Chapter 5: Basic finite element analysis of continua

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In this chapter,

- Generalized formulation of finite element procedure is introduced
- We will apply continuum mechanics learned in Chapter 4 to the development of finite element formulations for two- and three-dimensional continua.
- Uppercase letters X, Y, Z will relate to the initial coordinates, while lower-case x, y, z relate to the current configuration.

- Throughout the total Lagrangian formulation, such a distinction is unnecessary since we will always be referring to the initial configuration.

Review of chapter 4

Green strain

$$\mathbf{E}_2 = \frac{1}{2}\left[\mathbf{F}^T\mathbf{F} - \mathbf{I}\right] = \frac{1}{2}\left[\mathbf{D} + \mathbf{D}^T\right] + \frac{1}{2}\mathbf{D}^T\mathbf{D}$$

Where, $\mathbf{F} = \mathbf{I} + \mathbf{D}$ [eq. 5.1]

Virtual work principle

$$\delta\mathbf{E}_2 = \frac{1}{2}\mathbf{F}^T\delta\mathbf{D} + \frac{1}{2}\delta\mathbf{D}^T\mathbf{F} + \frac{1}{2}\delta\mathbf{D}^T\delta\mathbf{D}$$ [eq. 5.2]

$$V = \int \mathbf{S}_2 : \delta\mathbf{E}_{v2} dV_0 - V_e = \int \mathbf{S}^T\delta\mathbf{E}_v dV_0 - V_e$$ [eq. 5.3]

'2' = second order tensor

$$\delta V = \int \left(\delta\mathbf{E}_{v2} : \mathbf{C}_{t4} : \delta\mathbf{E}_2 + \mathbf{S} : \delta\mathbf{D}_v^T\delta\mathbf{D}\right) dV_0 = \int \left(\delta\mathbf{E}^T\mathbf{C}_{t2}\delta\mathbf{E} + \mathbf{S}^T\delta\left(\delta\mathbf{E}_{v2}\right)\right) dV_0$$ [eq. 5.4]

4.4.1 Virtual work expressions using Green's strain

- Conjugate of the Green strain is the 2nd Piola-Kirchhoff stress

$$\mathbf{S}(or\ \mathbf{S}_2) : 2^{nd}\ \ Piola - Kirchhoff\ \ stress\ tensor$$

- Virtual work expression is given by:

$$V = V_i - V_e = \int \mathbf{S}^T \delta \mathbf{E}_v dV_0 - V_e = \int \mathbf{S}_2 : \delta \mathbf{E}_{v2} dV_0 - V_e \qquad \text{[eq. 4.76]}$$

by $\mathbf{E}_2 = \frac{1}{2}\left[\mathbf{F}^T\mathbf{F} - \mathbf{I}\right] = \frac{1}{2}\left[\mathbf{D} + \mathbf{D}^T\right] + \frac{1}{2}\mathbf{D}^T\mathbf{D}$,

$$\delta \mathbf{E}_2 = \frac{1}{2}\left[\delta\mathbf{D} + \delta\mathbf{D}^T\right] + \underbrace{\frac{1}{2}\mathbf{D}^T\delta\mathbf{D} + \frac{1}{2}\delta\mathbf{D}^T\mathbf{D}}_{=\mathbf{D}^T\delta\mathbf{D}} + \frac{1}{2}\delta\mathbf{D}^T\delta\mathbf{D} \qquad \text{[eq. 4.74]} \qquad \delta\mathbf{E}_{v2} = \frac{1}{2}\left[\delta\mathbf{D}_v + \delta\mathbf{D}_v^T\right] + \mathbf{D}^T\delta\mathbf{D}_v$$

or $\delta \mathbf{E}_2 = \frac{1}{2}\mathbf{F}^T\delta\mathbf{D} + \frac{1}{2}\delta\mathbf{D}^T\mathbf{F} + \frac{1}{2}\delta\mathbf{D}^T\delta\mathbf{D}$ [eq. 4.77] $\qquad \delta\mathbf{E}_{v2} = \frac{1}{2}\mathbf{F}^T\delta\mathbf{D}_v + \frac{1}{2}\delta\mathbf{D}_v^T\mathbf{F}$ [eq. 4.79]

$$\delta V = \int \left(\delta\mathbf{S}^T\delta\mathbf{E}_v + \mathbf{S}^T\delta\left(\delta(\mathbf{E}_v)\right)\right) dV_0 = \int \left(\delta\mathbf{S}_2 : \delta\mathbf{E}_{v2} + \mathbf{S}_2 : \delta\left(\delta(\mathbf{E}_{v2})\right)\right) dV_0 \qquad \text{[eq. 4.80]}$$

