Aeroelasticity

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Static Aeroelasticity

Static Aeroelasticity

- Definition ··· interaction of aerodynamic and elastic force
  intensive to rate and acceleration of the structural deflection
- Two class of problems

  1. Effects of elastic deformation on the airloads

     ··· normal operating condition → performance, handling qualities, structural load distribution

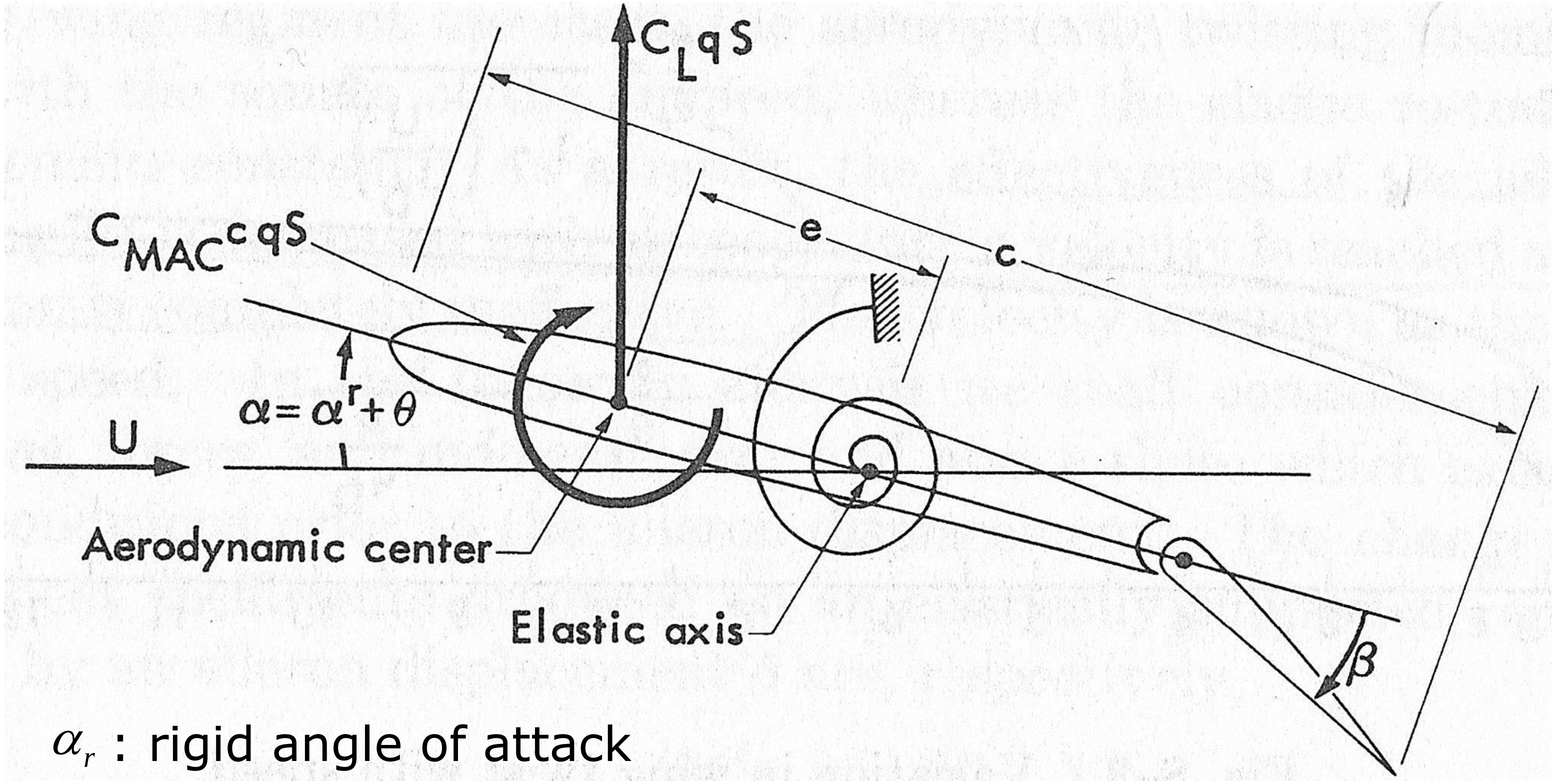
  2. Static stability ··· divergence

     [Note] Instability ··· tendency to move away from equilibrium

     static ··· $e^{(a+iw)t}, a > 0, w = 0$

Typical section of aircraft wing

··· consider a rigid airfoil mounted on an elastic support (in a wind tunnel)



α_r : rigid angle of attack

θ : elastic angle of attack

$\alpha = \alpha_r + \theta$

Typical section of aircraft wing

$\alpha_{i,}\alpha$: ↻ (+)

L: lift, ↑ (+), $L = \frac{1}{2}\rho U^2 S C_L$, where U: flow speed, ρ: flow density

W : weight

q : $\frac{1}{2}\rho U^2$, dynamic pressure

S : reference area = $c \cdot 1$

C_L: lift coefficient

M_{ac} : $qScC_{M_{ac}}$,where : pitching moment coefficient at a.c.

[Note] $C_L = fn.(\alpha, M, \text{airfoil shape})$

$C_{M_{ac}} = fn.(M, \text{airfoil shape})$

Typical section of aircraft wing

From a Taylor series

$$C_L = C_{L0} + \frac{\partial C_L}{\partial \alpha}\alpha + h.o.t.$$

$$= C_{L0} + \frac{\partial C_L}{\partial \alpha}\alpha$$

$$C_{M_{ac}} = C_{M_{ac0}} + \frac{\partial C_{M_{ac}}}{\partial \alpha}\alpha + h.o.t.$$

$$= C_{M_{ac0}} + \frac{\partial C_{M_{ac}}}{\partial \alpha}\alpha$$

[Note] for a flat plate

$$C_{L_\alpha} = \frac{\partial C_L}{\partial \alpha} = 2\pi$$

$$C_{M_{ac_\alpha}} = \frac{\partial C_{M_{ac}}}{\partial \alpha} = 0$$

Typical section of aircraft wing

For a rigid support

$$L_{rigid} = \frac{1}{2}\rho U^2 S C_{L_\alpha} \cdot \alpha_r$$

However, for an elastic support

$$L_{elastic} = L = qSC_{L_\alpha} \cdot \alpha$$

$\alpha(\alpha_r + \alpha_e)$

[Note]

$$L_{rigid} \neq L_{elastic}$$

$$L_{rigid} < L_{elastic}$$

Consider the system in equilibrium, all applied moments at support point must equal the torsional reaction at the point

Torsional divergence

Consider the system in equilibrium, all applied moments at support point must equal the torsional reaction at the point

$$L(x_{ea}-x_{ac})-W(x_{ea}-x_{cg})+M_{ac}=K_{\alpha}\alpha_{e}$$

$$qSC_{L\alpha}(\alpha_{r}+\alpha_{e})(x_{ea}-x_{cg})-W(x_{ea}-x_{cg})+qScC_{M_{ac_{\alpha}}}=K_{\alpha}\alpha_{e}$$

$$\rightarrow \alpha_{e}=\frac{-W(x_{ea}-x_{cg})+qScC_{M_{ac_{\alpha}}}+qSC_{L\alpha}\alpha_{r}(x_{ea}-x_{cg})}{K_{\alpha}-qScC_{M_{ac}}(x_{ea}-x_{cg})}$$

0 ··· torsional divergence

For a given q and α_r, we can evaluate L

Torsional divergence

[Note] equilibrium eqn.

$$\underbrace{(K - qSC_{L\alpha} \cdot e)}_{A}\underbrace{\alpha_e}_{x} = \underbrace{qScC_{Mac_\alpha} + qSC_{L_\alpha}\alpha_r e - W \cdot d}_{B}$$

Ax=B → Kx=F

The stability is associated with the homogeneous part of the eqn.
: Ax=0, A ≡ 0 → divergence condition

Torsional Divergence (BAH)

