Field and Wave Electromagnetic

Chapter9

Theory and Applications of Transmission Lines

Transmission Line

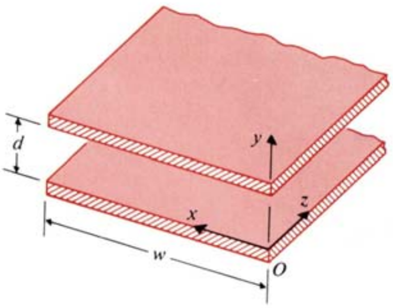
- TEM (Transverse electromagnetic) waves guided by transmission lines.
  - $\bar{E} \perp \bar{H} \perp \bar{k}$ (along the guiding line)

- The three most common types of guiding structures that support TEM waves.
  - (a) Parallel-plate transmission line ⇒ striplines
  - (b) Two wire transmission line
  - (c) Coaxial cable : No stray fields

(a) Parallel-plate transmission line. (b) Two-wire transmission line. (c) Coaxial transmission line.

- TEM wave solution of Maxwell's equations for the parallel-plate guiding structure ⇒ A pair of transmission line equation.

TEM Wave along a Parallel-Plate Transmission Line (1)



① y polarized

② Propagating in the +z direction

$$\overline{E} = \hat{y}E_y = \hat{y}E_0 e^{-\gamma z}$$

$$\overline{H} = \hat{x}H_x = -\hat{x}\frac{E_0}{\eta}e^{-\gamma z}$$

γ : propagating constant

η : intrinsic impedance

cf) Fringe fields at the edges of the plates are neglected.

TEM Wave along a Parallel-Plate Transmission Line (2)

③ Assuming perfect conductor and a lossless dielectric

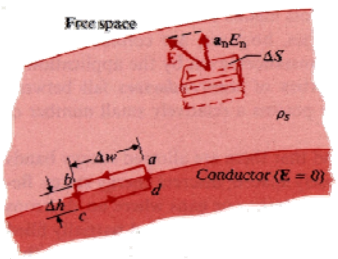
$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

④ Boundary conditions

- At $y = 0$ and $y = d$

$$E_t = 0, H_n = 0 \quad \Rightarrow \quad E_x = E_z = 0, H_y = 0$$



$$\begin{cases} E_{1t} = E_{2t} \\ \hat{n}_2 \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s \end{cases} \qquad \begin{cases} \hat{n}_2 \cdot (\overline{D}_1 - \overline{D}_2) = \rho_s \\ B_{1n} = B_{2n} \end{cases}$$

TEM Wave along a Parallel-Plate Transmission Line (3)

- At $y=0, \quad \hat{n}=\hat{y}$

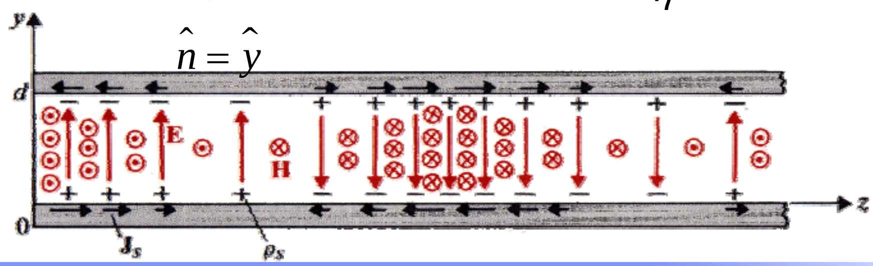
$$\hat{y}\cdot\bar{D}=\rho_{sl} \quad \Rightarrow \quad \rho_{sl}=\varepsilon E_y=\varepsilon E_0 e^{-j\beta z}$$

$$\hat{y}\times\bar{H}=\bar{J}_{sl} \quad \Rightarrow \quad \bar{J}_{sl}=-\hat{z}H_x=\hat{z}\frac{E_0}{\eta}e^{-j\beta z}$$

- At $y=d, \quad \hat{n}=-\hat{y}$

$$-\hat{y}\cdot\bar{D}=\rho_{su} \quad \Rightarrow \quad \rho_{sl}=-\varepsilon E_y=-\varepsilon E_0 e^{-j\beta z}$$

$$-\hat{y}\times\bar{H}=\bar{J}_{su} \quad \Rightarrow \quad \bar{J}_{su}=\hat{z}H_x=-\hat{z}\frac{E_0}{\eta}e^{-j\beta z}$$



TEM Wave along a Parallel-Plate Transmission Line (4)

⑤ $\bar{E}$ & $\bar{H}$ satisfy Maxwell's equation

$$\begin{cases}\nabla\times\bar{E}=-j\omega\mu\bar{H}\\ \nabla\times\bar{H}=j\omega\varepsilon\bar{E}\end{cases}$$

$$\bar{E}=\hat{y}E_y, \quad \bar{H}=\hat{x}H_x$$

$$\nabla\times\bar{E}=-j\omega\mu\bar{H} \quad \Rightarrow \quad \frac{dE_y}{dz}=j\omega\mu H_x \quad \text{——— ①}$$

$$\nabla\times\bar{H}=j\omega\varepsilon\bar{E} \quad \Rightarrow \quad \frac{dH_x}{dz}=j\omega\varepsilon E_y \quad \text{——— ②}$$

(cf) $\frac{\partial}{\partial z}\Rightarrow\frac{d}{dz}$ $\because$ E_y and H_x are functions of z only.

TEM Wave along a Parallel-Plate Transmission Line (5)

Integrating ① over y from 0 to d,

$$\frac{d}{dz}\int_0^d E_y dy = j\omega\mu\int_0^d H_x dy$$

cf) $V_{d0}(z) = -\int_0^d E_y dy = -E_y(z)d$; Potential difference from the lower plate to the upper plate

$H_x = J_{su}(z)$ assuming $\bar{J}_{su} = \hat{z}J_{su}(z)$

$\underline{I(z) = J_{su}w}$ Where w is the width of the plate

The total current flowing in the +z direction

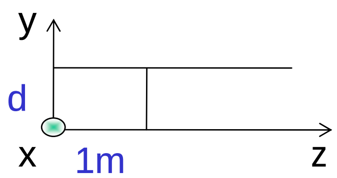
TEM Wave along a Parallel-Plate Transmission Line (6)

Then $$-\frac{dV(z)}{dz} = j\omega\mu J_{su}(z)d = j\omega\left(\mu\frac{d}{w}\right)\left[J_{su}(z)w\right]$$

i.e. $$-\frac{dV(z)}{dz} = j\omega LI(z) \quad \text{———— ①'}$$

where $L = \mu\frac{d}{w}$ (H/m) : inductance per unit length of the parallel-plate transmission line

Flux linkage per unit current



$$= \frac{\Phi}{I} = \frac{BS}{I}$$

$$= \frac{\mu Hd1}{Hw} = \mu\frac{d}{w}$$

Integrating ② over x from 0 to w,

$$\frac{d}{dz}\int_0^w H_x dx = j\omega\varepsilon\int_0^w E_y dx$$

TEM Wave along a Parallel-Plate Transmission Line (7)

$$cf)\ \int_0^w H_x dx = I(z), \quad \int_0^w E_y dx = E_y(z) w$$

$$E_y(z) d = -V(z)$$

$$\therefore \frac{d}{dz} I(z) = j\omega\varepsilon E_y(z) w$$

$$= j\omega\left(\varepsilon \frac{w}{d}\right) E_y(z) d = -j\omega\left(\varepsilon \frac{w}{d}\right) V(z)$$

i.e. $-\frac{d}{dz} I(z) = j\omega C V(z)$ ——— ②'

where $C = \varepsilon \frac{w}{d} (F/m)$: capacitance per unit length of the parallel-plate transmission lines

①' & ②' : Time-harmonic transmission line equations.

