\varepsilon_A dV_n = \sigma_A\left(\frac{l_0^2}{l_n^3}\delta u\right)A_n l_n = q_A^T \delta u$$

$q_L = q_A$

➡ $$\sigma_A = {}'\sigma'\frac{l_n^2}{l_0^2}$$ [eq. 3.44]

➡ $$b_A = \frac{l_0^2}{l_n^3}$$

⬇ $$q_A = \int b_A \sigma_A dV_n$$

➡ $$q_A = \sigma_A A_n \frac{l_0^2}{l_n^2}$$

- **Almansi** strain can be contrasted with Green's strain

$$\varepsilon_A = \frac{1}{2}\left(1-\left(\frac{l_n}{l_0}\right)^{-2}\right) = \frac{l_n^2 - l_0^2}{2l_n^2}$$

$$d\mathbf{r}_n^2 - d\mathbf{r}_o^2 = d\mathbf{x}^2 - d\mathbf{X}^2 = 2d\mathbf{X}^T\mathbf{E}d\mathbf{X}$$ $\mathbf{E}_2 : Green(-Lagrange)\ strain$ [eq. 4.68]

$$d\mathbf{r}_n^2 - d\mathbf{r}_o^2 = d\mathbf{x}^2 - d\mathbf{X}^2 = 2d\mathbf{x}^T\mathbf{A}d\mathbf{x}$$ $\mathbf{A} : Almansi\ strain$ [eq. 4.89]

$$d\mathbf{X} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}d\mathbf{x} = \mathbf{F}^{-1}d\mathbf{x} = \left[\mathbf{I} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right]d\mathbf{x}$$ [eq. 4.90]

Note: $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \neq \mathbf{D} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$

➡ $$\mathbf{A} = \frac{1}{2}\left[\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right] = \frac{1}{2}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right] - \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)$$ [eq. 4.91]

Relationship with Green's strain: $$\mathbf{F}^T\mathbf{A}\mathbf{F} = \frac{1}{2}\left[\mathbf{F}^T\mathbf{F} - \mathbf{I}\right] = \mathbf{E}$$ [eq. 4.92]

- In the updated coordinate system by maintaining the directions of the original rectangular cartesian system (refer to section 3.3.6), we will need to use **a stress measure that relates to this new (or current) system.**

- Even if we adopt a "Green strain – 2nd Piola-Kirchhoff stress" system, we may wish to interpret our final stresses in relation to the final geometry.

➡ **True (or Cauchy) stress**

- In section 3.2, it is shown that for true stress, corresponding virtual strain should be:

$$\delta\varepsilon_L = \frac{\delta u}{l_n}$$

- In 3D, it is generalized by:

$$\delta\boldsymbol{\varepsilon}_v = \frac{1}{2}\left[\frac{\partial\delta\mathbf{u}_v}{\partial\mathbf{x}} + \frac{\partial\delta\mathbf{u}_v^T}{\partial\mathbf{x}}\right] \quad \text{[eq. 4.93]}$$

- Relation between the Cauchy stress and the 2nd Piola-Kirchhoff stress
- Equivalent work concept

$$V_i = \int \mathbf{S} : \delta \mathbf{E}_v dV_0 = \int \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon}_v dV$$ [eq. 4.94]

