# Chapter 4: Basic continuum mechanics Part 3

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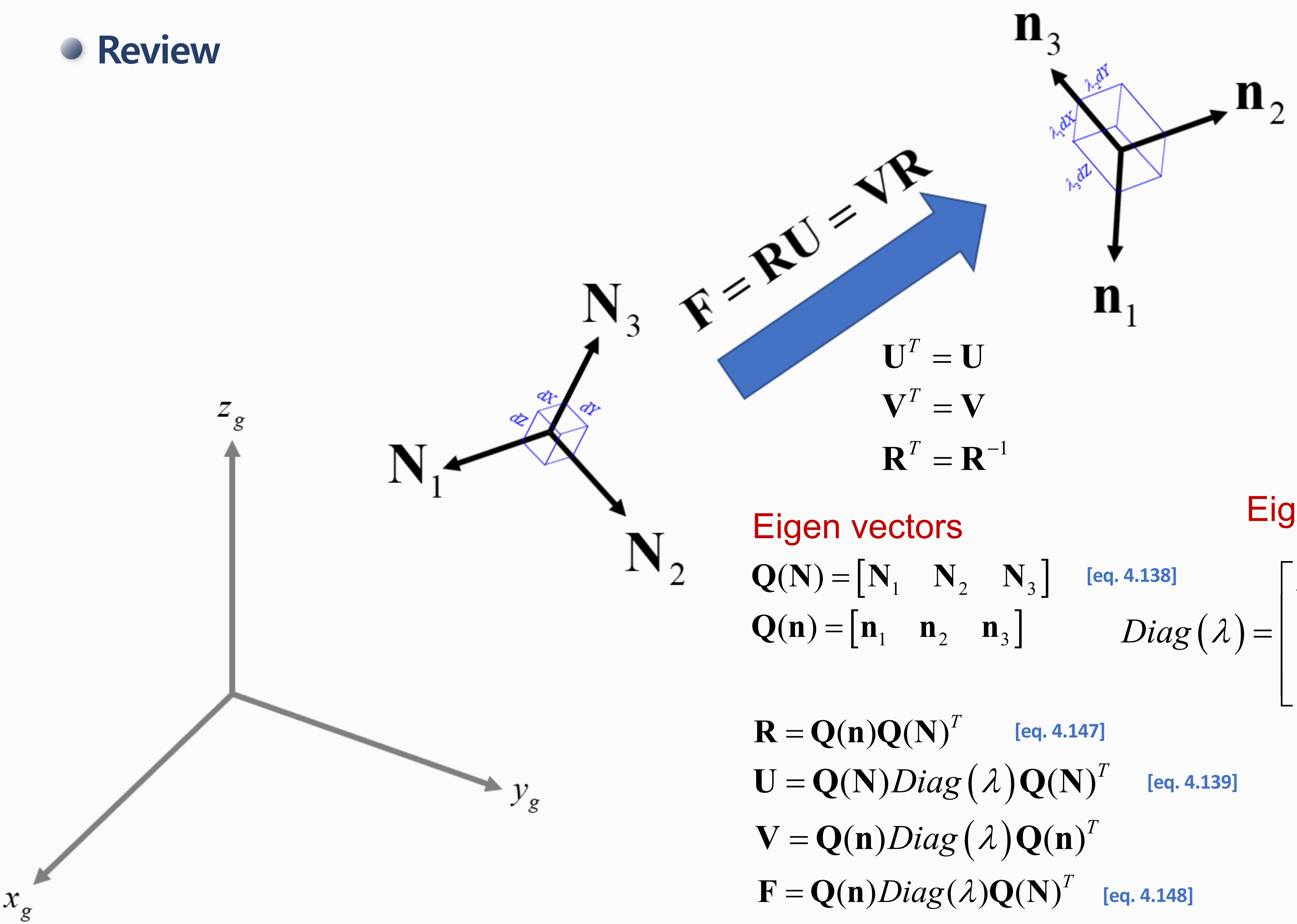
## Review




$$\mathbf{U}^T = \mathbf{U}$$
$$\mathbf{V}^T = \mathbf{V}$$
$$\mathbf{R}^T = \mathbf{R}^{-1}$$

