• Mean flow and turb kinetic energy

$$\begin{cases} \dfrac{\bar{D}\bar{E}}{\bar{D}t} + \dfrac{\partial}{\partial x_j}\bar{T}_j = -P - \bar{\varepsilon} \\[2ex] \dfrac{\bar{D}k}{\bar{D}t} + \dfrac{\partial}{\partial x_j}T_j' = P - \varepsilon \end{cases}$$

$\bar{E} = \frac{1}{2}\langle \underline{U}\rangle \cdot \langle \underline{U}\rangle$

$k = \frac{1}{2}\langle u_i u_i\rangle$

$\bar{T}_j = \cdots$

$T_j' = \cdots$

$$P = -\langle u_i u_j\rangle \frac{\partial \langle U_i\rangle}{\partial x_j}$$: production

$$\bar{\varepsilon} = 2\nu \bar{S}_{ij}\bar{S}_{ij} = 2\nu \langle S_{ij}\rangle\langle S_{ij}\rangle$$

$$\varepsilon = 2\nu \langle s_{ij} s_{ij}\rangle \qquad s_{ij} = S_{ij} - \langle S_{ij}\rangle$$

[dissipation due to fluctuating strain rate

• dissipation ε

$$\varepsilon = 2\nu \langle s_{ij} s_{ij}\rangle \geq 0$$

$$2\nu \left\langle s_{ij} \cdot \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\rangle$$

$$= 2\nu \langle s_{ij} \frac{\partial u_i}{\partial x_j} \rangle$$: fluctuating vel. grads. working against the fluctuating deviatoric stresses
transfer kinetic energy to internal energy.

$$\hat{P} = \frac{P}{U_0^3/r_{\frac{1}{2}}} \simeq \frac{-\langle uv \rangle}{U_0^2} \frac{r_{1/2}}{U_0} \frac{\partial \langle U \rangle}{\partial r}$$ self-similar

$$\hat{\varepsilon} = \frac{\varepsilon}{U_0^3/r_{\frac{1}{2}}}$$

are self-similar and indep. of Re.

$\sim$ from k-eq

- $\boxed{\text{Kolmogorov Scales}}$: characteristic scales of the smallest turbulent motions formed from ε and ν.

Ch. 6

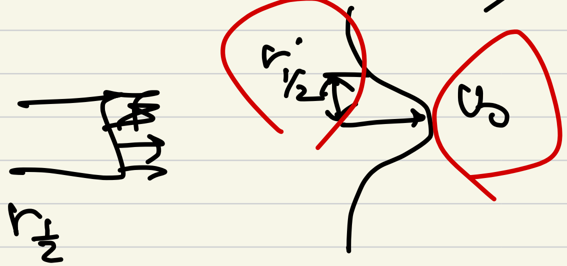
length scale : $\eta = (\nu^3/\varepsilon)^{1/4}$ $\qquad \eta = \varepsilon^a \nu^b$

time scale : $\tau_\eta = (\nu/\varepsilon)^{1/2}$

velocity scale : $u_\eta = (\nu\varepsilon)^{1/4}$

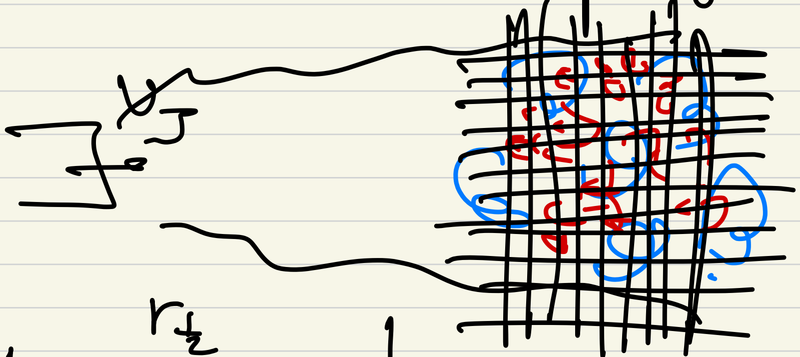
Comparison with mean flow scales (U_0 and $r_{1/2}$)

$$\left\{\begin{array}{l}
\cdot\ \dfrac{\eta}{r_{\frac{1}{2}}} = \dfrac{(\nu^3/\varepsilon)^{1/4}}{r_{\frac{1}{2}}} = \boxed{Re_0^{-\frac{3}{4}}}\ \hat{\varepsilon}^{-\frac{1}{4}} \\[2ex]
\cdot\ \dfrac{\tau_\eta}{r_{\frac{1}{2}}/U_0} = Re_0^{-\frac{1}{2}}\ \hat{\varepsilon}^{-\frac{1}{2}} \\[2ex]
\cdot\ \dfrac{u_\eta}{U_0} = Re_0^{-\frac{1}{4}}\ \hat{\varepsilon}^{\frac{1}{4}}
\end{array}\right.$$



$$Re_0 = \frac{U_0\, r_{\frac{1}{2}}}{\nu}$$

$\hat{\varepsilon}$ is non-dimensional and indep. of Re.



