457.644 Advanced Bridge Engineering

Aerodynamic Design of Bridges

Part V: Aeroelastic and Aerodynamic Analysis

Structural Design Lab.(Prof. Ho-Kyung Kim)
Dept. of Civil & Environmental Eng.
Seoul National University

1. Single-mode Flutter and Buffeting Theory

Concept of single mode single component

Basic assumption

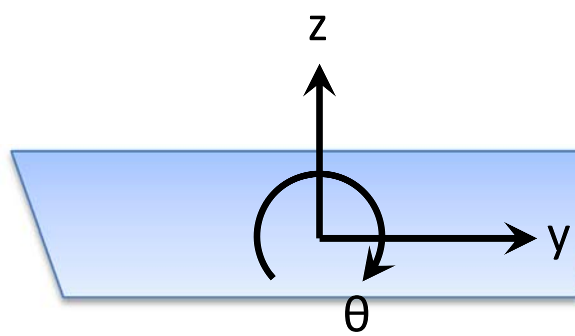
- The eigen-frequencies are well spaced out on the frequency axis.
- The cross sectional shear center is assumed to coincide (or nearly coincide) with the centroid.
- There are no other significant source of mechanism or flow induced coupling between the three displace component (horizontal, vertical or torsion).

Concept of single mode single component

Features

- Coupling between modes may be ignored.
- Each mode shape only contains one component, i.e. any of the N_{mod} mode shapes is purely horizontal, vertical or torsion.
- The variance of a displacement component is the sum of all variance contributions from excited modes containing displacement components exclusively in the y, z or θ direction.
  - σ_y^2 is the sum of all variances associated with the relevant number of modes containing only horizontal displacement.

$$\begin{bmatrix} \sigma_y^2 \\ \sigma_z^2 \\ \sigma_\theta^2 \end{bmatrix} = \begin{bmatrix} \sum_{i_y} \boldsymbol{\sigma}_{i_y}^2 \\ \sum_{i_z} \boldsymbol{\sigma}_{i_z}^2 \\ \sum_{i_\theta} \boldsymbol{\sigma}_{i_\theta}^2 \end{bmatrix}$$



Equation of motion for lateral direction

- **The modal time domain equilibrium equation for a lateral single mode**

$$\tilde{M}_y \cdot \ddot{\eta}_y(t) + \tilde{C}_y \cdot \dot{\eta}_y(t) + \tilde{K}_y \cdot \eta_y(t) = \tilde{Q}_{bu}(t) + \tilde{Q}_{se}(t, \eta_y, \dot{\eta}_y, \ddot{\eta}_y)$$

- **By the assumption and features for single mode, single component**
  - Mode shape: $\phi_{z_i} = \phi_{\theta_i} = 0$
  - Aerodynamic force (for drag): $\tilde{Q}_{bu}(t) = \frac{1}{2}\rho U^2 B^2 L \int_{deck} \left[2C_D \frac{u}{U} \phi_y \right] \frac{dx}{l}$

$$\tilde{Q}_{se}(t, \dot{\eta}_y) = \frac{1}{2}\rho U^2 B^2 L \int_{deck} \frac{KB}{U} \left[\phi_y^2 P_1^* \dot{\eta}_y \right] \frac{dx}{l}$$

- **As a result, the load per unit length can be derived as follows:**

$$\tilde{M}_y \left[\ddot{\eta}_y + 2\omega_y \zeta_y \cdot \dot{\eta}_y + \omega_y^2 \cdot \eta \right] = \frac{1}{2}\rho U^2 B^2 L \left[\int_{L_{\exp}} \Lambda_D \phi_y \frac{dx}{L} + \frac{KB}{U} P_1^* \int_{L_{\exp}} \phi_y^2(x) \frac{dx}{L} \cdot \dot{\eta}_y \right]$$

  - L_{exp} is the flow exposed part of the structure.
  - $\Lambda_D = 2C_D u/U$
  - $\tilde{Q}_b$ and $\tilde{Q}_{se}$ are modal aerodynamic loads. Each term is buffeting forces and unsteady self-excited force, respectively. (*Ref*, Part.IV: Wind loads)

Equation of Motion for Lateral Motion

- **Gathering all motion dependent load on the left hand side:**

$$\tilde{M}_y\left[\ddot{\eta}_y+2\omega_y\zeta_y\cdot\dot{\eta}_y+\omega_y^2\cdot\eta\right]=\frac{1}{2}\rho U^2B^2L\left[\int_{L_{\exp}}\Lambda_D\phi_y\frac{dx}{L}+\frac{KB}{U}P_1^*\int_{L_{\exp}}\phi_y^2(x)\frac{dx}{L}\cdot\dot{\eta}_y\right]$$

$$\rightarrow\tilde{M}_y\left[\ddot{\eta}_y+2\omega_y\zeta_y\cdot\dot{\eta}_y+\omega_y^2\cdot\eta\right]-\frac{1}{2}\rho U^2B^2L\frac{KB}{U}P_1^*\int_{L_{\exp}}\phi_y^2(x)\frac{dx}{L}\cdot\dot{\eta}_y=\frac{1}{2}\rho U^2B^2L\int_{L_{\exp}}\Lambda_D\phi_y\frac{dx}{L}$$

$$\rightarrow\tilde{M}_y\left[\ddot{\eta}_y+2\omega_y\zeta_y\cdot\dot{\eta}_y+\omega_y^2\cdot\eta\right]-\frac{1}{2}\rho U^2B^2L\frac{\omega B}{U}\frac{B}{U}P_1^*\int_{L_{\exp}}\phi_y^2(x)\frac{dx}{L}\cdot\dot{\eta}_y=\frac{1}{2}\rho U^2B^2L\int_{L_{\exp}}\Lambda_D\phi_y\frac{dx}{L}$$