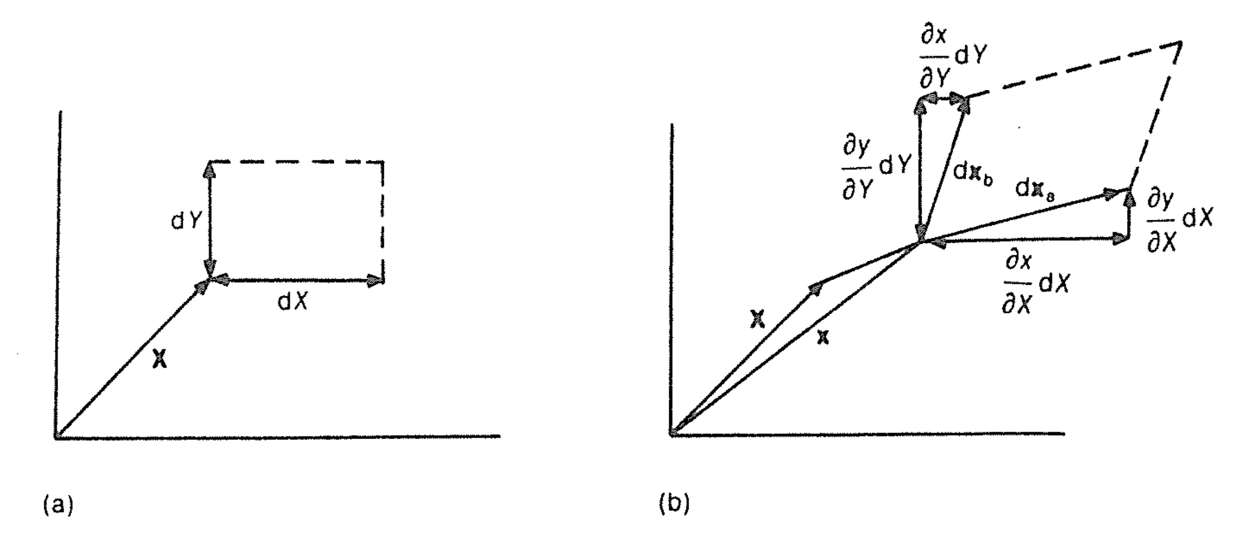
$\boldsymbol{\sigma} : Cauchy\ stress$

$$dV = \left(dxdydz\right) = J\left(dXdYdZ\right) = JdV_0$$ [eq. 4.95]

where $J = \det(\mathbf{F}) = \det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)$ [eq. 4.96]

- Two-dimensional form: $dA\mathbf{i}_3 = d\mathbf{x}_a \times d\mathbf{x}_b = \begin{pmatrix} \frac{\partial x}{\partial X} \\ \frac{\partial y}{\partial X} \end{pmatrix} dX \times \begin{pmatrix} \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial Y} \end{pmatrix} dY = \det(\mathbf{F})\left(dXdY\right)\mathbf{i}_3 = \det(\mathbf{F})\left(dA_0\right)\mathbf{i}_3$

[eq. 4.97]



[Fig 4.11 Two-dimensional areas: (a) initial element (b) final element]

$$V_i = \int \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}_v dV = \int J\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}_v dV_0 = \int \boldsymbol{\tau} : \delta\boldsymbol{\varepsilon}_v dV_0 = \int \mathbf{S} : \delta\mathbf{E}_v dV_0$$ [eq. 4.98]

where $\boldsymbol{\tau} = J\boldsymbol{\sigma}$: *Kirchhoff (or nominal) stress*

$$\delta\boldsymbol{\varepsilon}_v = \frac{1}{2}\left[\frac{\partial\delta\mathbf{u}_v}{\partial\mathbf{x}} + \frac{\partial\delta\mathbf{u}_v^T}{\partial\mathbf{x}}\right]$$ [eq. 4.93]

$$\delta\mathbf{D}_v = \frac{\partial\delta\mathbf{u}_v}{\partial\mathbf{X}} = \frac{\partial\delta\mathbf{u}_v}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{X}} = \frac{\partial\delta\mathbf{u}_v}{\partial\mathbf{x}}\mathbf{F}$$ [eq. 4.100]

➡ $$\delta\mathbf{E}_v = \mathbf{F}^T\delta\boldsymbol{\varepsilon}_v\mathbf{F}$$ [eq. 4.101]

➡ $$V_i = \int J\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}_v dV_0 = \int \mathbf{S} : \left[\mathbf{F}^T\delta\boldsymbol{\varepsilon}_v\mathbf{F}\right] dV_0 = \int \left[\mathbf{F}\mathbf{S}\mathbf{F}^T\right] : \delta\boldsymbol{\varepsilon}_v dV_0$$ [eq. 4.102]

by using $\mathbf{AB} : \mathbf{C}^T = \mathbf{CA} : \mathbf{B}^T = \mathbf{BC} : \mathbf{A}^T$ [eq. 4.103]

➡ $$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T$$ or $$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$$

[eq. 4.104] [eq. 4.105]

$$\delta \boldsymbol{\varepsilon}_v = \frac{1}{2}\left[\frac{\partial \delta \mathbf{u}_v}{\partial \mathbf{x}} + \frac{\partial \delta \mathbf{u}_v^T}{\partial \mathbf{x}}\right]$$

- From the definition of Almansi strain, the virtual variation can be derived.

$$\mathbf{A} = \frac{1}{2}\left[\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right] = \frac{1}{2}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right] - \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)$$ [eq. 4.91]

$$\Rightarrow \quad \delta \mathbf{A}_v = \delta \boldsymbol{\varepsilon}_v - \left[\mathbf{A}^T\left(\frac{\partial \delta \mathbf{u}_v}{\partial \mathbf{x}}\right) + \left(\frac{\partial \delta \mathbf{u}_v}{\partial \mathbf{x}}\right)^T \mathbf{A}\right] \neq \delta \boldsymbol{\varepsilon}_v$$ [eq. 4.106]

