Chap. 20

PROFIT MAXIMIZATION
Introduction

- A model of how the firm chooses the amount to produce and the method of production to employ
- Profit maximization problem of a firm that faces competitive market for the factors of production it uses and the output goods it produces

Competitive market

- A collection of well-informed consumers
- Homogeneous product that is produced by a large number of firms
- Price-taking behavior
  - Exogenous variable: price
  - Endogenous variable: levels of outputs and inputs
Economic Profit

- A firm uses inputs $j = 1 \ldots, m$ to make products $i = 1, \ldots, n$.
- Output levels are $y_1, \ldots, y_n$.
- Input levels are $x_1, \ldots, x_m$.
- Product prices are $p_1, \ldots, p_n$.
- Input prices are $w_1, \ldots, w_m$.
- Profit = Revenue – Cost

\[ \pi = \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{m} w_i x_i \]

- Economic definition of profit requires that all inputs and outputs are valued at their opportunity cost
Profit Maximization

- Profit maximization

\[
\text{Max } \pi = \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{m} w_i x_i
\]

- Using production plan \( \tilde{y} \in Y \), where \( y_j \geq (\leq)0 \) if \( j \) is output (input)

\[
\text{Max } \pi(\tilde{p}) = \tilde{p} \cdot \tilde{y}
\]

such that \( \tilde{y} \in Y \),

where \( \tilde{p} \) is the vector of prices for all inputs and outputs

- 1-output case, we can use the production function \( y = f(\tilde{x}) \)

\[
\text{Max } \pi = p \cdot f(\tilde{x}) - \sum_{i=1}^{m} w_i x_i
\]
Fixed and Variable factors

- **Fixed factor**: a factor of production that is in a fixed amount for the firm
  - Fixed factor must be expensed even at the state of zero output

- **Variable factor**: a factor which can be used in different amounts

- **Short run**: there are some fixed factors

- **Long run**: all factors are variable factors
  - In the short run, the firm could make negative profits
  - But in the long run, the least profit is zero since the firm always free to decide to use zero inputs and produce zero output
Suppose the firm is in a short-run circumstance in which $x_2$ : a fixed factor

Its short-run production function is $f(x_1, x_2)$

Profit-max. problem

$$\max \pi = p \cdot f(x_1, x_2) - w_1 x_1 - w_2 x_2$$

F.O.C.

$$\frac{\partial \pi}{\partial x_1} = p \cdot \frac{\partial f(x_1^*, x_2)}{\partial x_1} - w_1 = 0$$

$p \cdot MP_1(x_1^*, x_2) = w_1$

"The value of marginal product of factor 1 should equal its price"
Short-run Profit Maximization (1-output & 2-inputs)

- **Iso-profit curves**
  - Given profit function \( \pi = py - w_1 x_1 - w_2 x_2 \)
  - The level set of profit function
    \[
    y = \frac{w_1}{p} x_1 + \frac{1}{p} (\bar{\pi} + w_2 x_2)
    \]
  - all combinations of inputs and outputs that give a constant level of profit

![Graph showing iso-profit curves and increasing profit]

Slopes \( = \frac{w_1}{p} \)
Short-Run Profit-Maximization

Short-run production function

Increasing profit
Short-Run Profit-Maximization

Optimality condition

\[
\frac{d}{dx_1} f(x_1^*, x_2) = \frac{w_1}{p}
\]

\[
p \cdot MP_1(x_1^*, x_2) = w_1
\]

Slopes = \( \frac{w_1}{p} \)

Short-run production function

\[ f(x_1, x_2) \]

Second order condition

\[
\frac{d^2 f(x_1^*, x_2)}{dx_1^2} \leq 0 : \text{locally concave}
\]
Short-Run Profit-Maximization

- Short-run Cobb-Douglas production function
Now allow the firm to vary all input levels.

Since no input level is fixed, there are no fixed costs.