$$\delta V = \int \left(\delta \mathbf{S}^T \delta \mathbf{E}_v + \mathbf{S}^T \delta\left(\delta(\mathbf{E}_v)\right)\right) dV_0 = \int \left(\delta \mathbf{S}_2 : \delta \mathbf{E}_{v2} + \mathbf{S}_2 : \delta\left(\delta(\mathbf{E}_{v2})\right)\right) dV_0$$ [eq. 4.80]

$$\delta \mathbf{E}_{2v} = \frac{1}{2}\left[\delta \mathbf{D}_v + \delta \mathbf{D}_v^T\right] + \mathbf{D}^T \delta \mathbf{D}_v$$

$$\delta\left(\delta(\mathbf{E}_{v2})\right) = \frac{1}{2}\left[\delta \mathbf{D}_v^T \delta \mathbf{D} + \delta \mathbf{D}^T \delta \mathbf{D}_v\right] = \delta \mathbf{D}_v^T \delta \mathbf{D}$$ [eq. 4.81]

δ^2 is neglected for virtual change

$$\delta \mathbf{S} = \mathbf{C}_{t2} \delta \mathbf{E}, \quad \delta \mathbf{S}_2 = \mathbf{C}_{t4} : \delta \mathbf{E}_2$$ [eq. 4.82]

➡ $$\delta V = \int \left(\delta \mathbf{E}_v^T \mathbf{C}_{t2} \delta \mathbf{E} + \mathbf{S} : \left(\delta \mathbf{D}_v^T \delta \mathbf{D}\right)\right) dV_0$$ [eq. 4.83]

5.1.1 Element formulation

- Two-dimensional formulation where displacements are related to nodal values via shape functions

$$u = \mathbf{h}(\xi,\eta)^T \mathbf{u} \qquad v = \mathbf{h}(\xi,\eta)^T \mathbf{v}$$

$$x = \mathbf{h}(\xi,\eta)^T \mathbf{x} \qquad y = \mathbf{h}(\xi,\eta)^T \mathbf{y}$$

[eq. 5.5]

- Differentiation in two coordinates systems

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \end{bmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}$$

[eq. 5.6]

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \end{bmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}$$

- For example, $\dfrac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \mathbf{J}^{-1}(1,1)\frac{\partial u}{\partial \xi} + \mathbf{J}^{-1}(1,2)\frac{\partial u}{\partial \eta}$$

$$= \mathbf{J}^{-1}(1,1)\mathbf{h}(\xi,\eta)_{\xi}^{T}\mathbf{u} + \mathbf{J}^{-1}(1,2)\mathbf{h}(\xi,\eta)_{\eta}^{T}\mathbf{u} \qquad \text{[eq. 5.7]}$$

- Vectorized form of $\mathbf{D}$ can be obtained as:

$$\boldsymbol{\theta} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \mathbf{J}^{-1}(1,1)\mathbf{h}_{\xi}^{T} + \mathbf{J}^{-1}(1,2)\mathbf{h}_{\eta}^{T} & \mathbf{0}^{T} \\ \mathbf{J}^{-1}(2,1)\mathbf{h}_{\xi}^{T} + \mathbf{J}^{-1}(2,2)\mathbf{h}_{\eta}^{T} & \mathbf{0}^{T} \\ \mathbf{0}^{T} & \mathbf{J}^{-1}(1,1)\mathbf{h}_{\xi}^{T} + \mathbf{J}^{-1}(1,2)\mathbf{h}_{\eta}^{T} \\ \mathbf{0}^{T} & \mathbf{J}^{-1}(2,1)\mathbf{h}_{\xi}^{T} + \mathbf{J}^{-1}(2,2)\mathbf{h}_{\eta}^{T} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{G}\mathbf{p} \qquad \text{[eq. 5.8]}$$

- Incremental form is obtained by: $\delta\boldsymbol{\theta} = \mathbf{G}\delta\mathbf{p}$ [eq. 5.9]

- Using eq. 5.8, Greens strain can be written in vector form as:

$$\mathbf{E} = \mathbf{E}_l + \mathbf{E}_{nl} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial x} & 0 & \frac{\partial v}{\partial x} & 0 \\ 0 & \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \mathbf{E}_l + \frac{1}{2}\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\theta} \qquad \text{[eq. 5.10]}$$

or $$\mathbf{E} = \mathbf{E}_l + \mathbf{E}_{nl} = \left[\mathbf{H} + \frac{1}{2}\mathbf{A}(\boldsymbol{\theta})\right]\boldsymbol{\theta} \qquad \text{[eq. 5.11]}$$

where $$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \text{[eq. 5.12]}$$

$$\delta\boldsymbol{\theta} = \mathbf{G}\delta\mathbf{p}$$

- Incremental form is:

$$\delta\mathbf{E} = \delta\mathbf{E}_l + \frac{1}{2}\mathbf{A}(\boldsymbol{\theta})\delta\boldsymbol{\theta} + \frac{1}{2}\delta\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\theta} + O\left(\delta\boldsymbol{\theta}^2\right) \quad \text{[eq. 5.13]}$$