TEM Wave along a Parallel-Plate Transmission Line (8)

Combining ①' & ②'

$$\left.\begin{aligned} \frac{d^2 V(z)}{dz^2} &= -\omega^2 LCV(z) \\ \frac{d^2 I(z)}{dz^2} &= -\omega^2 LCI(z) \end{aligned}\right\rangle \text{Wave equations}$$

The solutions of the above wave equations are waves propagating in the +z direction.

$$\begin{cases} V(z) = V_0 e^{-j\beta z} \\ I(z) = I_0 e^{-j\beta z} \end{cases}$$

where $\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\frac{d}{w}\cdot\varepsilon\frac{w}{d}} = \omega\sqrt{\mu\varepsilon}$

TEM Wave along a Parallel-Plate Transmission Line (9)

$$Z_0 = \frac{V(z)}{I(z)} = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} \qquad (\Omega)$$

$$-\frac{dV}{dz} = j\omega LI$$

$$j\beta V_0 = j\omega LI_0$$

$$\frac{V_0}{I_0} = \frac{\omega L}{\beta} = \frac{\omega L}{\omega\sqrt{LC}} = \sqrt{\frac{L}{C}}$$

The impedance at any location that looks toward an infinitely long transmission line

⇒ Characteristic impedance of the line

$$Z_0 = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}} = \frac{d}{w}\eta$$

The propagating velocity

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} \quad (m/s)$$

Lossy Parallel-plate Transmission Lines (1)

❖ Two loss mechanism $\left\langle \begin{array}{l}\text{dielectric loss}\\ \text{ohmic loss}\end{array}\right.$

① Dielectric loss : dielectric medium have a non-vanishing loss tangent

i.e. $\left\langle \begin{array}{l}\text{permitivity } \varepsilon\\ \text{conductivity } \sigma \text{ of the dielectric medium}\end{array}\right.$

(cf) Reminding

$$C = \frac{Q}{V} = \frac{\oint \bar{D}\cdot d\bar{s}}{-\int_L \bar{E}\cdot d\bar{l}} = \frac{\oint \varepsilon\bar{E}\cdot d\bar{s}}{-\int_L \bar{E}\cdot d\bar{l}}$$

Integration over a surface enclosing the positive conductor

Line integration from the lower potential to the higher potential

Lossy Parallel-plate Transmission Lines (2)

$$R = \frac{V}{I} = \frac{-\int_L \overline{E} \cdot d\overline{l}}{\oint \overline{J} \cdot d\overline{s}} = \frac{-\int_L \overline{E} \cdot d\overline{l}}{\oint \sigma \overline{E} \cdot d\overline{s}}$$

$$\therefore RC = \frac{C}{G} = \frac{\varepsilon}{\sigma}$$

$$\therefore G = \frac{\sigma}{\varepsilon} C = \frac{\sigma}{\varepsilon} \cdot \varepsilon \frac{w}{d} = \sigma \frac{w}{d}$$

: Conductance per unit length (dielectric medium)

Lossy Parallel-plate Pransmission Lines (3)

② Ohmic loss

If the parallel-plate conductors have a very large but finite conductivity σ_c, ohmic power will be dissipated in the plates.

⇒ Nonvanishing axial electric field $\hat{z}E_z$ at the plate surfaces (conduction current)

$$\overline{P}_{av} = \hat{y} p_\sigma = \underbrace{\frac{1}{2} \mathrm{Re}(\hat{z}E_z \times \hat{x}H_x^*)}$$

: y component (loss)

The average power per unit area dissipated in each of the conducting plates

Lossy Parallel-plate Pransmission Lines (4)

Consider the upper plate

$$J_{su} = H_x$$

Surface impedance of an imperfect conductor : Z_s

$$Z_s = \frac{E_t}{J_s}(\Omega)$$

: The ratio of the tangential component of the electric field to the surface current density at the conductor surface

For upper plate

$$Z_s = \frac{E_z}{J_{su}} = \frac{E_z}{H_x} = \eta_c$$

: Intrinsic impedance of the plate conductor

cf) $\sigma_c >> 1,\ f >> 1$ ⇒ only surface current flows

Lossy Parallel-plate Pransmission Lines (5)

Intrinsic impedance of good conductor

$$Z_s = R_s + jX_s = (1+j)\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

$$P_\sigma = \frac{1}{2}\mathrm{Re}(|J_{su}|^2 Z_s)$$

$$= \frac{1}{2}|J_{su}|^2 R_s \quad \left(W/m^2\right)$$

$$\boxed{\begin{aligned} E_z &= J_{su} Z_s \\ H_x &= J_{su} \end{aligned}}$$

∴ The ohmic power dissipated in a unit length of the plate having a width w is wP_σ

Lossy Parallel-plate Pransmission Lines (6)

$$P_\sigma = wp_\sigma = \frac{1}{2} I^2 \left(\frac{R_s}{w}\right) \quad (W/m)$$ ⇒ Power loss in upper plate only

$$\text{where} \quad I = wJ_{su}$$

→ The power dissipated when a sinusioidal current of amplitude I flows through a resistance R_s/w

$$R = 2\left(\frac{R_s}{w}\right) = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega/m)$$

→ Effective series resistance per unit length for both plates of a parallel-plate transmission line of width w

General Transmission Line Equations (1)

cf) Difference between transmission lines and ordinary electric networks

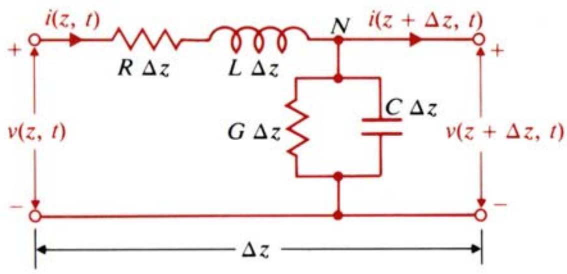
Electric Network	T.L.
Physical dimensions ≪ λ	Physical dimension ~ λ
Discrete circuit elements (lumped parameters)	Distributed-parameter
No standing wave	Standing wave except under matched conditions

General Transmission Line Equations (2)

- Distributed parameters
  - For differential length Δz

Series element: R : resistance per unit length(for both conductors) (Ω/m)
L : inductance per unit length (for both conductors) (H/m)

Shunt element: G : conductance per unit length (S/m)
C : capacitance per unit length (F/m)



General Transmission Line Equations (3)

Kirchhoff's voltage law

$$v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$$

$$-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}$$

let $\Delta z \to 0$ $\quad -\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}$ ——— ①

Kirchhoff's current law at node N

$$i(z,t) - G\Delta z v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

let $\Delta z \to 0$ $\quad -\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t}$ ——— ②

①, ② : General transmission line equations.

General Transmission Line Equations (4)

For time harmonic,

$$v(z,t) = \mathrm{Re}\left[V(z)e^{j\omega t}\right]$$
$$i(z,t) = \mathrm{Re}\left[I(z)e^{j\omega t}\right]$$

cf) cosine reference

V(z), I(z) : functions for the space coordinate z only, both may be complex

then, $\left.\begin{aligned} -\frac{dV(z)}{dz} &= (R + j\omega L)I(z) \\ -\frac{dI(z)}{dz} &= (G + j\omega C)V(z) \end{aligned}\right]$ $\Rightarrow$ Time-harmonic transmission line equations

Wave Characteristics on an Infinite T.L. (1)

From the coupled time-harmonic T.L. equations

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \quad \text{——— ①'}$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z) \quad \text{——— ②'}$$

where $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (m^{-1})$

γ : propagation constant

α : attenuation constant (Np/m)

β : phase constant (rad/m)

Wave Characteristics on an Infinite T.L. (2)

The solutions of ①' and ②'

$$V(z) = V^+(z) + V^-(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I^+(z) + I^-(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

wave amplitudes $(V_0^+, I_0^+)(V_0^-, I_0^-)$

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}$$

For an infinite line (semi-infinite line with the source at the left end)

$e^{\gamma z}$ → vanishes. (no reflected waves)

only waves traveling in the +z direction

Wave Characteristics on an Infinite T.L. (3)

$$V(z) = V^+(z) = V_0^+ e^{-\gamma z}$$

$$I(z) = I^+(z) = I_0^+ e^{-\gamma z}$$

$$\underline{Z_0 = \frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}}$$

→ Characteristic impedance

; independent of z

cf) uniform plane waves in a lossy medium

$$\gamma = \alpha + j\beta = \sqrt{(\omega\mu'' + j\omega\mu')(\omega\varepsilon'' + j\omega\varepsilon')}$$

$$\eta_c = \sqrt{\frac{\mu'' + j\mu'}{\varepsilon'' + j\varepsilon'}}$$

Wave Characteristics on an Infinite T.L. (4)

1. Lossless line
   a. Propagation constant

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$
$$\alpha = 0$$
$$\beta = \omega\sqrt{LC}$$ (A linear function of ω)

   b. Phase velocity

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$ (Non-dispersive)

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Wave Characteristics on an Infinite T.L. (5)

   c. Characteristic impedance

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}$$
$$R_0 = \sqrt{\frac{L}{C}}$$ (constant)
$$X_0 = 0$$ (Non-reactive line)

Wave Characteristics on an Infinite T.L. (6)

2. Low-loss line
   a. Propagation constant

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{1/2}$$

$$\cong j\omega\sqrt{LC}\left(1+\frac{R}{2j\omega L}\right)\left(1+\frac{G}{2j\omega C}\right)$$

$$\cong j\omega\sqrt{LC}\left[1+\frac{1}{2j\omega}\left(\frac{R}{L}+\frac{G}{C}\right)\right]$$

$$\therefore \alpha \cong \frac{1}{2}\left(R\sqrt{\frac{C}{L}}+G\sqrt{\frac{L}{C}}\right), \quad \beta \cong \omega\sqrt{LC}$$

