CHAPTER 6. LAPLACE TRANSFORMS

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서 유 택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

6.1 Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

- Laplace Transform (**라플라스 변환**): $F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)\,dt$

- **적분 과정을 통해 주어진 함수를 새로운 함수로 변환**하는 것

- Inverse Transform (역 변환): $\mathcal{L}^{-1}(F) = f(t)$

☑ Ex.1 Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$.

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s} \qquad (s > 0)$$

☑ **Ex.2** Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant. Find L(f).

$$\mathcal{L}\left(e^{at}\right) = \int_0^{\infty} e^{-st} e^{at} dt = \frac{1}{a-s} e^{-(s-a)t} \Big|_0^{\infty} = \frac{1}{s-a}$$

❖ **Theorem 1 Linearity of the Laplace Transform**

The Laplace transform is a linear operation;

that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b, the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}\left(af(t)+bg(t)\right)=a\mathcal{L}\left(f(t)\right)+b\mathcal{L}\left(g(t)\right)$$

☑ Ex.3 Find the transform of $\cosh at$ and $\sinh at$.

$$\cosh at=\frac{1}{2}\left(e^{at}+e^{-at}\right),\ \sinh at=\frac{1}{2}\left(e^{at}-e^{-at}\right)\quad\Rightarrow\quad \mathcal{L}\left(e^{at}\right)=\frac{1}{s-a},\ \ \mathcal{L}\left(e^{-at}\right)=\frac{1}{s+a}$$

$$\Rightarrow\quad \mathcal{L}\left(\cosh at\right)=\frac{1}{2}\left[\mathcal{L}\left(e^{at}\right)+\mathcal{L}\left(e^{-at}\right)\right]=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^2-a^2}$$

$$\mathcal{L}\left(\sinh at\right)=\frac{1}{2}\left[\mathcal{L}\left(e^{at}\right)-\mathcal{L}\left(e^{-at}\right)\right]=\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^2-a^2}$$

6.1 Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

Table **Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$**

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

❖ Theorem 2 First Shifting Theorem (제 1 이동 정리), s-shifting (S-이동)

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k),

then $e^{at}f(t)$ has the transform $F(s-a)$ (where $s-a > k$). In formulas,

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

or, if we take the inverse on both sides, $e^{at} f(t) = \mathcal{L}^{-1}\left\{F(s-a)\right\}$.

Proof) We obtain $F(s-a)$ by replacing s with $s-a$ in the integral, so that

$$F(s-a) = \int_0^{\infty} e^{-(s-a)t} f(t)dt = \int_0^{\infty} e^{-st}[e^{at} f(t)]dt = \mathcal{L}\{e^{at} f(t)\}$$

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

$$e^{at} f(t) = \mathcal{L}^{-1}\left\{F(s-a)\right\}$$

☑ **Ex. 5** Formulas 11 and 12 in Table 6.1,

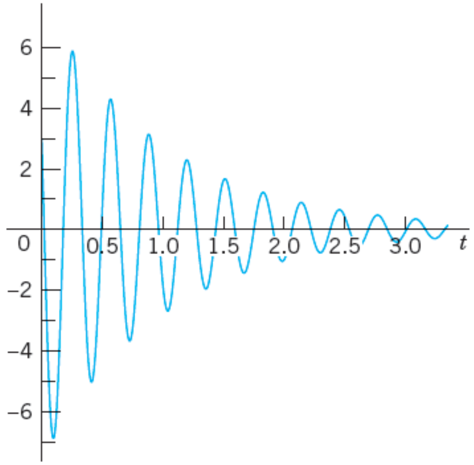
$$\mathcal{L}\left(e^{at}\cos\omega t\right) = \frac{s-a}{(s-a)^2+\omega^2}, \quad \mathcal{L}\left(e^{at}\sin\omega t\right) = \frac{\omega}{(s-a)^2+\omega^2}$$

For instance, use these formulas to find the inverse of the transform.

$$\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}$$

$(a = -1,\ \omega = 20)$

$$f = \mathcal{L}^{-1}\left(\frac{3(s+1)-140}{(s+1)^2+400}\right) = 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+400}\right) - 7\mathcal{L}^{-1}\left(\frac{20}{(s+1)^2+400}\right) = e^{-t}\left(3\cos 20t - 7\sin 20t\right)$$



6.1 Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

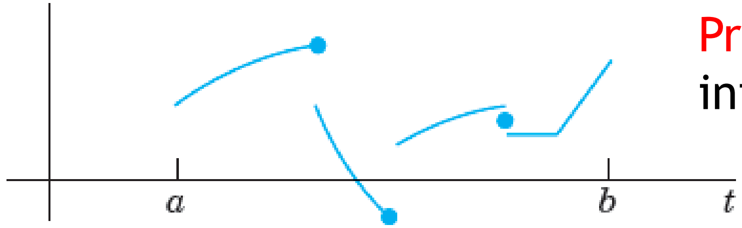
❖ **Theorem 3 Existence Theorem for Laplace Transform**

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$

and satisfies $|f(t)| \leq Me^{kt}$ for all $t \geq 0$ and some constants M and k,

that is, $f(t)$ does not grow too fast ("growth restriction (증가제한)"),

then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.



Example of a piecewise continuous function $f(t)$.
(The dots mark the function values at the jumps.)

Proof) Piecewise continuous → $e^{-st} f(t)$ is integrable over any finite interval on t-axis.

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t) dt \right| \leq \int_0^\infty |f(t)| e^{-st} dt$$

$$\leq \int_0^\infty Me^{kt} e^{-st} dt = \frac{M}{s-k}$$

$|f(t)| \leq Me^{kt}$ for all $t \geq 0$? $\quad$ (1) $f(t) = \cosh t$, (2) $f(t) = t^n$, (3) $f(t) = e^{t^2}$

$\cosh t \leq e^t, t^n < n! e^t, e^{t^2} = ?$

$$\cosh t = \frac{1}{2}(e^t + e^{-t}) < \frac{1}{2}(e^t + e^t) = e^t \qquad e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \Rightarrow e^t > \frac{t^n}{n!} \Rightarrow t^n < n! e^t$$

$e^{t^2} > Me^{kt}$ (not satisfied)

6.2 Transform of Derivatives and Integrals. ODEs

❖ **Theorem 1 Laplace Transform of Derivatives (도함수의 라플라스 변환)**

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0),$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Proof)

(integration by parts)

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$
$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)dt$$
$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$
$$= s[s\mathcal{L}(f) - f(0)] - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0)$$

$$\int g(t)f'(t)dt = g(t)f(t) - \int g'(t)f(t)dt$$

$$\left\{e^{-st} f(t)\right\}' = -se^{-st} f(t) + e^{-st} f'(t)$$
$$e^{-st} f(t)\Big|_0^\infty = \int_0^\infty -se^{-st} f(t)dt + \int_0^\infty e^{-st} f'(t)dt$$
$$e^{-st} f(t)\Big|_0^\infty + \int_0^\infty se^{-st} f(t)dt = \int_0^\infty e^{-st} f'(t)dt$$

6.2 Transform of Derivatives and Integrals. ODEs

❖ **Theorem 2 Laplace Transform of Derivatives**

Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $t \ge 0$ and satisfy the growth restriction $|f(t)| \le Me^{kt}$.

