Optimality Conditions and Convex Problem

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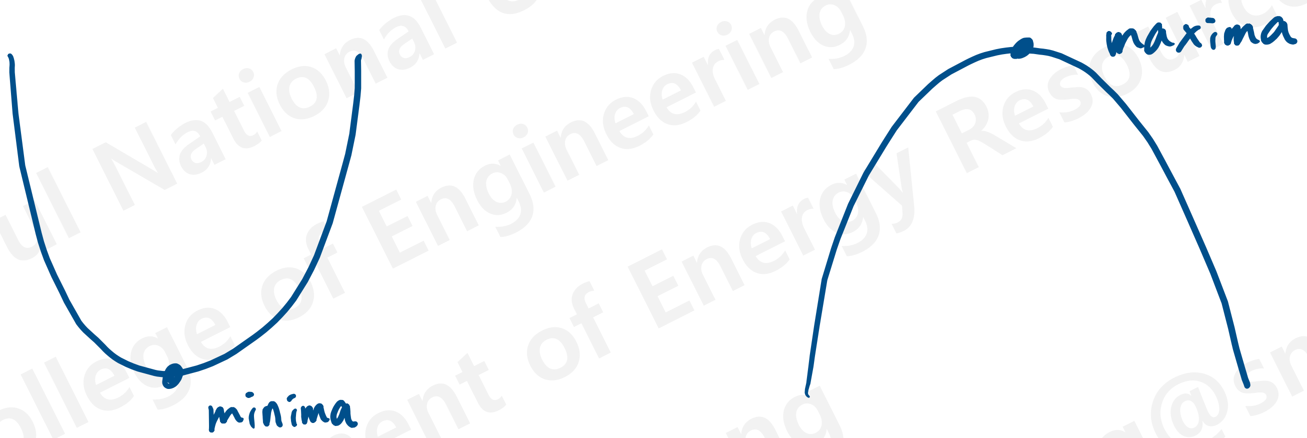
Key Questions

- What conditions do you need to see if a candidate is optimal for a univariate function?
- What conditions do you need to see if a candidate is optimal for a multivariate function?
- What is a convex problem?

Terminology

- 극대/극소
  - ✓ Local maxima
  - ✓ Local minima
- 극점
  - ✓ Extrema
- 변곡점
  - ✓ Inflection point
  - ✓ Saddle point (안장점)
- 극점+변곡점
  - ✓ Stationary point
- 아래로 볼록
  - ✓ Convex
- 위로 오목
  - ✓ Concave

Necessity for Extrema of a Univariate Function



- Necessity condition for an extreme point at $x = x^*$
  - ✓ $f'(x^*) = 0$

Optimality Conditions for Local Minima of a Univariate Function

- Taylor series
  - ✓ $f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + R$
  - ✓ $f(x) = f(x^*) + f'(x^*)\Delta x + \frac{f''(x^*)}{2!}\Delta x^2 + R$
  - ✓ Approximate the function values near $x = x^*$
- $\Delta f(x) = f(x) - f(x^*) = f'(x^*)\Delta x + \frac{f''(x^*)}{2!}\Delta x^2 + R$
- What condtions are needed if $(x^*, f(x^*))$ is a local minimum?
  - ✓ $\Delta f(x) \geq 0$ regardless of Δx
    - $f'(x^*) = 0$ regardless of Δx
    - $f''(x^*) > 0$

Optimality Conditions for Local Minima of a Multivariate Function

- Taylor series
  - ✓ $f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \mathbf{H}(\mathbf{x}^*)\Delta\mathbf{x} + R$
  - ✓ Approximate the function values near $\mathbf{x} = \mathbf{x}^*$
- $\Delta f(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^*) = \nabla f(\mathbf{x}^*)^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \mathbf{H}(\mathbf{x}^*)\Delta\mathbf{x} + R$
- What conditions are needed if $(\mathbf{x}^*, f(\mathbf{x}^*))$ is a local minimum?
  - ✓ $\Delta f(\mathbf{x}) \geq 0$ regardless of $\Delta\mathbf{x}$
    - $\nabla f(\mathbf{x}^*) = 0$
    - $\frac{1}{2}\Delta\mathbf{x}^T \mathbf{H}(\mathbf{x}^*)\Delta\mathbf{x} > 0$ ➔ $\mathbf{H}(\mathbf{x}^*)$ is positive definite

$$\frac{1}{2}\Delta \mathbf{x}^T \mathbf{H}(\mathbf{x}^*)\Delta \mathbf{x} > 0 \Rightarrow \mathbf{H}(\mathbf{x}^*) \text{ is positive definite}$$