- Since virtual variation for conjugate strain of Cauchy stress is $\delta \boldsymbol{\varepsilon}_v$, the **Almansi strain is not conjugate to the Cauchy stress.**

- Strictly, a stress-strain conjugate pair should be derived from:

$$\dot{V} = (stress):(strain\ rate)$$

$\dot{V}$: *virtual internal power*

- The strain-rate tensor, can be considered as:

$$\dot{\boldsymbol{\varepsilon}} = \frac{\delta\boldsymbol{\varepsilon}}{dt} = \frac{1}{dt}\left(\frac{1}{2}\left[\left(\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right)+\left(\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right)^T\right]\right) = \frac{1}{2}\left[\mathbf{L}+\mathbf{L}^T\right]$$

[eq. 4.107, 4.108]

where $\mathbf{L} = \frac{1}{dt}\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}} = \frac{\partial\dot{\mathbf{x}}}{\partial\mathbf{x}} = \frac{\partial\mathbf{v}}{\partial\mathbf{x}}$ [eq. 4.109]

or

$$\mathbf{L} = \begin{bmatrix} \frac{\partial\dot{x}_1}{\partial x_1} & \frac{\partial\dot{x}_1}{\partial x_2} & \frac{\partial\dot{x}_1}{\partial x_3} \\ \frac{\partial\dot{x}_2}{\partial x_1} & \frac{\partial\dot{x}_2}{\partial x_2} & \frac{\partial\dot{x}_2}{\partial x_3} \\ \frac{\partial\dot{x}_3}{\partial x_1} & \frac{\partial\dot{x}_3}{\partial x_2} & \frac{\partial\dot{x}_3}{\partial x_3} \end{bmatrix}$$ [eq. 4.110]

Velocity gradient tensor

velocity gradient

- From the definition of Green strain,

$$\dot{\boldsymbol{\varepsilon}} = \frac{\delta\boldsymbol{\varepsilon}}{dt} = \frac{1}{dt}\left(\frac{1}{2}\left[\left(\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right)+\left(\frac{\partial\delta\mathbf{u}}{\partial\mathbf{x}}\right)^T\right]\right) = \frac{1}{2}\left[\mathbf{L}+\mathbf{L}^T\right]$$

$$\mathbf{E} = \frac{1}{2}\left[\mathbf{F}^T\mathbf{F}-\mathbf{I}\right]$$ [eq. 4.74]

$$\dot{\mathbf{E}}_v = \frac{1}{2}\left[\dot{\mathbf{F}}_v^T\mathbf{F}+\mathbf{F}^T\dot{\mathbf{F}}_v\right]$$ [eq. 4.111]

where

$$\dot{\mathbf{F}}_v^T = \frac{\partial\dot{\mathbf{x}}_v}{\partial\mathbf{X}} = \frac{\partial\dot{\mathbf{x}}_v}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{X}} = \mathbf{L}_v\mathbf{F}$$ [eq. 4.112]

$$\Rightarrow \quad \dot{\mathbf{E}}_v = \frac{1}{2}\left[\mathbf{F}^T\mathbf{L}_v\mathbf{F}+\mathbf{F}^T\mathbf{L}_v^T\mathbf{F}\right] = \frac{1}{2}\mathbf{F}^T\left[\mathbf{L}_v+\mathbf{L}_v^T\right]\mathbf{F} = \mathbf{F}^T\dot{\boldsymbol{\varepsilon}}_v\mathbf{F}$$ [eq. 4.113]

- Same formulation for stress-strain conjugates can be derived.

$$\Rightarrow \quad \delta\mathbf{E}_v = \mathbf{F}^T\delta\boldsymbol{\varepsilon}_v\mathbf{F}$$ [eq. 4.101]