$$\rightarrow\tilde{M}_y\left[\ddot{\eta}_y+2\omega_y\zeta_y\cdot\dot{\eta}_y+\omega_y^2\cdot\eta\right]-\frac{\rho B^4L}{2}\omega\cdot P_1^*\int_{L_{\exp}}\phi_y^2(x)\frac{dx}{L}\cdot\dot{\eta}_y=\frac{1}{2}\rho U^2B^2L\int_{L_{\exp}}\Lambda_D\phi_y\frac{dx}{L}$$

$$\rightarrow\tilde{M}_y\cdot\ddot{\eta}_y+\left(2\omega_y\zeta_y\tilde{M}_y-\frac{\rho B^4L}{2}\omega\cdot P_1^*\int_{L_{\exp}}\phi_y^2(x)\frac{dx}{L}\right)\cdot\dot{\eta}_y+\omega_y^2\tilde{M}_y\cdot\eta=\frac{\rho U^2B^2L}{2}\int_{L_{\exp}}\Lambda_D\phi_y\frac{dx}{L}$$

- **Substitute** $\bar{\zeta}_{ae}(\omega)=\omega\cdot P_1^*\int_{L_{\exp}}\phi_y^2(x)\frac{dx}{L}$ **(not exact aerodynamic damping) then:**

$$\tilde{M}_y\cdot\ddot{\eta}_y+\left(2\omega_y\zeta_y\tilde{M}_y-\frac{\rho B^4L}{2}\bar{\zeta}_{ae}(\omega)\right)\cdot\dot{\eta}_y+\omega_y^2\tilde{M}_y\cdot\eta=\frac{\rho U^2B^2L}{2}\int_{L_{\exp}}\Lambda_D\phi_y\frac{dx}{L}$$

Transition into the Frequency Domain

- **Taking the Fourier transform:**

$$\left[\omega_y^2\tilde{M}_y-\omega^2\tilde{M}_y+i\omega\left(2\omega_y\zeta_y\tilde{M}_y-\frac{\rho B^4L}{2}\bar{\zeta}_{ae}\left(\omega_y\right)\right)\right]a_{\eta_y}\left(\omega\right)=\frac{\rho U^2B^2L}{2}\int_{L_{\exp}}a_{\Lambda_D}\phi_y\frac{dx}{L}$$

$$\rightarrow\left[\omega_y^2\tilde{M}_y-\omega^2\tilde{M}_y+i\omega\left(2\omega_y\zeta_y\tilde{M}_y-\frac{\rho B^4L}{2}\bar{\zeta}_{ae}\left(\omega_y\right)\right)\right]a_{\eta_y}\left(\omega\right)=\frac{\rho U^2B^2L}{2}\int_{L_{\exp}}\frac{2C_D}{U}a_u\phi_y\frac{dx}{L}$$

$$\rightarrow\omega_y^2\tilde{M}_y\left[1-\left(\frac{\omega}{\omega_y}\right)^2+2i\left(\zeta_y-\frac{\rho B^4L}{4\omega_y\tilde{M}_y}\bar{\zeta}_{ae}\left(\omega_y\right)\right)\cdot\frac{\omega}{\omega_y}\right]a_{\eta_y}\left(\omega\right)=\frac{\rho U^2B^2L}{2}\int_{L_{\exp}}\frac{2C_D}{U}a_u\phi_y\frac{dx}{L}$$

$$a_{\eta_y}\left(\omega\right)=\frac{\rho UB^2LC_D}{\omega_y^2\tilde{M}_y}\frac{1}{\left[1-\left(\frac{\omega}{\omega_y}\right)^2+2i\left(\zeta_y-\frac{\rho B^4L}{4\omega_y\tilde{M}_y}\bar{\zeta}_{ae}\left(\omega_y\right)\right)\cdot\frac{\omega}{\omega_y}\right]}\int_{L_{\exp}}a_u\phi_y\frac{dx}{L}$$

- a_η and a_{Λ_D} are the Fourier amplitudes of $\eta(t)$ and Λ_D

Frequency Response Function

- From the relationship between response and load in frequency domain:

$$a_{\eta_y}(\omega) = \frac{\rho U B^2 L C_D}{\omega_y^2 \tilde{M}_y} \frac{1}{\left[1 - \left(\frac{\omega}{\omega_y}\right)^2 + 2i\left(\zeta_y - \frac{\rho B^4 L}{4\omega_y \tilde{M}_y}\bar{\zeta}_{ae}(\omega_y)\right)\cdot\frac{\omega}{\omega_y}\right]} \int_{L_{\exp}} a_u \phi_y \frac{dx}{L}$$

$$= \frac{H_y(\omega)}{\tilde{K}_y} a_{Q_y}(\omega)$$

where:

Aerodynamic damping (ζ_{ae})

$$H_y = \left[1 - \left(\frac{\omega}{\omega_y}\right)^2 + 2i\left(\zeta_y - \boxed{\frac{\rho B^4 L}{4\omega_y \tilde{M}_y}\bar{\zeta}_{ae}(\omega_y)}\right)\cdot\frac{\omega}{\omega_y}\right]^{-1}$$: Non-dimensional modal frequency response function

$$\tilde{K}_y = \omega_y^2 \tilde{M}_y$$: Generalized stiffness

$$a_{Q_y} = \rho U B^2 L C_D \int_{L_{\exp}} a_u \phi_y \frac{dx}{L}$$: Fourier amplitudes of buffeting load Q_{bu}

Spectral Densites

- General single-sided spectral density of x(t) can be defined by form of square of Fourier constant.