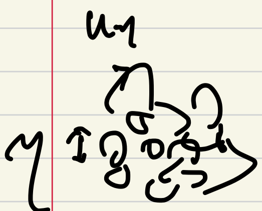
Re_1

$\wedge$

Re_2

$$\eta \sim Re^{-\frac{3}{4}}$$

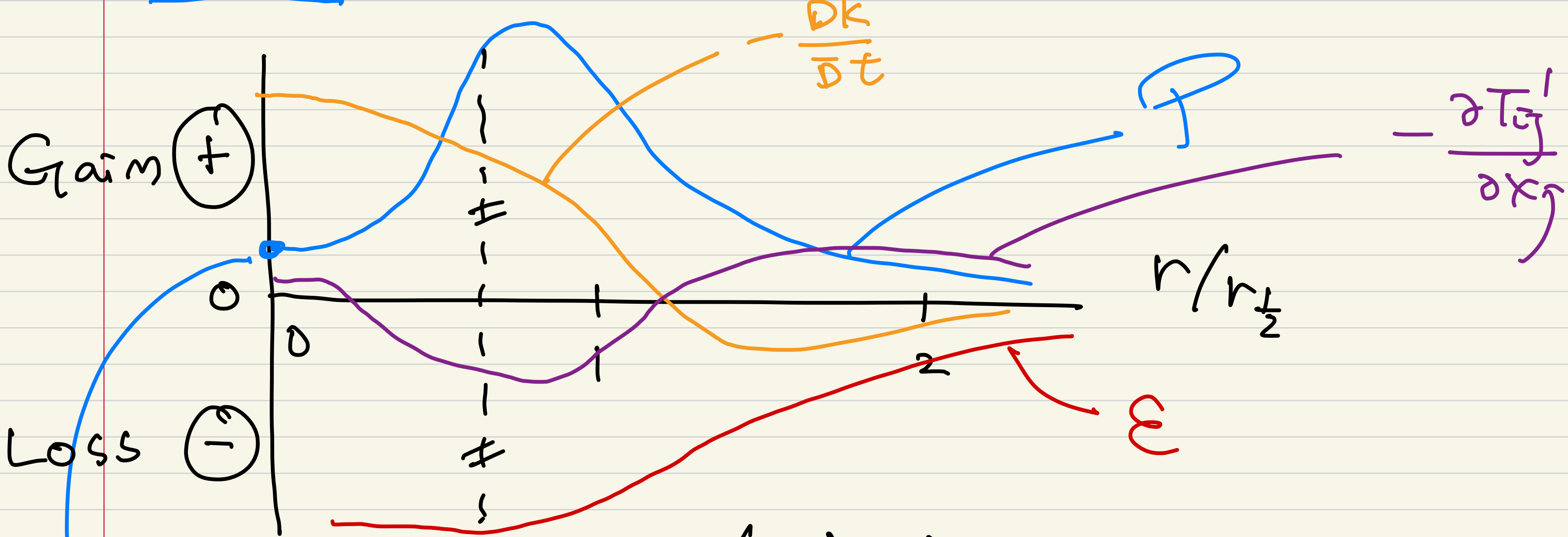
$$\Delta x \sim Re^{-\frac{3}{4}} \rightarrow N \sim \frac{r_{\frac{1}{2}}}{\Delta x} \sim \frac{1}{\Delta x}$$



$$\boxed{N^3 \sim Re^{\frac{9}{4}}} \quad \text{(DNS)}$$

$\cdot\ \dfrac{\eta\, u_\eta}{\nu} = 1$: Reynolds number based on the Kolmogorov scales is unity $\Rightarrow$ motions on these scales are strongly affected by viscosity.

- Budget of k (normalized by $U_0^3 / r_{\frac{1}{2}}$)

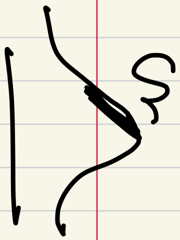


$P \simeq \varepsilon$ @ $r/r_{\frac{1}{2}} \doteq 0.6$

non-zero @ $r=0$ due to $(\langle u^2\rangle - \langle v^2\rangle)\frac{\partial \langle U \rangle}{\partial x}$.