Furthermore, let $f^{(n)}$ be continuous on every finite interval on the semi-axis $t \ge 0$.

$$\mathcal{L}\left(f^{(n)}\right) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

☑ **Ex. 1** Let $f(t) = t\sin\omega t$. Find the transform of $f(t)$.

$$f(0) = 0,$$

$$f'(t) = \sin\omega t + \omega t\cos\omega t, \quad f'(0) = 0,$$

$$f''(t) = 2\omega\cos\omega t - \omega^2 t\sin\omega t$$

$$\Rightarrow \quad \mathcal{L}(f'') = 2\omega\frac{s}{s^2+\omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\Rightarrow \quad \therefore \mathcal{L}(f) = \mathcal{L}(t\sin\omega t) = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

6.2 Transform of Derivatives and Integrals. ODEs

❖ **Theorem 3 Laplace Transform of Integral (적분의 라플라스 변환)**

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for all $t \geq 0$

and satisfies a growth restriction $|f(t)| \leq Me^{kt}$. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left(f(t)\right) = F(s) \quad \Rightarrow \quad \mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s), \quad \int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$$

Proof) $g(t) = \int_0^t f(\tau)d\tau$

$$|g(t)| = \left|\int_0^t f(\tau)d\tau\right| \leq \int_0^t |f(\tau)|d\tau \leq M\int_0^t e^{k\tau}d\tau = \frac{M}{k}(e^{kt}-1) \leq \frac{M}{k}e^{kt} \quad (k>0)$$

→ $g(t)$ also satisfies a growth restriction.
$g'(t) = f(t)$, except at points at which $f(t)$ is discontinuous.
$g'(t)$ is piecewise continuous on each finite interval.
$g(0) = 0$.

From Theorem 1

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\} \qquad \therefore \mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

6.2 Transform of Derivatives and Integrals. ODEs

❖ **Theorem 3** **Laplace Transform of Integral**

$$\mathcal{L}\left(f(t)\right)=F(s) \quad \Rightarrow \quad \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)=\frac{1}{s}F(s), \quad \int_0^t f(\tau)\,d\tau=\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$$

	$f(t)$	$\mathcal{L}(f)$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

$$\mathcal{L}\{\sin \omega t\}=\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos \omega t\}=\frac{s}{s^2+\omega^2}$$

☑ **Ex. 3** Find the inverse of $\frac{1}{s(s^2+\omega^2)}$ and $\frac{1}{s^2(s^2+\omega^2)}$.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right)=\frac{1}{\omega}\sin \omega t \quad \Rightarrow \quad \mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right)=\int_0^t \frac{\sin \omega\tau}{\omega}d\tau=\frac{1}{\omega^2}(1-\cos \omega t)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right)=\frac{1}{\omega^2}\int_0^t (1-\cos \omega\tau)\,d\tau=\frac{t}{\omega^2}-\frac{\sin \omega t}{\omega^3}$$

6.2 Transform of Derivatives and Integrals. ODEs

❖ Differential Equations, Initial Value Problems:

Step 1 Setting up the subsidiary equation (보조방정식).

$$y''+ay'+by=r(t), \quad y(0)=K_0, \quad y'(0)=K_1$$

$$\left(Y=\mathcal{L}(y), \quad R=\mathcal{L}(r)\right)$$

$$\left[s^2Y-sy(0)-y'(0)\right]+a\left[sY-y(0)\right]+bY=R(s) \quad \Rightarrow \quad \left(s^2+as+b\right)Y=(s+a)y(0)+y'(0)+R(s)$$

Step 2 Solution of the subsidiary equation by algebra.

Transfer Function (전달함수): $Q(s)=\dfrac{1}{s^2+as+b}=\dfrac{1}{\left(s+\frac{1}{2}a\right)^2+b-\frac{1}{4}a^2}$

Solution of the transfer function: $Y(s)=\left[(s+a)y(0)+y'(0)\right]Q(s)+R(s)Q(s)$

Step 3 Inversion of *Y* to obtain *y*.

$y(t)=\mathcal{L}^{-1}(Y)$.

6.2 Transform of Derivatives and Integrals. ODEs

☑ **Ex. 4** Solve $y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1$

Step 1 $$s^2 Y - sy(0) - y'(0) - Y = \frac{1}{s^2} \quad \Rightarrow \quad (s^2 - 1)Y = s + 1 + \frac{1}{s^2}$$

Step 2 $$Q = \frac{1}{s^2 - 1} \quad \Rightarrow \quad Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

Step 3 $$y(t) = L^{-1}(Y) = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{1}{s^2-1}\right) - L^{-1}\left(\frac{1}{s^2}\right) = e^t + \sinh t - t$$

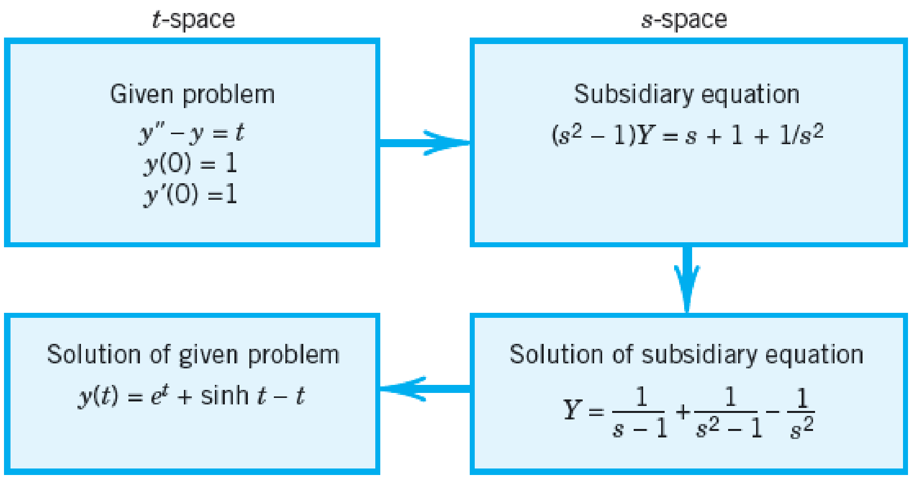


Fig. 116. Steps of the Laplace transform method

Table Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

6.2 Transform of Derivatives and Integrals. ODEs

☑ **Ex. 5** Solve $y'' + y' + 9y = 0. \quad y(0) = 0.16 \quad y'(0) = 0$

Step 1 $s^2Y - 0.16s + sY - 0.16 + 9Y = 0. \quad (s^2 + s + 9)Y = 0.16(s+1)$

Step 2 $$Y = \frac{0.16(s+1)}{s^2+s+9} = \frac{0.16(s+1/2)+0.08}{(s+1/2)^2 + \frac{35}{4}}$$

