Linear Piezoelectric

❖ **Linear Piezoelectric**

$$(M_S + M_P)\ddot{r} + (K_S + K_P{}^E)r - \theta V = \overbrace{B_f \vec{f}}^{Q_P} + Q_B + Q_S$$

$$(\theta^T)r + (c_s + c_p{}^s)V = B_q q$$

vector of applied charges

$$\mathrm{M}_{\mathrm{s,p}} = \int_{V_S, V_P} \psi_r{}^T \rho_{\mathrm{s,p}}(x)\psi_r dV$$

$$K_{\mathrm{s,p}} = \int_{V_S, V_P} N_r{}^T c_{\mathrm{s,p}}{}^E N_r \, dV$$

$$c_{\mathrm{s,p}} = \int_{V_S, V_P} N_V{}^T \varepsilon_{\mathrm{s,p}}{}^E N_V \, dV$$ piezoelectric capacitance matrix

$$\theta = \int_{V_P} N_R{}^T e_t N_V \, dV$$ electromechanical coupling matrix

$$\mathrm{B}_f = \begin{bmatrix} \psi_{r_1}{}^T(x_{f_1}) & \cdots & \psi_{r_1}{}^T(x_{f\ell}) \\ \vdots & & \vdots \\ \psi_{r_n}{}^T(x_{f_1}) & \cdots & \psi_{r_n}{}^T(x_{f\ell}) \end{bmatrix}$$ n: mechanical DOF

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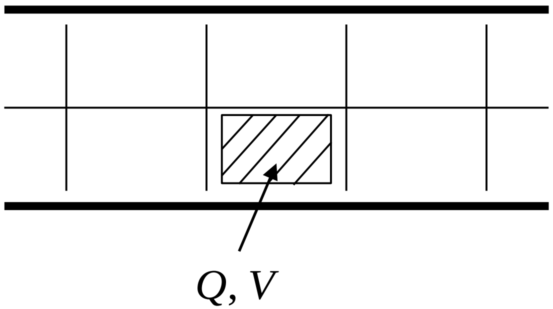
$$\mathrm{B}_q = \begin{bmatrix} \psi_{V_1}(x_{q_1}) & \cdots & \psi_{V_1}(x_{q_k}) \\ \vdots & & \vdots \\ \psi_{V_m}(x_{q_1}) & \cdots & \psi_{V_m}(x_{q_k}) \end{bmatrix}$$

m : electrical DOF

$$Q_S = \int_S \psi_r^{\;T}(x) f^S(x)\, dS$$

$$Q_V = \int_V \psi_r^{\;T}(x) f^B(x)\, dV$$



$$\begin{Bmatrix} F \\ Q \end{Bmatrix} = \begin{bmatrix} K_{uu} & K_{uv} \\ -K_{uv}^{\;T} & K_{vv} \end{bmatrix} \begin{Bmatrix} u \\ V \end{Bmatrix}$$

K_{vv} → capacitance $Q = CV$

$U = 0 \qquad F = K_{uv} V$

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Piezo: poled in x-direction

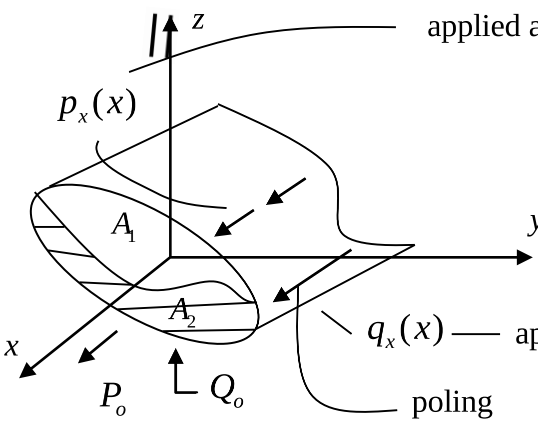
- Kinematic assumptions

$$\varphi=\varphi(x,t) \;\Rightarrow\; E_1=-\varphi'(x,t)$$

$$E_2=E_3=0$$

$$U_1=U_1(x,t) \Rightarrow S_1=U_1'(x,t)$$

$$U_2=U_3=0 \qquad S_2=S_6=0$$



$$\int_{t_1}^{t_2}\left[\delta T-\delta U_1^{\,M}+\delta U_1^{\,E}+\delta W_1^{\,M}+\delta W_1^{\,E}\right]dt=0$$

