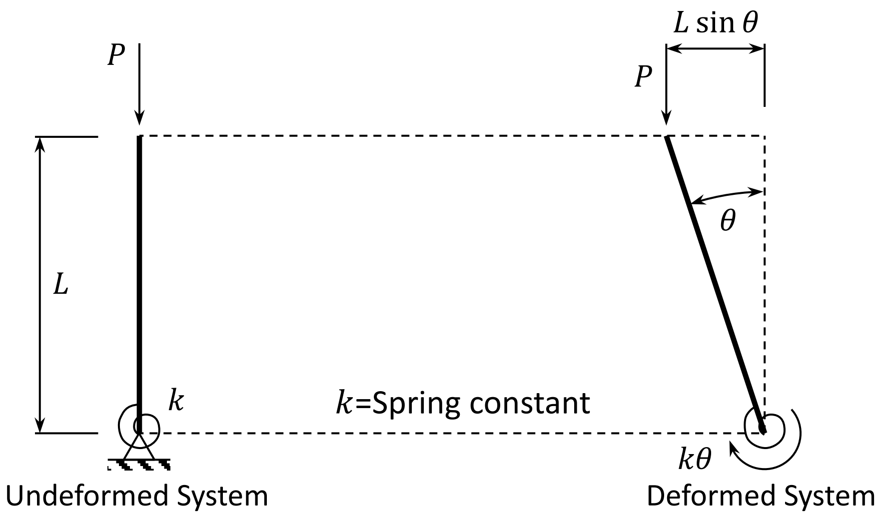
457.649 Advanced Structural Analysis

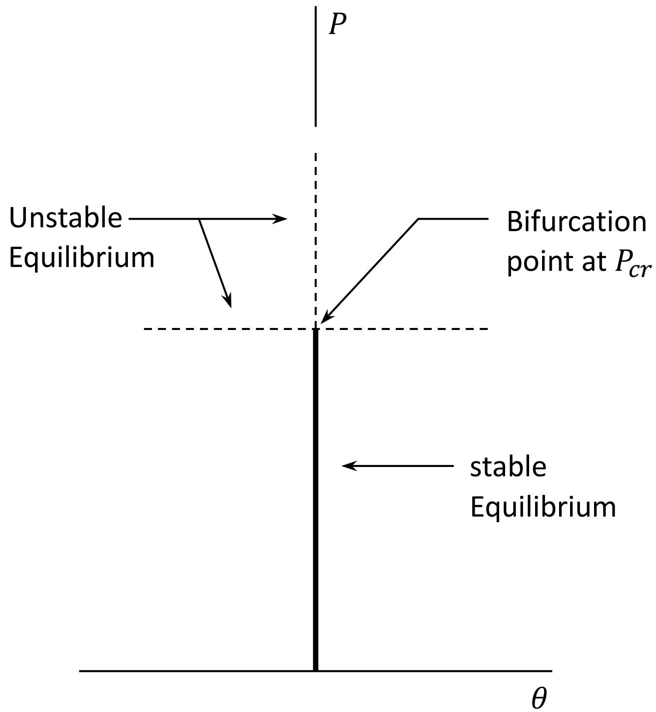
# Part VIII:

# 1. Fundamentals of Stability Theory

Structural Design Lab.(Prof. Ho-Kyung Kim)
Dept. of Civil & Environmental Eng.
Seoul National University

# Critical Load for a Simple Spring-Bar System–Equilibrium Approach




- **Equilibrium at the slightly disturbed with a rotation $\theta$**

$$\sum M_A = 0 = PL\sin\theta - k\theta$$

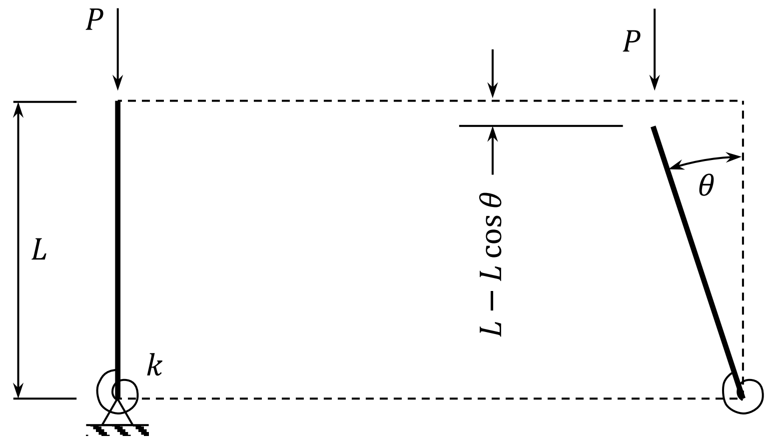
$$\Rightarrow P_{cr} = \frac{k\theta}{L\sin\theta}$$

- **Small displacement theory**

$$\sin\theta = \theta$$
$$\tan\theta = \theta$$
$$\cos\theta = 1$$

$$\Rightarrow P_{cr} = \frac{k\theta}{L\theta} = \frac{k}{L}$$

# Critical Load for a Simple Spring-Bar System–Energy approach

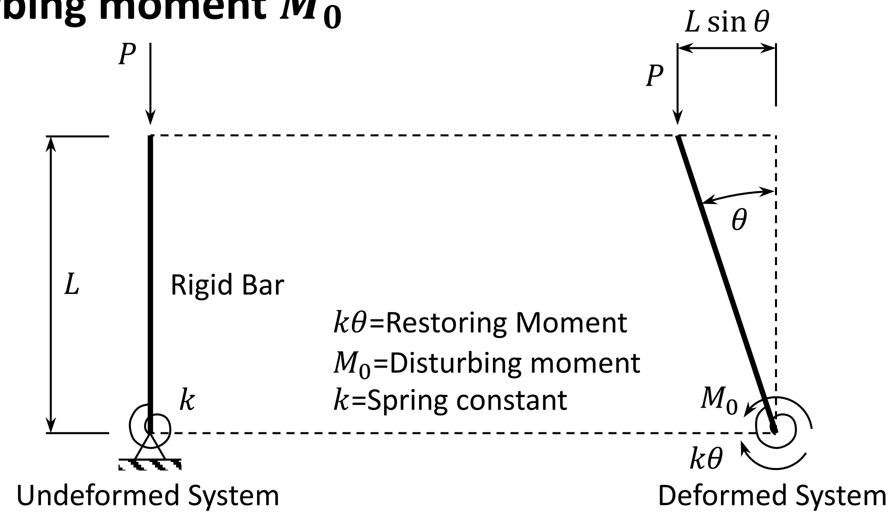



- **Total potential;** $\boldsymbol{\Pi = U + V_p}$
  - $U = W_i = \frac{1}{2}k\theta^2$
  - $V_p = -W_e = -PL(1-\cos\theta)$
  - $\Pi = U + V_p = \frac{1}{2}k\theta^2 - PL(1-\cos\theta)$
- **Stationary total potential for equilibrium condition;** $\boldsymbol{\frac{d\Pi}{d\theta} = 0}$
  - $\frac{d\Pi}{d\theta} = 0 = k\theta - PL\sin\theta$

$$\Rightarrow P_{cr} = \frac{k\theta}{L\sin\theta}$$

# Post-Buckling Behavior

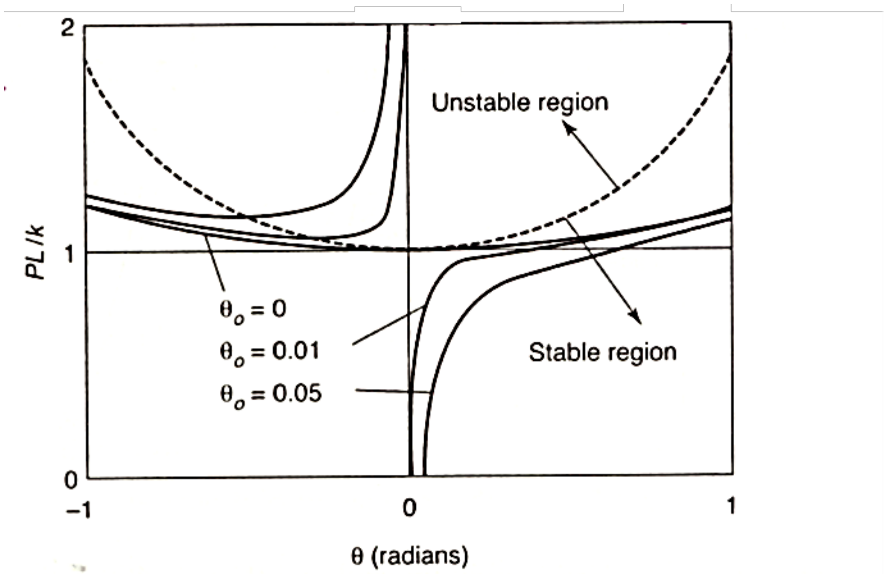
- **Introduce a disturbing moment $M_0$**