(Approximately a linear function of ω)

Wave Characteristics on an Infinite T.L. (7)

b. Phase velocity

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{LC}} \qquad \text{(Non-dispersive)}$$

c. Characteristic impedance

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left[1+\frac{1}{2j\omega}\left(\frac{R}{L}-\frac{G}{C}\right)\right]$$

$$R_0 \cong \sqrt{\frac{L}{C}}, \quad X_0 \cong \underbrace{-\sqrt{\frac{L}{C}}\frac{1}{2\omega}\left(\frac{R}{L}-\frac{G}{C}\right)}_{\text{Capacitive reactance}} \cong 0$$

Wave Characteristics on an Infinite T.L. (8)

3. Distortionless line $\left(\frac{R}{L}=\frac{G}{C}\right)$

a. Propagation constant

$$\gamma=\alpha+j\beta=\sqrt{(R+j\omega L)\left(\frac{RC}{L}+j\omega C\right)}$$

$$=\sqrt{\frac{C}{L}}(R+j\omega L)$$

$$\therefore \alpha\equiv R\sqrt{\frac{C}{L}},\quad \beta=\omega\sqrt{LC}$$ (A linear function of ω)

b. Phase velocity

$$u_p=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}}$$ (constant)

Wave Characteristics on an Infinite T.L. (9)

c. Characteristic impedance

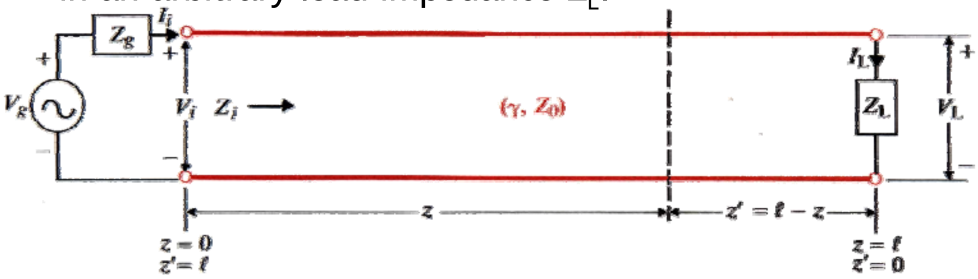
$$Z_0=R_0+jX_0=\sqrt{\frac{R+j\omega L}{\frac{RC}{L}+j\omega C}}=\sqrt{\frac{L}{C}}$$

$$R_0=\sqrt{\frac{L}{C}}$$ (constant)

$$X_0=0$$

Wave Characteristics on Finite Transmission Line (1)

The general case of a finite transmission line (Z_0) terminated in an arbitrary load impedance Z_L.



A sinusoidal voltage source $V_g \angle 0°$ with internal impedance Z_g is connected to the line at z=0.

$\left(\frac{V}{I}\right)_{z=l} = \frac{V_L}{I_L} = Z_L$: Cannot be satisfied without $e^{\gamma z}$ term unless $Z_L = Z_0$

$\Rightarrow$ Reflected waves exist on unmatched lines

Wave Characteristics on Finite Transmission Line (2)

General solutions for the time-harmonic one-dimensional Helmholtz equations

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$where \quad \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0$$

(cf) - circuit theory

matched condition ($Z_g = Z_L^*$) $\rightarrow$maximum transfer of power

- T.L.

line is matched when $Z_L = Z_0$. $\rightarrow$ no $e^{\gamma z}$ term

Wave Characteristics on Finite Transmission Line (3)

Four unknowns

$V_0^+, I_0^+, V_0^-, I_0^-$: from the wave equation solutions

cf) not independent because of the constraint by the relations at z=0 and z=l

Let z=l

$$\begin{cases} V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l} \\ I_L = \dfrac{V_0^+}{Z_0} e^{-\gamma l} - \dfrac{V_0^-}{Z_0} e^{\gamma l} \end{cases} \Rightarrow \begin{cases} V_0^+ = \dfrac{1}{2}\left(V_L + I_L Z_0\right) e^{\gamma l} \\ V_0^- = \dfrac{1}{2}\left(V_L - I_L Z_0\right) e^{-\gamma l} \end{cases}$$

and $\dfrac{V_L}{I_L} = Z_L$

Wave Characteristics on Finite Transmission Line (4)

$$\therefore V(z) = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma(l-z)} + (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

$$I(z) = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma(l-z)} - (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

New variable $z' = l - z$: distance measured backward from the load

$$V(z') = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right]$$

$$I(z') = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}\right]$$

Wave Characteristics on Finite Transmission Line (5)

In order to simplify the above equations, using hyperbolic functions

$$e^{\gamma z'} + e^{-\gamma z'} = 2\cosh \gamma z' \qquad e^{\gamma z'} - e^{-\gamma z'} = 2\sinh \gamma z'$$

$$\boxed{\begin{aligned} \therefore V(z') &= I_L \left(Z_L \cosh \gamma z' + Z_0 \sinh \gamma z' \right) \\ I(z') &= \frac{I_L}{Z_0}\left(Z_L \sinh \gamma z' + Z_0 \cosh \gamma z' \right) \end{aligned}}$$

Two equations can provide the voltage and current at any point along a transmission line in terms of I_L, Z_L, γ and Z_0 .

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Wave Characteristics on Finite Transmission Line (6)

$$\underline{Z(z')} = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L \cosh \gamma z' + Z_0 \sinh \gamma z'}{Z_L \sinh \gamma z' + Z_0 \cosh \gamma z'}$$

$$= Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'}$$

Impedance when look toward the load end of the line at a distance z' from the load

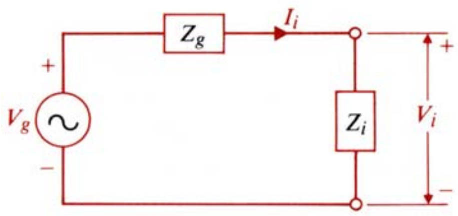
Wave Characteristics on Finite Transmission Line (7)

$z' = l(z=0)$ (at the source end of the line)

the generator looking into the line sees an input impedance Z_i

$$Z_i = (Z)_{\substack{z=0 \\ z'=l}} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l}$$

(cf) Impedance transformation



$$\left[\begin{array}{l} V_i = \dfrac{Z_i}{Z_g + Z_i} V_g \\ I_i = \dfrac{V_g}{Z_g + Z_i} \end{array}\right.$$

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Wave Characteristics on Finite Transmission Line (8)

The average power delivered to the input terminals of the line

$$(P_{av})_i = \frac{1}{2} \mathrm{Re}\left[V_i I_i^*\right]_{z=0, z'=l}$$

The average power delivered to the load

$$(P_{av})_L = \frac{1}{2} \mathrm{Re}\left[V_L I_L^*\right]_{z=l, z'=0} = \frac{1}{2} \left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2} |I_L|^2 R_L$$

For a lossless line

$$(P_{av})_i = (P_{av})_L$$

Wave Characteristics on Finite Transmission Line (9)

- If $Z_L = Z_0, \quad Z(z') = Z_0$

$$\left.\begin{aligned} \therefore V(z) &= (I_L Z_0 e^{\gamma l})e^{-\gamma z} = V_i e^{-\gamma z} \\ I(z) &= (I_L e^{\gamma l})e^{-\gamma z} = I_i e^{-\gamma z} \end{aligned}\right\}$$

Waves traveling in +z direction

$\Rightarrow$ No reflected waves

Transmission Line as Circuit Elements (1)

Transmission line having inductive or capacitive impedance

$\Rightarrow$ impedance matching between a generator and a load.

Frequency band : 300 MHz ~ 3GHz

cf) f < 300MHz : line's physical dimension is too long

f > 3GHz : waveguide is preferred

For lossless T.L.

$$\gamma = j\beta, \quad Z_0 = R_0, \quad \tanh \gamma l = \tanh(j\beta l) = j \tan \beta l$$

Input impedance at distance l from the load(Z_L) end

$$Z_i = R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l}$$

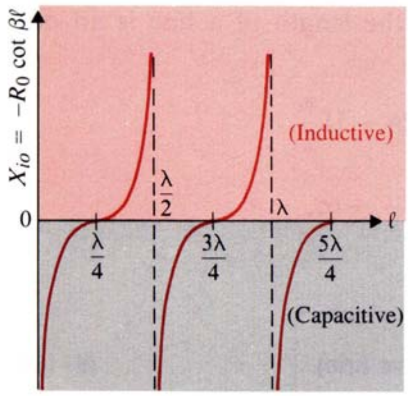
Impedance transformations by lossless transmission line

Transmission Line as Circuit Elements (2)

❖ Special cases

1. Open-circuit termination $(Z_L \to \infty)$

$$Z_{i0} = jX_{i0} = -j\frac{R_0}{\tan\beta l} = -jR_0\cot\beta l \qquad cf)\quad \beta l = \frac{2\pi}{\lambda}l$$



Transmission Line as Circuit Elements (3)

X_{i0} can be either capacitive or inductive depending on βl.