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} a_x^*(\omega) \cdot a_x(\omega)$$

- Therefore, the single-sided spectrum of generalized coordinate $\eta_y(t)$ is

$$S_{\eta_y}(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} a_\eta^*(\omega) \cdot a_\eta(\omega) = \frac{\left|H_y(\omega)\right|^2}{\tilde{K}_y^2} \lim_{T\to\infty} \frac{1}{\pi T} a_{Q_y}^*(\omega) \cdot a_{Q_y}(\omega)$$

$$= \frac{\left|H_y(\omega)\right|^2}{\tilde{K}_y^2} \cdot S_{Q_y}(\omega)$$

- The single-sided spectrum of buffeting load $Q_{bu}(t)$ is also given by

$$S_{Q_y}(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} a_{Q_y}^*(\omega) \cdot a_{Q_y}(\omega)$$

Spectral Densities

- **Substitutes the load term of wind fluctuation term:**

$$S_{Q_y}(\omega) = \lim_{T\to\infty}\frac{1}{\pi T} a^*_{Q_y}(\omega)\cdot a_{Q_y}(\omega)$$

$$= \lim_{T\to\infty}\frac{1}{\pi T}\left\{\rho U B^2 L C_D \int_{L_{exp}} \frac{a^*_u \phi_y dx}{L}\right\}\cdot\left\{\rho U B^2 L C_D \int_{L_{exp}} \frac{a_u \phi_y dx}{L}\right\}$$

$$= (\rho U B^2 L C_D)^2 \lim_{T\to\infty}\frac{1}{\pi T}\iint_{L_{exp}} \phi_y(x_1)\phi_y(x_2) a^*_u(x_1,\omega)\, a_u(x_2,\omega)\frac{dx_1}{L}\frac{dx_2}{L}$$

- **the single-sided spectrum of wind fluctuation $u_i(t)$ is given by**

$$S_u(\omega) = \lim_{T\to\infty}\frac{1}{\pi T} a^*_u(\omega)\cdot a_u(\omega)\,,\;\; S_u(x_1,x_2,\omega) = S_u(\omega) e^{-\frac{C\cdot\omega|x_1-x_2|}{U}}$$

- **Finally, spectrum of wind load Q_{bu} can be defined as follows.**

$$S_{Q_y}(\omega) = \left(\rho U B^2 L C_D\right)^2 \iint_{L_{exp}} \phi_y(x_1)\phi_y(x_2) e^{-\frac{C\cdot\omega\cdot|x_1-x_2|}{U}} \frac{dx_1}{L}\frac{dx_2}{L}\cdot S_u(\omega)$$

The RMS of Response

- The response may simply be obtained by recognizing that due to linearity the Fourier amplitude at arbitrary position x is given by:

$$a_y(\omega) = \phi_i(x) \cdot a_\eta(\omega)$$

- Therefore, the response spectrum for the displacement response is given by

$$S_y(x,\omega) = \phi_y^2(x) \cdot S_\eta(\omega) = \frac{\phi_y^2(x)}{\tilde{K}_y^2} \cdot \left|H_y(\omega)\right|^2 \cdot S_{Q_y}(\omega)$$

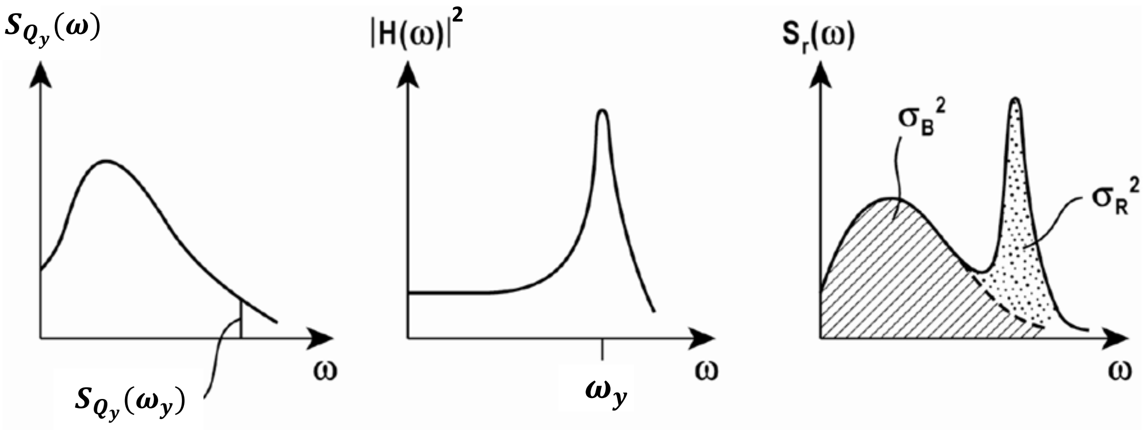
- The variance of the displacement response can be calculated by integration:

$$\sigma_y^2(x) = \frac{\phi_y^2(x)}{\tilde{K}_y^2} \int_0^\infty \left|H_y(\omega)\right|^2 \cdot S_{Q_y}(\omega)\, d\omega$$

$$\sigma_y^2(x_r) = \left[\frac{\phi_y^2(x_r)}{\omega_y^2 \tilde{M}_y}\right]^2 \int_0^\infty \left[\begin{array}{l} \left[\left(1-\left(\frac{\omega}{\omega_y}\right)^2\right)^2 + \left(2\frac{\omega}{\omega_y}\left(\zeta_y - \frac{\rho B^4}{4\tilde{M}_y} P_1^* \int_{L_{\exp}} \phi_y^2 dx\right)\right)^2\right]^{-1} \\ \times \iint_{L_{\exp}} \rho U B^2 C_D S_u(\omega) \left[\phi_y(x_1)\phi_y(x_2) \cdot e^{-\frac{C \cdot \omega \cdot |x_1 - x_2|}{U}}\right] dx_1 dx_2 \end{array}\right] d\omega$$

Background and resonant part

- In structural engineering, the response spectrum has been customary to split the response calculations into a background and a resonant part as illustrated as follows.