- **Equilibrium approach**

$$\sum M_A = 0 = PL\sin\theta + M_0 - k\theta$$

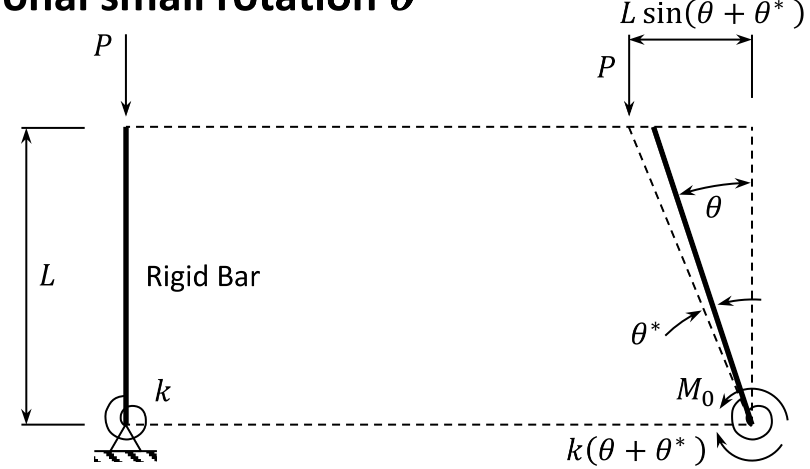
$$\theta_0 = \frac{M_0}{k} \quad \Rightarrow \quad \frac{PL}{k} = \frac{\theta - \theta_0}{\sin\theta}$$




Load-deflection relations for spring-bar system with disturbing moment.

# Stability Criteria by an Equilibrium Approach

- **Introduce an additional small rotation $\theta^*$**




- **Equilibrium equation**

$$\sum M_A = 0 = PL\sin(\theta+\theta^*) + M_0 - k(\theta+\theta^*)$$

$$\theta_0 = \frac{M_0}{k} \Rightarrow \frac{PL}{k} = \frac{\theta+\theta^*-\theta_0}{\sin(\theta+\theta^*)} = \frac{\theta+\theta^*-\theta_0}{\sin\theta + \theta^*\cos\theta}$$

$$\sin(\theta+\theta^*) = \sin\theta\cos\theta^* + \cos\theta\sin\theta^*$$
$$\cos\theta^* \approx 1$$
$$\sin\theta^* \approx \theta^*$$

$$\Rightarrow \frac{PL}{k}\sin\theta - \theta + \theta_0 + \theta^*\left(\frac{PL}{k}\cos\theta - 1\right) = 0$$

=0 due to equilibrium condition

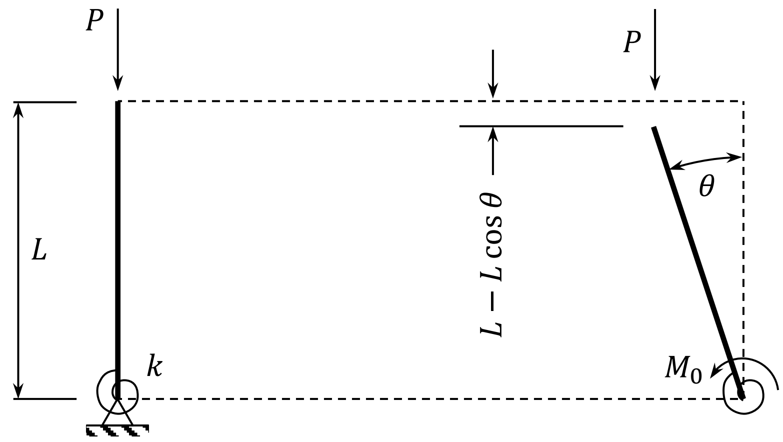
# Stability Criteria by an Equilibrium Approach

- **Locus of points for which $\theta^* \neq 0$ while equilibrium is just maintained, that is the equilibrium is *neutral*:**

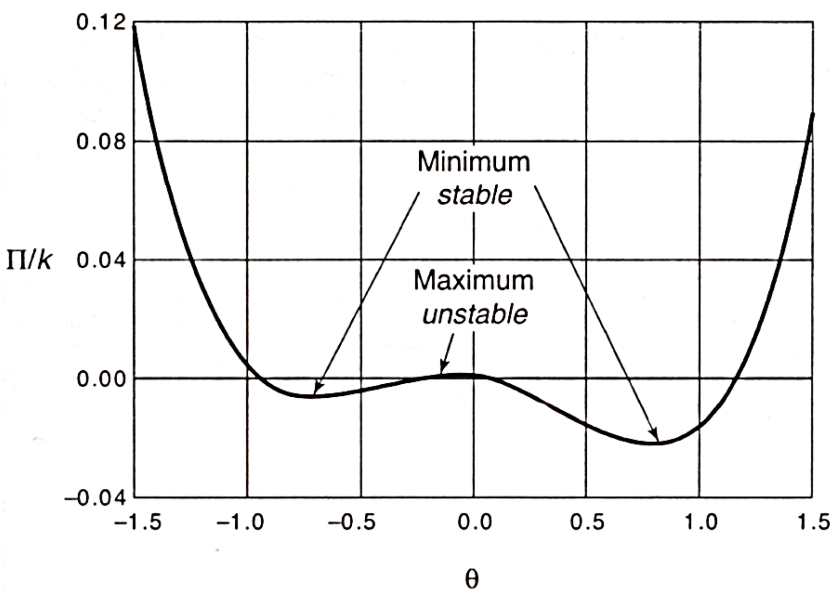
$$\frac{PL}{k}\cos\theta - 1 = 0$$

- $\cos\theta < \frac{1}{PL/k}$ the equilibrium is *stable* -that is, the bar returns to its original position when $\theta^*$ is removed; energy must be added.
- $\cos\theta = \frac{1}{PL/k}$ the equilibrium is *neutral* -that is, no force is required to move the bar a small rotation $\theta^*$.
- $\cos\theta > \frac{1}{PL/k}$ the equilibrium is *unstable* -that is, the configuration will snap from an unstable to a stable shape; energy is released.