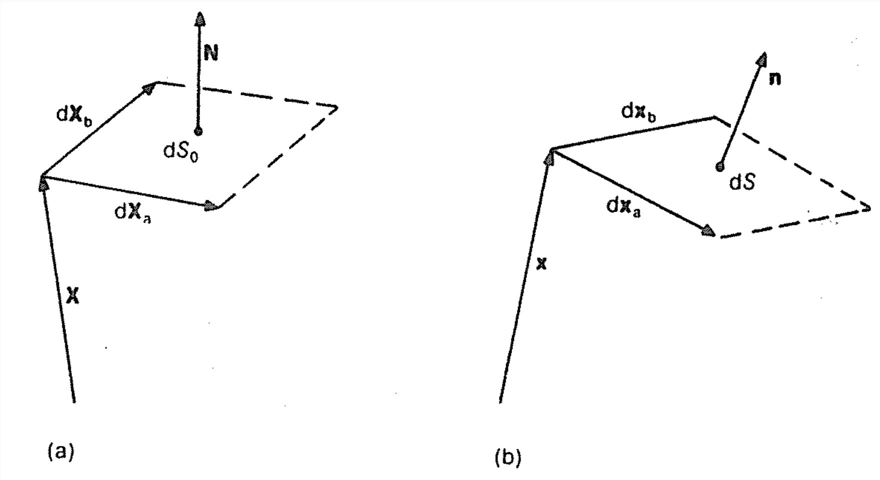
$$\Rightarrow \quad V_i = \int J\boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}_v dV_0 = \int \mathbf{S}:\left[\mathbf{F}^T\delta\boldsymbol{\varepsilon}_v\mathbf{F}\right]dV_0 = \int\left[\mathbf{F}\mathbf{S}\mathbf{F}^T\right]:\delta\boldsymbol{\varepsilon}_v dV_0$$ [eq. 4.102]

by using $\mathbf{AB}:\mathbf{C}^T = \mathbf{CA}:\mathbf{B}^T = \mathbf{BC}:\mathbf{A}^T$ [eq. 4.103]

$$\Rightarrow \quad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T$$ [eq. 4.104]

or

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$$ [eq. 4.105]



[Fig4.12 Areas in the initial and final three-dimensional configurations. (a) initial (b) final]

c.f Nanson's formula

- Let infinitesimal volume dv made by parallelepiped of $d\mathbf{x}_a, d\mathbf{x}_b, d\mathbf{x}_c$
- And infinitesimal volume dV made by parallelepiped of $d\mathbf{X}_a, d\mathbf{X}_b, d\mathbf{X}_c$

$$\begin{cases} (\mathbf{Ac})^T\left((\mathbf{Aa})\times(\mathbf{Ab})\right)=\det(\mathbf{A})\left(\mathbf{c}^T(\mathbf{a}\times\mathbf{b})\right) \\ \mathbf{A}: 2^{nd}\ order\ tensor \\ \mathbf{a},\mathbf{b},\mathbf{c}: 1^{st}\ order\ tensor \end{cases} \quad \Rightarrow \quad dv = JdV$$

$$d\mathbf{a} \doteq \mathbf{n}dS = d\mathbf{x}_a \times d\mathbf{x}_b \qquad d\mathbf{A} \doteq \mathbf{N}dS_o = d\mathbf{X}_a \times d\mathbf{X}_b$$

$$dv = d\mathbf{x}_c^T d\mathbf{a} = \left(\mathbf{F}d\mathbf{X}_c\right)^T d\mathbf{a} = d\mathbf{X}_c^T\left(\mathbf{F}^T d\mathbf{a}\right)$$
$$= JdV = Jd\mathbf{X}_c^T d\mathbf{A} \quad \Rightarrow \quad d\mathbf{a} = J\mathbf{F}^{-T}d\mathbf{A}$$

- External tractions can be expressed either in terms of the original configuration or the final configuration, by 1st Piola-Kirchhoff stress.

$$\mathbf{f}_e = \mathbf{t}_o dS_o = \mathbf{P}\left(\mathbf{N}dS_o\right) = \mathbf{t}dS = \boldsymbol{\sigma}\mathbf{n}dS \qquad \text{[eq. 4.114]}$$

$$\mathbf{P}: 1^{st}\ Piola-Kirchhoff\ stress$$

$$\mathbf{N}dS_o = d\mathbf{X}_a \times d\mathbf{X}_b = \left(\mathbf{F}^{-1}d\mathbf{x}_a\right)\times\left(\mathbf{F}^{-1}d\mathbf{x}_b\right)$$
$$= \frac{1}{J}\mathbf{F}^T\left(d\mathbf{x}_a \times d\mathbf{x}_b\right) = \frac{1}{J}\mathbf{F}^T\mathbf{n}dS \qquad \text{[eq. 4.115]}$$

$$\Rightarrow \quad \boxed{\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{FS}} \qquad \text{[eq. 4.116]}$$

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{FS}$$ [eq. 4.116]