$$\Leftarrow \quad \mathbf{A}(\boldsymbol{\theta})\delta\boldsymbol{\theta} = \delta\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\theta} \quad \text{[eq. 5.14]}$$

$$= \delta\mathbf{E}_l + \delta\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\theta} + O\left(\delta\boldsymbol{\theta}^2\right)$$

$$= \left[\mathbf{H} + \mathbf{A}(\boldsymbol{\theta})\right]\mathbf{G}\delta\mathbf{p} + O\left(\delta\boldsymbol{\theta}^2\right) \quad \text{[eq. 5.15]}$$

$$\Leftarrow \quad \boldsymbol{\theta} = \mathbf{G}\mathbf{p} \quad \text{[eq. 5.8]} \quad \text{and} \quad \mathbf{B}_l \doteq \mathbf{H}\mathbf{G}$$

$$= \underbrace{\left[\mathbf{B}_l + \mathbf{A}(\mathbf{G}\mathbf{p})\mathbf{G}\right]}_{\mathbf{B}_{nl}}\delta\mathbf{p} + O\left(\delta\mathbf{p}^2\right) \quad \text{[eq. 5.16]}$$

$$\Rightarrow \quad \delta\mathbf{E}_v = \mathbf{B}_{nl}(\mathbf{p})\delta\mathbf{p}_v \quad \text{[eq. 5.17]}$$

- Virtual work is:

$$V = \int \mathbf{S}^T \delta\mathbf{E}_v dV_0 - V_e \quad \text{[eq. 5.3]}$$

$$= \delta\mathbf{p}_v^T \int \mathbf{B}_{nl}^T(\mathbf{p})\mathbf{S}dV_o - \delta\mathbf{p}_v^T\mathbf{q}_e = \delta\mathbf{p}_v^T\mathbf{g} \quad \text{[eq. 5.18]}$$

$$\mathbf{S} = \begin{pmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{pmatrix}$$

- Out-of-balance force is:

$$\mathbf{g} = \int \mathbf{B}_{nl}^T(\mathbf{p})\mathbf{S}dV_o - \mathbf{q}_e = \int \mathbf{G}^T\left[\mathbf{H} + \mathbf{A}(\boldsymbol{\theta})\right]\mathbf{S}dV_o - \mathbf{q}_e \quad \text{[eq. 5.19]}$$

5.1.2 The tangent stiffness matrix

$$V = \delta \mathbf{p}_v^T \mathbf{g}$$ [eq. 5.21]

$$\delta V = \int \left(\delta \mathbf{E}_v^T \mathbf{C}_{t2} \delta \mathbf{E} \;\; + \mathbf{S} : \left(\delta \mathbf{D}_v^T \delta \mathbf{D} \right) \right) dV_0$$ [eq. 4.83]

$$\delta V = \delta \mathbf{p}_v^T \frac{\delta \mathbf{g}}{\delta \mathbf{p}} \delta \mathbf{p} = \delta \mathbf{p}_v^T \mathbf{K}_t \delta \mathbf{p}$$

$$= \int \left(\delta \mathbf{E}_v^T \mathbf{C}_t \delta \mathbf{E} \;\; + \mathbf{S}^T \delta \mathbf{D}_v^T \delta \mathbf{D} \right) dV_0$$ [eq. 4.83]

$$\delta \mathbf{E}_v = \mathbf{B}_{nl}(\mathbf{p}) \delta \mathbf{p}_v$$ [eq. 5.17]

$$= \delta \mathbf{p}_v^T \underbrace{\left(\int \delta \mathbf{B}_{nl}^T(\mathbf{p}) \mathbf{C}_t \mathbf{B}_{nl}(\mathbf{p}) dV_0 \right)}_{\mathbf{K}_{t1}} \delta \mathbf{p} + \int \mathbf{S} : \left(\delta \mathbf{D}_v^T \delta \mathbf{D} \right) dV_0$$ [eq. 5.22]

- Second term is given by:

$$\int \mathbf{S} : \left(\delta \mathbf{D}_v^T \delta \mathbf{D} \right) dV_0$$

Vectorized $\delta \mathbf{D}$ is $\delta \boldsymbol{\theta}$

$$\delta \boldsymbol{\theta} = \mathbf{G} \delta \mathbf{p}$$ [eq. 5.9]

$$\cdots = \int \delta \boldsymbol{\theta}_v^T \hat{\mathbf{S}} \delta \boldsymbol{\theta} dV_o$$ [eq. 5.24]

$$= \delta \mathbf{p}_v^T \underbrace{\left(\int \mathbf{G}^T \hat{\mathbf{S}} \mathbf{G} dV_o \right)}_{\mathbf{K}_{t\sigma}} \delta \mathbf{p}$$

where

$$\hat{\mathbf{S}} = \begin{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \end{bmatrix}$$

$$\mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t\sigma}$$

Thank you!