Step 3 $$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16\cos\frac{\sqrt{35}}{2}t + \frac{0.08}{\sqrt{35}/2}\sin\frac{\sqrt{35}}{2}t\right)$$ $(a = -1/2,\ \omega = \sqrt{\frac{35}{4}})$

$$= e^{-t/2}\left(0.16\cos 2.96t + 0.027\sin 2.96t\right)$$

Table Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

6.2 Transform of Derivatives and Integrals. ODEs

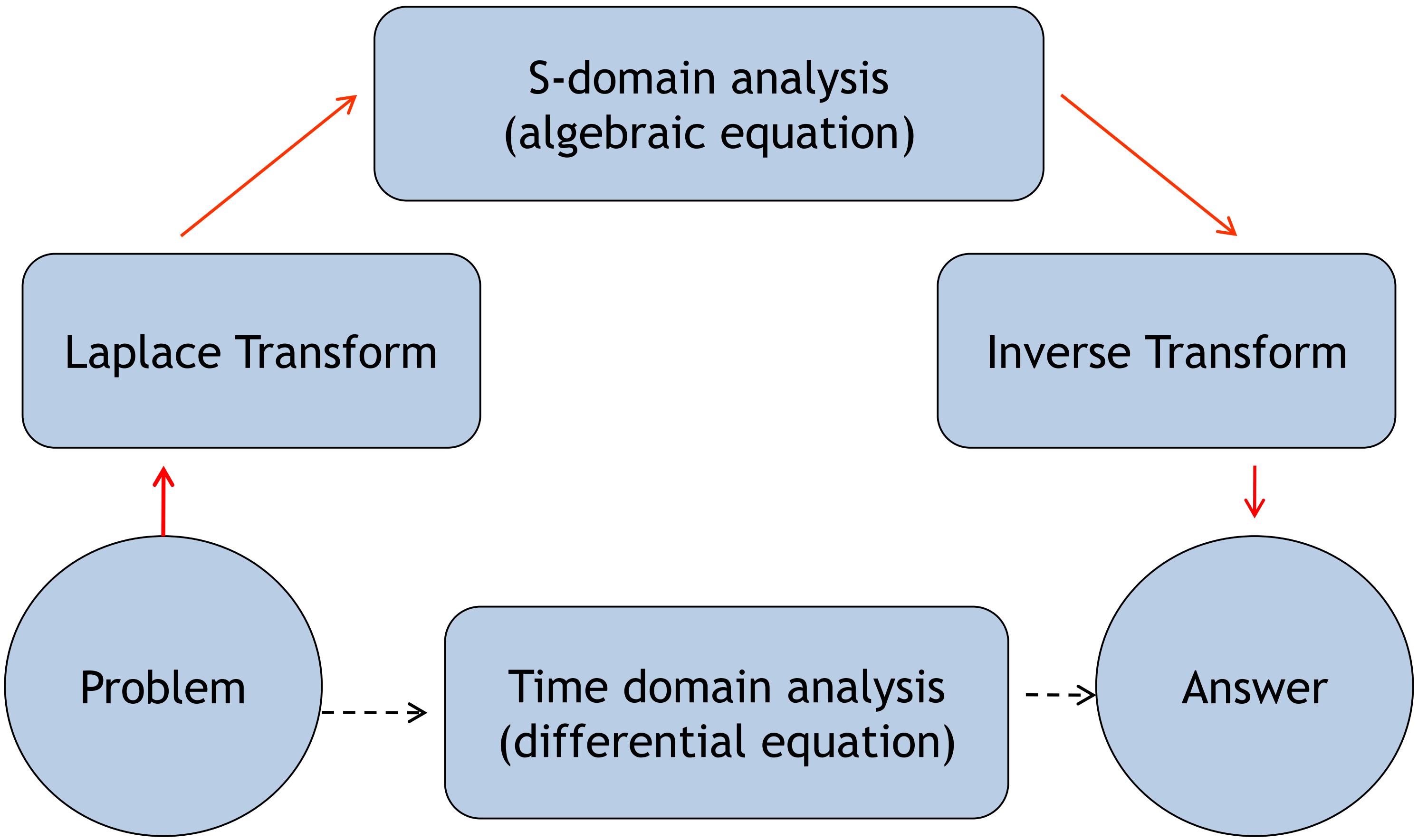
Q? Solve

$$y'' + 7y' + 12y = 21e^{3t},\ y(0) = 3.5,\ y'(0) = -10$$

Table Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

6.2 Transform of Derivatives and Integrals. ODEs



[Relation of Time domain and s-Domain]

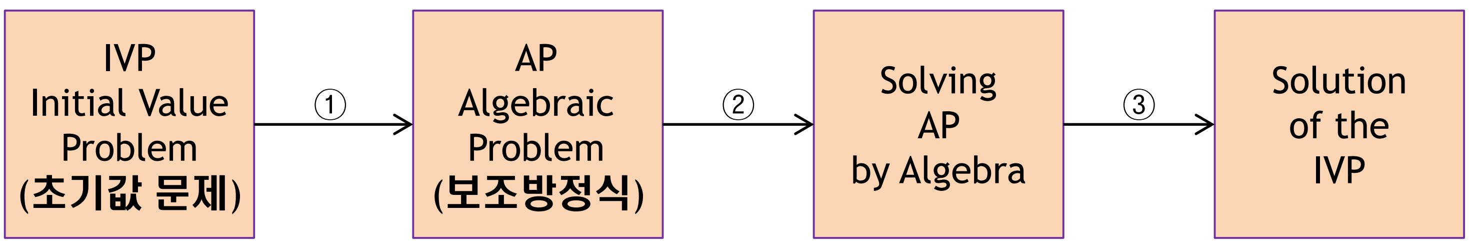
6.2 Transform of Derivatives and Integrals. ODEs

❖ The process of solution consists of three steps.

Step 1 The given ODE is transformed into an algebraic equation ("subsidiary equation").

Step 2 The subsidiary equation is solved by purely algebraic manipulations.

Step 3 The solution in Step 2 is transformed back, resulting in the solution of the given problem.



Solving an IVP by Laplace transforms

6.2 Transform of Derivatives and Integrals. ODEs

❖ Advantages of the Laplace Method

1. Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
2. Initial values are automatically taken care of.
3. Complicated inputs can be handled very efficiently.

6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

❖ Unit Step Function (단위 계단 함수)

- Unit Step Function (Heaviside function): $u(t-a)=\begin{cases}0 & (t<a)\\ 1 & (t>a)\end{cases}$ "on off function"
- Laplace transform of unit step function:

$$\mathcal{L}\{u(t-a)\}=\int_0^{\infty} e^{-st}u(t-a)dt=\int_a^{\infty} e^{-st}\cdot 1dt=-\frac{e^{-st}}{s}\bigg|_{t=a}^{\infty}=\frac{e^{-as}}{s}$$

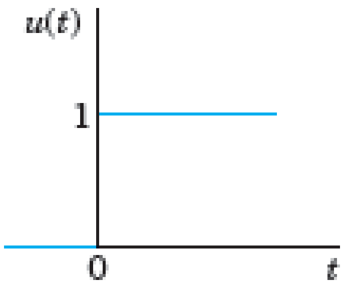


Fig. 118. Unit step function $u(t)$

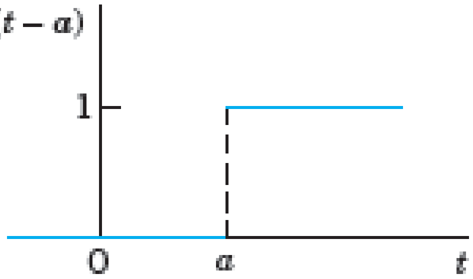
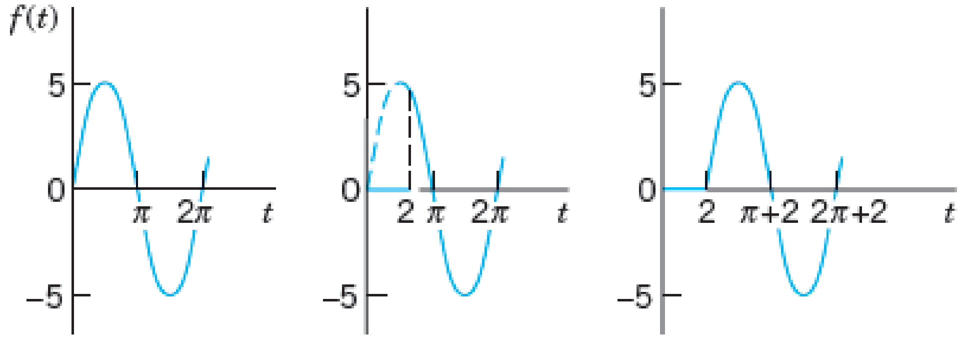