- The **1st Piola-Kirchhoff stress, which is non-symmetric**, is work-conjugate to the infinitesimal virtual displacement gradient, $\mathbf{D}_v$

$$\mathbf{D} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$ [eq. 4.72]

$$V_i = \int \mathbf{S} : \delta \mathbf{E}_v dV_o = \frac{1}{2}\int \mathbf{S} : \left[\mathbf{F}^T \delta \mathbf{D}_v + \delta \mathbf{D}_v^T \mathbf{F}\right] dV_o$$
$$= \int \mathbf{S} : \left[\mathbf{F}^T \delta \mathbf{D}_v\right] dV_o = \int \mathbf{FS} : \delta \mathbf{D}_v dV_o = \int \mathbf{P} : \delta \mathbf{D}_v dV_o$$ [eq. 4.117]

- In summary,

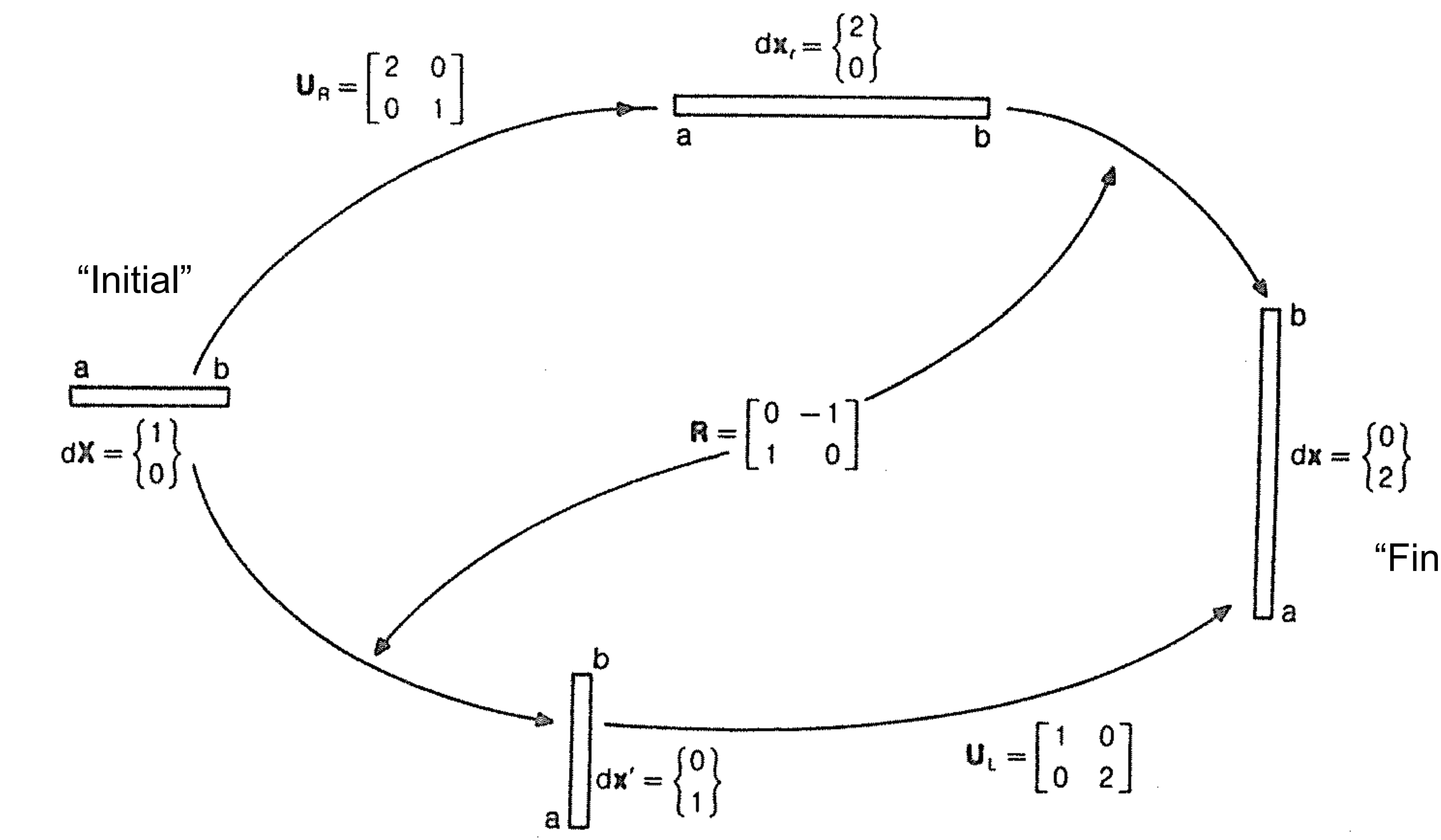
$$\dot{V} = \frac{V_i}{dt} = \frac{1}{dt}\int \mathbf{S} : \delta \mathbf{E}_v dV_o = \int \mathbf{S} : \dot{\mathbf{E}}_v dV_o$$
$$= \frac{1}{dt}\int \mathbf{P} : \delta \mathbf{D}_v dV_o = \int \mathbf{P} : \dot{\mathbf{F}}_v dV_o$$
$$= \frac{1}{dt}\int \boldsymbol{\tau} : \delta \boldsymbol{\varepsilon}_v dV_o = \int \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}_v dV_o$$
$$= \frac{1}{dt}\int \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon}_v dV = \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_v dV$$ [eq. 4.118]