- The variance of the displacement response split into a background and a resonant part is given by

$$\sigma_y^2(x) = \frac{\phi_y^2(x)}{\tilde{K}_y^2} \int_0^\infty \left|H_y(\omega)\right|^2 \cdot S_{Q_y}(\omega)\, d\omega$$

$$\simeq \frac{\phi_y^2(x)}{\tilde{K}_y^2} \left[\left|H_y(0)\right|^2 \int_0^\infty S_{Q_y}(\omega)\, d\omega + S_{Q_y}(\omega_y) \int_0^\infty \left|H_y(\omega)\right|^2 d\omega \right]$$

Background and resonant part

- It is in the following taken for granted that

$$|H(0)| = 1$$

$$\int_0^\infty S_{Q_y}(\omega)\, d\omega = \sigma_{Q_y}^2$$

$$\int_0^\infty \left|H_y(\omega)\right|^2 d\omega = \frac{\pi\omega_y}{4\zeta_{tot}}$$

- **where , $\zeta_{tot} = \zeta_y - \zeta_{ae}$, following is obtained:**

$$\sigma_y^2(x) = \sigma_{B_y}^2 + \sigma_{R_y}^2 = \frac{\phi_y^2(x)}{\tilde{K}_y^2} \cdot \left[\sigma_{Q_y}^2 + \frac{\pi\omega_y S_{Q_y}(\omega_y)}{4\zeta_{tot}}\right]$$

Single-Mode Flutter and Buffeting
(Theory)

There will be no single-mode flutter unless one of the principal flutter derivatives (such as A_2^*) takes on positive values for some range of reduced velocity $2\pi/K = U/nB$, where $K = B\omega/U$. [Notation will be listed at the end of these notes.]

Equation of motion (with limited choice of flutter derivatives)

$$I_i\left[\ddot{\xi}_i + 2\zeta_i\omega_i\dot{\xi}_i + \omega_i^2\xi_i\right] = Q_i \tag{1}$$

$$\begin{aligned} Q_i &= \frac{1}{2}\rho U^2 B^2 \ell \left\{ \frac{KB}{U}\left[H_1^* G_{h_i h_i} + P_1^* G_{p_i p_i} + A_2^* G_{\alpha_i \alpha_i}\right]\dot{\xi}_i \right. \\ &\quad \left. + K^2 A_3^* G_{\alpha_i \alpha_i}\xi_i + \int_{deck}\left[\mathcal{L}h_i + \mathcal{D}p_i + \mathcal{M}\alpha_i\right]\frac{dx}{\ell}\right\} \end{aligned} \tag{2}$$

Note that this form implies full coherence of flutter derivative action along the span.

$$G_{q_i q_i} = \int_{deck} q_i^2(x)\frac{dx}{\ell} \quad [q_i = h_i,\ p_i \text{ or } \alpha_i] \tag{3}$$

Flutter

Dropping $\int_{deck}$ and treating the homogeneous part of (1) and (2), we proceed as follows. If ξ_i is sinusoidal, i.e. $\xi_i = \xi_{i\omega}\, e^{i\omega t}$, (1) and (2) yield:

$$I_i[\omega_i^2 - \omega^2] = \frac{1}{2}\rho U^2 B^2 \ell K^2 A_3^* G_{\alpha_i\alpha_i} \tag{4}$$

$$2I_i\zeta_i\omega_i\omega = \frac{1}{2}\rho U^2 B^2 \ell \left\{ \frac{KB\omega}{U} [H_1^* G_{h_ih_i} + P_1^* G_{p_ip_i} + A_2^* G_{\alpha_i\alpha_i}] \right\} \tag{5}$$

These lead to the relation

$$\frac{\omega_i}{\omega} = \left[1 + \frac{\rho B^4 \ell A_3^*}{2I_1} G_{\alpha_1\alpha_1}\right]^{\frac{1}{2}} \tag{6}$$

and the flutter (zero or negative damping) condition:

$$H_1^* G_{h_ih_i} + P_1^* G_{p_ip_i} + A_2^* G_{\alpha_i\alpha_i} \geq \frac{4\zeta_i I_i}{\rho B^4 \ell}\left[1 + \frac{\rho B^4 \ell}{2I_i} A_3^* G_{\alpha_i\alpha_i}\right]^{\frac{1}{2}} \tag{7}$$

Note that only the principal flutter derivatives H_1^*, P_1^*, A_2^* (damping derivatives) are retained here. The influence of A_3^* is usually negligible. The flutter derivative A_2^* is usually the principal actor in s.d.o.f. flutter, as it affects torsional damping. Sway (P_1^*) and vertical bending (H_1^*) are often negative, tending to increase overall system damping.