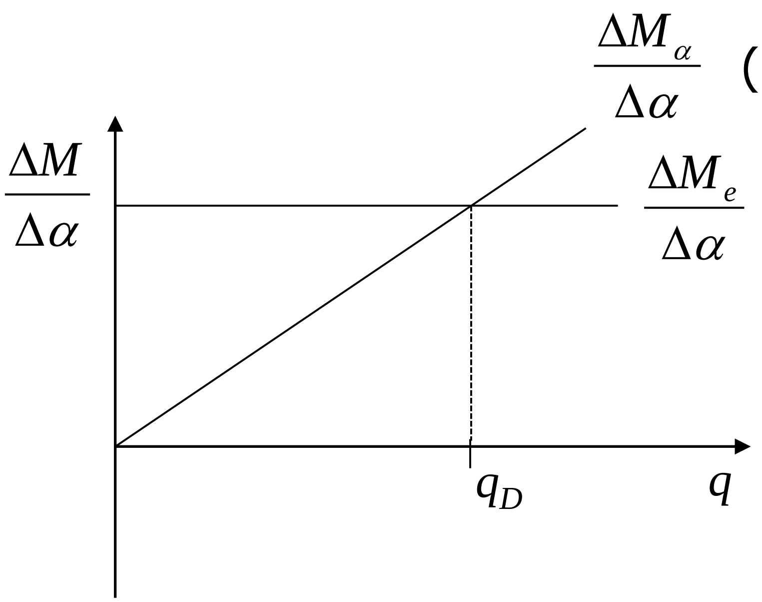
$$1-\underbrace{C^{\theta\theta}}_{1/K_\alpha}\frac{\partial C_L}{\partial \alpha}qSe=0$$

If e=0 (a.c. falls on e.a.)

or aft of it, wing is stable at all speeds.

- Divergence: independent of α_r (initial a.o.a) and airfoil camber $C_{M_{ac}}$
  the increase in aerodynamic moment about the elastic axis due to an arbitrary change in a.o.a is equal to the corresponding increase in elastic restoring moment

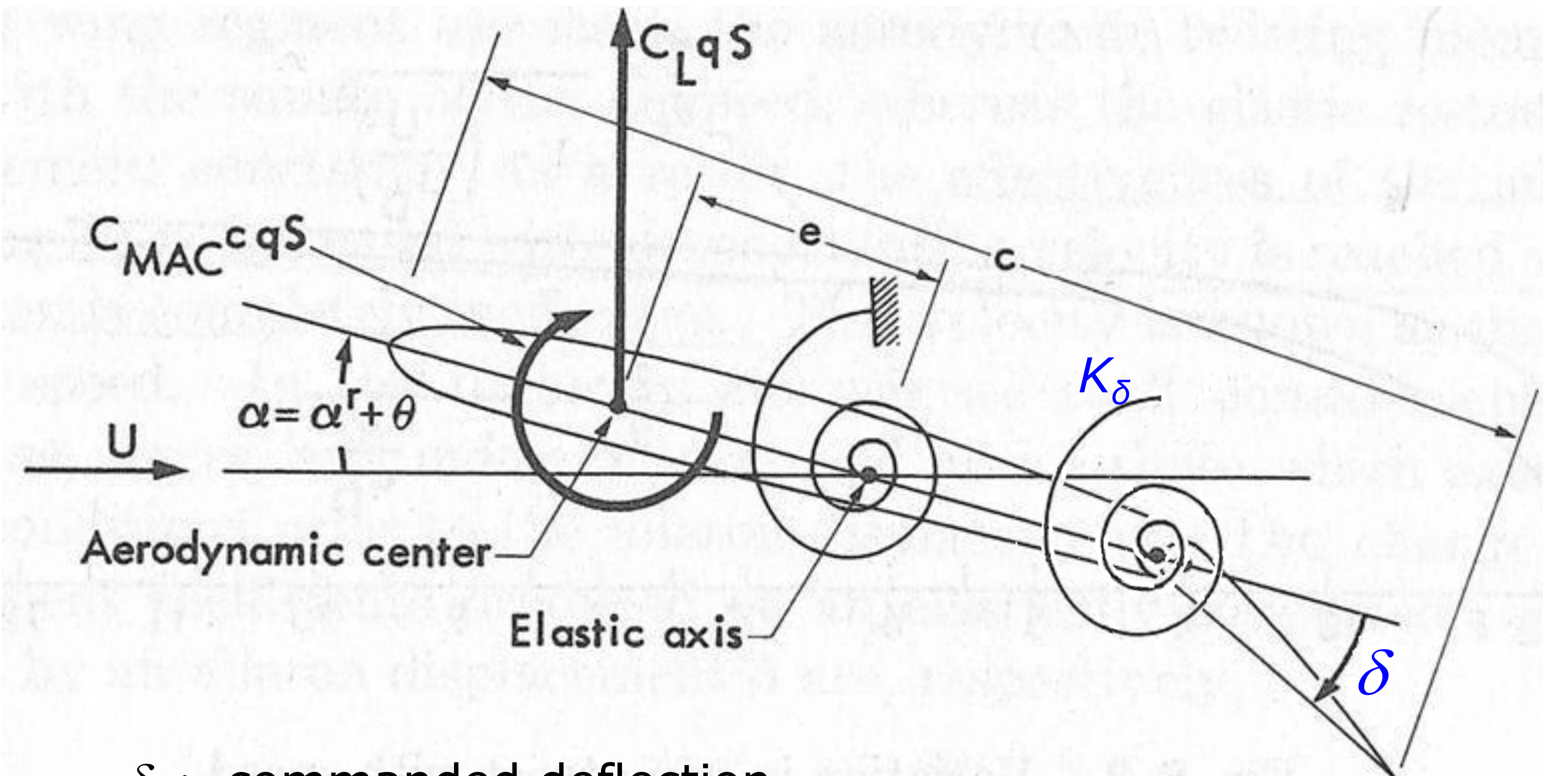
Torsional Divergence (BAH)



$$\Delta M_e = \frac{\Delta\alpha}{C^{\theta\theta}}$$

$$\Delta M_a = \frac{\partial C_L}{\partial\alpha}\Delta\alpha qSe$$

Control Reversal



δ_0 : commanded deflection

δ_e : $\delta - \delta_0$

Control Reversal

By adding a control surface

$$L = qcC_L$$

$$C_L = C_{L0} + \frac{\partial C_L}{\partial \alpha}\alpha + H.O.T + \frac{\partial C_L}{\partial \delta}\delta + H.O.T$$

$$= C_{L0} + \underbrace{\frac{\partial C_L}{\partial \alpha}}_{C_{L_\alpha}}\alpha + \underbrace{\frac{\partial C_L}{\partial \delta}}_{C_{L_\delta}}\delta$$

$$C_{M_{ac}} = C_{M_{ac0}} + \frac{\partial C_{M_{ac}}}{\partial \delta}\delta$$

H : moment about the hinge line of control surface

$$= q \cdot S_H \cdot C_H \cdot \boldsymbol{C}_H$$

$$= q \cdot S_H \cdot C_H (\boldsymbol{C}_{H\alpha} \cdot \alpha + \boldsymbol{C}_{H\delta} \cdot \delta)$$

Control Reversal

① $L(x_{ea} - x_{ac}) + M_{ac} = K_\alpha \alpha_e$

$$eqSC_{L_\alpha} + eqSC_{L_\delta}\delta + qScC_{mac} = K_\alpha \alpha_e$$

② $H = K_\delta \delta_e = K_\delta(\delta - \delta_0)$

$$q \cdot S_H \cdot C_H \cdot \mathbf{C}_H = q \cdot S_H \cdot C_H(\mathbf{C}_{H\alpha} \cdot \alpha + \mathbf{C}_{H\delta} \cdot \delta) = K_\delta(\delta - \delta_0)$$

$$\underbrace{\begin{bmatrix} eqC_{L\alpha} - K_\alpha & sqSC_{L\alpha} + qScC_{Mac_\delta} \\ qS_H C_H \mathbf{C}_H & qS_H C_H \mathbf{C}_{H_\delta} - K_\delta \end{bmatrix}}_{\Delta} \underbrace{\begin{Bmatrix} \alpha \\ \delta \end{Bmatrix}} = \underbrace{\begin{Bmatrix} 0 \\ -K_\delta \delta_0 \end{Bmatrix}}$$