Fig. 119. Unit step function $u(t-a)$

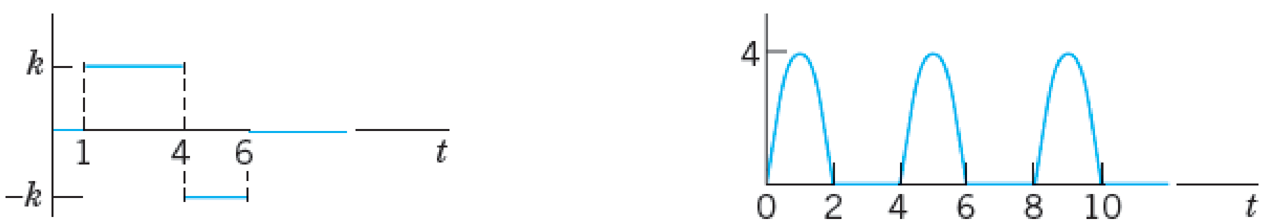
6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

❖ Unit Step Function



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Fig. 120. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 121. Use of many unit step functions.

6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)dt$$

$$\mathcal{L}\{u(t-a)\} = \int_0^{\infty} e^{-st} u(t-a)dt = \int_a^{\infty} e^{-st} \cdot 1 dt = -\frac{e^{-st}}{s}\Big|_{t=a}^{\infty} = \frac{e^{-as}}{s}$$

❖ **Theorem 1 Second Shifting Theorem (제 2이동 정리); Time Shifting**

$$\hat{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$$\mathcal{L}(f(t)) = F(s) \quad \Rightarrow \quad \mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s), \quad f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

Proof) $$e^{-as}F(s) = e^{-as}\int_0^{\infty} e^{-s\tau} f(\tau)d\tau = \int_0^{\infty} e^{-s(\tau+a)} f(\tau)d\tau$$

$$\tau + a = t, \text{ thus } \tau = t - a, \ d\tau = dt$$

$$e^{-as}F(s) = \int_a^{\infty} e^{-st} f(t-a)dt$$

$$= \int_0^{\infty} e^{-st} f(t-a)u(t-a)dt = \int_0^{\infty} e^{-st} \hat{f}(t)dt$$

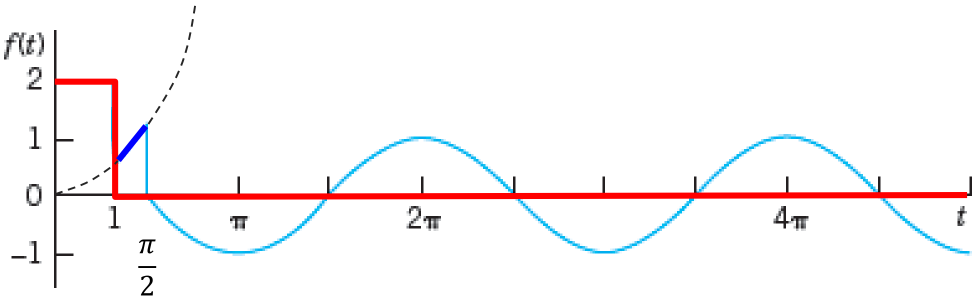
6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

☑ Ex. 1 Write the following function using unit step functions and find its transform.

$$f(t)=\begin{cases} 2 & (0<t<1) \\ t^2/2 & (1<t<\pi/2) \\ \cos t & (t>\pi/2) \end{cases}$$

Step 1 Using unit step functions:

$$f(t)=2\left(1-u(t-1)\right)+\frac{1}{2}t^2\left(u(t-1)-u\left(t-\frac{1}{2}\pi\right)\right)+(\cos t)u\left(t-\frac{1}{2}\pi\right)$$



6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

$$\mathcal{L}\left(f(t-a)u(t-a)\right)=e^{-as}F(s)$$

☑ Ex. 1 Write the following function using unit step functions and find its transform.

$$\mathcal{L}\left(t^n\right)=\frac{n!}{s^{n+1}}$$

Step 1 Using unit step functions:

$$f(t)=2\left(1-u(t-1)\right)+\frac{1}{2}t^2\left(u(t-1)-u\left(t-\frac{1}{2}\pi\right)\right)+(\cos t)u\left(t-\frac{1}{2}\pi\right)$$

Step 2

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=\mathcal{L}\left\{\left(\frac{1}{2}(t-1)^2+(t-1)+\frac{1}{2}\right)u(t-1)\right\}=\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{\left(\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2+\frac{\pi}{2}\left(t-\frac{1}{2}\pi\right)+\frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\right\}=\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{-\left(\sin\left(t-\frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right\}=-\frac{1}{s^2+1}e^{-\pi s/2}$$

$$\Rightarrow\quad \therefore\ \mathcal{L}(f)=\frac{2}{s}-\frac{2}{s}e^{-s}+\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}-\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}-\frac{1}{s^2+1}e^{-\pi s/2}$$

6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

☑ Ex. 2 Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$$

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

❖ **Time shifting**

$$\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{\sin \pi t}{\pi} \quad \Rightarrow \quad \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+\pi^2}\right) = \frac{1}{\pi}\sin\left(\pi(t-1)\right)u(t-1)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+\pi^2}\right) = \frac{1}{\pi}\sin\left(\pi(t-2)\right)u(t-2)$$

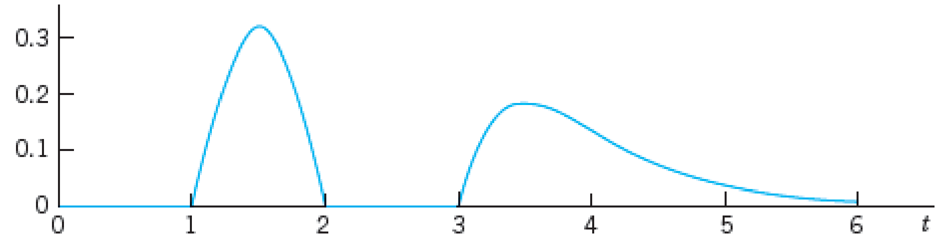
❖ **S-shifting**

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \quad \Rightarrow \quad \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = te^{-2t}$$

$$\Rightarrow \quad f(t) = \frac{1}{\pi}\sin\left(\pi(t-1)\right)u(t-1) + \frac{1}{\pi}\sin\left(\pi(t-2)\right)u(t-2) + (t-3)e^{-2(t-3)}u(t-3)$$

$$= \begin{cases} 0 & (0<t<1) \\ -\dfrac{(\sin \pi t)}{\pi} & (1<t<2) \\ 0 & (2<t<3) \\ (t-3)e^{-2(t-3)} & (t>3) \end{cases}$$



where, $\sin(\pi(t-1)) = \sin(\pi t - \pi) = -\sin \pi t, \sin(\pi(t-2)) = \sin(\pi t - 2\pi) = \sin \pi t$