Buffeting

Rewrite (1) and (2) in the form

$$\ddot{\xi}_i + 2\gamma_i\omega_{i0}\dot{\xi}_i + \omega_{i0}^2\xi_i = \frac{\rho U^2 B^2 \ell}{2I_i}\int_{deck}\left[\mathcal{L}h_i + \mathcal{D}p_i + \mathcal{M}\alpha_i\right]\frac{dx}{\ell} \tag{8}$$

where

$$\omega_{i0}^2 = \omega_i^2 - \frac{\rho B^4 \ell}{2I_i}\omega^2 A_3^* G_{\alpha_i\alpha_i} \tag{9}$$

and

$$2\gamma_i\omega_{i0} = 2\zeta_i\omega_i - \frac{\rho B^4 \ell}{2I_i}\omega\left[H_1^* G_{h_ih_i} + P_1^* G_{p_ip_i} + A_2^* G_{\alpha_i\alpha_i}\right] \tag{10}$$

We assume that the oscillator (8) responds in random amplitudes around the frequency ω_{i0}. Setting $\omega = \omega_{i0}$ in (9) yields

$$\omega_{i0} = \frac{\omega_i}{\left[1 + \frac{\rho B^4 \ell}{2I_i} A_3^* G_{\alpha_i\alpha_i}\right]^{\frac{1}{2}}} \tag{11}$$

and therefore, from (10):

$$\gamma_i = \zeta_i\frac{\omega_i}{\omega_{i0}} - \frac{\rho B^4 \ell}{4I_i}\left[H_1^*(\overset{K}{\cancel{k}}_{i0})G_{h_ih_i} + P_1^*(K_{i0})G_{p_ip_i} + A_2^*(K_{i0})G_{\alpha_i\alpha_i}\right] \tag{12}$$

where $K_{i0} = B\omega_{i0}/U$.

The Fourier transform of ξ_i for quiescent (or distant) initial conditions is

$$\bar{\xi} = \int_0^\infty \xi(t) e^{-i\omega t} dt \tag{13}$$

so that the F.T. of (8) is

$$\left[\omega_{i0}^2 - \omega^2 + 2i\gamma_i\omega_{i0}\omega\right] \bar{\xi}_i = \frac{\rho U^2 B^2 \ell}{2I_i} \int_{deck} \left[\overline{\mathcal{L}} h_i + \overline{\mathcal{D}} p_i + \overline{\mathcal{M}} \alpha_i\right] \frac{dx}{\ell} \tag{14}$$

Multiplying (14) by its complex conjugate yields

$$\left[(\omega_{i0}^2 - \omega^2)^2 + (2\gamma_i\omega_{i0}\omega)^2\right] \bar{\xi}_i \bar{\xi}_i^* = \left[\frac{\rho U^2 B^2 \ell}{2I_i}\right]^2 \iint_{deck} \Pi(x_A, x_B, \omega) \frac{dx_A}{\ell} \frac{dx_B}{\ell} \tag{15}$$

where $\overline{(\ )}^* =$ complex conjugate of $\overline{(\ )}$ and

$$\begin{aligned} \Pi(x_A, x_B, \omega) &= \left[\overline{\mathcal{L}}(x_A) h_i(x_A) + \overline{\mathcal{D}}(x_A) p_i(x_A) + \overline{\mathcal{M}}(x_A) \alpha_i(x_A)\right] \\ &\times \left[\overline{\mathcal{L}}^*(x_B) h_i(x_B) + \overline{\mathcal{D}}^*(x_B) p_i(x_B) + \overline{\mathcal{M}}^*(x_B) \alpha_i(x_B)\right] \end{aligned} \tag{16}$$

The lift ($\mathcal{L}$), drag ($\mathcal{D}$) and moment ($\mathcal{M}$) factors above depend on the horizontal and vertical (u, w) components of gusting:

$$\mathcal{L} = 2C_\ell \frac{u}{U} + (C_\ell' + C_D)\frac{w}{U} \tag{17}$$

$$\mathcal{D} = 2C_D \frac{u}{U} \tag{18}$$

$$\mathcal{M} = 2C_M \frac{u}{U} + C_M' \frac{w}{U} \tag{19}$$

Hence

$$\overline{\mathcal{L}}h_i + \overline{\mathcal{D}}p_i + \overline{\mathcal{M}}\alpha_i = \varphi(x)\frac{\bar{u}}{U} + \psi(x)\frac{\bar{w}}{U} \tag{20}$$

where

$$\varphi(x) = 2\left[C_\ell h_i(x) + C_D p_i(x) + C_M \alpha_i(x)\right] \tag{21}$$

$$\psi(x) = (C_\ell' + C_D)h_i(x) + C_M' \alpha_i(x) \tag{22}$$

Hence

$$\begin{aligned} \Pi(x_A,\ x_B,\ \omega) &= \left[\varphi(x_A)\bar{u}(x_A,\ \omega) + \psi(x_A)\bar{w}(x_A,\ \omega)\right] \\ &\times \left[\varphi(x_B)\bar{u}^*(x_B,\ \omega) + \psi(x_B)\bar{w}^*(x_B,\ \omega)\right]\frac{1}{U^2} \end{aligned} \tag{23}$$

In the limit, as $T \to \infty$:

$$\lim \frac{2}{T}\bar{\xi}_i\bar{\xi}_i^* = S_{\xi_i\xi_i}(\omega) \tag{24}$$

the auto-power spectral density of ξ_i, and analogously for u, w. Therefore (15) becomes

$$\omega_{i0}^4 \left[\left(1-\left(\frac{\omega}{\omega_{i0}}\right)^2\right)^2 + \left(2\gamma_i\frac{\omega}{\omega_{i0}}\right)^2\right] S_{\xi_i\xi_i}(\omega)$$

$$= \left[\frac{\rho U^2 B^2 \ell}{2I_i}\right]^2 \iint_{deck} \frac{1}{U^2} \{\varphi(x_A)\varphi(x_B)S_u(x_A,\ x_B,\ \omega)$$

$$+ \quad \psi(x_A)\psi(x_B)S_w(x_A,\ x_B,\ \omega)] \frac{dx_A}{\ell}\frac{dx_B}{\ell} \tag{25}$$

[neglecting S_{uw}, S_{wu} cross spectra].