If $\beta l << 1, \quad \tan\beta l \cong \beta l$

$$\therefore Z_{i0} = jX_{i0} \cong -j\frac{R_0}{\beta l} = -j\frac{\sqrt{L/C}}{\omega\sqrt{LC}l} = -j\frac{1}{\omega Cl}$$

; Impedance of a capacitance of Cl farads

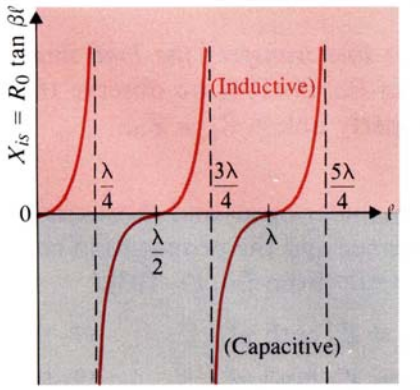
In practice, it is impossible to have an infinite load impedance at the end of a transmission line.

⇒ At high freq. ⇒ coupling and radiation

Transmission Line as Circuit Elements (4)

2. Short circuit termination $(Z_L = 0)$

$$Z_{is} = jX_{is} = jR_0 \tan \beta l$$



$$\beta l << 1$$

$Z_{is} = j\omega L l$: Impedance of inductance

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Transmission Line as Circuit Elements (5)

3. Quarter-wave section : $\left(l = \frac{\lambda}{4}, \ \beta l = \frac{\pi}{2} \right)$

$$l = (2n-1)\frac{\lambda}{4} \quad (n = 1,2,3,\cdots)$$

$$\beta l = \frac{2\pi}{\lambda}(2n-1)\frac{\lambda}{4} = (2n-1)\frac{\pi}{2}$$

$$\tan \beta l = \pm\infty$$

$Z_i = \dfrac{R_0^2}{Z_L}$ Quarter wave line $\Rightarrow$ impedance inverter.

quarter wave transformer.

Transmission Line as Circuit Elements (6)

4. Half-wave section $\left(l = \frac{\lambda}{2}, \quad \beta l = \pi\right)$

$$l = n \cdot \frac{\lambda}{2}, \quad \beta l = n\pi$$

$$\tan \beta l = 0$$

$$\therefore \underline{Z_i = Z_L}$$ (Half-wave line)

Only for lossless.
For lossy case, this properties are valid only for $Z_L = Z_0$

cf) The characteristic impedance and the propagation constant

Open-circuited line, $Z_L \to \infty \quad : \quad Z_{io} = Z_0 \coth \gamma l$

Short-circuited line, $Z_L \to 0 \quad : \quad Z_{is} = Z_0 \tanh \gamma l$

Transmission Line as Circuit Elements (7)

$$\therefore Z_0 = \sqrt{Z_{io} Z_{is}}$$

$$\gamma = \frac{1}{l} \tanh^{-1} \sqrt{\frac{Z_{is}}{Z_{io}}} \quad (m^{-1})$$

5. Lossy line with a short-circuit termination

$$Z_{is} = Z_0 \tanh \gamma l = Z_0 \frac{\sinh(\alpha + j\beta)l}{\cosh(\alpha + j\beta)l}$$

$$= Z_0 \frac{\sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l}{\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l}$$

Transmission Line as Circuit Elements (8)

For $l = n \cdot \dfrac{\lambda}{2} \Rightarrow \beta l = n\pi, \quad \sin\beta l = 0, \quad \cos\beta l = (-1)^n$

$\therefore Z_{is} = Z_0 \tanh\alpha l \cong Z_0 \left(\alpha l\right)$ assuming $\alpha l << 1$

$\tanh\alpha l \cong \alpha l$

: Series resonant circuit condition

For $l = n \cdot \dfrac{\lambda}{4} \Rightarrow \beta l = \dfrac{n\pi}{2}, \quad \left(n = odd\ number\right)$

$\cos\beta l = 0$

$\therefore Z_{is} = \dfrac{Z_0}{\tanh\alpha l} \cong \dfrac{Z_0}{\alpha l}$: Very large

: Parallel-resonant circuit condition

Reference & Homework

- Ref. Microwave engineering by David M. Pozar page 330-336

Lines with Resistive Termination (1)

$Z_L \neq Z_0$ both incident and reflected wave exist

$$V(z) = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma(l-z)} + (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

$$I(z) = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma(l-z)} - (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

$$\Rightarrow V(z') = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right]$$

$$I(z') = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}\right]$$

where $z' = l - z \quad \Rightarrow \quad e^{\gamma z'}$: right traveling wave (incident wave)

$e^{-\gamma z'}$: left traveling wave (reflected wave)

Lines with Resistive Termination(2)

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right]$$

$$= \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \Gamma e^{-2\gamma z'}\right]$$

where $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma}$: Voltage reflection coefficient of the load impedance Z_L

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}\left[1 - \Gamma e^{-2\gamma z'}\right]$$

cf) current reflection coefficient of the load impedance Z_L

$$\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma$$

Lines with Resistive Termination(3)

- For a lossless transmission line, $\gamma = j\beta$

$$V(z') = \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}\left[1 + \Gamma e^{-j2\beta z'}\right]$$
$$= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}\left[1 + |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right]$$
$$I(z') = \frac{I_L}{2R_0}(Z_L + R_0)e^{j\beta z'}\left[1 - |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right]$$

- From the expression using hyperbolic functions

$$\boxed{\begin{aligned} V(z') &= I_L(Z_L \cosh \gamma z' + Z_0 \sinh \gamma z') \\ I(z') &= \frac{I_L}{Z_0}(Z_L \sinh \gamma z' + Z_0 \cosh \gamma z') \end{aligned}}$$

Lines with Resistive Termination(4)

- For lossless line

$$\gamma = j\beta,\ V_L = I_L Z_L,\ \ \cosh j\theta = \cos\theta,\ \ \sinh j\theta = j\sin\theta$$

$$\boxed{\begin{aligned} V(z') &= V_L \cos\beta z' + jI_L R_0 \sin\beta z' \\ I(z') &= I_L \cos\beta z' + j\frac{V_L}{R_0}\sin\beta z' \end{aligned}}$$

- If $Z_L = R_L,\ \ V_L = I_L R_L$

$$|V(z')| = V_L\sqrt{\cos^2\beta z' + \left(\frac{R_0}{R_L}\right)^2 \sin^2\beta z'}$$
$$|I(z')| = I_L\sqrt{\cos^2\beta z' + \left(\frac{R_L}{R_0}\right)^2 \sin^2\beta z'}, \quad where\ R_0 = \sqrt{\frac{L}{C}}$$

Lines with Resistive Termination(5)

- Standing-wave ratio (SWR)

$$s = \frac{|V_{max}|}{|V_{min}|} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad |\Gamma| = \frac{s-1}{s+1}$$

- For a lossless transmission line

$\Gamma = 0,\ s = 1$ when $Z_L = Z_0$ (Matched load)

$\Gamma = -1,\ s \to \infty$ when $Z_L = 0$ (Short circuit)

$\Gamma = +1,\ s \to \infty$ when $Z_L \to \infty$ (Open circuit)

cf) $|V_{max}|$ and $|I_{min}|$ occur when

$$\theta_\Gamma - 2\beta z'_M = -2n\pi, \quad n = 0,\ 1,\ 2,\ \ldots$$

$|V_{min}|$ and $|I_{max}|$ occur together when

$$\theta_\Gamma - 2\beta z'_m = -(2n+1)\pi, \quad n = 0,\ 1,\ 2,\ \ldots$$

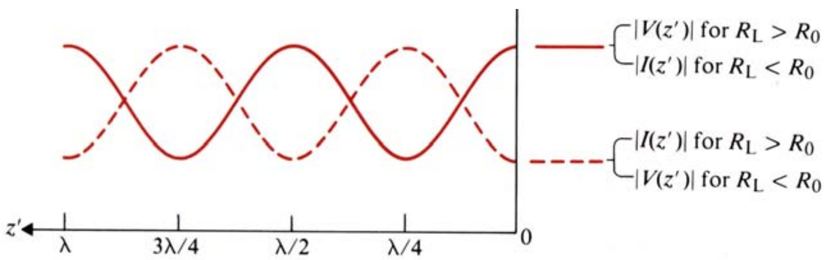
Lines with Resistive Termination(6)

cf) For resistive terminations on a lossless line,

$$Z_L = R_L,\ Z_0 = R_0,\ \Gamma = \frac{R_L - R_0}{R_L + R_0}$$

① $R_L > R_0,\ \Gamma > 0$ and real $(\theta_\Gamma = 0)$

② $R_L < R_0,\ \Gamma < 0$ and real $(\theta_\Gamma = -\pi)$



Lines with Resistive Termination(7)

cf) $R_L > R_0:\ |V_{\max}| = V_L,\qquad |V_{\min}| = V_L\dfrac{R_0}{R_L}\qquad \therefore s = \dfrac{R_L}{R_0}$