# Stability Criteria by an Energy Approach



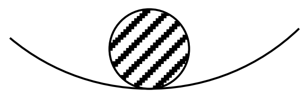
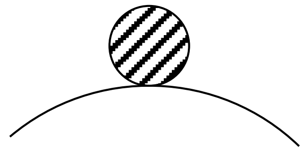
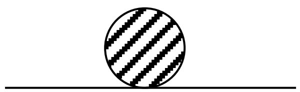

- $U = W_i = \frac{1}{2}k\theta^2$ $\quad\quad M_0 = k\theta_0$
- $V_p = W_e = -PL(1 - \cos\theta) - M_0\theta$
- $\frac{\Pi}{k} = \frac{\theta^2}{2} - \frac{PL}{k}(1 - \cos\theta) - \theta_0\theta$
- **For $\theta_0 = 0.01$ and $PL/k = 1.0$**



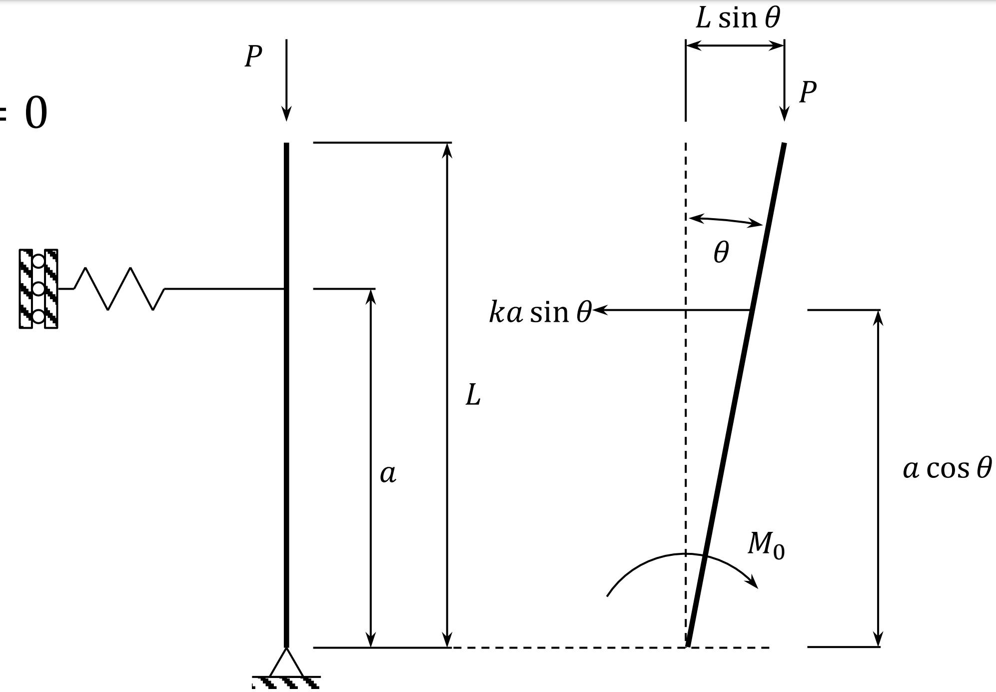

Total potential for $\theta_o = 0.01$ and $PL/k = 1.10$.

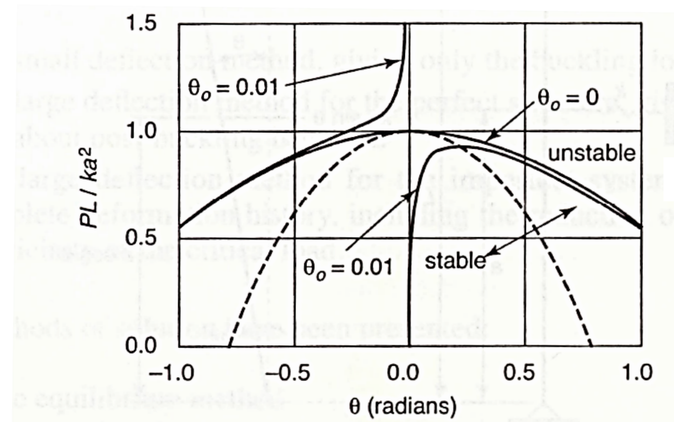
# Stability Criteria by an Energy Approach

- $\frac{\Pi}{k} = \frac{\theta^2}{2} - \frac{PL}{k}(1 - \cos\theta) - \theta_0\theta$
  - → *total potential*
- $\frac{d(\Pi/k)}{d\theta} = \theta - \theta_0 - \frac{PL}{k}\sin\theta = 0$
  - $\frac{PL}{k} = \frac{\theta - \theta_0}{\sin\theta}$
  - → *Equilibrium*
- $\frac{d^2(\Pi/k)}{d\theta^2} = 1 - \frac{PL}{k}\cos\theta = 0$
  - $\frac{PL}{k} = \frac{1}{\cos\theta}$
  - → *Stability*

| | | | |
|---|---|---|---|
| • Minimum of $\Pi$<br>• **Stable equalibrium**<br>• Energy must be added to change configuration. | $\frac{d^2\Pi}{d\theta^2} > 0$ |  | Ball in cup can be disturbed, but it will return to the center. |
| • Maximum of $\Pi$<br>• **Unstable equalibrium**<br>• Energy is released as configuration is changed | $\frac{d^2\Pi}{d\theta^2} < 0$ |  | Ball will roll down if disturbed. |
| • Transition from minimum to maximum<br>• **Neutral equalibrium**<br>• There is no change in energy | $\frac{d^2\Pi}{d\theta^2} = 0$ |  | Ball is free to roll. |

- **Equilibrium**
  - $(ka\sin\theta)a\cos\theta - M_0 - PL\sin\theta = 0$
  - $\theta_0 = M_0/(ka^2)$
  - $\frac{PL}{ka^2} = \frac{\sin\theta\cos\theta - \theta_0}{\sin\theta}$
- **Small deflection-ideal geometry**
  - $\theta_0 = 0;\ \sin\theta = \theta;\ \cos\theta = 1$
  - $P_{cr} = \frac{ka^2}{L}$
- **Large deflection-ideal geometry**
  - $\theta_0 = 0$
  - $P_{cr} = \frac{ka^2}{L}\cos\theta$
- **Load-rotation curves**
  - Perfect case of $\theta_0 = 0$
  - Imperfect case of $\theta_0 = 0.01$
  - ***Softening*** - The load is decreased as rotation increases.
  - Theoretical buckling load is upper bound.




Load-rotation curves for a softening system.

# Stability Criteria for Softening Case

- **Equilibrium equation for an additional small rotation $\theta^*$**

$$[ka\sin(\theta+\theta^*)]a\cos(\theta+\theta^*) - M_0 - PL\sin(\theta+\theta^*) = 0$$

$$\sin\theta^* \approx \theta^* \qquad \cos\theta^* = 1$$

$$\sin(\theta+\theta^*) = \sin\theta\cos\theta^* + \cos\theta\sin\theta^* = \sin\theta + \theta^*\cos\theta$$

$$\cos(\theta+\theta^*) = \cos\theta\cos\theta^* - \sin\theta\sin\theta^* = \cos\theta - \theta^*\sin\theta$$