We make the following assumptions concerning the lateral coherence of turbulence [neglecting the imaginary part]:

$$S_u(x_A,\ x_B\ \omega) \cong S_u(\omega)e^{-c|x_A-x_B|/\ell} \tag{26}$$

$$S_w(x_A,\ x_B\ \omega) \cong S_w(\omega)e^{-c|x_A-x_B|/\ell} \tag{27}$$

where

$$\frac{5n\ell}{U} \leq C \leq \frac{20n\ell}{U} \qquad \left[n = \frac{\omega_{i0}}{2\pi}\right] \tag{28}$$

We thus encounter integrals of the type

$$R_\varphi = \iint_{deck} \varphi(x_A)\varphi(x_B)e^{-C|x_A - x_B|/\ell}\,\frac{dx_A}{\ell}\frac{dx_B}{\ell} \tag{29}$$

$$R_\psi = \iint_{deck} \psi(x_A)\psi(x_B)e^{-C|x_A - x_B|/\ell}\,\frac{dx_A}{\ell}\frac{dx_B}{\ell} \tag{30}$$

to be evaluated from modal and force coefficient data.

Thus

$$S_{\xi_i\xi_i}(\omega) = \frac{\left[\frac{\rho B^4 \ell}{2 I_i K_{i0}^2}\right]^2}{\left[1 - \left(\frac{\omega}{\omega_{i0}}\right)^2\right]^2 + \left[2\gamma_i \frac{\omega}{\omega_{i0}}\right]^2}\{R_\varphi S_u + R_\psi S_w\}\frac{1}{U^2} \tag{31}$$

We now recall that, for the single mode i:

$$\begin{aligned} h(x,\ t) &= h_i(x)B\xi_i(t) \\ p(x,\ t) &= p_i(x)B\xi_i(t) \\ \alpha(x,\ t) &= \alpha_i(x)\xi_i(t) \end{aligned} \tag{32}$$

so that, for example

$$S_{hh}(x,\ \omega) = h_i^2(x)B^2 S_{\xi_i\xi_i}(\omega) \tag{33}$$

The variance σ_n^2 of h is

$$\sigma_h^2(x) = \int_0^\infty S_{hh}(x,\ n)dn \qquad \left[n = \frac{\omega}{2\pi}\right] \tag{34}$$

Now, for any P.S.D. $S(n)$:

$$\int_0^\infty \frac{S(n)dn}{\left[1-\left(\frac{n}{n_0}\right)^2\right]^2+\left[2\gamma\frac{n}{n_0}\right]^2} \cong \int_0^\infty S(n)dn + \frac{\pi n_0 S(n_0)}{4\gamma} \tag{35}$$

which will apply to the wind spectra S_u, S_w. Kaimal-Simiu [Simiu & Scanlan *Wind Effects* 1986, Ch.2] offer the following:

$$S_u(n) = \frac{200zu_*^2}{U\left(1+\frac{50nz}{U}\right)^{5/3}} \tag{36}$$

$$S_w(n) = \frac{3.36zu_*^2}{U\left[1+10\left(\frac{nz}{U}\right)^{5/3}\right]} \tag{37}$$

$$U = 2.5\ell n\frac{z}{z_0} \tag{38}$$

so that

$$\int_0^\infty S_u(n)dn = 6u_*^2 \tag{39}$$

$$\int_0^\infty S_w(n)dn = 1.7u_*^2 \tag{40}$$

Therefore the variance of ξ_i may be calculated from

$$\sigma_{\xi_i}^2 = \left[\frac{\rho B^4 \ell}{2I_i K_{i0}^2}\right]^2 \left\{ R_\varphi \left[\frac{\pi n_{i0} S_u(n_{i0})}{4\gamma_i} + 6u_*^2\right] + R_\psi \left[\frac{\pi n_{i0} S_w(n_{i0})}{4\gamma_i} + 1.7u_*^2\right]\right\} \frac{1}{U^2} \tag{41}$$

and the standard deviation of each component, from

$$\begin{aligned} \sigma_h(x) &= h_i(x) B \sigma_\xi \\ \sigma_p(x) &= p_i(x) B \sigma_\xi \\ \sigma_\alpha(x) &= \alpha_i(x) \sigma_\xi \end{aligned} \tag{42}$$

The max excursion may be taken as 3σ to 4σ and the max peak-to-peak, twice that. This outlines the buffeting analysis. In the above, the aerodynamic admittance is conservatively kept at unit value. [This writer believes that use of the Sears airfoil admittance function in this context is improper.] Note that in this writing wind cross spectra are conservatively neglected (they are negative, according to Kaimal).

Summary up buffeting analysis procedure

- Start with motion of equation.

- Define wind-induced forces i.e. buffeting forces and unsteady self-excited forces (aerodynamic stiffness and damping).

- Apply Fourier transform.

- Develop the response spectrum from wind turbulence spectrum with Joint acceptance function and frequency response function.

- Derive the variance of response from response spectrum by integration.

Natural modal freq. vs. Actual vibration freq.

- **Remind the self-excited force**
  - Self-excited force is motion-induced force.
  - Functions of displacement, velocity and acceleration of deck.
  - We define aerodynamic stiffness and damping with flutter derivatives.

- **Natural modal freq. is defined by mass and stiffness**

$$\omega_i = \sqrt{\frac{K_i}{M_i}}$$

- **Actual vibration freq. ($\omega_i(U)$) under wind condition (in terms of the resonance frequency in the textbook) is a function of the mean wind velocity, U. At $U = 0$, $\omega_i(U)$ is the natural modal freq. At $U \neq 0$, K_{ae} have the effect of changing the total stiffness then $\omega_i(U)$ is no more same with the natural modal freq.**

Variable ω

Go back to Part.IV: Wind loads

- Buffeting forces and unsteady self-excited forces are function of freq.