$R_L < R_0:\ |V_{\max}| = V_L\dfrac{R_0}{R_L},\quad |V_{\min}| = V_L\qquad \therefore s = \dfrac{R_0}{R_L}$

Lines With Arbitrary Termination(1)

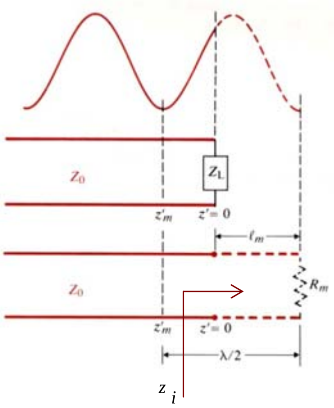
❖ Lines With Arbitrary Termination

Let $Z_L = R_L + jX_L$

– Neither a voltage maximum nor a voltage minimum appears at the load $(at\ z' = 0)$

– If we let the standing wave continue by an extra distance, it will reach a minimum

Lines With Arbitrary Termination(2)



$$Z_L = Z_i\Big|_{\substack{at\ z'=0 \\ onto\ the\ right}} = R_i + jX_i = R_0 \frac{R_m + jR_0 \tan \beta l_m}{R_0 + jR_m \tan \beta l_m}$$

1. Find $|\Gamma|$ from s. use $|\Gamma| = \dfrac{s-1}{s+1}$

2. Find θ_Γ from z'_m. use $\theta_\Gamma = 2\beta z_m' - \pi$ for $n = 0$.

3. Find Z_L, which is the ratio of $\dfrac{V(z')}{I(z')}$ at $z' = 0$.

$$Z_m + l_m = \frac{\lambda}{2}$$

$$Z_L = R_L + jX_L = R_0 \frac{1 + |\Gamma| e^{j\theta_\Gamma}}{1 - |\Gamma| e^{j\theta_\Gamma}}$$

$$R_m = \frac{R_0}{s}$$

Transmission Line Circuits (1)

- Transmission Line Circuits
  - Constraint at the load side (Boundary condition)

    $V_L = I_L Z_L \quad at\ z = l\ or\ z' = 0$
  - Constraint at the generator end where $z = 0$ *and* $z' = l$

    Voltage generator : V_g

    Internal impedance: Z_g

$\therefore\ V_i = V_g - I_i Z_g$ –① $at\ z = 0\ and\ z' = l$

and from the condition of load impedance

$$V_i = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma l}\left[1 + \Gamma e^{-2\gamma l}\right] - ②$$

$$I_i = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma l}\left[1 - \Gamma e^{-2\gamma l}\right] - ③$$

Transmission Line Circuits (2)

put ② and ③ into ①

$$\frac{I_L}{2}(Z_L+Z_0)e^{\gamma l}\left[1+\Gamma e^{-2\gamma l}\right]=V_g-\frac{I_L Z_g}{2Z_0}(Z_L+Z_0)e^{\gamma l}\left[1-\Gamma e^{-2\gamma l}\right]$$

$$\frac{I_L}{2}(Z_L+Z_0)e^{\gamma l}=\frac{Z_0 V_g}{Z_0+Z_g}\frac{1}{\left[1-\Gamma_g\Gamma e^{-2\gamma l}\right]}$$

$$where\ \Gamma_g=\frac{Z_g-Z_0}{Z_g+Z_0}:\ \text{Voltage reflection coefficient}$$

$[H.W]$ Derive the above expression.

$$\therefore\quad V(z')=\frac{Z_0 V_g}{Z_0+Z_g}e^{-\gamma z}\left(\frac{1+\Gamma e^{-2\gamma z'}}{1-\Gamma_g\Gamma e^{-2\gamma l}}\right)$$

$$I(z')=\frac{V_g}{Z_0+Z_g}e^{-\gamma z}\left(\frac{1-\Gamma e^{-2\gamma z'}}{1-\Gamma_g\Gamma e^{-2\gamma l}}\right)$$

Transmission Line Circuits (3)

Furthermore

$$V(z')=\frac{Z_0 V_g}{Z_0+Z_g}e^{-\gamma z}(1+\Gamma e^{-2\gamma z'})(1-\Gamma_g\Gamma e^{-2\gamma l})^{-1}$$

$$=\frac{Z_0 V_g}{Z_0+Z_g}e^{-\gamma z}(1+\Gamma e^{-2\gamma z'})(1+\Gamma_g\Gamma e^{-2\gamma l}+\Gamma_g{}^2\Gamma^2 e^{-4\gamma l}+\cdots)$$

$$=\frac{Z_0 V_g}{Z_0+Z_g}\left[e^{-\gamma z}+(\Gamma e^{-\gamma l})e^{-\gamma z'}+\Gamma_g(\Gamma e^{-2\gamma l})e^{-\gamma z}+\cdots\right]$$

$$=V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots$$

Transmission Line Circuits (4)

$$\text{where } V_1^+ = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z} = V_M e^{-\gamma z}$$

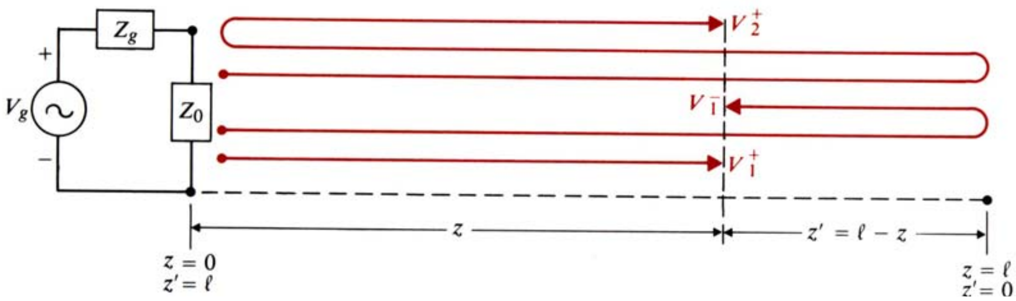
$$V_1^- = \Gamma (V_M e^{-\gamma l}) e^{-\gamma z'}$$

$$V_2^+ = \Gamma_g (\Gamma V_M e^{-2\gamma l}) e^{-\gamma z}$$

$$\vdots$$

$$\text{and } V_M = \frac{Z_0}{Z_0 + Z_g} V_g$$

Transmission Line Circuits (5)



① V_1^+ : the initial wave traveling in the $+z$-direction.

cf) Before this wave reaches the load impedance

it sees Z_0 of the line as if the line were infinitely long.

② When V_1^+ reaches Z_L at $z = l$, it is reflected because of impedance mismatch

$\rightarrow$ reflected wave $V_1^- : \Gamma(V_M e^{-\gamma l}) e^{-\gamma z'}$ traveling in the $-z$-direction.

Transmission Line Circuits (6)

③ As the wave V_1^- returns to the generator at $Z=0$,
it is reflected for $Z_g \neq Z_0 \Rightarrow V_2^+ = \Gamma_g(\Gamma V_M e^{-2\gamma l})$
traveling in the $+z$-direction.

④ This process continues indefinitely with reflections at both ends, and the resulting standing wave $V(z')$ is the sum of all the waves traveling in both directions. $\rightarrow$ Steady state, single frequency, time harmonic sources and signals.

Transmission Line Circuits (7)

cf) special cases

① $Z_L = Z_0$: matched load.
$\Gamma = 0 \Rightarrow$ only V_1^+ exists.

② If $Z_L \neq Z_0$, but $Z_g = Z_0$
$\Gamma \neq 0$ and $\Gamma_g = 0$
$\therefore$ V_1^+ and V_1^- exit.
V_2^+, V_2^- and all higher-order reflections vanish.

Transients on Transmission Lines (1)

Transient Conditions => reactance X, wave length λ, wave number k, and phase constant β would lose their meaning.

Examples of non time harmonic and non steady-state signals are digital pulse signals in computer networks and sudden surges in power and telephone lines.