Assumption: $\alpha_r = 0,\ C_{Mac} = 0,\ C_{L_0} = 0$

If $|\Delta| = 0$ → q_{D_1} : divergence $\qquad q = lowest + \{q_{D_1}, q_{D_2}\}$

q_{D_2}

Control Reversal

If Δ is non-singular

$$\begin{Bmatrix} \alpha \\ \delta \end{Bmatrix} = \Delta^{-1} \begin{Bmatrix} 0 \\ -K_\delta \delta_0 \end{Bmatrix}$$

If Δ is singular

det(Δ)=0 → quadratic eqn. in q_D

…lowest positive q_D is the "divergence dynamic pressure"

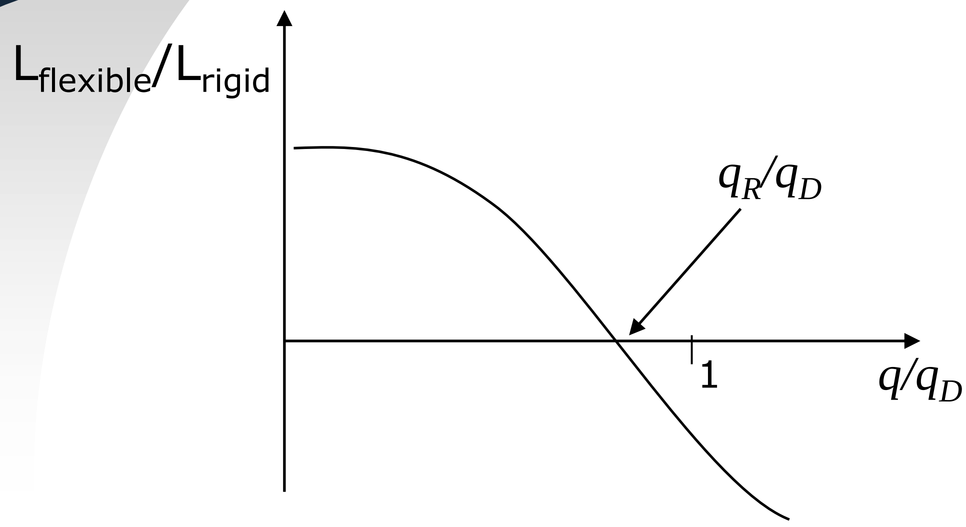
If system is rigid

$$L_{rigid} = qC_{L_\delta}\delta_0 S\left(\alpha_r = 0\right)$$

However, with flexible attachment

$$L_{flexible} = qS(C_{L_\alpha}\alpha + C_{L_\delta}\delta)$$

Control Reversal



q_R : "control surface reversal"

$$\rightarrow \frac{dL}{d\delta} = 0$$

$$\frac{dL}{d\delta} = qS(C_{L_\alpha} \underbrace{\frac{d\alpha}{d\delta}} + C_{L_\delta})$$

unknown value

From the first equilibrium eqn.

$$(eqSC_{L_\alpha} - K_\alpha)\alpha_e + (eqSC_{L_\delta} + qScCM_{ac_\delta})\delta = 0$$

$$(eqSC_{L_\alpha} - K_\alpha)\frac{d\alpha}{d\delta} + (eqSC_{L_\delta} + qScC_{mac_\delta}) = 0$$

$$\frac{d\alpha}{d\delta} = -\frac{(eqSC_{L_\delta} + qScC_{mac_\delta})}{(eqSC_{L_\alpha} - K_\alpha)}$$

Control Reversal

For $\dfrac{dL}{d\delta} = 0,$

$$qS\left\{C_{L_\alpha}(-)\cdot\frac{eqSC_{L_\delta}+qScC_{mac_\delta}}{eqC_{L_\alpha}-K_\alpha}+C_{L_\delta}\right\}=0$$

$$qS\left\{C_{L_\alpha}(-)\cdot\frac{eqSC_{L_\delta}+qScC_{mac_\delta}}{eqC_{L_\alpha}-K_\alpha}+\frac{eqSC_{L_\alpha}C_{L_\delta}-C_{L_\delta}K_\alpha}{eqC_{L_\alpha}-K_\alpha}\right\}=0$$

$$-C_{L_\alpha}qScC_{mac_\delta}-C_{L_\delta}K_\alpha=0$$

$$\therefore q_R=-\frac{K\alpha}{S\cdot c}\cdot\frac{C_{L_\delta}/C_{L_\alpha}}{C_{m_{ac_\delta}}}$$

Aileron Reversal (BAH)

depends upon $C_{L_\alpha}, C_{L\beta}, C_{M_{ac\,\beta}}$

independent e

$$\frac{\partial C_L}{\partial \beta} = \frac{1}{(\partial C_L / \partial \alpha) C^{\theta\theta} q S} + C \frac{\partial M_{ac}}{\partial \beta} = 0$$

δ

⋯ dependent upon aerodynamic quantities, $C_{L_\alpha}, C_{L_\beta}, C_{Mac\,\beta}$

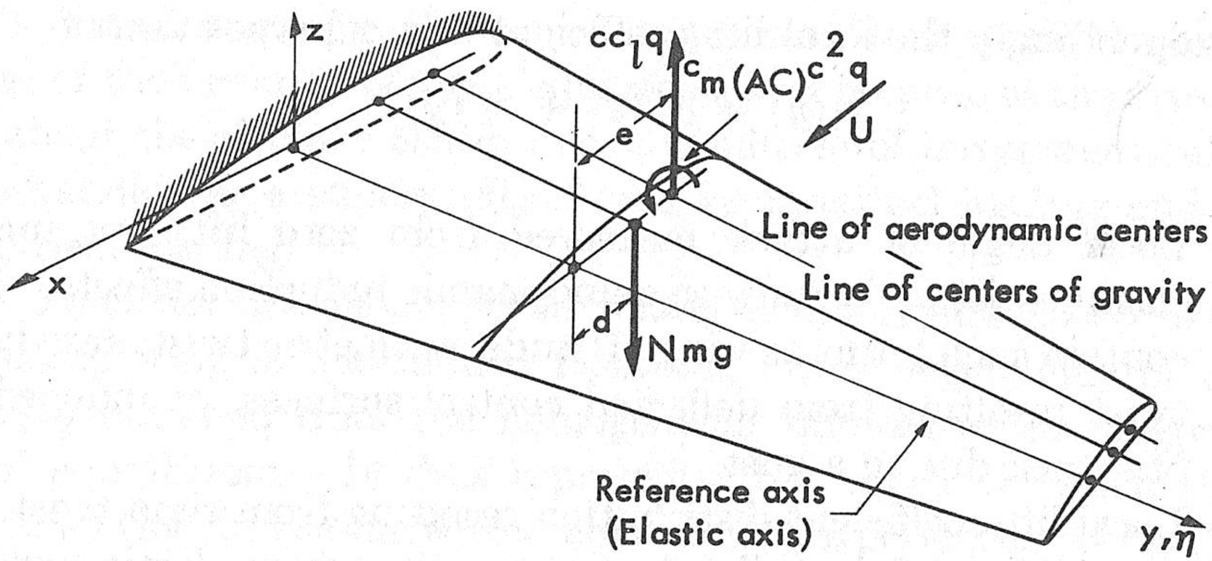
independent of *e* ← aerodynamic moment providing twist at U_R is a pure couple, independent of the e.a. position

$\cdot\ U_R = U_D$ ⋯ $\frac{\partial C_\partial}{\partial \beta} = -\frac{c}{e}\frac{\partial C_{mAC}}{\partial \beta}$

→ aileron retains its full effectiveness at all speeds

⋯ nose-down pitching moment due to the deflection aileron is cancelled by nose-up pitching moment due to the lift.