- $\tau = \frac{k}{\varepsilon}$: turbulence decaying time scale
  or time to dissipate an amount of energy k
  at constant rate ε.

- $\tau_p = \dfrac{k}{P}$: turbulence producing time scale
  or time to produce k at the rate P



$\tau \simeq \tau_p \simeq 3\, S^{-1}$: three times the time scale of the imposed mean shear S^{-1}

→ turbulence is long-lived.

- Pseudo-dissipation $\tilde{\varepsilon} = \nu \left\langle \dfrac{\partial u_i}{\partial x_j} \dfrac{\partial u_i}{\partial x_j} \right\rangle$

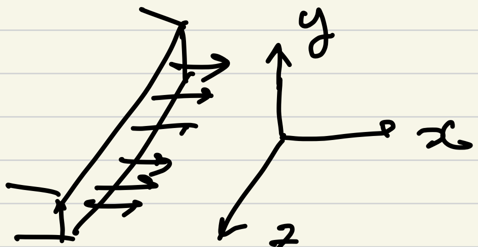
$$\not\simeq$$

$$\varepsilon = 2\nu \langle s_{ij} s_{ij} \rangle$$

$$\underbrace{\tilde{\varepsilon}}_{\text{pseudo-dissipation}} = \underbrace{\varepsilon}_{\text{true dissipation}} - \underbrace{\nu \frac{\partial^2}{\partial x_i \partial x_j} \langle u_i u_j \rangle}_{\text{quite small}}$$

5.4 Other self-similar flows

⊙ Plane jet



$U_0(x) = \langle U(x, y=0, z) \rangle$: centerline vel.

$y_{\frac{1}{2}}$: jet half width

↳ $\langle U(x, y=y_{\frac{1}{2}}, z) \rangle = \frac{1}{2} U_0(x)$

mtm flux conservation & self similarity

$$\rightarrow \quad \frac{dy_{\frac{1}{2}}}{dx} = S \simeq 0.10 \quad \rightarrow \quad y_{\frac{1}{2}} \sim x$$

$$U_0(x) \sim x^{-\frac{1}{2}}$$

$$\hat{\nu}_T(\xi) = \frac{\nu_T}{U_0 y_{\frac{1}{2}}} \quad \rightarrow \quad \nu_T \sim x^{\frac{1}{2}} \qquad \left(\xi = \frac{y}{y_{\frac{1}{2}}} \right)$$

$$Re_0 = \frac{U_0(x)\, y_{\frac{1}{2}}(x)}{\nu} \sim x^{\frac{1}{2}}$$

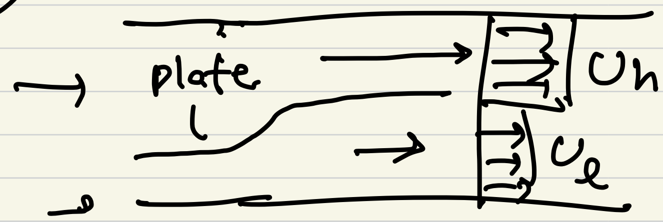
(35 for round jet)

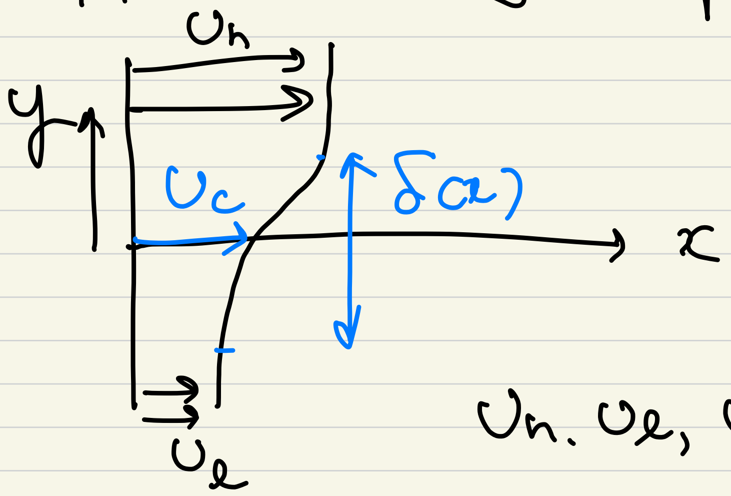
$$Re_T = \frac{U_0\, y_{\frac{1}{2}}}{\nu_T}$$ indep. of x, $Re_T \simeq 31$

bdry layer approx. & self-similarity

$$\rightarrow \bar{f}(\xi) = \frac{\langle U \rangle}{U_0} = \mathrm{sech}^2(\alpha \xi)$$

⊙ Plane mixing layer





$U_c = \frac{1}{2}(U_h + U_l)$: char. conv. vel.