$$\delta T=\delta\int_V\left\{\int_0^{\bar{\varsigma}}\rho\dot{U}\dot{U}'\,\mathrm{d}\zeta\right\}dV=\int_\ell\left[\bar{m}\dot{U}\delta\dot{U}\right]dx$$

$$\frac{\partial\dot{U}}{\partial\zeta} \qquad \bar{m}=A_1\rho_1+A_2\rho_2$$

$$\delta U_1^{\,M}=\delta\int_V\left\{\int_0^{\bar{\zeta}}T\,S'\,dS\right\}dV=\int_V\left[T_1\,\delta S_1\right]dV$$

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Introduce constitutive relationship

$$\text{structure} \ - \begin{bmatrix} T_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}_{11} & 0 \\ 0 & \varepsilon_{11} \end{bmatrix} \begin{bmatrix} \mathrm{S}_1 \\ \mathrm{E}_1 \end{bmatrix}$$

$$\text{piezo} \ - \begin{bmatrix} T_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} c_{11}{}^{E} & -e_{11} \\ -e_{11} & \varepsilon_{11}{}^{S} \end{bmatrix} \begin{bmatrix} S_1 \\ E_1 \end{bmatrix}$$

$$\delta U_1{}^{M} = \int_{\ell} \left[\overline{c} S_1 - \overline{e} E_1 \right] \delta S_1 \, dx$$

$$\overline{c} = A_1 C_{11} + A_2 C_{11}{}^{E}$$

$$\overline{e} = A_2 e_{11}$$

likewise

$$\delta U_1{}^{E} = \int_{V} D \delta E dV = \int_{\ell} \left[\overline{e} S_1 + \overline{\varepsilon} E_1 \right] \delta E_1 dx$$

$$\overline{e} = A_2 \varepsilon_{11}{}^{S} + A_1 \varepsilon_{11}$$

Works,

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- Classical plate

$$\delta W_1^{\,M} = P_o \delta U_1(e_{x=\ell}) + \int_\ell p_x \delta U(x)\,\mathrm{dx}$$

$$\delta W_1^{\,E} = Q_o \delta\varphi(e_{x=\ell}) + \int_\ell q(x)\delta\varphi\,\mathrm{dx}$$

stuff-in the variational principle

$$\int_{t_1}^{t_2}\left\{\int_0^\ell \left[\bar{m}\dot{U}\delta\dot{U} - (\bar{c}S_1 - \bar{e}E_1)\delta S_1 + (\bar{e}S_1 + \bar{e}E_1)\delta E_1 + p_x\delta U - q_x\delta\varphi\right]dx + P_o\delta U_{(x=\ell)} - Q_o\delta\varphi_{(x=\ell)}\right\}dt$$

$$= 0$$

Decisions:

i) Go continuous

$$S_1 = \frac{dU}{dx},\ \mathrm{E}_1 = -\frac{d\varphi}{dx}$$

$$\int_{t_1}^{t_2}\int_0^\ell \{[\qquad]\delta U + [\qquad]\delta\varphi\}dxdt + \text{B.C's terms}$$

ii) Discrete Representation

Rayleigh - Ritz

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$$U(x) = U_o\left(\frac{x}{\ell}\right) \Rightarrow \delta S_1 = \frac{\delta U_o}{\ell}$$