Straight High-aspect-ratio Wing



l' : lift per unit span

$$e = x_{ea} - x_{ac}$$

$$d = x_{ea} - x_{cg}$$

$$\Lambda = \frac{b^2}{S_W} = \frac{l}{c}$$

Clamped-free B.C.

$$W' = Nmg = 1 + \frac{\ddot{z}}{g},$$ N: normal load factor

The total moment applied at the elastic axis

$$M_{ea}' = L'e + M'_{ac} - W'd = t(y)$$: applied torque/span

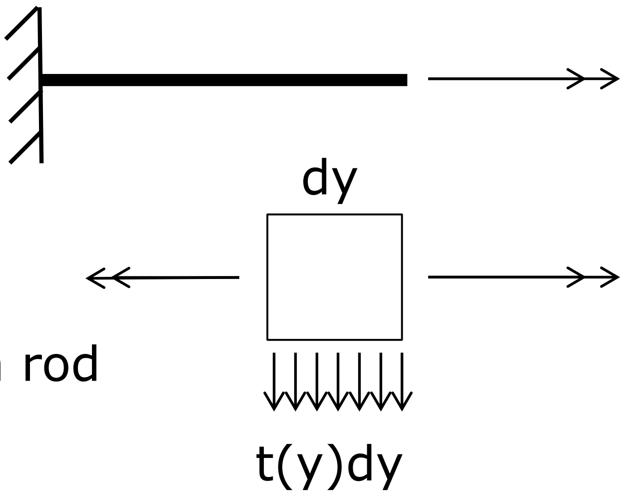
Straight High-aspect-ratio Wing

For the equilibrium eqn., consider a rod in torsion (St. Venant torsion)

$$T + dT = T + \frac{\partial T}{\partial y} dy$$

$$T = T + dT + t(y)dy$$

$$\frac{\partial T}{\partial y} = -t(y)$$



Constitutive relation for a uncoupled torsion rod

$$T = GJ \frac{d\theta}{dy}$$

where, GJ : torsional stiffness

J : function of geometry (geometry of aircraft)

$J = I_p$: polar moment of inertia for circular, solid, cross section

Straight High-aspect-ratio Wing

Combining the two eqns

$$GJ\frac{d^2\theta}{dy^2} = -t(y) = -L'e - M'_{ac} + Nmgd$$

Using a "strip theory," where both lift and moment at a given Spanwise station depend only on $\alpha(y)$ at given station.

$$C_l(y) = f(\alpha) = f(\alpha(y))$$

$$\therefore L' = qcC_{L\alpha}\alpha(y)$$

$$M'_{ac} = qc^2 C_{maco}$$

where, $C_{L\alpha}$: 2-D lift coefficient slope

Straight High-aspect-ratio Wing

[Note] Assume $C_{L_\alpha},\ C_{M_{ac_O}} \neq f(y)$, constant airfoil chord

Therefore,

$$\frac{d^2\theta}{d\bar{y}^2} + \lambda^2\theta = \kappa$$

where,

$$\bar{y} = y/l$$

$$\lambda^2 = l^2 qecC_{L_\alpha} / GJ$$

$$\kappa = -l^2 / GJ(qcC_{L_\alpha} e\alpha_r + qc^2 C_{M_{aco}} - Nmgd)$$

B.C.

$\theta(0) = 0$ ··· cantilever end

$T(1) = 0 = \frac{d\theta}{d\bar{y}}(1)$ ··· free end

Straight High-aspect-ratio Wing

The general sol. is

$$\theta(\bar{y}) = A\sin\lambda\bar{y} + B\cos\lambda\bar{y} + \frac{\kappa}{\lambda^2}$$

Applying B.C.s

$$\theta(0) = 0 \quad \rightarrow \quad B = -\frac{\kappa}{\lambda^2}$$

$$\frac{d\theta}{d\bar{y}}(1) = 0 \quad \rightarrow \quad \lambda(A\cos\lambda - B\sin\lambda) = 0$$

$$\rightarrow \quad A = B\tan\lambda$$

Finally,

$$\theta(\bar{y}) = \frac{\kappa}{\lambda^2}[1 - \tan\lambda\sin\lambda\bar{y} - \cos\lambda\bar{y}]$$

Straight High-aspect-ratio Wing

Divergence occurs when $\theta(\bar{y}) \rightarrow \infty$, and this happen when

$$\lambda_n \rightarrow (2n-1)\frac{\pi}{2} \quad (\mathrm{n} = 1, 2, \cdots)$$

The torsional divergence will occur for the lowest *n*(*n*=1), yielding

$$q_D = (\frac{\pi}{2l})^2 \frac{GJ}{ecC_{L_\alpha}}$$

[Note] Comparing with the typical section

$$q_D = \frac{K_\alpha}{C_{L\alpha}Se}$$

Straight High-aspect-ratio Wing

Using a homogeneous part of the governing eqn.

Applying B.C.

$$\begin{bmatrix} 0 & 1 \\ \lambda\cos\lambda & -\lambda\sin\lambda \end{bmatrix} \begin{Bmatrix} A \\ \hline B \end{Bmatrix} = \begin{Bmatrix} 0 \\ \hline 0 \end{Bmatrix}$$

$$\uparrow$$
$$\Delta$$

For nontrivial solution,

det(Δ)=0 → $\lambda\cos\lambda = 0$

$$\lambda_n = (2n-1)\frac{\pi}{2}$$

If $e < 0$, the characteristic eqn. becomes

$\cosh|\lambda| \neq 0$ → no divergence

Airload Distribution

- The spanwise lift distribution (L') as

$$L'(\bar{y}) = qcC_{L\alpha}(\alpha_r + \theta(\bar{y}))$$

where,

$$\theta(\bar{y}) = \frac{\kappa}{\lambda^2(1 - \tan\lambda \sin\lambda\bar{y} - \cos\lambda\bar{y})}$$

and $\kappa = \kappa(\alpha_r, N)$

For a given α_r

→ corresponding elastic twist distribution
→ particular airload distribution

··· integrate over the span: total lift *(L)*
But also $L=NW$,

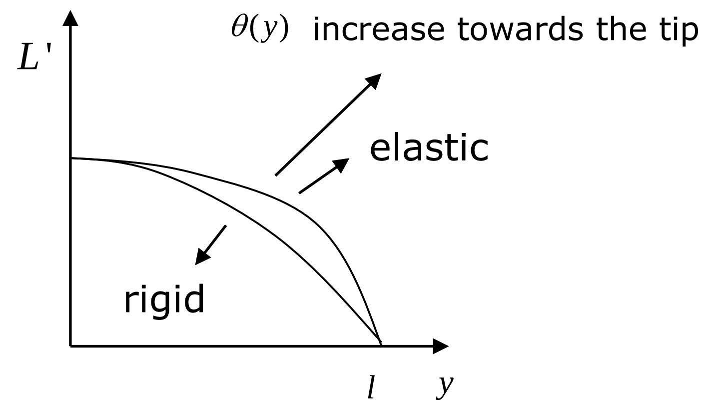
Therefore either α_r or N may be specified and the others are, determined from the total lift.