H : symmetric matrix → can be diagonalized using eigenvalue decomposition

$H = P \Lambda P^{-1} = P \Lambda P^T$

eigenvalues are diagonal elements

eigenvectors are column vectors

$$\Delta x^T H \Delta x = \Delta x^T P \Lambda P^T \Delta x = y^T \Lambda y = [y_1 \cdots y_n]\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$P^T \Delta x = y$

$$= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

⇒ if H is positive definite ($\lambda_i > 0$),

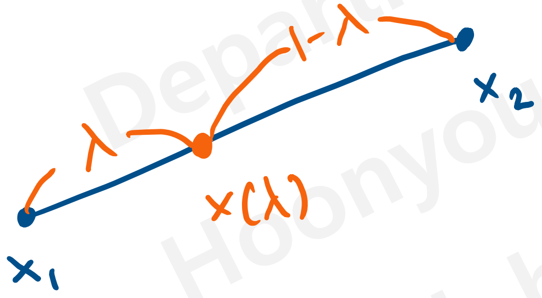
then $\Delta x^T H \Delta x > 0$ ⇒ $\Delta f \geq 0$

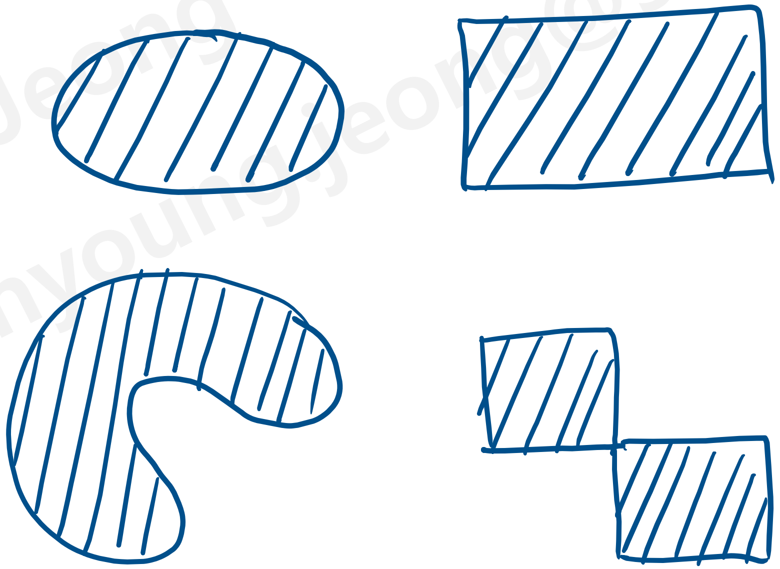
What is a Convex Problem?

- Conditions
  - ✓x belongs to a convex set
  - ✓f(x) is a convex function
- A convex problem has a global minimum

What is a Convex Set?

- A set (S) is convex if the following condition is satisfied
  - ✓For all $\mathbf{x}_1$ and $\mathbf{x}_2$ that belong to S
  - ✓$\mathbf{x}(\lambda) = (1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2, 0 \le \lambda \le 1$
  - ✓$\mathbf{x}(\lambda) \in S$
- Geometric meaning





→ Convex set

→ Non-convex set

What is a Convex Function?

- $f(\mathbf{x})$ is a convex function if the following conditions are satisfied
  - ✓For all $\mathbf{x}_1$ and $\mathbf{x}_2$ that belong to a convex set, S
  - ✓$\mathbf{x}(\lambda) = (1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2, 0 \le \lambda \le 1$
  - ✓$f\big((1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2\big) \le \lambda f(\mathbf{x}_2) + (1-\lambda)f(\mathbf{x}_1)$
- Geometric meaning
  - ✓아래로 볼록한 함수