6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

Q? Solve using Laplace transform.

$$y'' + 3y' + 2y = 1 \text{ if } 0 < t < 1 \text{ and } 0 \ \textit{if} \ t > 1$$

$$y(0) = 0, \ y'(0) = 0$$

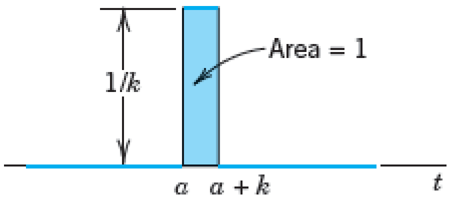
6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

❖ Dirac delta function or the unit impulse function: $\delta(t-a)=\begin{cases}\infty & (t=a)\\ 0 & (\text{otherwise})\end{cases}$

- Definition of the Dirac delta function

$$f_k(t-a)=\begin{cases}1/k & (a\le t\le a+k)\\ 0 & (\text{otherwise})\end{cases}\quad\Rightarrow\quad \delta(t-a)=\lim_{k\to 0} f_k(t-a)$$

❖ Laplace transform of the Dirac delta function



$$\int_0^\infty f_k(t-a)\,dt=\int_a^{a+k}\frac{1}{k}\,dt=1\quad\Rightarrow\quad\int_0^\infty \delta(t-a)\,dt=1$$

$$f_k(t-a)=\frac{1}{k}\left[u(t-a)-u\left(t-(a+k)\right)\right]$$

$$\Rightarrow\ \mathcal{L}\left(f_k(t-a)\right)=\frac{1}{ks}\left[e^{-as}-e^{-(a+k)s}\right]=e^{-as}\frac{1-e^{-ks}}{ks}\quad\Rightarrow\quad\mathcal{L}\left\{\delta(t-a)\right\}=e^{-as}$$

$$\left(\lim_{k\to0}\frac{1-e^{-ks}}{ks}=-\frac{1}{s}\lim_{k\to0}\frac{e^{-ks}-e^{-0s}}{k-0}=-\frac{1}{s}\left.\frac{de^{-ks}}{dk}\right|_{k=0}=-\frac{1}{s}(-s)=1\right)$$

☑ **Ex. 1** Mass Spring System Under a Square Wave

Step 1 $y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2).\quad y(0) = 0 \quad y'(0) = 0$

Step 2 $s^2Y - 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}) \quad Y(s) = \frac{1}{s(s^2+3s+2)}(e^{-s} - e^{-2s})$

Step 3 $F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{(s+1)} + \frac{1/2}{s+2}$

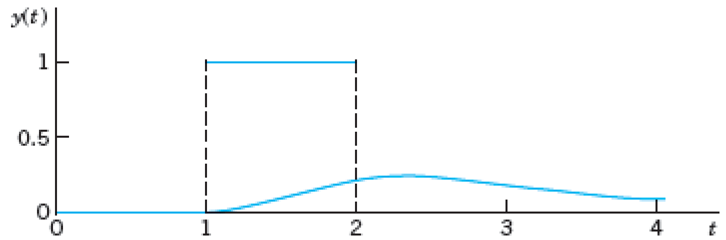
$$f(t) = \mathcal{L}^{-1}(F) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$y = \mathcal{L}^{-1}(F(s)e^{-s} - F(s)e^{-2s})$$
$$= f(t-1)u(t-1) - f(t-2)u(t-2)$$

$$= \begin{cases} 0 & (0 < t < 1) \\ 1/2 - e^{-(t-1)} + 1/2e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{(t-2)} + 1/2e^{-2(t-1)} - 1/2e^{-2(t-2)} & (t > 2) \end{cases}$$

❖ **Time shifting**

$$\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}F(s)$$



❖ **Time shifting**

$$\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}F(s)$$

❖ **S-shifting**

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

☑ **Ex. 2** Find the response of the system in Example 1 with the square wave replaced by a unit impulse at time t = 1.

Initial value problem: $y''+3y'+2y=\delta(t-1), \quad y(0)=0, \quad y'(0)=0$

Subsidiary equation: $s^2Y+3sY+2Y=e^{-s}$

$$\Rightarrow \quad Y(s)=\frac{e^{-s}}{(s+1)(s+2)}=\left(\frac{1}{s+1}-\frac{1}{s+2}\right)e^{-s}$$

$$= f(t-1)u(t-1)-g(t-1)(u-1)$$

❖ **Dirac's Delta**

$$\mathcal{L}\{\delta(t-a)\}=e^{-as}$$

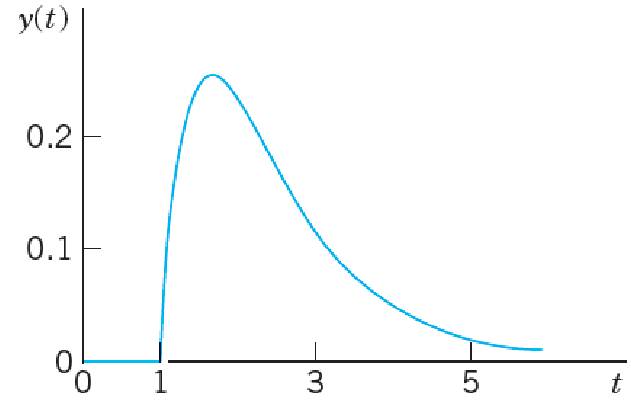
❖ **Time shifting**

$$\mathcal{L}\,(f(t-a)u(t-a))=e^{-as}F(s)$$

$$\mathcal{L}\,(e^{at})=\frac{1}{s-a} \quad \Rightarrow f(t)=e^{-t}, g(t)=e^{-2t}$$

$$= e^{-(t-1)}u(t-1)-e^{-2(t-1)}(u-1)$$

$$\therefore y(t)=\mathcal{L}^{-1}(Y)=\begin{cases} 0 & (0<t<1) \\ e^{-(t-1)}-e^{-2(t-1)} & (t>1) \end{cases}$$



6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

Q? Solve

$$y'' + 9y = \delta(t - \pi / 2),\ y(0) = 2,\ y'(0) = 0$$

6.5 Convolution. Integral Equations

❖ Transform of a product is generally different from the product of the transforms of the factors.

$$\mathcal{L}(f+g)=\mathcal{L}(f)+\mathcal{L}(g) \qquad \mathcal{L}(fg)\neq\mathcal{L}(f)\mathcal{L}(g)$$

❖ Convolution (합성곱): $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$

$$\Rightarrow \mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)$$

$$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$$

$$\mathcal{L}\{\sin\omega t\}=\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos\omega t\}=\frac{s}{s^2+\omega^2}$$

☑ **Ex.** Transform of a Convolution

Evaluate $\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}=\mathcal{L}\left(e^t*\sin t\right)$

Solution) $f(t)=e^t,\ \ g(t)=\sin t$

$$\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}=F(s)G(s)=\mathcal{L}\{e^t\}\mathcal{L}\{\sin t\}$$

$$=\frac{1}{s-1}\cdot\frac{1}{s^2+1}=\frac{1}{(s-1)(s^2+1)}$$

6.5 Convolution. Integral Equations

❖ Convolution: $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$

❖ **Theorem 1 Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on [0, ∞) and of exponential order, then

$$\mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)$$

☑ **Ex. 1** Let $H(s)=\dfrac{1}{(s-a)s}$. Find $h(t)$.