$[ka\sin\theta\cos\theta - M_0 - PL\sin\theta]$ =0 due to equilibrium condition

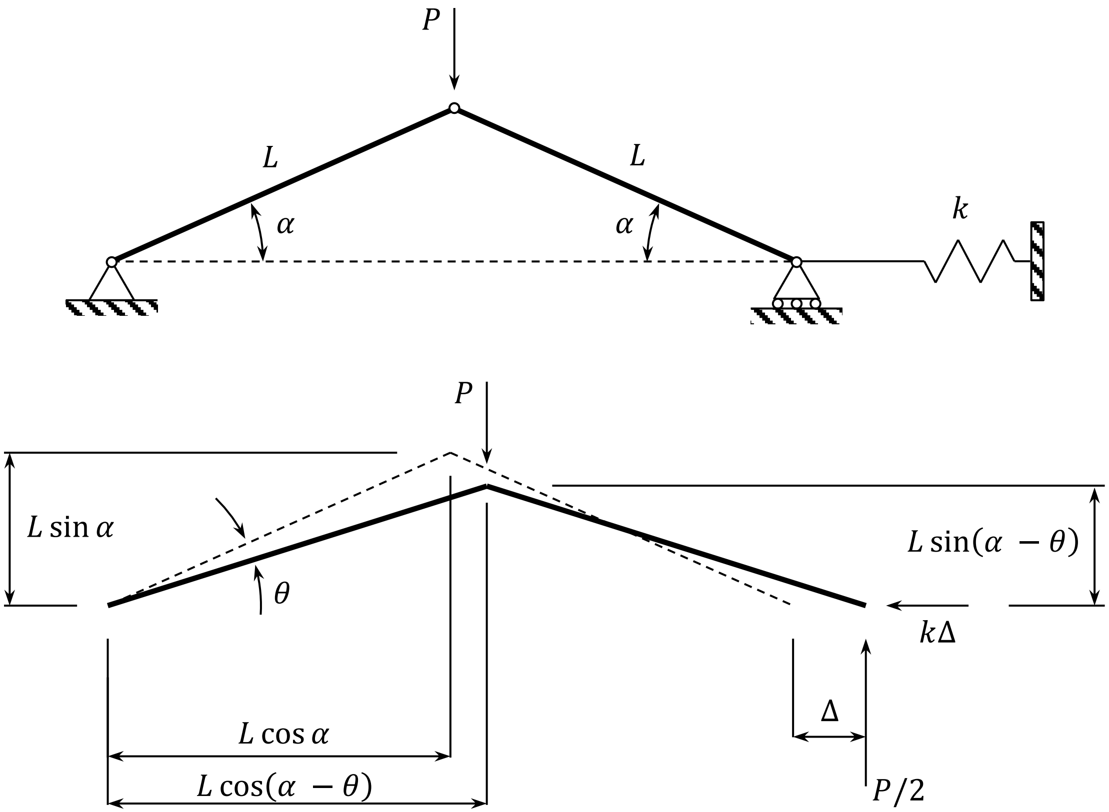
$$+\theta^*[ka^2(cos^2\theta - sin^2\theta) - PL\cos\theta] - {\theta^*}^2[ka^2\cos\theta\sin\theta] = 0$$

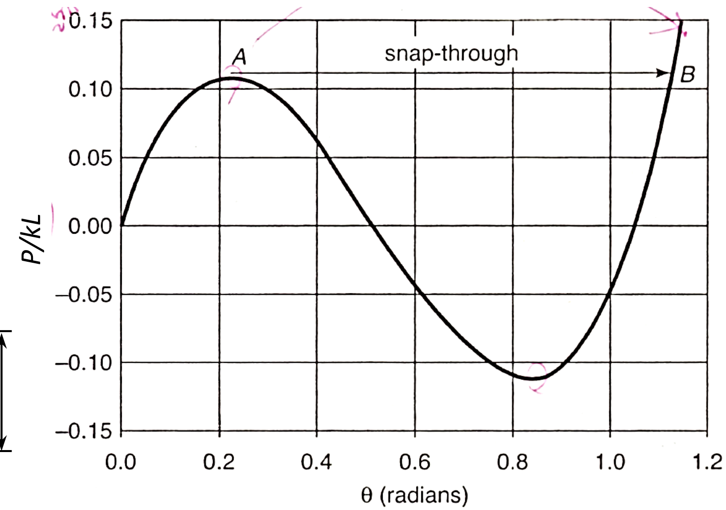
$$\Rightarrow \frac{PL}{ka^2} = \frac{cos^2\theta - sin^2\theta}{\cos\theta} = \frac{2cos^2\theta - 1}{\cos\theta}$$

- **In energy approach**

  - Total potential: $\Pi = \frac{k(a\sin\theta)^2}{2} - M_0\theta - PL(1-\cos\theta)$
  - Equilibrium: $\frac{\partial\Pi}{\partial\theta} = ka^2\sin\theta\cos\theta - M_0 - PL\sin\theta = 0$ -> $\frac{PL}{ka^2} = \frac{\sin\theta\cos\theta - \theta_0}{\sin\theta}$
  - Stability: $\frac{\partial^2\Pi}{\partial\theta^2} = ka^2[cos^2\theta - sin^2\theta] - PL\cos\theta = 0$ -> $\frac{PL}{ka^2} = \frac{2cos^2\theta - 1}{\cos\theta}$

# Snap-Through Buckling



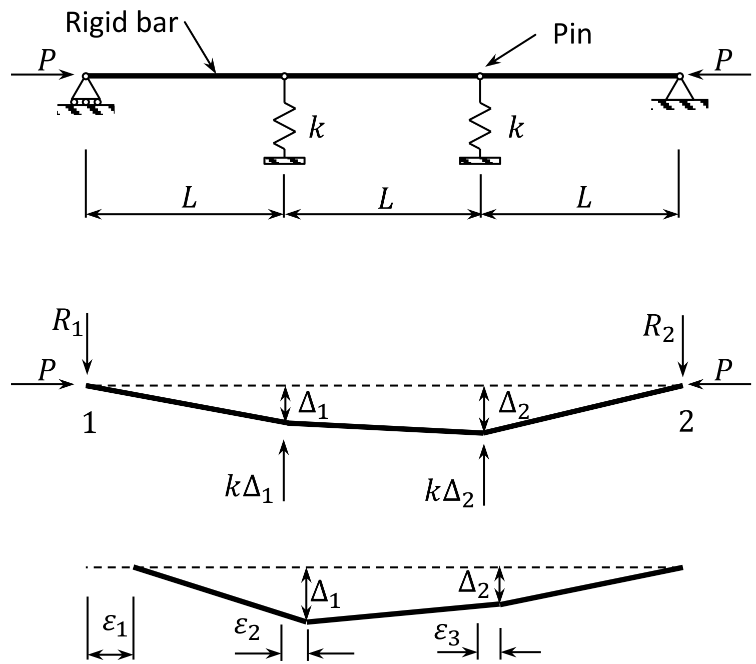

Load-rotation curve for snap-through structure for α = 30°.

Total potential : $\Pi = \frac{1}{2}k\{2L[\cos(\alpha-\theta) - \cos\alpha]\}^2 - PL[\sin\alpha - \sin(\alpha-\theta)]$

Equilibrium : $\frac{\partial \Pi}{\partial \theta} = 0 \rightarrow \frac{P}{kL} = 4[\sin(\alpha-\theta) - \tan(\alpha-\theta)\cos\alpha]$

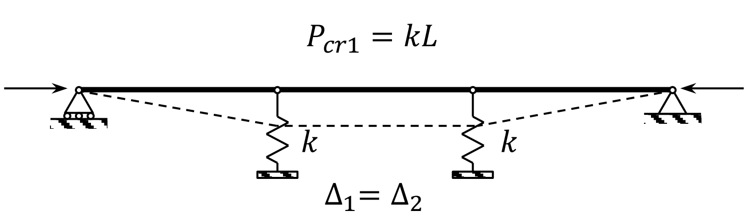
Stability : $\frac{\partial^2 \Pi}{\partial \theta^2} = 0 \rightarrow \frac{P}{kL} = 4\left[\frac{1-2cos^2(\alpha-\theta)+\cos(\alpha-\theta)cos\alpha}{sin(\alpha-\theta)}\right]$

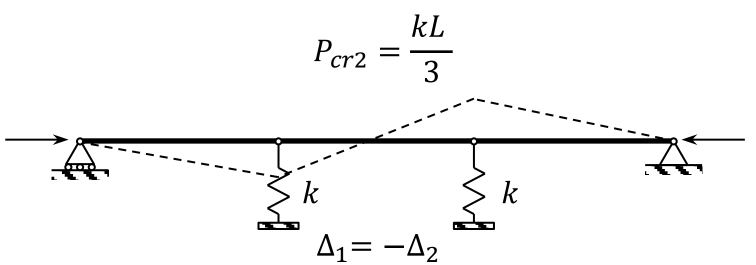
# Multi-Degree-of-Freedom System




- $\Delta_1 = \psi L$ and $\Delta_2 = \theta L$
- $\frac{\Delta_1 - \Delta_2}{L} = \gamma = \psi - \theta$
- $\varepsilon_3 = L - L\cos\theta \approx \frac{L\theta^2}{2}$
- $\varepsilon_2 = \varepsilon_3 + L[1 - \cos(\psi - \theta)] = \frac{L}{2}(2\theta^2 + \psi^2 - 2\psi\theta)$
- $\varepsilon_1 = \varepsilon_2 + \frac{L\psi^2}{2} = L(\theta^2 + \psi^2 - \psi\theta)$