– Transient behavior of lossless transmission lines.

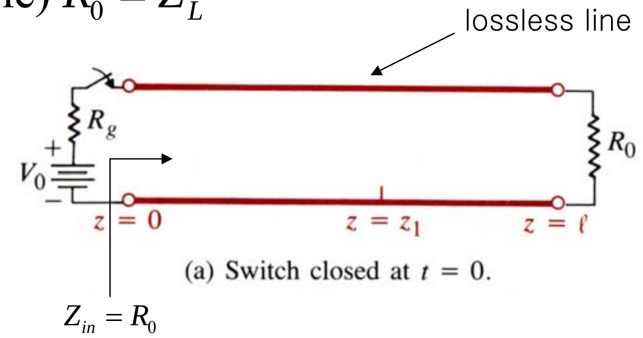
$R = 0,\ G = 0$

Characteristic impedance, Z_0 $\qquad Z_0 = R_0 = \sqrt{\dfrac{L}{C}}$

Propagation velocity, u $\qquad u = \dfrac{1}{\sqrt{LC}}$

Transients on Transmission Lines (2)

Example) $R_0 = Z_L$



(a) Switch closed at $t = 0$.

– Magnitude of voltage wave $\qquad V_1^+ = \dfrac{R_0}{R_0 + R_g} V_0$

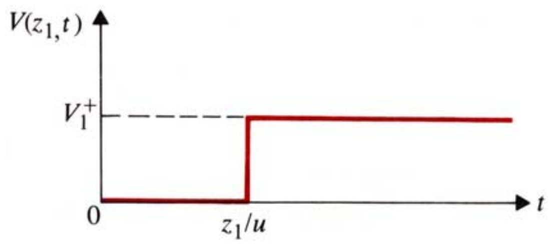
– Voltage wave travels down the line in the +z-direction with a velocity $u = 1/\sqrt{LC}$

– Magnitude of the current wave $\qquad I_1^+ = \dfrac{V_1^+}{R_0} = \dfrac{V_0}{R_0 + R_g}$

Transients on Transmission Lines (3)

– Plot of the voltage across at $z = z_1$, as a function of time

$\Rightarrow$ Delayed unit step functions at $t = z_1 / u$.



(b) Voltage at $z = z_1$.

– When the voltage and current wave reach the termination at $z = l$

$\Rightarrow$ no reflected waves.($\because \Gamma = 0$)

– Steady state $\Rightarrow$ the entire line is charged to a voltage equal to V_1^+.

Transients on Transmission Lines (4)

Example)

– $R_o \neq Z_g$, $R_o \neq Z_L (R_L)$

– Switch is closed at $t = 0$ $\Rightarrow$ the d - c source sends a voltge wave of magnitude

$$V_1^+ = \frac{R_0}{R_0 + R_g} V_0 \text{ in the } +z \text{ direction with a velocity } u = \frac{1}{\sqrt{LC}}$$

– At $t = T = \dfrac{l}{u}$, this wave reaches the load end $z = l$.

$R_L \neq R_0 \Rightarrow$ reflected wave travels in the $-z$ direction

with a magnitude $V_1^- = \Gamma_L V_1^+$

$$\Gamma_L = \frac{R_L - R_0}{R_L + R_0}$$

Transients on Transmission Lines (5)

– At $t = 2T$, this reflected wave reaches the input end where it is reflected by $R_g \neq R_0$

– New voltage wave having a magnitude V_2^+.

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+ \quad \text{where } \Gamma_g = \frac{R_g - R_0}{R_g + R_0}$$

– This process will go on indefinitely

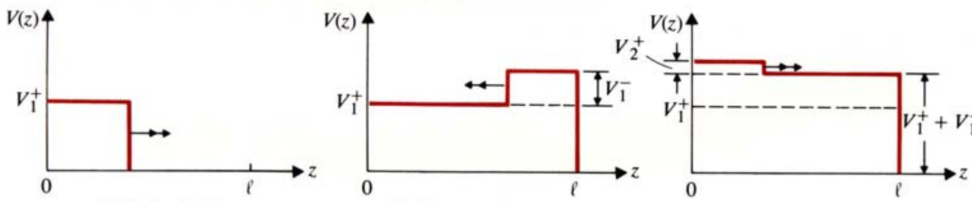
Transients on Transmission Lines (6)

cf) First: Some of the reflected waves traveling in either direction may have a negative amplitude

Second: except for an open circuit or a short circuit

$\Gamma_L, \Gamma_g < 1$

cf) For $R_L = 3R_0 (\Gamma_L = \frac{1}{2}),\ R_g = 2R_0 (\Gamma_g = \frac{1}{3})$



(a) $0 < t < T$, $V_1^+ = V_0/3$ (b) $T < t < 2T$, $V_1^- = V_1^+/2 = V_0/6$ (c) $2T < t < 3T$, $V_2^+ = V_1^-/3 = V_0/18$

note $I_1^- = -\frac{V_1^-}{R_0} = -\frac{V_0}{6R_0}$

Transients on Transmission Lines (7)

The voltage and current at any particular location on the line in any particular time interval are just the algebraic sums $(V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots)$ and $(I_1^+ + I_1^- + I_2^+ + I_2^- + \cdots)$, respectively

Ultimate value of the voltage across the load,

$$\begin{aligned}
V_L = V(l) &= V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + \cdots \\
&= V_1^+(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L{}^2 + \Gamma_g{}^2\Gamma_L{}^2 + \Gamma_g{}^2\Gamma_L{}^3 + \cdots) \\
&= V_1^+[(1 + \Gamma_g\Gamma_L + \Gamma_g{}^2\Gamma_L{}^2 + \cdots) + \Gamma_L(1 + \Gamma_g\Gamma_L + \Gamma_g{}^2\Gamma_L{}^2 + \cdots)] \\
&= V_1^+[(\frac{1}{1 - \Gamma_g\Gamma_L}) + (\frac{\Gamma_L}{1 - \Gamma_g\Gamma_L})] \\
&= V_1^+(\frac{1 + \Gamma_L}{1 - \Gamma_g\Gamma_L})
\end{aligned}$$

Homework

[H.W.] 9-4, 9-9, 9-13, 9-15, 9-20,
9-25, 9-31, 9-36

The Smith Chart (1)

Smith chart

cf) – input impedance

$$Z_{in} = R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l}$$

– reflection coef.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma}$$

– load impedance

$$Z_L = R_L + jX_L = R_0 \frac{1 + |\Gamma| e^{j\theta_\Gamma}}{1 - |\Gamma| e^{j\theta_\Gamma}}$$

The Smith Chart (2)

Manipulations of complex numbers

$\Rightarrow$ The best known and most widely used graphical chart is the smith chart devised By P.H. Smith

Simth chart: A graphical plot of normalized resistance and reactance functions in the reflection coefficient plane

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\theta_\Gamma}$$

The Smith Chart (3)

Let the load impedance Z_L be normalized with respect to $R_0 = \sqrt{\frac{L}{C}}$

$$z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \ (Dimensionless), \quad \text{where} \begin{cases} r : \text{normalized resistance} \\ x : \text{normalized reactance} \end{cases}$$

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1}, \qquad \text{where} \begin{cases} \Gamma_r : \text{real part of } \Gamma \\ \Gamma_i : \text{imaginary part of } \Gamma \end{cases}$$

$$\therefore \quad z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$$

$$\text{or} \quad r + jx = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}$$

$$\text{i.e.} \quad r = \frac{1-\Gamma_r{}^2 - \Gamma_i{}^2}{(1-\Gamma_r)^2 + \Gamma_i{}^2}, \quad x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i{}^2}$$

The Smith Chart (4)

For a given value of r and x, their locus can be plotted in the Γ_r and Γ_i plane.

$$\begin{pmatrix} \Gamma_r : x \text{ axis} \\ \Gamma_i : y \text{ axis} \end{pmatrix}$$

$$\text{i.e.} \quad \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$$

$\Rightarrow$ Equation for a circle with a radius $\frac{1}{1+r}$ and

a center at $\Gamma_r = \frac{r}{1+r}$ and $\Gamma_i = 0$.

$|\Gamma| \le 1$ for a lossless line $\Rightarrow$ that part of the graph lying within unit circle is meaningful.