$$\mathcal{L}(h)=H(s) \qquad \mathcal{L}(h)=\mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)=\frac{1}{s-a}\frac{1}{s},\ \ \mathcal{L}(f)=\frac{1}{s-a},\ \mathcal{L}(g)=\frac{1}{s}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=e^{at},\quad \mathcal{L}^{-1}\left(\frac{1}{s}\right)=1 \quad\Rightarrow\quad h(t)=e^{at}*1=\int_0^t e^{a\tau}\cdot 1\,d\tau=\frac{1}{a}\left(e^{at}-1\right)$$

$$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}},\ \mathcal{L}\{e^{at}\}=\frac{1}{s-a}$$

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

☑ Ex. 2 Convolution

Evaluate $H(s)=\dfrac{1}{(s^2+\omega^2)^2}$, find $h(t) \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\}$

Solution) Let $F(s)=G(s)=\dfrac{1}{(s^2+\omega^2)}$

$$f(t)=g(t)=\frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}=\frac{1}{\omega}\sin\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\}=\frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)\,d\tau$$

$$=\frac{1}{\omega^2}\int_0^t \frac{1}{2}[\cos(\omega\tau-\omega t+\omega\tau)-\cos(\omega\tau+\omega t-\omega\tau)]\,d\tau$$

$$=\frac{1}{2\omega^2}\int_0^t [\cos(2\omega\tau-\omega t)-\cos(\omega t)]\,d\tau \quad =\frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin\omega(2\tau-t)-\tau\cos\omega t\right]_0^t$$

$$=\frac{1}{2\omega^2}\left[\frac{\sin\omega t}{\omega}-t\cos\omega t\right]$$

$$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$$

$$\mathcal{L}\{\sin\omega t\}=\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos\omega t\}=\frac{s}{s^2+\omega^2}$$

6.5 Convolution. Integral Equations

☑ **Ex. 4 Repeated Complex Factors. Resonance – Undamped mass-spring system**

$$y'' + \omega_0^2 y = K \sin \omega_0 t \qquad y(0) = 0 \quad y'(0) = 0$$

The subsidiary equation

$$s^2 Y + \omega_0^2 Y = \frac{K\omega_0}{s^2 + \omega_0^2} \Rightarrow Y(s) = \frac{K\omega_0}{(s^2 + \omega_0^2)^2}$$

In Ex. 2, we found

$$H(s) = \frac{1}{(s^2 + \omega^2)^2} \quad \Rightarrow \quad h(t) = \frac{1}{2\omega^2}\left[\frac{\sin \omega t}{\omega} - t\cos \omega t\right]$$

The solution

$$y(t) = \frac{K\omega_0}{2\omega_0^2}\left[\frac{\sin \omega_0 t}{\omega_0} - t\cos \omega_0 t\right]$$

6.5 Convolution. Integral Equations

❖ Property of convolution

- Commutative Law (교환법칙): $f * g = g * f$
- Distributive Law (분배법칙): $f * (g_1 + g_2) = f * g_1 + f * g_2$
- Associative Law (결합법칙): $(f * g) * v = f * (g * v)$
- Unusual Properties of Convolution: $f * 0 = 0 * f = 0$

☑ **Ex. 3 Unusual Properties of Convolution**

$$f * 1 \neq f$$

$$t * 1 = \int_0^t \tau \cdot 1 d\tau = \frac{1}{2} t^2 \neq t$$

6.5 Convolution. Integral Equations

❖ Application to Nonhomogeneous Linear ODEs by convolution

$$y'' + ay' + by = r(t), \quad y(0) = K_0, \quad y'(0) = K_1 \qquad (Y = \mathcal{L}(y), \quad R = \mathcal{L}(r))$$

$$\left[s^2 Y - sy(0) - y'(0)\right] + a\left[sY - y(0)\right] + bY = R(s)$$

$$\Rightarrow \quad (s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s)$$

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{\left(s + \frac{1}{2}a\right)^2 + b - \frac{1}{4}a^2}$$

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$$

$$f * g = g * f$$

If $y(0) = 0 \quad y'(0) = 0$ ⇨ $Y = RQ$

⇨ $y(t) = \int_0^t q(t - \tau) r(\tau)\, d\tau$

6.5 Convolution. Integral Equations

❖ **Theorem 1 Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on [0, ∞) and of exponential order, then

$$\mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)$$

Proof) $$F(s)=\int_0^\infty e^{-s\tau}f(\tau)d\tau \quad and \quad G(s)=\int_0^\infty e^{-sp}g(p)dp$$

$$\mathcal{L}(f*g)=\mathcal{L}(h)=H(s)$$

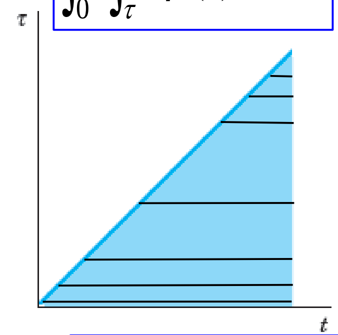
$$t=p+\tau, \quad p=t-\tau, \quad t=[\tau,\infty)$$

$$G(s)=\int_\tau^\infty e^{-s(t-\tau)}g(t-\tau)dt=e^{s\tau}\int_\tau^\infty e^{-st}g(t-\tau)dt$$

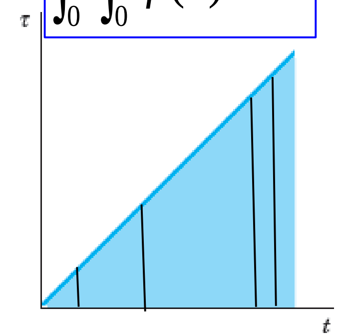
$$F(s)G(s)=\int_0^\infty e^{-s\tau}f(\tau)e^{s\tau}\int_\tau^\infty e^{-st}g(t-\tau)dtd\tau=\int_0^\infty f(\tau)\int_\tau^\infty e^{-st}g(t-\tau)dtd\tau$$

$$F(s)G(s)=\int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)d\tau dt=\int_0^\infty e^{-st}h(t)dt=\mathcal{L}(h)=H(s)$$

$$\int_0^\infty\int_\tau^\infty f(t)dtd\tau$$



$$\int_0^\infty\int_0^t f(\tau)d\tau dt$$



6.5 Convolution. Integral Equations

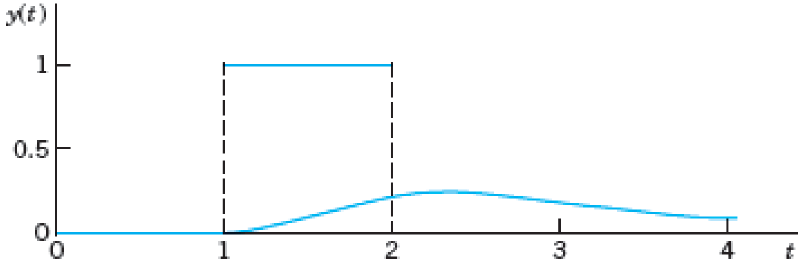
☑ **Ex. 5** **Response of a Damped Vibrating System to a Single Square Wave**

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2). \quad y(0) = 0 \quad y'(0) = 0$$

$$s^2Y - 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}) \quad \Rightarrow Q(s) = \frac{1}{(s^2 + 3s + 2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$q(t) = e^{-t} - e^{-2t}$$

$$r(t) = 1, \text{ only for } 1 < t < 2$$



For $t < 1, \ y = 0$

For $1 < t < 2, \ y = \int_1^t q(t-\tau)\cdot 1 d\tau = \int_1^t e^{-(t-\tau)} - e^{-2(t-\tau)} d\tau$

$$= \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right]_1^t = \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

For $2 < t, \ y = \int_0^t q(t-\tau)\cdot 1 d\tau = \int_1^2 q(t-\tau)\cdot 1 d\tau = \int_1^2 e^{-(t-\tau)} - e^{-2(t-\tau)} d\tau$

$$= \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right]_1^2 = e^{-(t-2)} - \frac{1}{2} e^{-2(t-2)} - \left(e^{-(t-1)} - \frac{1}{2} e^{-2(t-1)} \right)$$

6.5 Convolution. Integral Equations (적분방정식)

❖ Integral Equation: Equation in which the unknown function appears in an integral.