- This strain energy equals $U_P = \frac{k}{2}(\Delta_1^2 + \Delta_2^2) = \frac{kL^2}{2}(\psi^2 + \theta^2)$
- The Potential of the external forces equals $V_P = -P\varepsilon_1 = -PL(\theta^2 + \psi^2 - \psi\theta)$
- The total potential is then
  - $\Pi = U + V_P = \frac{kL^2}{2}(\psi^2 + \theta^2) - PL(\theta^2 + \psi^2 - \psi\theta)$
- For equilibrium, we take the derivatives with respect to the two angular rotations:
  - $\frac{\partial \Pi}{\partial \psi} = 0 = \frac{kL^2}{2}(2\psi) - 2PL\psi + PL\theta$
  - $\frac{\partial \Pi}{\partial \theta} = 0 = \frac{kL^2}{2}(2\theta) - 2PL\theta + PL\psi$
- Rearranging, we get
  - $\begin{bmatrix} (kL^2 - 2PL) & PL \\ PL & (kL^2 - 2PL) \end{bmatrix} \begin{bmatrix} \psi \\ \theta \end{bmatrix} = 0$



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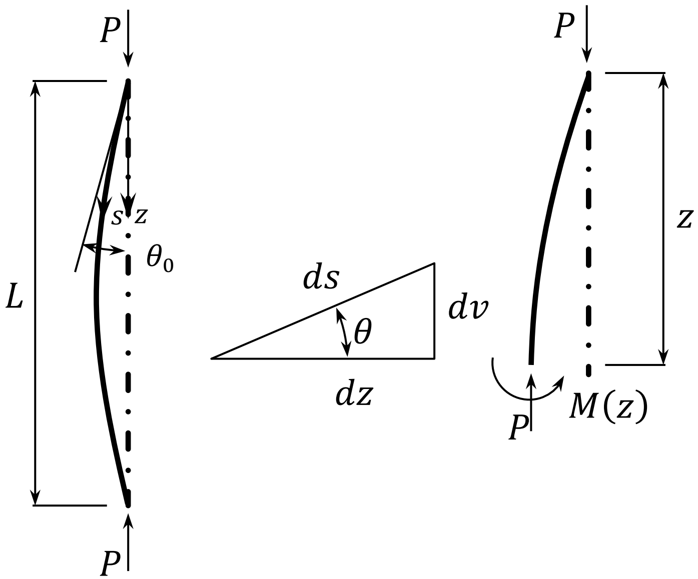
# Part VIII:

# 2. Elastic Buckling of Planar Columns

Structural Design Lab.(Prof. Ho-Kyung Kim)
Dept. of Civil & Environmental Eng.
Seoul National University

# 1. Large-Deflection Solution of an Elastic Column

- Perfectly straight
- Elastic
- Prismatic




- $M(z) = Pv = -EI\phi$ (1)

  $\phi = \frac{d\theta}{ds}\,;\, \frac{dv}{ds} = sin\theta$
- $\frac{d^2\theta}{ds^2} + k^2 sin\theta = 0$ (2)

  $k^2 = \frac{P}{EI}$ (3)
- Solution of Eq.(2)

$\frac{kL}{2} = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-P^2 sin^2\alpha}}$ (4)

$P = sin\frac{\theta_0}{2}$

Fig. p2.9c

$$y(0) = -\frac{T}{EI}\left(\frac{1}{3\alpha^2-\beta^2}\right); \quad \theta(0) = \frac{T}{EI}\left(\frac{2\alpha}{3\alpha^2-\beta^2}\right); \quad M(0) = T;$$

$$V(0) = T\left(\frac{-4\alpha(\alpha^2-\beta^2)}{3\alpha^2-\beta^2}\right)$$

## APPENDIX 2.1

Equation 2.2: $\frac{d^2\theta}{ds^2} + K^2 \sin\theta = 0$

Multiply by $d\theta$ and integrate: $\int \frac{d^2\theta}{ds^2} d\theta + K^2 \int \sin\theta d\theta = 0$

$$\int \frac{d^2\theta}{ds^2}\frac{d\theta}{ds} ds + K^2 \int \sin\theta d\theta = 0$$

$$\frac{1}{2}\int_0^s \frac{d}{ds}\left(\frac{d\theta}{ds}\right)^2 ds + K^2 \int_{\theta_o}^{\theta} \sin\theta d\theta = 0 = \left[\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2\right]_0^s - K^2[\cos\theta]_{\theta_o}^{\theta}$$

at $s = 0$, the curvature $\phi(0) = 0 = \frac{d\theta}{ds}(0)$

$$\therefore \left(\frac{d\theta}{ds}\right)^2 = 2K^2(\cos\theta - \cos\theta_o) = 4K^2\left(\sin^2\frac{\theta_o}{2} - \sin^2\frac{\theta}{2}\right)$$

$$\frac{d\theta}{ds} = \pm 2K\sqrt{\sin^2\frac{\theta_o}{2} - \sin^2\frac{\theta}{2}}$$

Use the negative sign because $\theta$ decreases from $z = 0$ to $z = L/2$

$$\therefore -2K ds = \frac{d\theta}{\sqrt{\sin^2\frac{\theta_o}{2} - \sin^2\frac{\theta}{2}}}$$



At $z = L/2$, $\theta = 0$; Integration from $s = L/2$ to $s = 0$ leads to

$$\int_0^{\theta_o} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_o}{2} - \sin^2\frac{\theta}{2}}} = \int_{L/2}^{0} -2K ds = 2K\int_0^{L/2} ds = KL$$

Let $\sin\frac{\theta}{2} = p\sin\alpha$, where $p = \sin\frac{\theta_o}{2}$

when $\theta$ varies from 1 to 0, $\sin\alpha$ varies from $\frac{\pi}{2}$ to 0 and $\alpha$ varies from $\frac{\pi}{2}$ to 0.

$$\theta = 2\sin^{-1}(p\sin\alpha); \quad d\theta = \frac{2p\cos\alpha\, d\alpha}{\sqrt{1-p^2\sin^2\alpha}}$$

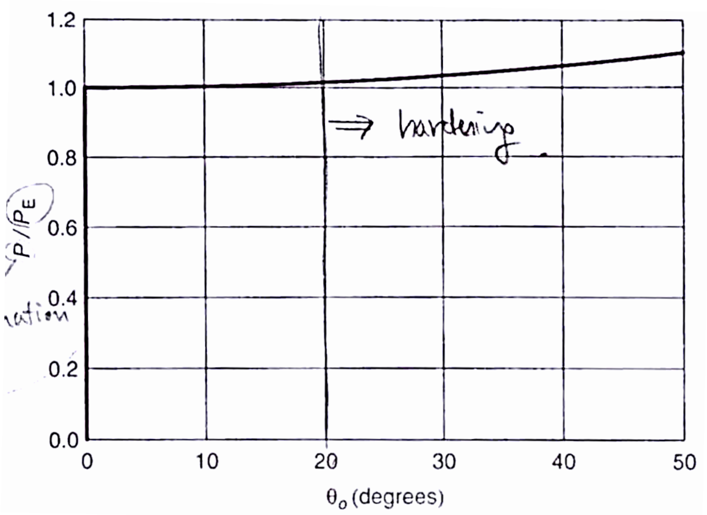
$$\text{then } KL = \int_0^{\theta_o} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_o}{2} - \sin^2\frac{\theta}{2}}} = \int_0^{\frac{\pi}{2}} \frac{2p\cos\alpha\, d\alpha}{\sqrt{1-p^2\sin^2\alpha}\sqrt{p^2 - p^2\sin^2\alpha}}$$

$$\text{Finally } \frac{KL}{2} = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-p^2\sin^2\alpha}};$$