The Smith Chart (5)

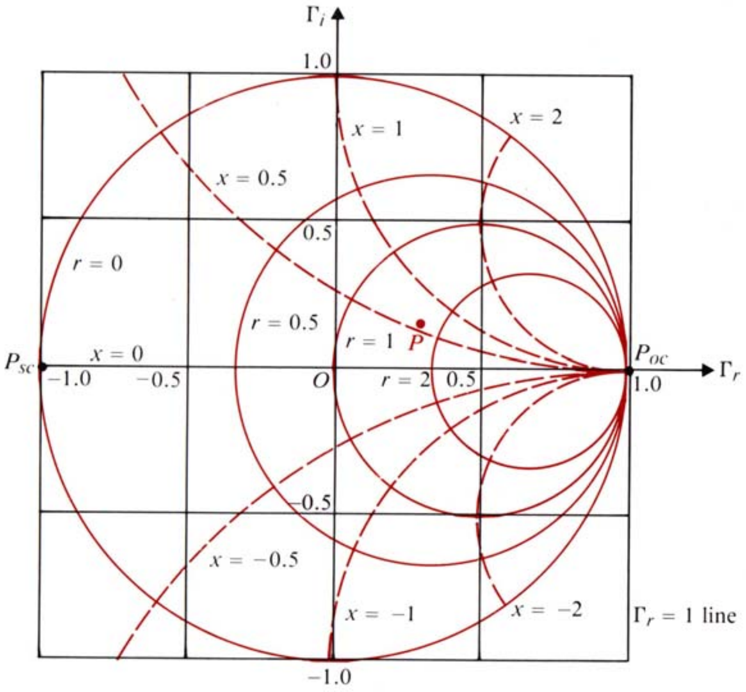


FIGURE 9–30
Smith chart with rectangular coordinates.

The Smith Chart (6)

* Properties of the r-circles are,

1. The centers of all r-circles lie on the Γ_r-axis.
2. The $r = 0$ circle, having a unity radius and centered at the origin, is the largest.
3. The r-circles become progressively smaller as r increase from 0 toward ∞, ending at the $(\Gamma_r = 1, \Gamma_i = 0)$ point for open circuit.
4. All r-circles pass through the $(\Gamma_r = 1, \Gamma_i = 0)$ point.

$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \quad \Rightarrow (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

This is the equation for a circle having a radius $\frac{1}{|x|}$ and centered at different positions on the $\Gamma_r = 1$ and $\Gamma_i = \frac{1}{x}$.

The Smith Chart (7)

* Properties of x-circles.

1. The centers of all x-circles line on the $\Gamma_r = 1$ line;
   those for x>0 (inductive reactance) lie above the Γ_r-axis and
   those for x<0 (capacitive reactance) lie below the Γ_r-axis.

2. The x=0 circle becom0es the Γ_r-axis. (i.e. $\Gamma_i = 0$)

3. The x-circle becomes progressively smaller as $|x|$ increase
   from 0 toward ∞, ending at the $(\Gamma_r = 1, \Gamma_i = 0)$ point for open circuit.
   $\Gamma = 1$

4. All x-circles pass through the $(\Gamma_r = 1, \Gamma_i = 0)$ point.

The Smith Chart (8)

cf) – A smith chart is a chart of r- and x- circles in the $\Gamma_r - \Gamma_i$ plane for $|\Gamma| \le 1$.

– The r-circle and x-circle are everywhere orthogonal to one another.

[H.W.] prove this.

– The intersection point of an r-cirlce and x-circle defines a point that represents a normalized load impedance $z_L = r + jx$.

$\Rightarrow$ Then $Z_L = R_0(r + jx)$.

The Smith Chart (9)

* $|\Gamma|$ Circle

The smith chart can be marked with polar coordinates.

=> i.e. every point in the Γ-plane is specified by

a magnitude $|\Gamma|$ and a phase angle θ_Γ.

cf) – Γ-cirlces are centered at the origin.

– The fractional distance from the center to the point: $|\Gamma|$

– The angle that the line to the point makes with the real axis: θ_Γ

The Smith Chart (10)

Note

– Γ-cirlces intersects the real axis at two points.

– P_M on the positive axis and P_m on the negative axis where x=0, along the real axis.

– $P_M \Rightarrow R_L > R_0$ and $r > 1$.

– $P_m \Rightarrow R_L < R_0$ and $r < 1$.

Remind

– $R_L = SR_0$ for lines with resistive termination and $R_L > R_0$.

– The value of the r-circle passing through the point $P_M = s = \dfrac{R_L}{R_0} = r$(lossless line).

– $R_L = \dfrac{R_0}{s}$ for $R_L < R_0$.

– The value of the r-circle passing through the point $P_m = \dfrac{1}{s}$.

The Smith Chart (11)

Summary

1. All $|\Gamma|$ circles are centered at the origin.

2. Their radii vary uniformly from 0 to 1.

3. The angle measured from the positive real axis of the line drawn from the origin through the point representing z_L equals θ_Γ

4. The value of the r-circle passing through the intersection of the $|\Gamma|$-circle and the positive-real axis = s. (cf. 1/s)

The Smith Chart (12)

* Input impedance and smith chart

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0\left[\frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}\right]$$

$\therefore$ Normalized input impedance

$$z_i = \frac{Z_i}{Z_0} = \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}$$

$$= \frac{1+|\Gamma| e^{j\phi}}{1-|\Gamma| e^{j\phi}} \quad \text{where } \phi = \theta_\Gamma - 2\beta z'$$

Reminding $z_L = \dfrac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}} \Rightarrow$ analogy to z_i except $\phi = \theta_\Gamma - 2\beta z'$

The Smith Chart (13)

note: – The magnitude, $|\Gamma|$, of the reflection coefficient and therefore the standing-wave ratio S, are not changed by the additional line length z'.

– rotation $2\beta z' = 4\pi \dfrac{z'}{\lambda}$

then $\phi = \theta_\Gamma - \boxed{2\beta z'} \rightarrow$ another scale on the $|\Gamma| = 1$ circle.

Reminding

$$z_L = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}, \qquad z_i = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}}$$

– We can use the Smith chart to find $|\Gamma|$ and θ_Γ.

– We can use the Smith chart to find $|\Gamma|$ and ϕ then we can determine z_i.

The Smith Chart (14)

Fig 9-32.

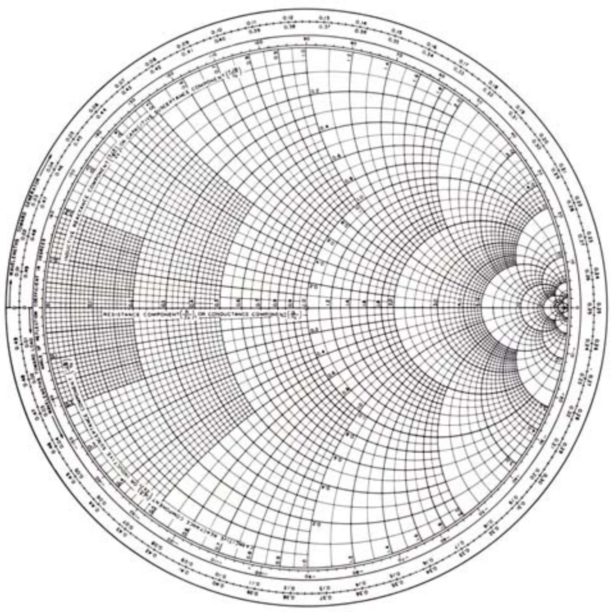


FIGURE 9–32
The Smith chart.

The Smith Chart (15)

cf) – ϕ is the rotated angle from θ_Γ in the clockwise direction

by an angle of $2\beta z' = 4\pi \dfrac{z'}{\lambda}$.

- outer circle : wavelengths to the generator.
- inner circle: wavelength to the load.
- Half wavelength: 2π change in ϕ.
- Complete revolution around a $|\Gamma|$-circle returns to the same point and results in no change in impedance.

The Smith Chart (16)

Ex. 9-14

A lossless transmission line of length 0.434λ and characteristic impedance 100 (Ω), load impedance $260 + j180$ (Ω).

$\Rightarrow z' = 0.434\lambda,\ Z_0 = 100(\Omega),$

$Z_L = 260 + j180(\Omega)$

Find (a) Γ (b) s (C) Z_{in}

(d) z_M (location of voltage maxima

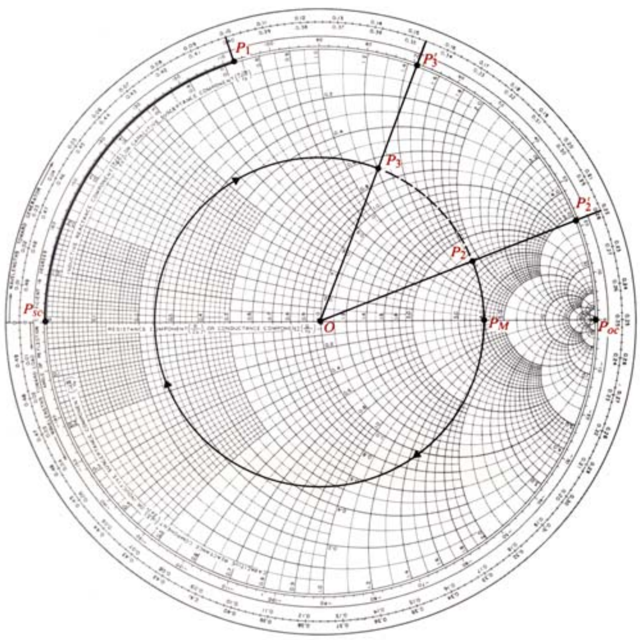


FIGURE 9–33
Smith-chart calculations for Examples 9–13 and 9–14.