Airload Distribution

$$L = 2l\int_0^1 L' d\bar{y}$$
$$= 2qcC_{L\alpha} l[\alpha_r + \frac{\kappa(\alpha, N)}{\lambda^2}(1 - \frac{\tan\lambda}{\lambda})] = NW$$

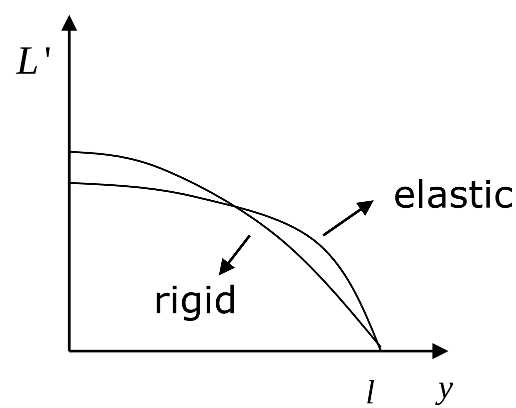
a) α_r is specified by aerodynamics or performance engineering

$$L_{rigid} < L_{elastic}$$



b) N is specified by structural engineer

$$L_{rigid} = L_{elastic}$$



Symmetrical Lift Distribution (BAH)

1) α_r is specified ··· total lift and lift distribution remains to be computed

- lift distribution is not known explicitly until a numerical value of N has been obtained.

$$c_l = c_{l_\alpha}\alpha^r(0) + c_{l_\alpha}\theta(y)$$

$$N = \frac{L}{W} = \frac{2g}{W}\int_0^l cc_l dy$$

$$N = \frac{\sim}{\sim}$$

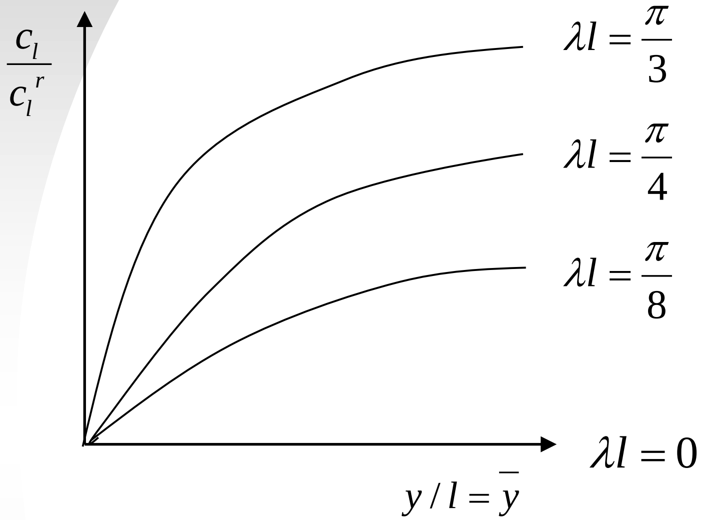
- rigid lift coefficient distribution due to the specified attitude

$$c_l^r = c_{L_\alpha}\alpha^r(0)$$

- lift coefficient distribution resulting from elastic twist

$$c_l^e(y) = c_{l_\alpha}\alpha^r(0)[\cos\lambda y + \tan\lambda y \sin\lambda y - 1]$$

Symmetrical Lift Distribution (BAH)



⋯ profound effect of elastic twist on spanwise lift distribution.

⋯ when $\lambda l = \frac{\pi}{2}$, divergence and cl / cl^2 approaches infinity

2) N is specified ⋯ structural design, (V-n diagram, N and speed specified

→ attitude and lift distribution must both be computed

- assumption

$$q\int_0^l c c_l^r dy = \frac{NW}{2}$$

Symmetrical Lift Distribution (BAH)

- corrective elastic distribution ⋯ will be superposed which produces no additional total lift

$$\int_0^l c c_l^e \, dy = 0$$

Different form of κ^r

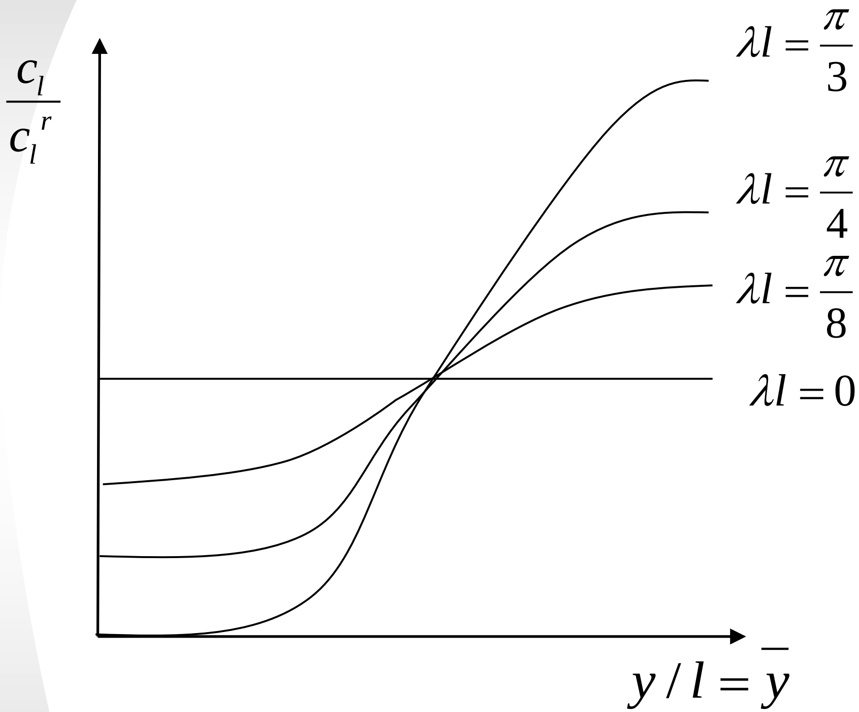
$$\kappa^r = -\frac{1}{GJ}\left(\frac{WNe}{2l} + q c_{mac} c^2 - mNgd\right)$$

- corrective a.o.a. distribution $\alpha^e(y)$

$$\alpha^e(y) = \alpha^e(0) + \frac{\kappa^2}{\lambda^2}(1 - \tan\lambda l \sin\lambda y - \cos\lambda y)$$

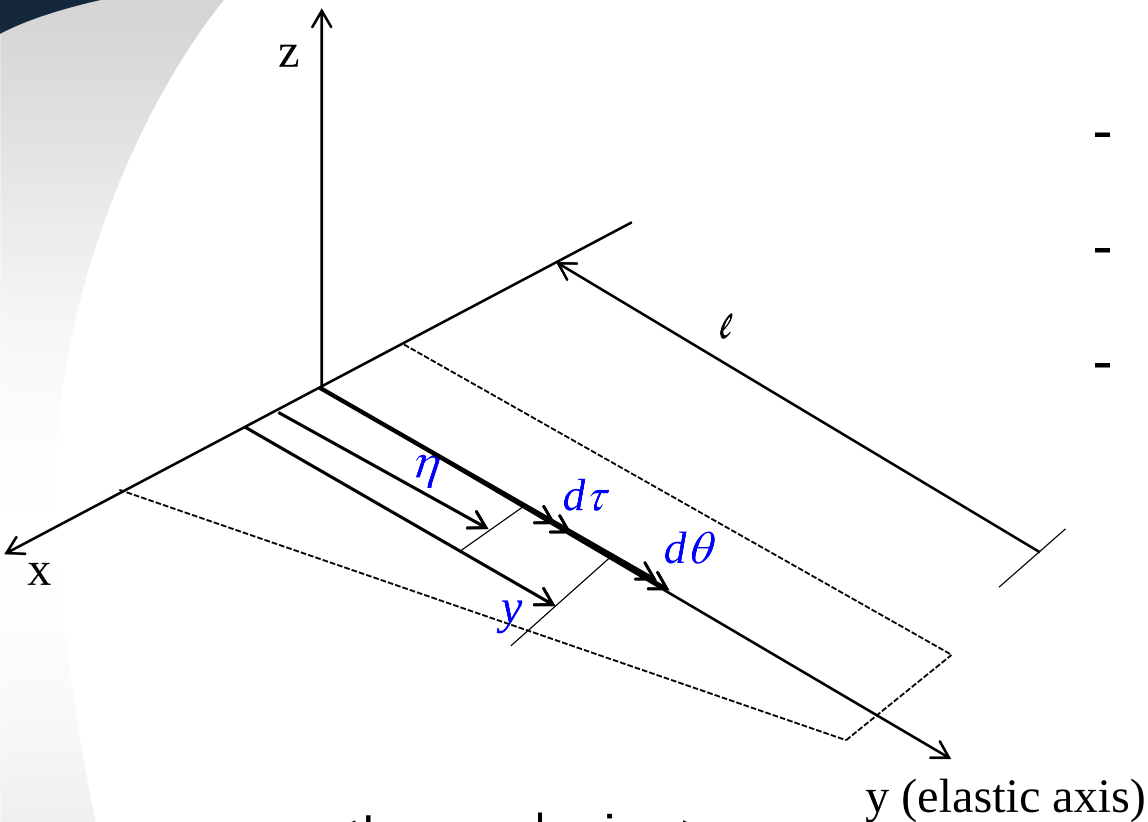
→ nose-down pitching attitude that must be assumed by the wing root in order that the total lift resulting from elastic twist be zero