complete elliptic integral of the first kind

# 2. Buckling Load

- Large-deflection solution




- Post-buckling behavior
- Hardening type

- Small-deflection assumption

$$sin\theta \approx \theta \; ; \; \frac{dv}{ds} \approx \theta$$

$$EI\phi \approx -EI\frac{d^2v}{dS^2}$$

$$\rightarrow EI\frac{d^2v}{ds^2} + Pv = 0$$

$$\therefore v'' + k^2v = 0$$

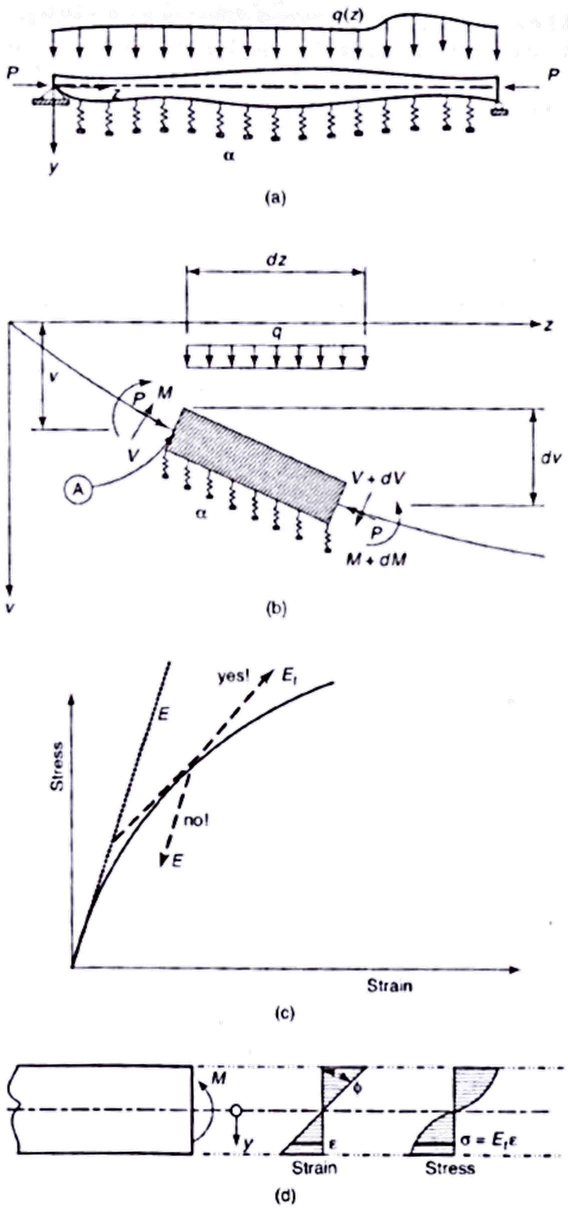
- B.C.

$$v(0) = 0 \; ; \; v(L) = 0$$

$$\rightarrow AsinKL = 0 \therefore KL = \pi$$

- $P_{cr} = \frac{\pi^2 EI}{L^2}$

# 3. Differential Equation of Beam-Column




$$\sum M_A = 0 \sim V + P\frac{dV}{dz} - \frac{dM}{dz} = 0 \text{ (1)}$$

$$\sum F_v = 0 \sim \frac{dV}{dz} = \alpha v - q \text{ (2)}$$

- From (1) & (2)

$$-\frac{d^2M}{dz^2} + P\frac{d^2v}{dz^2} + \alpha v = q \quad \text{(3)}$$

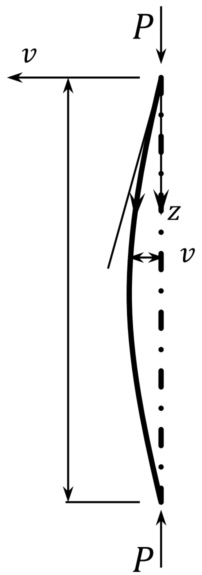
- $M = \int_{Area}(\sigma y)dA = \phi \int_{Area}(E_t y^2)dA$ (4)

  $(\sigma = E_t\epsilon = E_t\phi \cdot y)$

- Elastic material and prismatic column

  $EI_x v'''' + Pv'' + \alpha v = q$ (5)

# 4. Pin-Ended Column




Differential equation:
$EIv^{(4)} + Pv = 0$
Boundary conditions:
$@z = 0;\ v = 0,\ M = 0$
$@z = L;\ v = 0,\ M = 0$

- $\alpha = 0\,; q = 0 \ \rightarrow \ v'''' + k^2 v'' = 0$ (1)

$v = C_1 e^{r_1} + C_2 e^{r_2} + C_3 e^{r_3} + C_4 e^{r_4}$

$= C_1 + C_2 z + C_3 e^{ikz} + C_4 e^{-ikz}$ (2)

- $v = A + Bz + C \sin kz + D \cos kz$ (3)

$v'' = -Ck^2 \sin kz - Dk^2 \cos kz$ (4)

B.C.

$v(0) = 0 = A(1) + B(0) + c(0) + D(1)$

$v''(0) = 0 = A(0) + B(0) + C(0) + D(k^2)$

$v(L) = 0 = A(1) + B(L) + C(\sin kl) + D(\cos kl)$

$v''(L) = 0 = A(0) + B(0) + C(-k^2 \sin kL) + D(-k^2 \cos kL)$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin KL & \cos kL \\ 0 & 0 & -k^2 \sin kL & -k^2 \cos kL \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

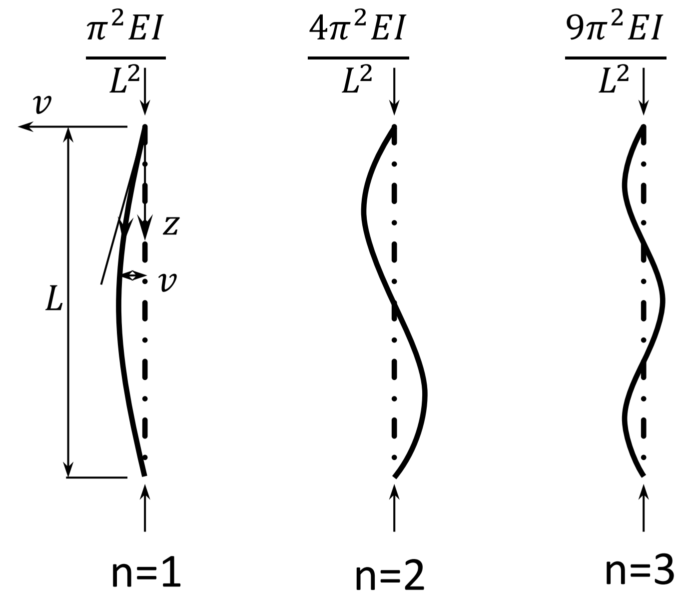
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin KL & \cos kL \\ 0 & 0 & -k^2 \sin kL & -k^2 \cos kL \end{bmatrix} = 0$$

$$\rightarrow Lk^4 \sin kL = 0$$

$$\therefore KL = \sqrt{\frac{PL^2}{EI}} = n\pi, \qquad n = 1,2,3 \ldots$$

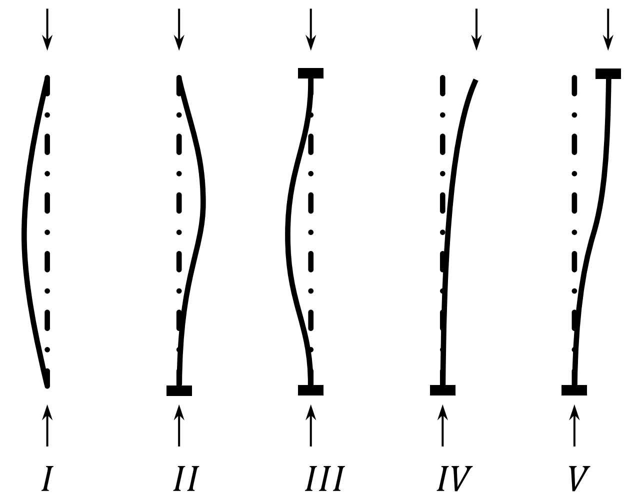
$$P_{cr} = \frac{n\pi^2 EI}{L^2}$$

$$A = B = D = 0 \;\leadsto v = C \sin\frac{n\pi z}{L}$$



# 6. Five Fundamental Cases of Column Buckling

I. Pined-Pined
II. Pined-fixed
III. Fixed-fixed
IV. Free-fixed
V. Fixed-fixed+sidesway




B.C.