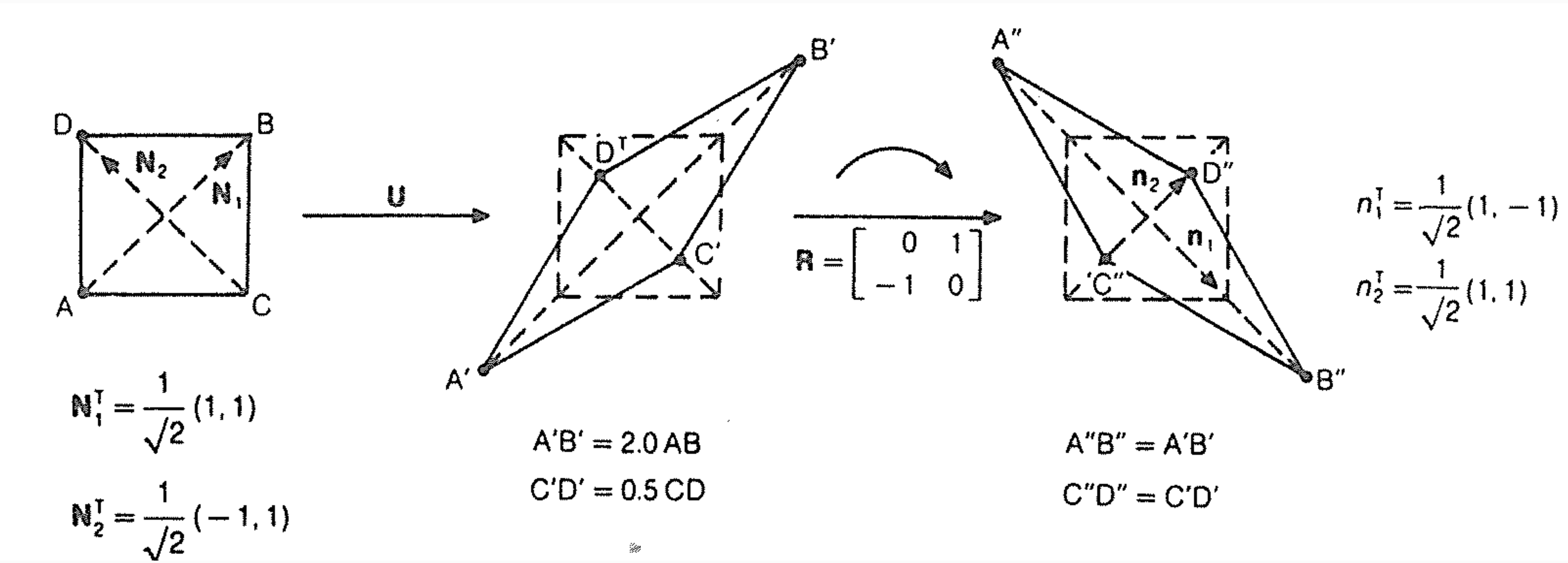
$$\mathbf{P} = \mathbf{FS} = \left[\mathbf{I} + \mathbf{D}\right]\mathbf{S} = \det(\mathbf{F})\boldsymbol{\sigma}\mathbf{F}^{-T}$$ [eq. 4.119]