Profit maximization

\[
\max_{\{x_1, x_2\}} \pi = p \cdot f(x_1, x_2) - w_1 x_1 - w_2 x_2
\]

F.O.C.

\[
\frac{\partial \pi}{\partial x_1} = p \cdot \frac{\partial f(x_1^*, x_2^*)}{\partial x_1} - w_1 = 0
\]

\[
\frac{\partial \pi}{\partial x_2} = p \cdot \frac{\partial f(x_1^*, x_2^*)}{\partial x_2} - w_2 = 0
\]

- Optimality condition

\[
p \cdot MP_1 = w_1, \quad p \cdot MP_2 = w_2
\]

- Solution

\[
x_1^*(w_1, w_2, p)
\]

\[
x_2^*(w_1, w_2, p) : \text{Factor demand function}
\]
Cobb-Douglas production function \( y = x_1^a x_2^b \)

- Profit-max. problem
  \[
  \max \pi(x_1, x_2) = px_1^a x_2^b - w_1 x_1 - w_2 x_2
  \]

- F.O.C.
  \[
  \begin{align*}
  \frac{\partial \pi}{\partial x_1} &= pax_1^{a-1} x_2^b - w_1 = 0 \\
  \frac{\partial \pi}{\partial x_2} &= pbx_1^a x_2^{b-1} - w_2 = 0
  \end{align*}
  \]

- Multiplying \( x_i \)
  \[
  \begin{align*}
  pax_1^a x_2^b - w_1 x_1 &= 0 \\
  pbx_1^a x_2^b - w_2 x_2 &= 0
  \end{align*}
  \]
  \[
  \begin{align*}
  pay &= w_1 x_1 \\
  pby &= w_2 x_2
  \end{align*}
  \]

- Factor demand function
  \[
  \begin{align*}
  x_1^* (w_1, w_2, p) &= \frac{apy}{w_1} \\
  x_2^* (w_1, w_2, p) &= \frac{bpy}{w_2}
  \end{align*}
  \]
Long-Run Profit-Maximization (1-output, 2-inputs)

• Inserting factor demand functions into the production function gives

\[ y = \left(\frac{apy}{w_1}\right)^a \left(\frac{bpw}{w_2}\right)^b = \left(\frac{ap}{w_1}\right)^a \left(\frac{bp}{w_2}\right)^b y^{a+b} \]

\[ ∴ y^{1-a-b} = \left(\frac{ap}{w_1}\right)^a \left(\frac{bp}{w_2}\right)^b \]

• Supply function

\[ y(p, w_1, w_2) = \left(\frac{ap}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{bp}{w_2}\right)^{\frac{b}{1-a-b}} \]
Long-Run Profit-Maximization (1-output, n-inputs)

- Output $y$, Input bundle $\tilde{x}$
- Profit maximization
  \[
  \max_{\tilde{x}} \pi(\tilde{x}) = pf(\tilde{x}) - \tilde{w} \cdot \tilde{x}
  \]
- F.O.C.
  \[
p \cdot \frac{\partial f(\tilde{x})}{\partial x_i} = w_i \quad i = 1, \ldots, n
  \]

$\tilde{x}^*(p, \tilde{w})$: factor demand function

$f(\tilde{x}^*(p, \tilde{w}))$: supply function
Exceptional case

1) When production function is not differentiable
   - Leontief technology

2) Corner (boundary) solution case ($x_i^* = 0$ for some $i$)
   - F.O.C.

\[
p \cdot \frac{\partial f(\tilde{x})}{\partial x_i} - w_i = 0 \quad \text{if } x_i^* > 0
\]

\[
p \cdot \frac{\partial f(\tilde{x})}{\partial x_i} - w_i \leq 0 \quad \text{if } x_i^* = 0
\]
Optimization with constraints

- When equality constraints

\[ \max f(\tilde{x}) \]

\[ s.t. \quad h(\tilde{x}) = c \]

- Lagrangian function

\[ L(\tilde{x}, \lambda) = f(\tilde{x}) - \lambda[h(\tilde{x}) - c] \]

- Kuhn-Tucker condition

Suppose that \( \tilde{x}^* = (x_1^*, ..., x_n^*) \) is a solution.