A. Pined end : $v = v'' = 0$ (1)

B. Fixed end : $v = v' = 0$ (2)

C. Free end : zero moment $\rightsquigarrow$ $v'' = 0$ (3)

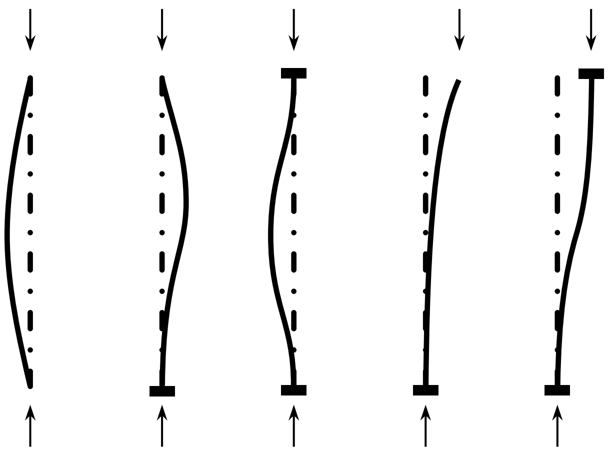
$$\text{zero shear} \rightsquigarrow V + Pv' + EI\frac{dv''}{dz} = 0$$

$$\rightarrow V = -EIv''' - Pv' = 0$$

# 7. Five Fundamental Cases of Column Buckling

- Deflection Eqs

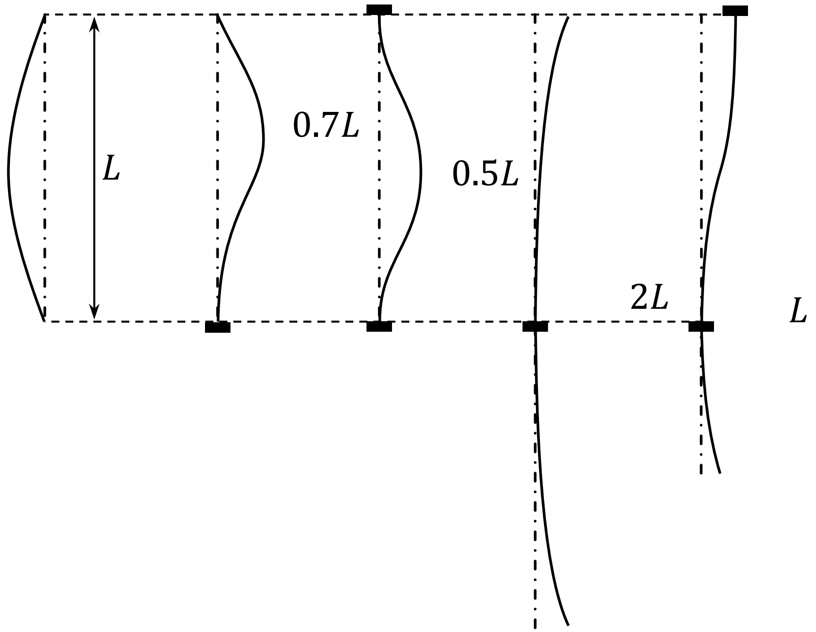
$$\upsilon = A + Bz + C\sin kz + D\cos kz$$
$$\upsilon' = B + Ck\cos kz - Dk\sin kz$$
$$\upsilon'' = -ck^2\sin kz - Dk^2\cos kz$$
$$\upsilon''' = -Ck^3\cos kz + \mathrm{D}k^3\sin kz$$




| Case | Boundary Conditions | Buckling Determinant | Eigenfunction Eigenvalue Buckling Load | Effective Length Factor |
|---|---|---|---|---|
| *I* | $v(0) = v''(0) = 0$<br>$v(L) = v''(L) = 0$ | $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2\sin kL & -k^2\cos kL \end{vmatrix}$ | $\sin kL = 0$<br>$kL = \pi$<br>$P_{cr} = P_E$ | 1.0 |
| *II* | $v(0) = v''(0) = 0$<br>$v(L) = v'(L) = 0$ | $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$ | $\tan kl = kl$<br>$kl = 4.493$<br>$P_{cr} = 2.045\,P_E$ | 0.7 |
| *III* | $v(0) = v'(0) = 0$<br>$v(L) = v'(L) = 0$ | $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & k & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$ | $\sin \frac{kL}{2} = 0$<br>$kL = 2\pi$<br>$P_{cr} = 4\,P_E$ | 0.5 |
| *IV* | $v'''(0) + k^2v' = v''(0) = 0$<br>$v(L) = v'(L) = 0$ | $\begin{vmatrix} 0 & 0 & 0 & -k^2 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$ | $\cos kL = 0$<br>$kL = \frac{\pi}{2}$<br>$P_{cr} = \frac{P_E}{4}$ | 2.0 |
| *V* | $v'''(0) + k^2v' = v'(0) = 0$<br>$v(L) = v'(L) = 0$ | $\begin{vmatrix} 0 & 1 & k & 0 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k\cos kL & -k\sin kL \end{vmatrix}$ | $\sin kL = 0$<br>$kL = \pi$<br>$P_{cr} = P_E$ | 1.0 |

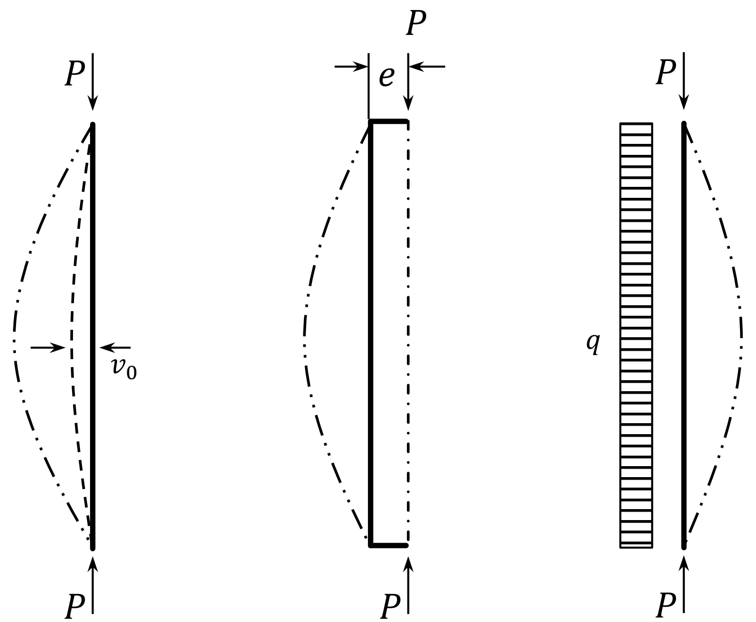
# 8. Effective Buckling Length

- $P_{cr} = \frac{P_E}{K^2} = \frac{\pi^2 EI}{(KL)^2}$ ; k=effective length factor