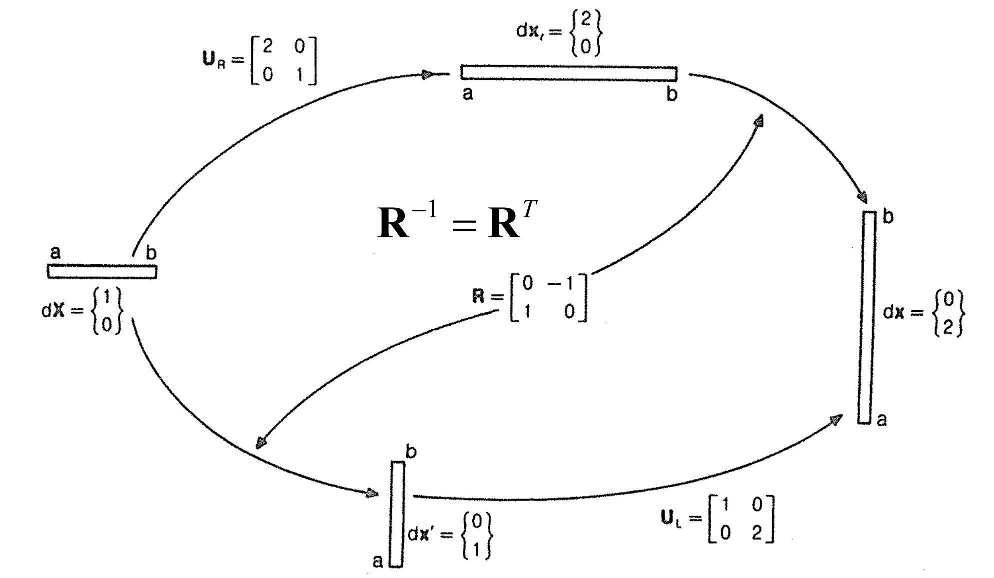
$$\mathbf{S} = \det(\mathbf{F})\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$$ [eq. 4.120]

$$\boldsymbol{\sigma} = \frac{1}{\det(\mathbf{F})}\mathbf{FSF}^T$$ [eq. 4.121]

$$\boldsymbol{\tau} = \mathbf{FSF}^T$$ [eq. 4.122]


$dx_r = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$
$U_R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
a
b
"Initial"
a
b
$dX = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$
$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
b
$dx = \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$
"Final"
a
b
$dx' = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$
a
$U_L = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$


D
B
A
C
N₂
N₁
U
D′
B′
C′
A′
R = [0 1 ; −1 0]
A″
D″
C″
B″
n₂
n₁
$N_1^T = \frac{1}{\sqrt{2}}(1, 1)$
$N_2^T = \frac{1}{\sqrt{2}}(-1, 1)$
A′B′ = 2.0 AB
C′D′ = 0.5 CD
A″B″ = A′B′
C″D″ = C′D′
$n_1^T = \frac{1}{\sqrt{2}}(1, -1)$
$n_2^T = \frac{1}{\sqrt{2}}(1, 1)$



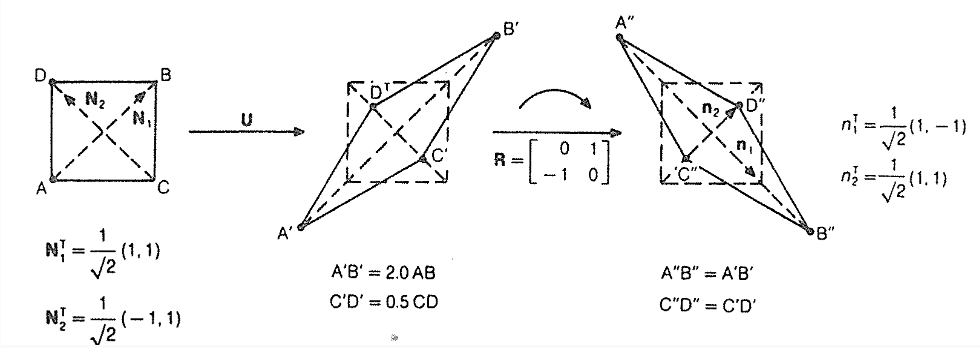
[Fig4.13 A simple example of polar decomposition]

$$d\mathbf{x}_r = \frac{\partial \mathbf{x}_r}{\partial \mathbf{X}} d\mathbf{X} = \mathbf{U}_R d\mathbf{X}$$ [eq. 4.123] Stretch

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_r} d\mathbf{x}_r = \mathbf{R} d\mathbf{x}_r$$ [eq. 4.124] Rotation

$$d\mathbf{x} = \mathbf{R}\mathbf{U}_R d\mathbf{x}$$ [eq. 4.125]

$$\mathbf{F} = \mathbf{R}\mathbf{U}_R = \mathbf{U}_L\mathbf{R} = \mathbf{V}\mathbf{R}$$ [eq. 4.126]



[Fig4.14 A more complex example of polar decomposition]

$$\mathbf{U} = 2.0\mathbf{N}_1\mathbf{N}_1^T + 0.5\mathbf{N}_2\mathbf{N}_2^T = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$$ [eq. 4.129]

$\mathbf{N}_1, \mathbf{N}_2$: *eigenvector of* $\mathbf{U}$

$2.0, 0.5$: *eigenvalue of* $\mathbf{U}$

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$$ [eq. 4.130]