$U_s = U_h - U_l$: " vel. difference.

U_h, U_l, U_c, U_s are indep. of x

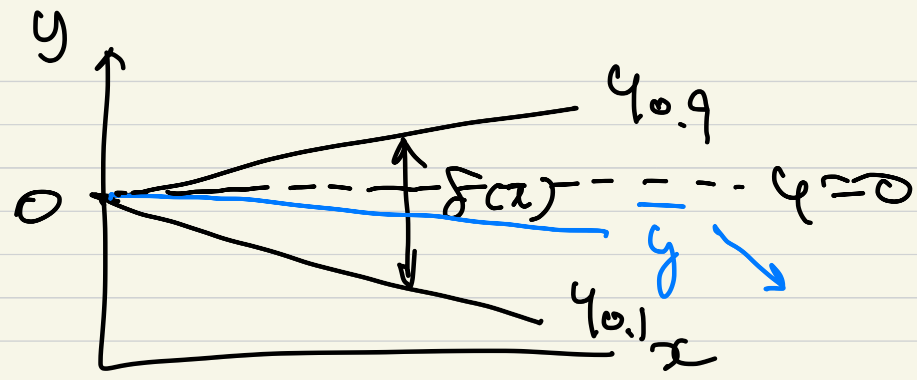
char. length $\delta(x)$?

$$y_\alpha : \langle U(x, y = y_\alpha(x), z) \rangle = U_l + \alpha (U_h - U_l)$$

$$\delta(x) \equiv y_{0.9}(x) - y_{0.1}(x)$$

$$\bar{y}(x) \equiv \frac{1}{2}[y_{0.9}(x) + y_{0.1}(x)]$$

$$\xi = \frac{y - \bar{y}(x)}{\delta(x)}, \quad f(\xi) = \frac{\langle U \rangle - U_c}{U_s}$$



$$\xi \to \infty, \quad \langle U \rangle = U_h$$

$$f(\infty) = \frac{U_h - \frac{1}{2}(U_h + U_e)}{U_h - U_e} = \frac{1}{2}$$

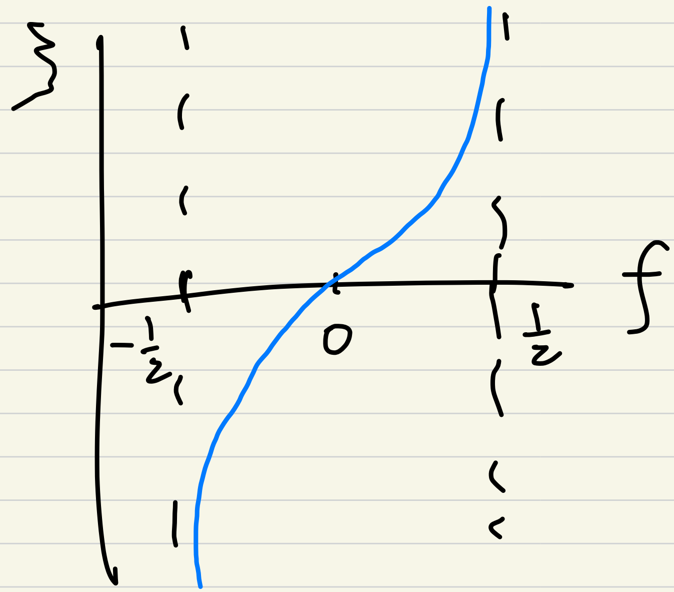
$$f(-\infty) = \frac{U_e - ''}{''} = -\frac{1}{2}$$

$$\bar{y} = \frac{1}{2}(y_{0.9} + y_{0.1}) \to f(0) = \frac{\langle U(\bar{y}) \rangle - \frac{1}{2}(U_h + U_e)}{U_h - U_e}$$

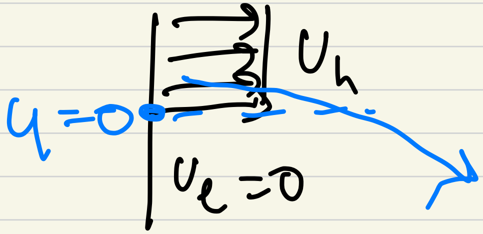
$$\langle U(\bar{y}) \rangle = \frac{1}{2}\left[\underline{U_e + 0.9(U_h - U_e)} + \underline{U_e + 0.1(U_h - U_e)} \right]$$

$$= \frac{1}{2}(U_h + U_e)$$

$$\to f(0) = 0$$



* $U_\ell / U_h = 0$: exp. results confirm that the mixing layer is self-similar



Flow is not symmetric about $y=0$

and spreads into the low-speed stream.