Suppose further that \( \frac{\partial h}{\partial x_i} \big|_{x_i = x_i^*} \neq 0 \) (critical point)

Then there exits a real number \( \lambda^* \) such that

\[ \frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial \lambda} = 0 \quad \text{at} \quad (\tilde{x}^*, \lambda^*) \]
Long-Run Profit-Maximization (1-output, n-inputs)

• When inequality constraints

\[
\begin{align*}
\text{max } & f(\tilde{x}) \\
\text{s.t. } & g(\tilde{x}) \leq b
\end{align*}
\]

Lagrangian function

\[
L(\tilde{x}, \mu) = f(\tilde{x}) - \mu [g(\tilde{x}) - b]
\]

• Kuhn-Tucker condition

Suppose that \(\tilde{x}^* = (x_1^*, ..., x_n^*)\) is a solution.

If \(g(x^*, y^*) = b\) (binding), then further suppose that \(\partial g / \partial x_i \big|_{x_i = x_i^*} \neq 0\).

Then there is a multiplier \(\mu^* \geq 0\) such that

\[
\frac{\partial L}{\partial x_i} = 0 \text{ at } (\tilde{x}^*, \mu^*)
\]

\[
\mu^* \left[ g(\tilde{x}^*) - b \right] = 0 \text{ (complementary slackness)}
\]

\[
g(x^*, y^*) \leq b
\]
Example

\[ \max f(x, y) = xy \]
\[ \text{s.t. } x^2 + y^2 \leq 1 \]
Long-Run Profit-Maximization

- Generalized optimality condition for 2-input

\[
\max p \cdot f(x_1, x_2) - (w_1 x_1 + w_2 x_2)
\]
\[s.t. \quad x_i \geq 0 \quad \Rightarrow \quad -x_i \leq 0\]

• Lagrangian

\[
L = p \cdot f(x_1, x_2) - (w_1 x_1 + w_2 x_2) + \mu_i x_i
\]

• K-T condition

\[
\frac{\partial L}{\partial x_i} = p \frac{\partial f}{\partial x_i} - w_i + \mu_i = 0,
\]
\[
\mu_i x_i = 0 \text{ (complementary slackness), } x_i \geq 0, \quad \mu_i \geq 0
\]

• Optimality condition

\text{If } x_i^* = 0, \text{ then } \mu_i^* \geq 0. \quad \text{If } x_i^* > 0, \text{ then } \mu_i^* = 0.

Thus, if \( x_i^* = 0, \text{ then } p \cdot \frac{\partial f}{\partial x_i} - w_i \leq 0 \)

Thus, if \( x_i^* > 0, \text{ then } p \cdot \frac{\partial f}{\partial x_i} - w_i = 0 \)
Exceptional case

3) No optimal solution case

- When \( f(x) = x \). If \( p > w \), then \( x^* = \infty \)

- When CRS technology

Let \( \tilde{x}^* \) be the optimal and assume that \( p \cdot f(\tilde{x}^*) - \tilde{w} \cdot \tilde{x}^* = \pi^* > 0 \)

Scale up production by \( t > 1 \)

Since CRS, \( f(t \tilde{x}^*) = tf(\tilde{x}^*) \)

Then \( p \cdot f(t \tilde{x}^*) - \tilde{w} \cdot (t \tilde{x}^*) = t \left[ p \cdot f(\tilde{x}^*) - \tilde{w} \cdot \tilde{x}^* \right] = t \pi^* > \pi^* \)

Contradiction!