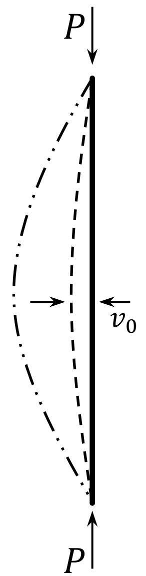
# 9.Effect of Imperfections

- Small initial crookedness
- Small load eccentricity
- Small lateral load




(a) Initial crookedness, $v_0$ (b) Load eccentricity, $e$ (c) Lateral load, $q$

## Half-sine curve




(a) Initial crookedness, $v_0$

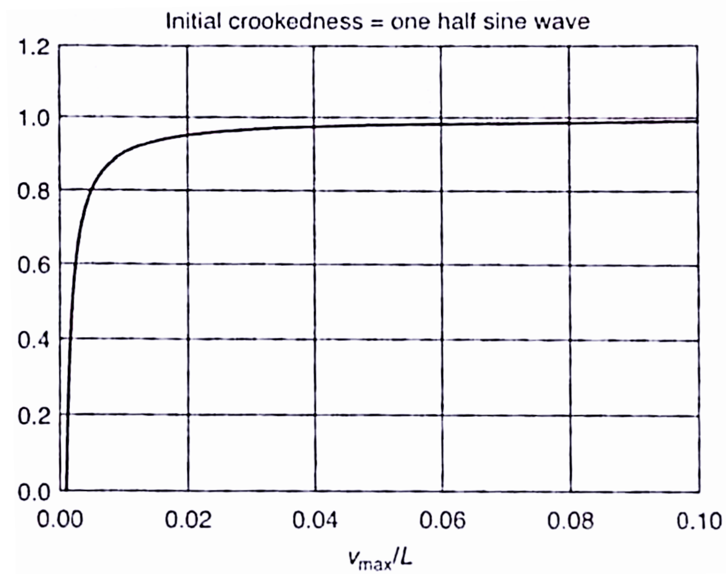
$$v_i = v_0 \sin\frac{\pi z}{L}$$

$$M_{int} = -EIv'' \qquad \text{-(1)}$$

$$M_{ext} = P(v_i + v)$$

$$\therefore EIv'' + pv = -pv_i \qquad \text{-(2)}$$

- $v'' + k^2 v = -k^2 v_0 \sin\frac{\pi z}{L}$ -(3)




## B.C $v(0) = v(L) = 0$

- $v_{total} = v_i + v = \dfrac{v_0 \sin\frac{\pi z}{L}}{1 - P/P_E}$ -(4)

- $\dfrac{v_{total}(\frac{L}{2})}{L} = \dfrac{v_0/L}{1 - P/P_E}$ -(5)

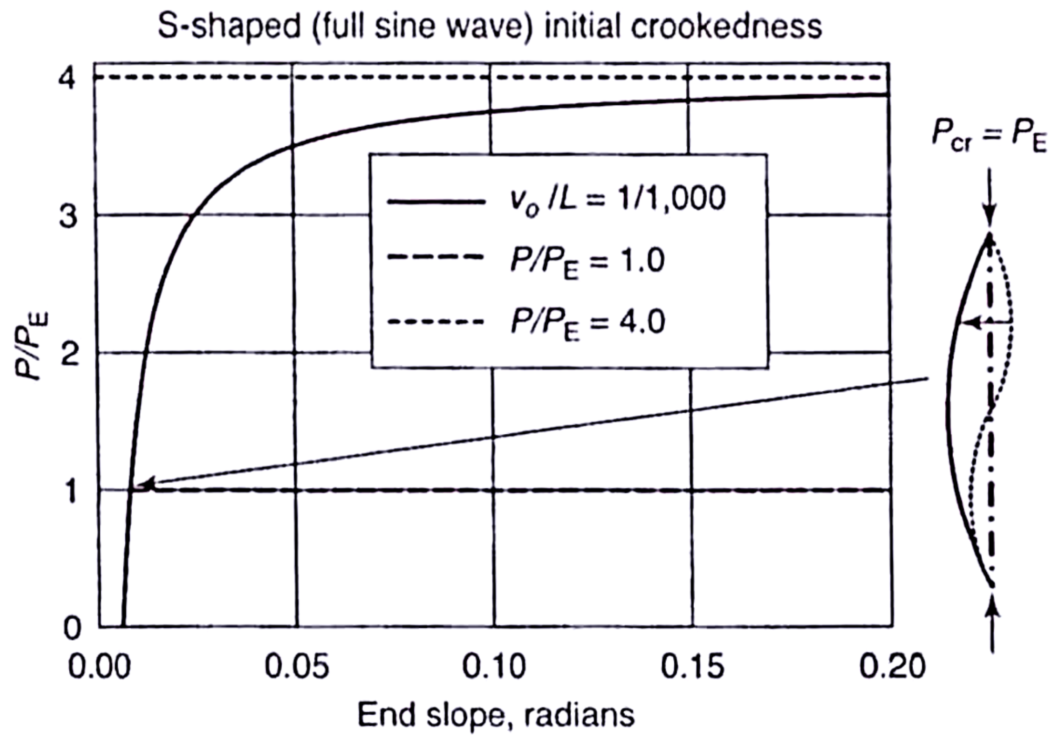
- **Full-sine curve**

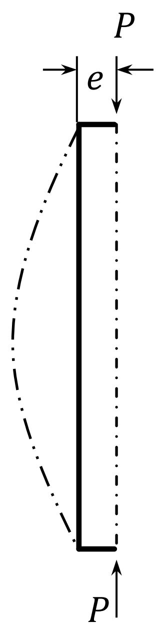
$$v_i = v_0 \sin\frac{2\pi z}{L} \qquad \text{-(1)}$$

$$v = v_0 \frac{P/P_E}{4 - P/P_E} \sin\frac{2\pi z}{L} \qquad \text{-(2)}$$

or

$$\theta_{0,total} = \frac{2\pi v_0}{L}\left(\frac{1}{1-\frac{P}{4P_E}}\right)$$






(b) Load eccentricity, $e$

$$-EIv'' = P(e+v)$$

$$v'' + k^2 v = -k^2 e \;\rightarrow v = A\sin kz + B\cos kz - e$$

B.C $v(0) = v(L)=0$; $k^2 = \frac{P}{EI}$

$$\rightarrow v = e(\cos kz + \frac{1-\cos kL}{\sin kL}\sin kz - 1)$$

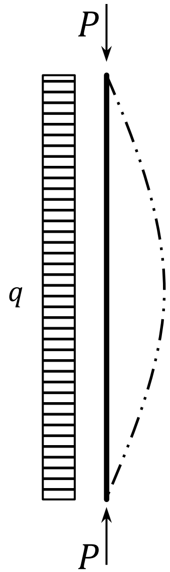
$$v\left(\frac{L}{2}\right) = e(\cos\frac{kL}{2} + \frac{1-\cos kL}{\sin kL}\sin\frac{kL}{2} - 1)$$

→2nd order analysis!

$$v\left(\frac{L}{2}\right) \text{ by 1st order analysis} = \frac{ML^2}{8EI} = \frac{PeL^2}{8EI}$$

$$= \frac{\pi^2}{8}\left(\frac{P}{P_e}\right)$$

$$\therefore MF = \frac{8}{\pi^2(P/P_e)}\left[\frac{1-\cos\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}{\cos\frac{\pi}{2}\sqrt{\frac{P}{P_E}}}\right]$$



(c) Lateral load, $q$

$$v'''' + k^2 v'' = \frac{q}{E}$$

$$\rightarrow v = \frac{q}{pk^2}\left[\left(\frac{1-\cos kL}{\sin kL}\right)\sin kz + \cos kz + \frac{(kz)^2}{2} - \frac{k^2 Lz}{2} - 1\right]$$

$$v\left(\frac{L}{2}\right) = \frac{q}{pk^2}\left(\frac{1}{\cos\frac{kL}{2}} - \frac{(kL)^2}{8} - 1\right)$$

$1^{st}$ order deflection: $\frac{5qL^4}{384EI}$

$$\therefore MF = \frac{384EI}{5k^4L^4}\left[\frac{1}{\cos\frac{kL}{2}} - \frac{(kL)^2}{8} - 1\right]$$

# 14.Magnification Factor

- $MF = \dfrac{1}{1-P/P_E}$