Eigen vectors

$$\mathbf{Q}(\mathbf{N}) = [\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3]$$ [eq. 4.138]

$$\mathbf{Q}(\mathbf{n}) = [\mathbf{n}_1 \quad \mathbf{n}_2 \quad \mathbf{n}_3]$$

Eigen values

$$Diag(\lambda) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{Q}(\mathbf{n})\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.147]

$$\mathbf{U} = \mathbf{Q}(\mathbf{N})Diag(\lambda)\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.139]

$$\mathbf{V} = \mathbf{Q}(\mathbf{n})Diag(\lambda)\mathbf{Q}(\mathbf{n})^T$$

$$\mathbf{F} = \mathbf{Q}(\mathbf{n})Diag(\lambda)\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.148]

$$\mathbf{U} = \mathbf{Q}(\mathbf{N}) \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mathbf{Q}(\mathbf{N})^T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_{\mathbf{N}_i}$$ [eq. 4.139]

"referenced with undeformed config."

- In principal strain space, the Green strain components can be written as:

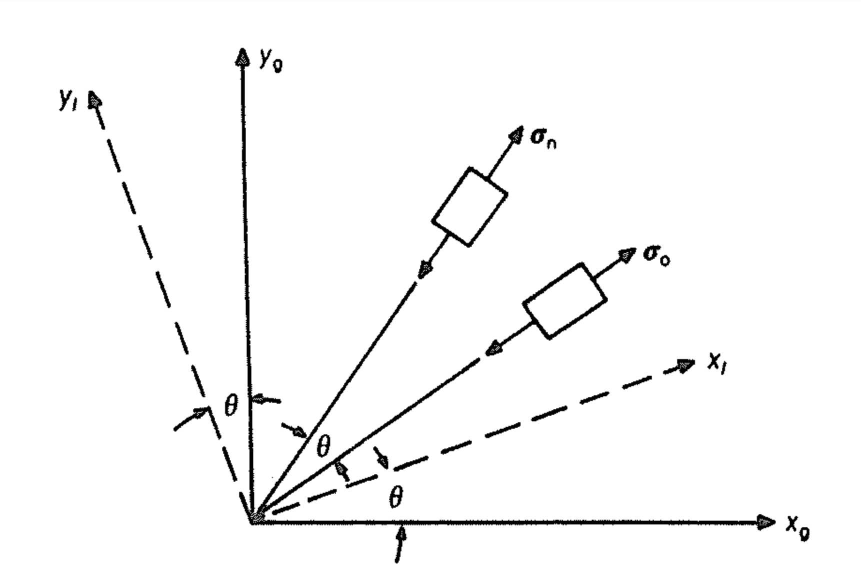
$$E_i = \frac{1}{2}\left(\lambda_i^2 - 1\right) \quad i = 1,2,3\ (\textit{principal strain space of } \mathbf{N})$$ [eq. 4.151]

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{U}^T\mathbf{U} - \mathbf{Q}(\mathbf{N})\mathbf{Q}(\mathbf{N})^T\right) = \mathbf{Q}(\mathbf{N})\,Diag\left(\frac{\lambda^2 - 1}{2}\right)\mathbf{Q}(\mathbf{N})^T$$ [eq. 4.153]

- Almansi strains can be written as:

$$A_i = \frac{1}{2}\left(1 - \frac{1}{\lambda_i^2}\right) \quad i = 1,2,3\ (\textit{principal strain space of } \mathbf{n})$$ [eq. 4.152]

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right) = \frac{1}{2}\left(\mathbf{Q}(\mathbf{n})\mathbf{Q}(\mathbf{n})^T - \mathbf{V}^{-T}\mathbf{V}^{-1}\right) = \mathbf{Q}(\mathbf{n})\,Diag\left(\frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)\right)\mathbf{Q}(\mathbf{n})^T$$ [eq. 4.154]




[Fig 4.8 Rotating a stress state]

$$\boldsymbol{\sigma}_{n,g} = \mathbf{R}\boldsymbol{\sigma}_{o,g}\mathbf{R}^T$$ [eq. 4.63]

- If the strains are small, $\lambda_i \simeq 1$ so

$$\mathbf{U} \simeq \mathbf{I}$$ [eq. 4.156]