Symmetrical Lift Distribution (BAH)



… area under the various load distribution curves remain the same

Non-Uniform Lifting Surface



<tapered wing>

- Variable cross section properties
- Structural (flexibility) influence function
- For a linear elastic structure

$$C^{\theta\theta}(y,\eta) = \frac{d\theta(y)}{d\tau(\eta)}$$

: resulting twist at point y due to torque at point η

(notation in textbook: $C^{\alpha\alpha}$)

$t(y)$=applied torque/span

Non-Uniform Lifting Surface

$$d\tau(\eta) = t(\eta)d\eta$$

$$\rightarrow \;\; \theta(y) = \int_0^y C^{\theta\theta}(y,\eta)t(\eta)d\eta$$

- Numerical integration by weighting numbers

$$I = \int_a^b f(y)dy \cong \sum_{i=1}^{N} f_i W_i$$

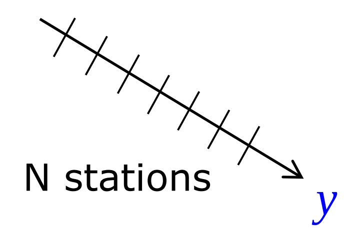
$$I = \lfloor W \rfloor \{f\} = \lfloor 1 \rfloor \begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix} \{f\}$$

[Ref] B.A.H. p809-812

Therefore, the elastic twist can be expressed as

$$\{\theta\} = \left[C^{\theta\theta}\right] \begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix} \{t\}$$

Nx1 NxN NxN Nx1



Non-Uniform Lifting Surface

- Structural (flexibility) influence coefficient

Discretized version of structural influence function

If $\Delta\tau_j$: variation of torque of station j

$\Delta\theta_j$: corresponding variation of twist at station i

Then for a linearly elastic structure, $C_{ij} = \dfrac{\Delta\theta_i}{\Delta\tau_i}$

$$\theta_i = \sum_{j=1}^{N} C_{ij}\tau_j = [C_{ij}]\{\tau_j\}$$

[Note] $\{\tau\} = [C]^{-1}\{\theta\}$

↑ stiffness matrix

Non-Uniform Lifting Surface

- Static aeroelastic equilibrium

$$\theta(y) = \int_0^y C^{\theta\theta}(y,\eta)t(\eta)d\eta$$

$$= \int_0^y C^{\theta\theta}(y,\eta)\{q[e(\eta)cc_l(\eta) + c^2(\eta)c_{mac}(\eta)] - Ngm(\eta)d(\eta)\}d\eta$$

Discretizing the span along N points and using an appropriate integration scheme.

$$\{\theta\} = q[E]\{CC_l\} + q[F]\{C_{Mac}\} - Ng[G]\{m\}$$

where,

$$[E] = \left[C^{\theta\theta}\right]\begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \backslash & & \\ & e & \\ & & \ddots \end{bmatrix} \quad [F] = \left[C^{\theta\theta}\right]\begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \ddots & & \\ & c^2 & \\ & & \ddots \end{bmatrix}$$

$$[G] = \left[C^{\theta\theta}\right]\begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \ddots & & \\ & d & \\ & & \ddots \end{bmatrix}$$

Non-Uniform Lifting Surface

- Aerodynamic influence coefficient

If the spanwise airload distribution can be linearly related to the angle of attack distribution, then

$$Aij = \frac{(CC_l)i}{\alpha_j}$$

[Note] For “Strip theory” → [A]: diagonal

Therefore, $\{cc_l\} = [A]\{\alpha\}$

where, $\{\alpha\} = \{\alpha_r\} + \{\theta\}$

$$\{\alpha_r\} = \alpha_r(0)\{1\} + \{\alpha_t\}$$

↓

wing root AOA

Finally, $[A]^{-1}\{cc_l\} - \{\alpha_r\}$ → replace $\{\theta\}$

Non-Uniform Lifting Surface

- Aeroelastic load distribution

$$[A]^{-1}\{cc_l\} = \alpha_r(0)\{1\} + \{\alpha_t\} + q[E]\{cc_l\} + q[F]\{c_{mac}\} - Ng[G]\{m\}$$

We can solve by specifying $\alpha_r(0)$ or N, and use the normal load eqn.

$$L/W = N$$

$$\rightarrow \quad L = NW = 2q\int_0^l cc_l\,dy$$

$$= 2q[1]\begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix}\{cc_l\}$$

$$0 = NW - 2q[1]\begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix}\{cc_l\}$$

Non-Uniform Lifting Surface

Specifying $\alpha_r(0)$, the two sets of equilibrium eqns together.

$$\begin{bmatrix} [A]^{-1} - q[E] & q[G]\{m\} \\ 2q[1]\begin{bmatrix} \ddots & & \\ & W & \\ & & \ddots \end{bmatrix} & -W \end{bmatrix} \begin{Bmatrix} cc_l \\ \hline N \end{Bmatrix} = \begin{Bmatrix} \alpha_r(0)\{1\} + \{\alpha_t\} + q[F]\{c_{mac}\} \\ \hline 0 \end{Bmatrix}$$

Torsional divergence

What dynamic pressure will yield a finite load distribution when there are no external disturbances

$$\underbrace{[[A]^{-1} - q[E]]}_{\Delta}\{cc_l\} = \{0\}$$

divergence occurs when det(Δ)=0

→ q_D is the lowest root

We could also write:

$$\underbrace{[A][E]}_{A}\underbrace{\{cc_l\}}_{x} = \underbrace{(\frac{1}{q})}_{\lambda}\underbrace{\{cc_l\}}_{x}$$

Structural Influence Function (BAH)

$$w(x,y)=\iint_s C\left(x,y;\xi,\eta\right)Z\left(\xi,\eta\right)d\xi d\eta$$

$$Z\left(\xi,\eta\right):\text{normal load}$$

- wing is sufficiently slender so that chordwise segments of the wing parallel to the x-axis can be assumed rigid.

→ $$C\left(x,y;\xi,\eta\right)=C^{zz}\left(y,\eta\right)-xC^{\theta z}\left(y,\eta\right)+\xi C^{\theta\theta}\left(y,\eta\right)-\xi C^{z\theta}\left(y,\eta\right)$$

Influence function definition

$C^{pq}\left(y,\eta\right)$ = linear or angular deflection in the *p*-direction at *y* due to a unit force or torque in the *q*-direction at *η*

Deflection

$$w(x,y)=w\left(y\right)-x\theta\left(y\right)$$

Structural Influence Function (BAH)

$$w(y)=\int_0^l C^{zz}\left(y,\eta\right)Z\left(\eta\right)d\eta+\int_0^l C^{z\theta}\left(y,\eta\right)t\left(\eta\right)d\eta$$

$$\theta(y)=\int_0^l C^{\theta z}\left(y,\eta\right)Z\left(\eta\right)d\eta+\int_0^l C^{\theta\theta}\left(y,\eta\right)t\left(\eta\right)d\eta$$

Where

$$Z\left(\eta\right)=\int_{chord} Z\left(\xi,\eta\right)d\xi$$

$$t\left(\eta\right)=-\int_{chord} \xi Z\left(\xi,\eta\right)d\xi$$

--- chordwise drag loads on the wing have been neglected.

This formula can be applied to unswept or swept wings with structural discontinuities.