Unknown

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

: Volterra integral equation for $f(t)$

☑ Ex. 6 Solve the Volterra integral equation of the second kind.

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)d\tau = t$$

Equation written as a convolution: $y - y * \sin t = t$ $\mathcal{L}(y) = Y(t)$

Apply the convolution theorem: $Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2}$

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \quad \Rightarrow \quad y(t) = t + \frac{t^3}{6}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

6.5 Convolution. Integral Equations

☑ **Example** **An Integral Equation**

Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}\,d\tau \quad for \;\; f(t)$

Solution)

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{t^2\} - \mathcal{L}\{e^{-t}\} - \mathcal{L}\left\{\int_0^t f(\tau)e^{t-\tau}\,d\tau\right\}$$

$$F(s) = 3\frac{2!}{s^3} - \frac{1}{s+1} - F(s)\cdot\frac{1}{s-1}$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$f(t) = 3\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 3t^2 - t^3 + 1 - 2e^{-t}$$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

6.5 Convolution. Integral Equations

Q? Solve $y(t)+2e^{t}\int_{0}^{t} y(\tau)e^{-\tau}d\tau = te^{t}$

Q? find $f(t)$ if $\mathcal{L}\{f(t)\}$ equals:

$$\mathcal{L}\{f(t)\} = \frac{9}{s(s+3)},\ f(t) = ?$$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

Differentiation of Transforms

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt \quad \Rightarrow \quad F'(s) = \frac{dF}{ds} = -\int_0^\infty e^{-st} tf(t)\,dt = -\mathcal{L}(tf)$$

$$\Rightarrow \quad \mathcal{L}\left(tf(t)\right) = -F'(s), \quad \mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

☑ Ex.1 We shall derive the following three formulas.

$\mathcal{L}(f)$	$f(t)$
$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

$\mathcal{L}(f)$	$f(t)$
$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
$\dfrac{s}{(s^2+\beta^2)^2}$ ⇐	$\dfrac{t}{2\beta}\sin\beta t$
$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

$$\mathcal{L}\{tf(t)\} = -F'(s),$$
$$\mathcal{L}^{-1}\{F'(s)\} = -f(t)$$

$$\mathcal{L}(\sin\beta t) = \frac{\beta}{s^2+\beta^2}$$

By differentiation

$$\Rightarrow \mathcal{L}(t\sin\beta t) = -F'(s) = \frac{2\beta s}{(s^2+\beta^2)^2} \Rightarrow \mathcal{L}\left(\frac{t}{2\beta}\sin\beta t\right) = \frac{s}{(s^2+\beta^2)^2}$$

$$\mathcal{L}(\cos\beta t) = \frac{s}{s^2+\beta^2}$$

By differentiation

$$\Rightarrow \mathcal{L}(t\cos\beta t) = -F'(s) = -\frac{1}{s^2+\beta^2} + \frac{2s^2}{(s^2+\beta^2)^2}$$

$$= -\frac{s^2+\beta^2-2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$\Rightarrow \mathcal{L}(t\cos\beta t) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

$\mathcal{L}(f)$		$f(t)$
$\dfrac{1}{(s^2+\beta^2)^2}$	⇐	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
$\dfrac{s}{(s^2+\beta^2)^2}$		$\dfrac{t}{2\beta}\sin\beta t$
$\dfrac{s^2}{(s^2+\beta^2)^2}$	⇐	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

$$\mathcal{L}\{tf(t)\} = -F'(s),$$
$$\mathcal{L}^{-1}\{F'(s)\} = -f(t)$$

$$\mathcal{L}(t\cos\beta t) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}(\sin\beta t) = \frac{\beta}{s^2+\beta^2}$$

$$\mathcal{L}\left\{\frac{1}{2}\left(\frac{1}{\beta}\sin\beta t + t\cos\beta t\right)\right\} = \frac{1}{2}\left(\frac{1}{(s^2+\beta^2)} + \frac{s^2-\beta^2}{(s^2+\beta^2)^2}\right) = \frac{1}{2}\frac{s^2+\beta^2+s^2-\beta^2}{(s^2+\beta^2)^2} = \frac{s^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}\left\{\frac{1}{2\beta^2}\left(\frac{1}{\beta}\sin\beta t - t\cos\beta t\right)\right\} = \frac{1}{2\beta^2}\left(\frac{1}{(s^2+\beta^2)} - \frac{s^2-\beta^2}{(s^2+\beta^2)^2}\right) = \frac{1}{2\beta^2}\frac{s^2+\beta^2-s^2+\beta^2}{(s^2+\beta^2)^2}$$

$$= \frac{1}{2\beta^2}\frac{2\beta^2}{(s^2+\beta^2)^2} = \frac{1}{(s^2+\beta^2)^2}$$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

❖ Integration of Transforms: $\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\int_{s}^{\infty}F(\tilde{s})d\tilde{s} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{\int_{s}^{\infty}F(\tilde{s})d\tilde{s}\right\}=\frac{f(t)}{t}$

Proof)

$$\int_s^\infty F(\tilde{s})d\tilde{s}=\int_s^\infty\left[\int_0^\infty e^{-\tilde{s}t}f(t)dt\right]d\tilde{s}$$

$$\int_s^\infty F(\tilde{s})d\tilde{s}=\int_0^\infty\left[\int_s^\infty e^{-\tilde{s}t}f(t)d\tilde{s}\right]dt=\int_0^\infty f(t)\left[\int_s^\infty e^{-\tilde{s}t}d\tilde{s}\right]dt$$

$$\int_s^\infty e^{-\tilde{s}t}d\tilde{s}=\left[-\frac{e^{-\tilde{s}t}}{t}\right]_s^\infty=\frac{e^{-st}}{t}$$

$$\int_s^\infty F(\tilde{s})d\tilde{s}=\int_0^\infty e^{-st}\frac{f(t)}{t}dt=\mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

$$\mathcal{L}\{tf(t)\} = -F'(s), \qquad \mathcal{L}^{-1}\{F'(s)\} = -f(t)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}\,, \qquad \mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}$$

☑ Ex. 2 Find the inverse transform of $\ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\frac{s^2+\omega^2}{s^2}$

$$\ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\frac{s^2+\omega^2}{s^2} \xrightarrow{\text{Derivative}} \frac{d}{ds}\left(\ln\left(s^2+\omega^2\right) - \ln s^2\right) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$