Coherence function (Textbook, p.67)

- To consider spatial properties, the single point spectrum, $S_u(\omega)$ and coherence function, $\mathrm{Coh}(\Delta x, \omega)$ are adopted.
- Not only spectrum but also coherence function have variable ω.

Re-call the buffeting analysis formulation (PPT, p10)

- To get the variance of response, all these terms are integrated together w.r.t variable, ω
- Triple integration !

$$\sigma_y^2(x_r) = \left[\frac{\phi_y^2(x_r)}{\omega_y^2 \tilde{M}_y}\right]^2 \underline{\int_0^\infty} \left[\begin{array}{l} \left[\left(1-\left(\frac{\boxed{\omega}}{\omega_y}\right)^2\right)^2 + \left(2\frac{\boxed{\omega}}{\omega_y}\left(\zeta_y - \frac{\rho B^4}{4\tilde{M}_y}\boxed{P_1^*}\int_{L_{\exp}} \phi_y^2 dx\right)\right)^2\right]^{-1} \\ \times \rho U B^2 C_D S_u(\omega) \underline{\iint_{L_{\exp}}} \left[\phi_y(x_1)\phi_y(x_2)\cdot e^{-\frac{C\cdot\boxed{\omega}\cdot|x_1-x_2|}{U}}\right] dx_1 dx_2 \end{array} \right] d\omega$$

- **Go back to the assumption in p.15**
  - We assume that the oscillator (8) responds in random amplitudes around the frequency ω_{i0}.
  - ω_{i0} is the result of 1st iteration $\omega_{\boldsymbol{i}}$ and $\omega_{\boldsymbol{i}}(\boldsymbol{U})$
  - Then, we adopt $\omega = \omega_{i0}$ as a constant, also we can use $P_1^*(\omega_{i0})$ as a constant value.

- **Assume** $\mathrm{Coh}(\Delta x, \omega)$ **to** $\mathrm{Coh}(\Delta x)$ **(PPT, p18)**
  - Coherence is a function of gap distance, Δx and turbulence freq., ω.
  - Assume that $\omega = \omega_{i0}$ as a constant.

$$\sigma_y^2(x_r) = \left[\frac{\phi_y^2(x_r)}{\omega_y^2 \tilde{M}_y}\right]^2 \left[\left(1-\left(\frac{\boxed{\omega_{i0}}}{\omega_y}\right)^2\right)^2 + \left(2\frac{\boxed{\omega_{i0}}}{\omega_y}\left(\zeta_y - \frac{\rho B^4}{4\tilde{M}_y}\boxed{P_1^*(\omega_{i0})}\int_{L_{\exp}}\phi_y^2 dx\right)\right)^2\right]^{-1}$$

$$\times \iint_{L_{\exp}}\left[\phi_y(x_1)\phi_y(x_2)\cdot e^{-\frac{C\cdot\boxed{\omega_{i0}}\cdot|x_1-x_2|}{U}}\right]dx_1 dx_2 \int_0^{\infty}\left[\rho U B^2 C_D S_u(\omega)\right]d\omega$$

Simplified method 2 - Textbook

- **Getting closed form solution of joint acceptance function – Analytically solve the double integration w.r.t. spatial coordinates.**

$$J_y^2(\omega) = \iint_{L_{exp}} \phi_y(x_1)\phi_y(x_2)e^{-\frac{C\omega|x_1-x_2|}{U}}dx_1dx_2$$

  - Assume the mode shape vectors as well-known function, i.e. $\sin(n\pi x)$, x^2 and etc.
  - Then, eliminate the double integration by closed form solution of joint acceptance function.
  - Example $\phi_y(x) = sin\frac{\pi x}{L}$

$$J_y^2(\omega) = \iint_0^1 sin\frac{\pi x_1}{L} sin\frac{\pi x_2}{L} e^{-\frac{C\omega|x_1-x_2|}{U}}dx_1dx_2$$

$$= \frac{\widehat{\omega}}{\widehat{\omega}^2+\pi^2} + 2\pi^2\frac{1+e^{-\widehat{\omega}}}{(\widehat{\omega}^2+\pi^2)^2}$$