Swept Wings (BAH)

- Swept-back wing ⋯ a.o.a. is reduced in the streamwise direction
  → tends to shift the pressure center of the aerodynamic loads inboard

  (swept-forward ⋯ outboard)

  → possibility of wing divergence reduced, possibility of control reversal increased
  → Produce large fore and after pressure center shifts
  → marked effect upon static longitudinal stability

- Integral eqn. (streamwise segments)

$$\theta(y) = \int_0^l C^{\theta Z}(y,\eta) Z(\eta) d\eta + \int_0^l C^{\theta\theta}(y,\eta) t(\eta) d\eta$$

$$= \int_0^l C^{\theta Z}(y,\eta)(gcc_l - mNg) d\eta + \int_0^l C^{\theta\theta}(y,\eta)(gecc_l + gc^2 c_{mAC} - mNgd) d\eta$$

Swept Wings (BAH)

substitute $c_l = c_l^r + c_l^e$

$$\theta(y) = q\int_0^l \bar{C}(y,\eta) c c_l d\eta + F(y)$$

→ c_l^e and θ are unknown, and all other quantities are regarded as known.

$$\bar{C}(y,\eta) = C^{\theta z}(y,\eta) + e(\eta) C^{\theta\theta}(y,\eta)$$ → non self-adjoint

nonsymmetrical function

- homogeneous form

$$\theta(y) = q\int_0^l \bar{C}(y,\eta) c c_l^e d\eta$$

since $\bar{C}(y,\eta)$ is not a symmetrical function, real characteristic values may not exist for a given case.

Swept Wings (BAH)

→ $[A]\{cc_l^{\,e}\} = q[\bar{E}]\{cc_l^e\}$

$[A]$: matrix of aerodynamic influence coefficients

$$[\bar{E}] = [\bar{C}]\begin{bmatrix} \ddots & & \\ & \bar{W} & \\ & & \ddots \end{bmatrix}$$

$$[\bar{C}] = [C^{\theta z}] + [C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & e & \\ & & \ddots \end{bmatrix}$$: matrix of flexibility influence coefficient

$$[A] = \frac{1}{dC_L / d\alpha}\begin{bmatrix} \ddots & & \\ & 1/e & \\ & & \ddots \end{bmatrix}$$: strip theory

Swept Wings (BAH)

- Solution in terms of matrices for arbitrary planform and stiffness

$$\mathcal{a}[ccl_e] = q\int_0^l \overline{C}(y,\eta)\overline{cc_l}^e d\eta + \overline{f}(y)$$

$\mathcal{a}$: appropriate linear aerodynamic operator accounting for spanwise aerodynamic effects

- In matrix form

symmetric

$$[A^s]\{cc_l^e\} = q[\overline{E}]\{cc_l^e\} + \{\overline{f}\}$$

$[A^s]$: Matrix of aerodynamic influence coeff. for symmetrical loading

$$\{\overline{f}\} = q[\overline{E}]\{cc_l^r\} + q[\overline{F}]\{c_{mAC}\} - N[\overline{G}]\{mg\}$$

$$[\overline{E}] = ([C^{\theta z}] + [C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & e & \\ & & \ddots \end{bmatrix})\begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$$

Swept Wings (BAH)

$$[\overline{F}] = ([C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & c^2 & \\ & & \ddots \end{bmatrix})\begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$$

$$[\overline{G}] = ([C^{\theta z} + [C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & d & \\ & & \ddots \end{bmatrix})\begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$$

Eq. (5-195): $[A^s]$ According to Weissinger's theory

- Alternative form

By putting $\{cc_l^{\,e}\} = \{cc_l\} - \{cc_l^{\,r}\}$

$$[A^s]\{cc_l\} = q[\overline{E}]\{cc_l\} + \{\alpha^r\} + q[\overline{F}]\{c_{mAC}\} - N[\overline{G}]\{mg\}$$

where $\{\alpha^r\} = [A^s]\{cc_l^{\,r}\}$

Swept Wings (BAH)

$\{cc_l^{\,e}\}$ and $\{cc_l\}$ are unknowns.

when $\{\alpha^r\}$ (attitude) is specified

$$[A^s]\{cc_l\} = q([\overline{E}] - \frac{2}{W}[\overline{G}]\{mg\}\lfloor 1 \rfloor \begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix})\{cc_l\} + \{\alpha^r\} + q[\overline{F}]\{c_{mAC}\}$$

… can be solved by matrix inversion, some cases by matrix iteration

Anti-symmetrical a.o.a. distribution (Rolling of a Wing, BAH)

B.A.H. 8.3(d) Anti-symmetrical lift distribution (rolling of a wing)

- anti-symmetrical a.o.a. distribution

$$\alpha(y) = \frac{\partial \alpha^r}{\partial \beta}\beta - \frac{py}{U} + \theta(y)$$

p : rolling velocity (rad/sec)

$\frac{py}{U}$: induced a.o.a. due to rolling velocity p

- Equation of equilibrium (integral eqn.)

$$c_{mAC}(y) = \frac{\partial c_{mAC}}{\partial \beta}\beta$$

$$c_l{}^r(y) = \frac{\partial c_l{}^r}{\partial \beta}\beta + \frac{\partial c_l{}^r}{\partial c_l l / U}\left(\frac{pl}{U}\right)$$

$$N(y) = \frac{y\dot{p}}{g}$$

Antisymmetrical a.o.a. distribution (BAH)

$$\theta(y) = q\int_0^l C^{\theta\theta}(y,\eta) ec{c_l}^e d\eta + f_a(y)$$

$$f_a(y) = \int_0^l C^{\theta\theta}(y,\eta)[qec(\frac{\partial {c_l}^r}{\partial\beta}\beta + \frac{\partial {c_l}^r}{\partial(pl/U)}(\frac{pl}{U})) + qc^2\frac{\partial c_{mAC}}{\partial\beta}\beta - m\eta\dot{p}d]d\eta$$

⋯ all terms except $\theta(y)$ and $c{c_l}^e$ are known

- Solution by matrices

$$[A^a]\{c{c_l}^e\} = q[E]\{c{c_l}^e\} + \{f_a\}$$

$[A^a]$: Matrix of aerodynamic influence coeff. for anti-symmetrical loading on a straight wing, refer to Eq. (5-157)

Antisymmetrical a.o.a. distribution (BAH)

$$\{f^a\} = q[E](\{c\frac{\partial c_l^r}{\partial \beta}\beta + c\frac{\partial c_l{}^r}{\partial (pl/U)}\}\frac{pl}{U}) + q[F]\{\frac{\partial c_{mAC}}{\partial \beta}\}\beta - [G]\begin{bmatrix} \ddots & & \\ & y & \\ & & \ddots \end{bmatrix}\{m\}\dot{p}$$

$$[E] = [C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & e & \\ & & \ddots \end{bmatrix})\begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$$

$$[F] = [C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & c^2 & \\ & & \ddots \end{bmatrix})\begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$$

$$[G] = [C^{\theta\theta}]\begin{bmatrix} \ddots & & \\ & d & \\ & & \ddots \end{bmatrix})\begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$$

Antisymmetrical a.o.a. distribution (BAH)

- Alternative form

  by substituting

$$\{cc_l^{\,e}\} = \{cc_l\} - \{cc_l^{\,r}\} = \{cc_l\} - \{c\frac{\partial c_l^{\,r}}{\partial \beta}\}\beta - \{c\frac{\partial c_l^{\,r}}{\partial (pl/U)}\}\frac{pl}{U}$$

$$[Aa]\{cc_l\} = q[E]\{cc_l\} + \{f_a{}'\}$$

$\cdots$ $\beta, p, \dot{p}$ are prescribed, can be solved by matrix inversion or iteration

$$\{cc_l^{\,e}\} = ([A^a] - q[E])^{-1}\{f_a\}$$

- Sudden aileron displacement

$\cdots$ if $\beta, p, \dot{p}$ are not prescribed, additional eqn. must be needed.