$$\delta U = \delta U_o\left(\frac{x}{\ell}\right)$$

$$\varphi(x) = V_o\left(\frac{x}{\ell}\right) \Rightarrow \delta E_1 = -\frac{\delta V_o}{\ell}, \quad \delta\varphi = \delta V_o\left(\frac{x}{\ell}\right)$$

can assume more complex distribution if you want better accuracy

$$\int_{t_1}^{t_2}\left\{\begin{array}{l}\int_0^{\ell}\left[m\dot{U}_o\delta\dot{U}_o\left(\frac{x}{\ell}\right)^3 - \left(\bar{c}\frac{V_o}{\ell} + \bar{e}\frac{V_o}{\ell}\right)\frac{\delta V_o}{\ell} + \left(\bar{e}\frac{V_o}{\ell} - \bar{\varepsilon}\frac{V_o}{\ell}\right)\left(-\frac{\delta V_o}{\ell}\right) + p_x\left(\frac{x}{\ell}\right)\delta U_o - q_x\left(\frac{x}{\ell}\right)\delta V_o\right]dx \\ + P_o\delta U_o - Q_o\delta V_o\end{array}\right\}dt = 0$$

$$\int_{t_1}^{t_2}\left\{\frac{\bar{m}\ell}{3}\dot{U}_o\delta\dot{U}_o - \left(\frac{\bar{c}}{\ell}U_o + \frac{\bar{e}}{\ell}V_o\right)\delta U_o + \left(\bar{e}\frac{U_o}{\ell} - \bar{\varepsilon}\frac{V_o}{\ell}\right)(-\delta V_o) + p_x\frac{\ell}{2}\delta U_o - g_x\frac{\ell}{2}\delta V_o + P_o\delta U_o - Q_o\delta V_o\right\}dt = 0$$

Integrating by parts

$$\int_{t_1}^{t_2}\frac{\bar{m}\ell}{3}\dot{U}_o\delta\dot{U}_o dt = \underbrace{\left.\frac{\bar{m}\ell}{3}\dot{U}_o\delta U_o\right|_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2}\frac{\bar{m}\ell}{3}\ddot{U}_o\delta U_o dt$$

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- for arbitrary variations in $\delta U_o \; \delta V_o$

$$\delta U_0 : \underbrace{\frac{\bar{m}\ell}{3}}_{M}\ddot{U}_o + \overbrace{\frac{\bar{c}}{\ell}}^{K}U_o + \overbrace{\frac{\bar{e}}{\ell}}^{\theta'}V_o = p_x\frac{\ell}{2} + P_o \text{ : Actuator}$$

$$\delta V_0 : \qquad -\frac{\bar{e}}{\ell}U_o + \frac{\varepsilon}{\ell}V_o = q_x\frac{\ell}{2} + Q_o \text{ : Sensor}$$

$$M = \frac{\bar{m}\ell}{3}$$

$$K = \frac{\bar{c}}{\ell}$$

$$\theta = -\frac{\bar{e}}{\ell}$$

$$c_p = \frac{\bar{\varepsilon}}{\ell}$$

- Simplifying case

  Free response to applied voltage, V_o at end electrode → quantitative

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$$U_o = -\frac{\bar{e}}{\bar{c}}\frac{V_o}{\ell}$$

$$\bar{\varepsilon} = A_1\,\varepsilon_{11} + A_2\,\varepsilon_{11}{}^S$$

$$\bar{c} = A_1\,c_{11} + A_2\,c_{11}{}^E$$

$$\bar{e} = A_2\,e_{11}$$

$$\frac{U_o}{\ell} = S_1 = -\left(\frac{A_2}{A_1\,c_{11} + A_2\,c_{11}{}^S}\right)e_{11}E_o$$

if you say $e_{11} / c_{11}{}^S \simeq d_{11},\;\; d_{11}E_o = \Lambda_o$

$$S_1 = -\left(\frac{1}{1+\psi}\right)\Lambda_o$$

$$\psi = \left(\frac{A_1\,c_{11}}{A_2\,c_{11}{}^S}\right)$$