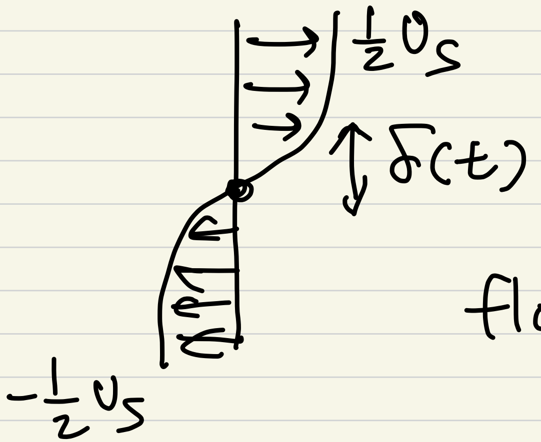
boundary layer eq. + self-similarity

$$\rightarrow \boxed{\delta \sim x}$$

spreading rate

$$S \equiv \frac{U_c}{U_s} \frac{d\delta}{dx} = \text{const} \quad (0.06 \sim 0.11)$$

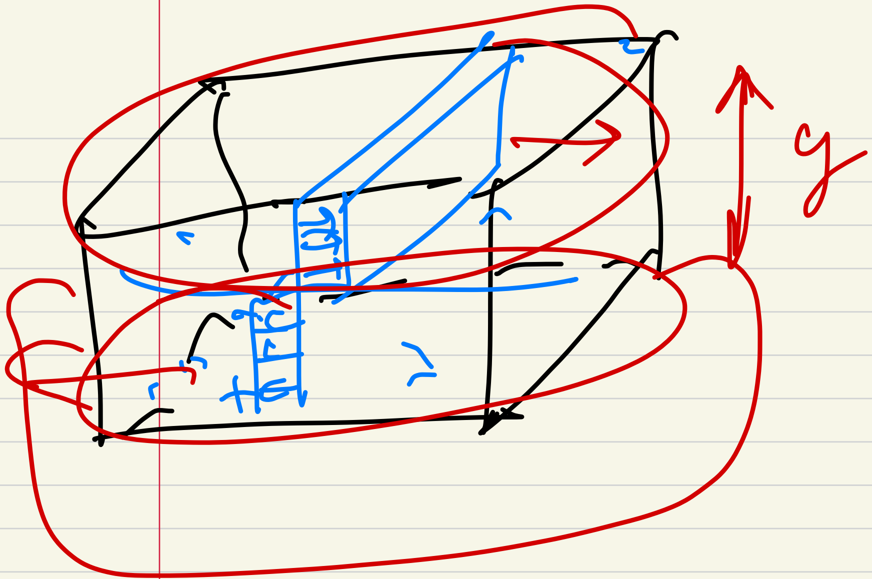
* To an observer travelling in x direction at U_c.



$$\frac{d\delta}{dt} = U_c \frac{d\delta}{dx} = S\, U_s = \text{const}$$

$$\rightarrow \delta \sim t$$

flow becomes statistically 1D (varying in y) and time-dependent



⇒ called 'temporal mixing layer'.

(as opposed to spatial mixing layer)

Flow is symmetric about $y=0$

Rogers & Moser (1994)

* Mixing layer

$U_s = U_h - U_l = \text{const}$, $\delta \sim x$

$Re_0 = \dfrac{U_s \delta}{\nu} \sim x$, $\nu_T = U_s \delta \hat{\nu}_T \sim x$

$Re_T = \dfrac{U_s \delta}{\nu_T} = \text{const}$

Flow rate of k : $\dot{k}(x) = \int_{-\infty}^{\infty} \langle U \rangle k \, dy \sim U_c U_s^2 \delta \sim x$

→ P should be bigger than ε.

cf. jet : $\int_0^\infty \langle U \rangle \cdot k \cdot 2\pi r \, dr \sim U_c^3 \delta^2 \sim x^{-1}$

wake : $\dot{k} \sim x^{-1}$