$$\dot{p} = \frac{2g}{I_x}\lfloor H \rfloor\{cc_l\} = \frac{2g}{I_x}\lfloor H \rfloor(\{cc_l^e\} + \{c\frac{\partial c_l^{\,r}}{\partial \beta}\}\beta)$$

row matrix $\lfloor H \rfloor = \lfloor 1 \rfloor \begin{bmatrix} \ddots & & \\ & y & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \overline{W} & \\ & & \ddots \end{bmatrix}$

$$p = 0$$

Antisymmetrical a.o.a. distribution (BAH)

- steady rolling motion

$$\lfloor H \rfloor \{cc_l\} = 0$$

- aileron effectiveness

$$\frac{\partial(pl/U)}{\partial\beta} = -\frac{\lfloor H \rfloor([A^a]-q[E])^{-1}(\{\partial\alpha^r/\partial\beta\}+q[F]\{\partial c_{mAC}/\partial\beta\})}{\lfloor H \rfloor([A^a]-q[E])^{-1}\{y/l\}} \quad \ldots \ (*)$$

lift distribution can be computed by $\dot{p} = 0$

pl/U and β cannot be both specified due to (*)

→ In design specifications, specified pl/U, then β from (*)

Antisymmetrical a.o.a. distribution (BAH)

-aileron reversal ⋯ numerator of (*) vanishes

$$q_R = -\frac{\lfloor H \rfloor([A^a]-q_R[E])^{-1}\{\partial\alpha^r/\partial\beta\}}{\lfloor H \rfloor([A^a]-q_R[E])^{-1}[F]\{\partial c_{mAC}/\partial\beta\}}$$

⋯ can be solved iteratively

- Alternative form of q_R

$$\lfloor H \rfloor\{cc_l^{\,e}\}+\lfloor H \rfloor\{cc_l^{\,r}\}=0$$

$$q_R = -\frac{c_{l_\beta}}{\frac{1}{sl}\lfloor H \rfloor([A^a]-q_R[E])^{-1}([E]\{c\frac{\partial c_l^{\,r}}{\partial\beta}\}+[F]\{\frac{2c_{mAC}}{\partial\beta}\})}$$

where $$c_{l_\beta} = \frac{1}{sl}\lfloor H \rfloor\{c\frac{\partial c_l^{\,r}}{\partial\beta}\}$$

$$c_{l_p} = \frac{1}{sl}\lfloor H \rfloor\{c^2\frac{\partial c_l^{\,r}}{\partial(pl/U)}\}$$

Sweep + Roll (BAH)

$$\theta(y) = q\int_0^l \bar{C}(y,\eta) c c_e^{\,e} d\eta + \bar{f}_a(y)$$

where

$$f_a(y) = q\int_0^l \bar{C}(y,\eta)[c\frac{\partial c_l^{\,r}}{\partial \beta} d\eta + c\frac{\partial c_l^{\,r}}{\partial (pl/U)}(\frac{pl}{U})]d\eta$$

$$+ q\int_0^l C^{\theta\theta}(y,\eta) c \frac{\partial c_{mAC}}{\partial \beta}\beta c^2 d\eta - p^{\circ}\int_0^l [C^{\theta Z}(y,\eta) + C^{\theta\theta}(y,\eta) d] m\eta d\eta$$

- Appropriate relation between antisymmetric lift distribution and twist distribution

$$a[cc_l] = q\int_0^l \bar{C}(y,\eta) c c_e^{\,e} d\eta + \bar{f}_a(y)$$

Sweep + Roll (BAH)

- Matrix form

$$[A^a][cc_l] - q[\bar{E}] + \{\bar{f}_a\}$$

$$\{\bar{f}_a\} = q[\bar{E}](\{c\frac{\partial c_l{}^r}{\partial \beta}\}\beta + \{c^2\frac{\partial c_l{}^r}{\partial (pl/U)}\}(\frac{pl}{U})) + q[\bar{F}]\{\frac{\partial c_{mAC}}{\partial \beta}\}\beta - [G]\begin{bmatrix} \ddots & & \\ & y & \\ & & \ddots \end{bmatrix}\{m\}p^{\circ}$$

- Aileron reversal

$$q_R = -\frac{c_{l_\beta}}{\frac{1}{sl}\lfloor H \rfloor([A^a] - q_R[E])^{-1}([E]\{c\frac{\partial c_l{}^r}{\partial \beta}\} + [F]\{\frac{2c_{mAC}}{\partial \beta}\})}$$

- Influence of sweep on aileron reversal

… for swept-forward wing – aileron deflected down → lift increase → nose-up deflection

⟷ opposite to nose-down twisting by the pitching moment

Sweep + Roll (BAH)

··· two effects tend to cancel each other aileron remain effective

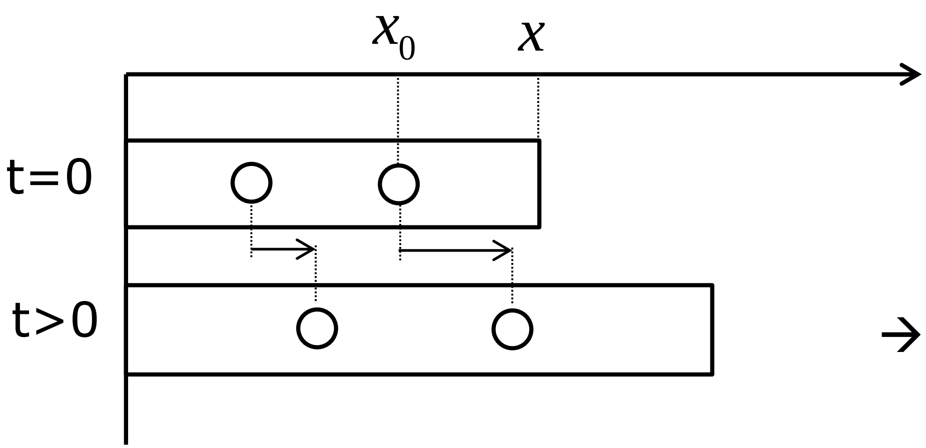
In the case of swept-back wing, two effects add together,
→ low aileron effectiveness

Low-speed Aerodynamics

#7 Katz & Plotkin

"Low-speed Aerodynamics"

1. Field description
   - Lagrangian method
   - Euler method



→ $F(t)$

$\delta_{tip}(t)$

Low-speed Aerodynamics

a) $\bar{\delta}(t, x_0)$ ··· L, particle point of view $x = x_p(x_0, y_0, z_0, t)$ ← material

b) $\delta(t, x)$ ··· E, $u = u(x, y, z, t)$ field point of view ← fluid

- Velocity

b) $$u(t,x) = \lim_{\Delta t \to 0} \frac{\delta(t+\Delta t, x+\Delta x)}{\Delta t} = \frac{\partial \delta}{\partial t} + u\frac{\partial \delta}{\partial x}$$

- Acceleration
- 3D

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + q \cdot \nabla(\ )$$

Low-speed Aerodynamics

2. Governing Eqn's

1) Integral form

① Mass

$$\frac{d}{dt}m_{c.v} = \frac{\partial}{\partial t}\int_{c.v.}\rho dV + \int_{c.s.}\rho(\vec{q}\cdot\vec{n})dS = 0$$

② Momentum

$$\frac{d}{dt}(m\vec{q})_{c.v.} = \frac{\partial}{\partial t}\int_{c.v.}\rho\vec{q}dV + \int_{c.s.}\rho\vec{q}(\vec{q}\cdot\vec{n})dS = \sum\vec{F}$$

$$(\sum\vec{F})_i = \int_{c.v.}\rho f_c dV + \int_{c.s.}n_j\cdot\tau_{ij}dS$$

③ Energy

Low-speed Aerodynamics

2) Differential form

① Mass

- divergence theorem

$$\int_{c.s.} n_j q_j dS = \int_{c.v.} \frac{\partial q_j}{\partial x_j} dV$$

$$\int_{c.v.} \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q}\right)}dV = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho}_{\frac{D\rho}{Dt}} + \rho \nabla \vec{q} = 0 \quad \rightarrow \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{q} = 0$$

incompressible: $\nabla \vec{q} = 0$