\( \rightarrow \) Thus the only nontrivial profit-max position for a CRS firm is zero-profits
Exceptional case

4) Multiple (Infinite) number of optimal solutions

Let $\tilde{x}^*$ be the optimal for a CRS technology which gives zero profit, i.e., $p \cdot f(\tilde{x}^*) - \tilde{w} \cdot \tilde{x}^* = \tilde{\pi}^* = 0$

Then scale up production by $t > 0$

$$p \cdot f(t\tilde{x}^*) - \tilde{w} \cdot (t\tilde{x}^*) = t\tilde{\pi}^* = 0$$

Thus $t\tilde{x}^*$ is also an optimal for any $t > 0$ !!
Comparative Statics

- One-input & One-output

\[ \max_x pf(x) - wx \]

F.O.C.: \( pf'(x^*(p, w)) - w = 0 \)
S.O.C.: \( pf''(x^*(p, w)) \leq 0 \)

- Differentiating F.O.C. with respect to \( w \)

\[ pf''(x^*(p, w)) \frac{dx^*(p, w)}{dw} - 1 \equiv 0 \]

- Assuming that \( f'' \neq 0 \)

\[ \frac{dx^*(p, w)}{dw} = \frac{1}{pf''(x^*(p, w))} \]

1) Sign
   ➢ Since \( f'' < 0 \), \( \frac{\partial x^*(p, w)}{\partial w} < 0 \)

2) Magnitude
   ➢ As \( \left| f'' \right| \) increases, \( \left| \frac{\partial x^*}{\partial w} \right| \) decreases.
Comparative Statics

- **Two-input & One-output**

  \[
  \text{Max } pf(x_1, x_2) - (w_1 x_1 + w_2 x_2) 
  \]

  - **F.O.C.**

    \[
    p \frac{\partial f [x_1(w_1, w_2), x_2(w_1, w_2)]}{\partial x_1} \equiv w_1
    \]

    \[
    p \frac{\partial f [x_1(w_1, w_2), x_2(w_1, w_2)]}{\partial x_2} \equiv w_2
    \]

  - **Differentiating F.O.C. with respect to** \( w_1 \) **and** \( w_2 \) (let \( p=1 \))

    \[
    f_{11} \frac{\partial x_1}{\partial w_1} + f_{12} \frac{\partial x_2}{\partial w_1} = 1 
    \]

    \[
    f_{11} \frac{\partial x_1}{\partial w_2} + f_{12} \frac{\partial x_2}{\partial w_2} = 0 
    \]

    \[
    f_{21} \frac{\partial x_1}{\partial w_1} + f_{22} \frac{\partial x_2}{\partial w_1} = 0 
    \]

    \[
    f_{21} \frac{\partial x_1}{\partial w_2} + f_{22} \frac{\partial x_2}{\partial w_2} = 1
    \]

\( x_i^* (w_1, w_2) \): factor demand function
Comparative Statics

• Rearranging by matrix form

\[
\begin{pmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{pmatrix}
\begin{pmatrix}
  \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\
  \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

• Note that \( \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \) is Hessian matrix.
  ➢ By Young’s theorem, Hessian matrix is symmetric \( f_{12} = f_{21} \)

• Then the substitution matrix can be obtained

\[
\begin{pmatrix}
  \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\
  \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2}
\end{pmatrix}
= \left( \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \right)^{-1}
= \frac{1}{\det H} \begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}
\]
Comparative Statics

- If we assume that S.O.C. is satisfied, it is equivalent to the fact that the Hessian matrix is ND

  ➢ Thus, \( f_{11} < 0, \quad |H| = f_{11}f_{22} - f_{12}f_{21} > 0 \)

- Comparative results

  ➢ The changes of factor demand with respect to the change of its own price

    \[
    \frac{\partial x_i}{\partial w_i} = \frac{f_{ii}}{|H|} < 0
    \]

  ➢ The changes of factor demand with respect to the change of other price

    \[
    \frac{\partial x_i}{\partial w_j} = -\frac{f_{ij}}{|H|} = -\frac{f_{ji}}{|H|} = \frac{\partial x_j}{\partial w_i}: \text{indeterminate and symmetric}
    \]