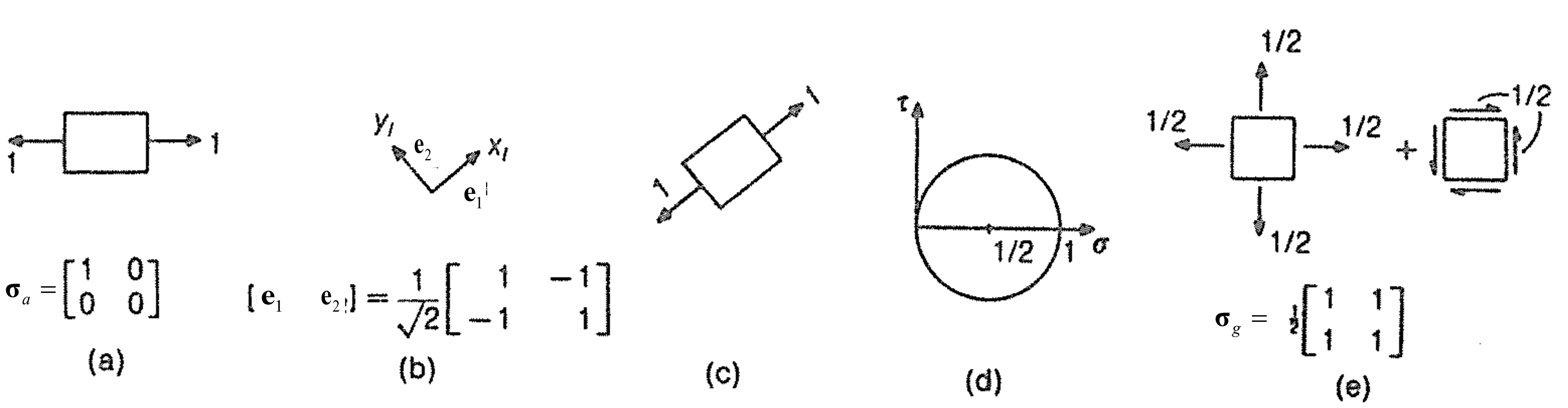
$$\mathbf{F} \simeq \mathbf{R}$$ [eq. 4.157]

$$J = \det(\mathbf{F}) \simeq 1$$ [eq. 4.158]

➡ $$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} \simeq \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$$

From Fig 4.8, $\mathbf{S} \simeq \mathbf{R}^T\boldsymbol{\sigma}_{n,g}\mathbf{R}$

➡ $$\mathbf{S} \simeq \boldsymbol{\sigma}_{o,g}$$ [eq. 4.160]




[Fig 4.15 Some concepts with rotating coordinates]

- Suppose the stress state in (a) is rotated to (c).

$$\boldsymbol{\sigma}_g = \mathbf{R}\boldsymbol{\sigma}_a\mathbf{R}^T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{[eq. 4.161]}$$

- **Hyperelastic model:** Higher-order form of linear elastic models in which the stresses are some functions of the total strains or stretches.

$$\delta\varphi = \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = \frac{\partial\varphi}{\partial\boldsymbol{\varepsilon}} : \delta\boldsymbol{\varepsilon} \qquad \varphi : strain\ energy / unit\ volume$$ [eq. 4.163]

$$\varphi = \int \delta\varphi = \int \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}$$ [eq. 4.164]

※ If plastic strain is involved, strain energy is not a state function.

$$\boldsymbol{\sigma} = \frac{\partial\varphi}{\partial\boldsymbol{\varepsilon}}$$ [eq. 4.165]

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}_t : \dot{\boldsymbol{\varepsilon}} = \frac{\partial^2\varphi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}$$ [eq. 4.166]

$$\varphi = \varphi\left(\lambda_1, \lambda_2, \lambda_3\right)$$

"constitutive modeling of hyperelastic materials"

- **Hypo-elastic model:**
  Strain and stress only have a rate(incremental) relationship.

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}_t : \dot{\boldsymbol{\varepsilon}} \quad \text{or} \quad \delta\boldsymbol{\sigma} = \mathbf{C}_t : \delta\boldsymbol{\varepsilon}$$ [eq. 4.170]

- $\mathbf{C}_t$ is not only required for the structural tangent stiffness matrix, but also it must be 'integrated' (at the Gauss-point level) to obtain the total stresses and hence the internal force vector.

- Plasticity leads to eq. 4.170.

# Thank you!