Case 1 Differentiation of transform

$$F(s) = \mathcal{L}(f) = \ln\left(1+\frac{\omega^2}{s^2}\right) \quad \Rightarrow \quad \mathcal{L}^{-1}\left(F'(s)\right) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}\right) = 2\cos\omega t - 2 = -tf(t)$$

$$\therefore \; f(t) = \frac{2\cos\omega t - 2}{-t} = \frac{2}{t}\left(1-\cos\omega t\right)$$

Case 2 Integration of transform

$$G(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s} \quad \Rightarrow \quad g(t) = \mathcal{L}^{-1}(G) = 2\left(\cos\omega t - 1\right)$$

$$\therefore \mathcal{L}^{-1}\left(\ln\left(1+\frac{\omega^2}{s^2}\right)\right) = \mathcal{L}^{-1}\left(\int_s^\infty G(\tilde{s})\,d\tilde{s}\right) = -\frac{g(t)}{t} = \frac{2}{t}\left(1-\cos\omega t\right)$$

Because the lower limit of the integral is "s"

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

❖ Special Linear ODEs with Variable Coefficients

$$\mathcal{L}(ty') = -\frac{d}{ds}\left[sY - y(0)\right] = -Y - s\frac{dY}{ds}$$

$$\mathcal{L}(ty'') = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] = -2sY - s^2\frac{dY}{ds} + y(0)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0),$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

☑ **Ex. 3 Laguerre's Equation, Laguerre Polynomials**

Laguerre's ODE: $ty'' + (1-t)y' + ny = 0 \quad (n = 0, 1, 2, \cdots), \; y(0) = y'(0) = 0$

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0$$

$$\Rightarrow \quad (s - s^2)\frac{dY}{ds} + (n+1-s)Y = 0$$

$$\Rightarrow \quad \frac{dY}{Y} = -\frac{n+1-s}{s-s^2}ds = \left(\frac{n}{s-1} - \frac{n+1}{s}\right)ds$$

$$\Rightarrow \quad \ln Y = n\ln(s-1) - (n+1)\ln s \Rightarrow \quad \ln Y = \ln\frac{(s-1)^n}{s^{n+1}} \quad \Rightarrow \quad Y = \frac{(s-1)^n}{s^{n+1}}$$

Rodrigues's formula

$$l_n = \mathcal{L}^{-1}(Y) = ?$$

❖ **S-shifting**

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

In the mean time, $\mathcal{L}\{t^n e^{-t}\} = \dfrac{n!}{(s+1)^{n+1}}$ ⇐

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

☑ **Ex.3 Laguerre's Equation, Laguerre Polynomials**

Laguerre's ODE: $ty'' + (1-t)y' + ny = 0 \quad (n = 0,1,2,\cdots), \ y(0) = y'(0) = 0$

$$\mathcal{L}\{t^n e^{-t}\} = \frac{n!}{(s+1)^{n+1}} \qquad f(t) = t^n e^{-t} \Rightarrow f(0) = f'(0) =, \dots, = f^{(n-1)}(0) = 0$$

$$\mathcal{L}\left(f^{(n)}\right) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) = s^n \mathcal{L}(f)$$

$$\therefore \mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n! s^n}{(s+1)^{n+1}}$$

Back to $\boxed{Y = \frac{(s-1)^n}{s^{n+1}}}$ $\boxed{l_n = \mathcal{L}^{-1}(Y) = ?}$

❖ **S-shifting**

$\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}$

$$\mathcal{L}\{l_n\} = \mathcal{L}\left\{\frac{e^t}{n!} \frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{1}{n!} \frac{n!(s-1)^n}{(s-1+1)^{n+1}} = \frac{(s-1)^n}{s^{n+1}} = Y \quad \Leftarrow$$

$$\therefore l_n(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 1, & n = 0 \\ \dfrac{e^t}{n!} \dfrac{d^n}{dt^n}(t^n e^{-t}), & n = 1, \ 2, \ \cdots \end{cases}$$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

Q? Solve $\mathcal{L}\left\{t^2 \sin 3t\right\} = ?$

Q? Solve $\mathcal{L}\left\{f(t)\right\} = \dfrac{s}{(s^2+16)^2},\ f(t) = ?$

6.7 Systems of ODEs

❖ The Laplace transform method may also be used for solving systems of ODE (연립상미분 방정식),
a first-order linear system with constant coefficients.

$$Y_1 = \mathcal{L}(y_1), \quad Y_2 = \mathcal{L}(y_2),$$
$$G_1 = \mathcal{L}(g_1), \quad G_2 = \mathcal{L}(g_2)$$

$$y_1' = a_{11}y_1 + a_{12}y_2 + g_1(t)$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + g_2(t)$$

$$sY_1 - y_1(0) = a_{11}Y_1 + a_{12}Y_2 + G_1(s)$$
$$sY_2 - y_2(0) = a_{21}Y_1 + a_{22}Y_2 + G_2(s)$$

$$(a_{11} - s)Y_1 + a_{12}Y_2 = -y_1(0) - G_1(s)$$
$$a_{21}Y_1 + (a_{22} - s)Y_2 = -y_2(0) - G_2(s)$$

$$\Rightarrow Y_1,\ Y_2 \Rightarrow y_1 = \mathcal{L}^{-1}(Y_1), \quad y_2 = \mathcal{L}^{-1}(Y_2)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0),$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

☑ Ex. 3 Model of Two Masses on Springs

The mechanical system consists of two bodies of mass 1 on three springs of the same spring constant *k*. Damping is assumed to be practically zero.

Model of the physical system:

$$y_1'' = -ky_1 + k(y_2 - y_1)$$
$$y_2'' = -k(y_2 - y_1) - ky_2$$

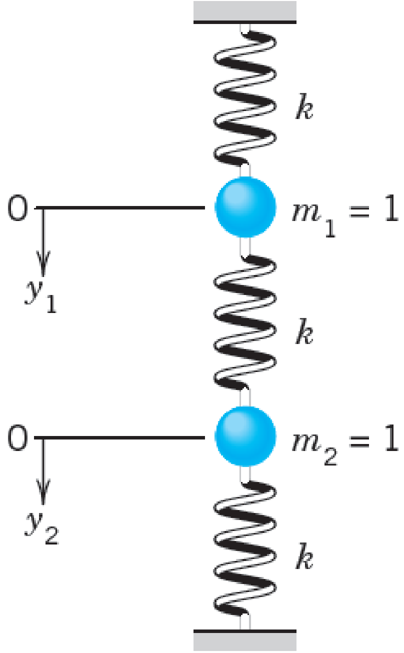
Initial conditions:

$$y_1(0) = 1, \quad y_2(0) = 1,$$
$$y_1'(0) = \sqrt{3k}, \quad y_2(0) = -\sqrt{3k}$$

⇩

Laplace transform

$$s^2Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2$$



6.7 Systems of ODEs

☑ Ex. 3 Model of Two Masses on Springs

$$s^2Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2$$

⇩ **Cramer's rule or Elimination**

$$Y_1 = \frac{\left(s+\sqrt{3k}\right)\left(s^2+2k\right)+k\left(s-\sqrt{3k}\right)}{\left(s^2+2k\right)^2-k^2} = \frac{s}{s^2+k} + \frac{\sqrt{3k}}{s^2+3k}$$

$$Y_2 = \frac{\left(s-\sqrt{3k}\right)\left(s^2+2k\right)+k\left(s-\sqrt{3k}\right)}{\left(s^2+2k\right)^2-k^2} = \frac{s}{s^2+k} - \frac{\sqrt{3k}}{s^2+3k}$$

⇩ **Inverse transform**

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$
$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t$$