  - Where, $\widehat{\omega} = \frac{C\omega}{U}$

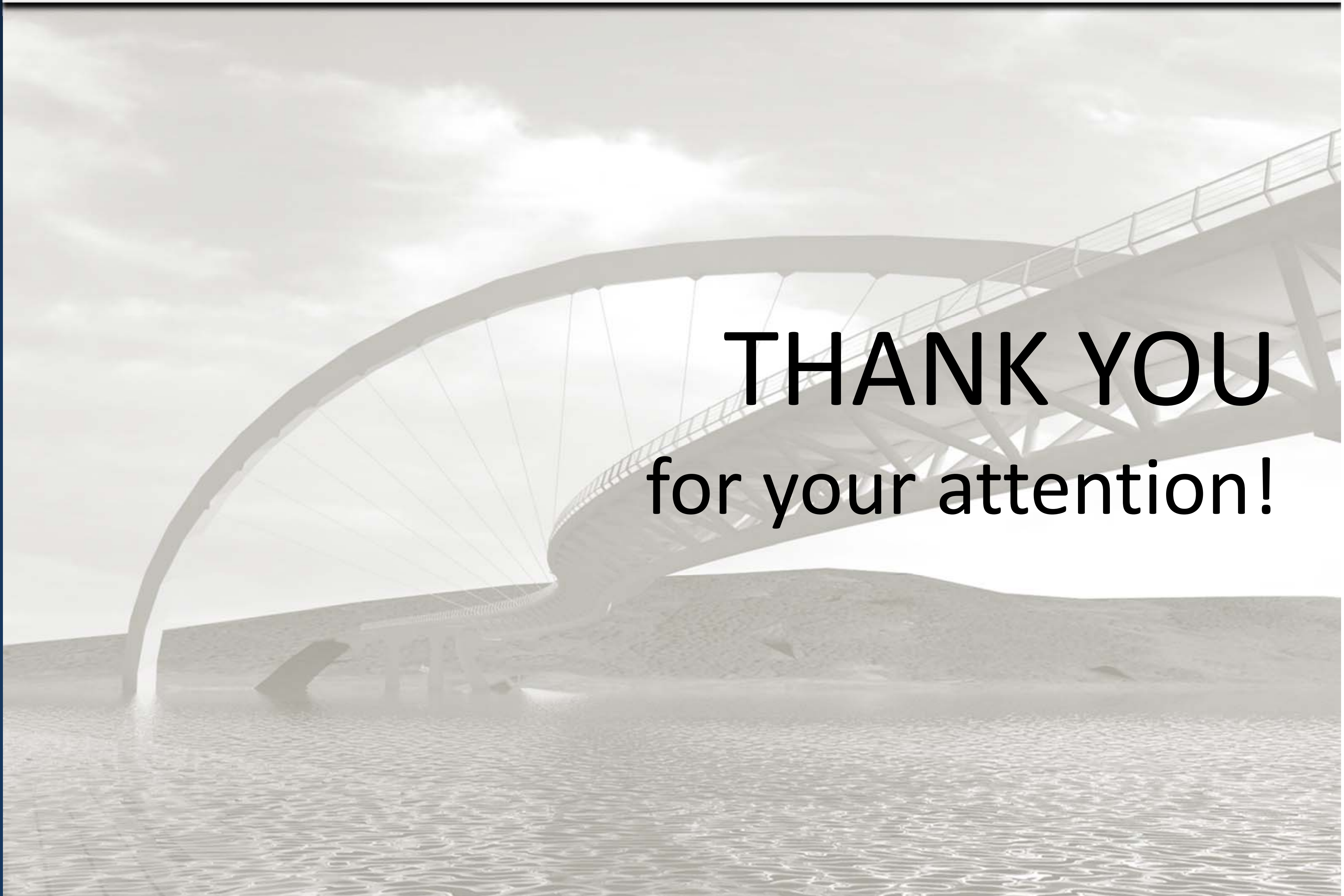
Simplified method 2 - Textbook

- Then, the only single integration remains

$$\sigma_y^2(x_r) = \left[\frac{\phi_y^2(x_r)}{\omega_y^2 \tilde{M}_y}\right]^2 \int_0^\infty \left[\begin{array}{l} \left[\left(1-\left(\dfrac{\omega}{\omega_y}\right)^2\right)^2 + \left(2\dfrac{\omega}{\omega_y}\left(\zeta_y - \dfrac{\rho B^4}{4\tilde{M}_y} P_1^* \displaystyle\int_{L_{\exp}} \phi_y^2 dx\right)\right)^2\right]^{-1} \\ \times \rho U B^2 C_D S_u(\omega) J_y^2(\omega) \end{array}\right] d\omega$$

Homework

- Follow up Example 6.2 (Textbook pp.122-127). It is a buffeting analysis for a typical single mode single component situation. Basically you are needed to reproduce all figures in the example by yourself.



THANK YOU
for your attention!

Equation of Motion

- Recall the equation of motion for ith mode is

$$I_i\left(\ddot{\xi}_i+2\zeta_i\omega_i\dot{\xi}_i+\omega_i^2\xi_i\right)=Q_i$$

- By definition, equation for 1st lateral mode is

$$I_1\left(\ddot{\xi}_1+2\zeta_1\omega_1\dot{\xi}_1+\omega_1^2\xi_1\right)=Q_1$$

$$I_1\left(\ddot{\xi}_1+2\zeta_1\omega_1\dot{\xi}_1+\omega_1^2\xi_1\right)=\frac{1}{2}\rho U^2B^2L\left[\frac{KB}{U}P_1^*\int_{L_{\exp}}p_y^2(x)\frac{dx}{L}\cdot\dot{\xi}_1+\int_{L_{\exp}}2C_D\frac{u}{U}\cdot p_y\frac{dx}{L}\right]$$

$$\rightarrow I_1\left(\ddot{\xi}_1+\left(2\zeta_1\omega_1-\frac{\rho B^4L}{2}\zeta_{ae}\right)\dot{\xi}_1+\omega_1^2\xi_1\right)=\frac{\rho U^2B^2L}{2}\int_{L_{\exp}}2C_D\frac{u}{U}\cdot p_y\frac{dx}{L}$$

Transform to Frequency Domain

Fourier transform

$$\left[\omega_1^2-\omega^2+i\omega\left(2\omega_1\zeta_1-\frac{\rho B^4 L}{2}\zeta_{ae}\left(\omega_1\right)\right)\right]a_{\xi_1}\left(\omega\right)=\frac{\rho U^2 B^2 L}{2I_1}\int_{L_{\exp}}\frac{2C_D}{U}a_u p_1\frac{dx}{L}$$

$$\rightarrow a_{\xi_1}\left(\omega\right)=\frac{\rho U B^2 L C_D}{\omega_1^2 I_1}\frac{1}{\left[1-\left(\frac{\omega}{\omega_1}\right)^2+2i\left(\zeta_1-\frac{\rho B^4 L}{4\omega_1}\zeta_{ae}\left(\omega_1\right)\right)\cdot\frac{\omega}{\omega_1}\right]}\int_{L_{\exp}}a_u p_1\frac{dx}{L}$$