Reading assignment

$$\lambda = \frac{dr_n}{dr_o} = \left(\frac{d\mathbf{x}^T d\mathbf{x}}{d\mathbf{X}^T d\mathbf{X}}\right)^{1/2} = \frac{\| d\mathbf{x} \|}{\| d\mathbf{X} \|}$$ [eq. 4.131] $\lambda : stretch\ measure$

$$\lambda^2 = \frac{d\mathbf{x}^T d\mathbf{x}}{d\mathbf{X}^T d\mathbf{X}} = \mathbf{N}^T\mathbf{F}^T\mathbf{F}\mathbf{N}$$ [eq. 4.132] where $$\mathbf{N} = \frac{d\mathbf{X}}{\left(d\mathbf{X}^T d\mathbf{X}\right)^{1/2}} = \frac{d\mathbf{X}}{\| d\mathbf{X} \|}$$ [eq. 4.133]

- It is useful to vary the directions of $\mathbf{N}$ and find the principal stretch values and their corresponding directions. Lagrangian multiplier method can be used to do this.

$$\phi = \mathbf{N}^T\mathbf{F}^T\mathbf{F}\mathbf{N} - \alpha\left(\mathbf{N}^T\mathbf{N} - 1\right)$$ [eq. 4.134]

$$\delta\phi = 0 \Rightarrow \left[\mathbf{F}^T\mathbf{F} - \alpha\mathbf{I}\right]\mathbf{N} = 0$$ [eq. 4.135]

$$\lambda^2 = \frac{d\mathbf{x}^T d\mathbf{x}}{d\mathbf{X}^T d\mathbf{X}} = \mathbf{N}^T\mathbf{F}^T\mathbf{F}\mathbf{N}$$ [eq. 4.132] $\Rightarrow$ $\alpha = \lambda^2$

$$\Rightarrow \left[\mathbf{F}^T\mathbf{F} - \lambda^2\mathbf{I}\right]\mathbf{N} = 0 \quad eigenvalue\ problem$$ [eq. 4.136]

$$\Rightarrow \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{U} = \lambda_1^2\mathbf{N}_1\mathbf{N}_1^T + \lambda_2^2\mathbf{N}_2\mathbf{N}_2^T + \lambda_3^2\mathbf{N}_3\mathbf{N}_3^T = \mathbf{Q}(\mathbf{N})Diag\left(\lambda^2\right)\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.137]

where $$\mathbf{Q}(\mathbf{N}) = \left[\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3\right]$$ [eq. 4.138]

$$\left[\mathbf{F}^T\mathbf{F}-\lambda^2\mathbf{I}\right]\mathbf{N}=0$$ [eq. 4.136]

$$\mathbf{F}^T\mathbf{F}=\mathbf{U}^T\mathbf{U}=\lambda_1^2\mathbf{N}_1\mathbf{N}_1^T+\lambda_2^2\mathbf{N}_2\mathbf{N}_2^T+\lambda_3^2\mathbf{N}_3\mathbf{N}_3^T=\mathbf{Q}(\mathbf{N})Diag\left(\lambda^2\right)\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.137]

where $\mathbf{Q}(\mathbf{N})=\left[\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3\right]$ [eq. 4.138]

➡ $$\mathbf{U}=\lambda_1\mathbf{N}_1\mathbf{N}_1^T+\lambda_2\mathbf{N}_2\mathbf{N}_2^T+\lambda_3\mathbf{N}_3\mathbf{N}_3^T=\mathbf{Q}(\mathbf{N})Diag\left(\lambda\right)\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.139]

➡ $$\left[\mathbf{U}-\lambda\mathbf{I}\right]\mathbf{N}=0$$ [eq. 4.140]

- Similarly, $\mathbf{V}$ can be obtained from:

$$\left[\mathbf{F}\mathbf{F}^T-\lambda^2\mathbf{I}\right]\mathbf{n}=0$$ [eq. 4.144]

$$\left[\mathbf{V}-\lambda\mathbf{I}\right]\mathbf{n}=0$$ [eq. 4.146]

- Rotation matrix $\mathbf{R}$ can be expressed by:

$$\mathbf{R}=\mathbf{Q}(\mathbf{n})\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.147]

➡ $$\mathbf{F}=\mathbf{R}\mathbf{U}=\mathbf{Q}(\mathbf{n})Diag(\lambda)\mathbf{Q}(\mathbf{N})^T=\lambda_1\mathbf{n}_1\mathbf{N}_1^T+\lambda_2\mathbf{n}_2\mathbf{N}_2^T+\lambda_3\mathbf{n}_3\mathbf{N}_3^T$$ [eq. 4.148]

Thank you!