The Smith Chart (17)

(sol) a. ① $z_L = \dfrac{Z_L}{R_0} = 2.6 + j1.8$ (point P_2 in Fig. 9-33)

② With the center at the origin, draw a circle passing through point $\overline{OP_2} = |\Gamma| = 0.6$

③ Extend $\overline{OP_2}$ line $+P_2'$ on the periphery. Read the phase angle form the line $\overline{OP_{oc}}$

i.e. $(0.25 - 0.22) \times 4\pi = 0.12\pi$ or $21°$ from the chart.

$\therefore\ \Gamma = |\Gamma| e^{j\theta_\Gamma} = 0.60\angle 21°$

The Smith Chart (18)

b. s $\Rightarrow |\Gamma| = 0.6$ circles intersects with the positive lead axis OP_{oc} at $r = s = 4$.

s = 4.

cf) $|\Gamma| = 0.6$ circles intersects with the negative real axis OP_{oc} at $r = \dfrac{1}{s} = 0.25$.

c. $Z_{in} \Rightarrow$ Rotate the point of Z_L Keeping $|\Gamma|$=0.6 as constant by an angle correspoinding to 0.434 wavelength toward generator(passing through P_{sc}) to the point P_3.

– read the point P_3

$$r = 0.69 \text{ and } x = 1.2$$

$\therefore\ Z_i = R_0 z_i = 100(0.69 + j1.2) = 69 + j120$

The Smith Chart (19)

d. location of voltage Maxima.

wavelength difference between P_2 and P_M

$= 0.030\lambda \Rightarrow$ voltage maxima appears at 0.030λ

from the load foward generator.

cf) Smith chart calculations for lossy lines.

$$z_i = \frac{Z_i}{Z_0} = \frac{1+\Gamma e^{-2\alpha z'} e^{-j2\beta z'}}{1-\Gamma e^{-2\alpha z'} e^{-j2\beta z'}}$$

$$= \frac{1+|\Gamma| e^{-2\alpha z'} e^{j(\theta_\Gamma - 2\beta z')}}{1-|\Gamma| e^{-2\alpha z'} e^{j(\theta_\Gamma - 2\beta z')}}$$

$\therefore$ $|\Gamma|$ circle shrinks as much as $e^{-2\alpha z'}$.

Transmission-Line Impedance Matching (1)

* Transmission line impedance matching.

– Impedance matching by quarter-wave transfomer.

$$Z_{in} = Z_0' \frac{Z_L + jZ_0' \tan \beta l}{Z_0' + jZ_L \tan \beta l}, \quad (Z_0' \text{ is the characteristic impedance of matching line})$$

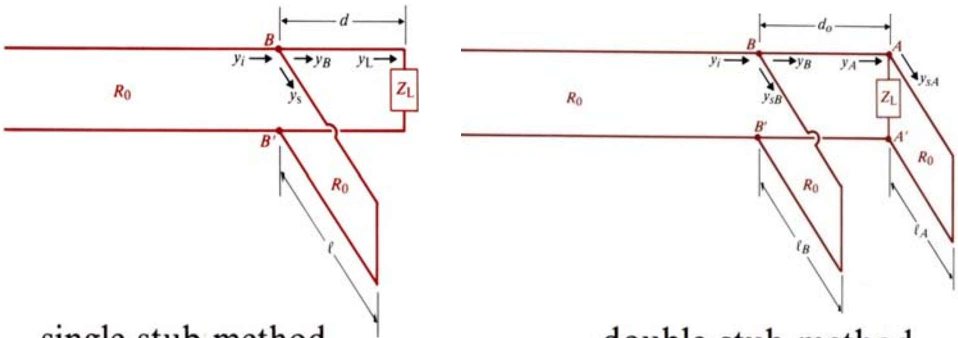
For many cases, $\underbrace{Z_0' = R_0'}_{\text{matching line}}$ (loss load) and $\underbrace{Z_0 = R_0}_{\text{main line}}$.

$\therefore Z_0' = \sqrt{Z_0 Z_L} \Rightarrow R_0' = \sqrt{R_0 Z_L}$

If Z_L is a complex number, it is impossible to construct a impedance matching.

Transmission-Line Impedance Matching (2)

* Impedance and Admittance.



single-stub method. double-stub method.

The shorted line setion (single stub) is connected in parallel with the main line.
It is more convenient to use admittance instead of impedance.

Transmission-Line Impedance Matching (3)

Let $Y_L = \dfrac{1}{Z_L}$

$$z_L = \frac{Z_L}{R_0} = \frac{1}{R_0 Y_L} = \frac{1}{y_L}$$

$$y_L = \frac{Y_L}{Y_0} = \frac{Y_L}{G_0} = R_0 Y_L = g + jb,$$ where g: normalized conductance.

b:normalized susceptance.

(cf) Quarter-wave line

$$Z_{in} = \frac{Z_0{}^2}{Z_L} \Rightarrow \frac{Z_{in}}{Z_0} = \frac{Z_0}{Z_L} = \frac{1}{z_L} = y_L$$

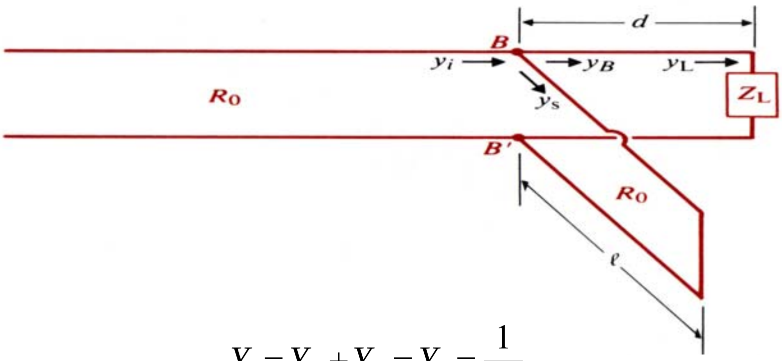
$\therefore$ A quarter wave line transform z_L to y_L.

quarter wave line correspoding to π radians on the Smith chart.

The points representing z_L and y_L are the diametrically opposite to each other on the $|\Gamma|$-circle.

Transmission-Line Impedance Matching (4)

* Single stub matching



$$Y_i = Y_B + Y_S = Y_0 = \frac{1}{R_0}$$

- In terms of normalized admittance,

$$y_B + y_S = 1 \quad \text{where } y_B = R_0 Y_B,\ y_S = R_0 Y_S.$$

- The input admittance of a short-circuit stub is purely subceptive, y_S is purely imaginary

Transmission-Line Impedance Matching (5)

$$\text{cf)}\ Z_{in} = Z_0 \frac{\cancelto{0}{Z_L} + jZ_0 \tan \beta l}{Z_0 + j\cancelto{0}{Z_L} \tan \beta l} = jZ_0 \tan \beta l$$

$$Y_{in} = \frac{-j}{Z_0 \tan \beta l}$$

$$\therefore\ y_B = 1 + jb_B \text{ to satisfy } y_B + y_S = 1 \text{ then } y_S = -jb_B$$

$\Rightarrow$ y_B has a unity real part and a imaginary part that cancel the imaginary part of the stub.

(cf) Smith chart can be used as an admittance chart, in which case the r- and x-circles could be g- and b-circles. The points representing an open- and a short- circuit termination could be the points on the extreme left and the extreme right, respectively, on an admittance chart.

Transmission-Line Impedance Matching (6)

* Using the Smith chart as an admittance chart,
we proceed as follows for single stub matching.

1. Enter the point representing the normalized load admittance, y_L

2. Draw the $|\Gamma| - circle$ for y_L,which will intersect the $g = 1$ circle at two points.
At three points, $y_{B1} = 1 + jb_{B1}$ and $y_{B2} = 1 + jb_{B2}$. Both are possible solutions.

3. Determine load-section lengths d_1 and d_2 from the angles between the point representing y_L and the points representing y_{B1} and y_{B2}.

4. Determine stub length l_{B1} and l_{B2} from the angles between the short-circuit point or the extreme right of the chart to the points representing $-jb_{B1}$and $-jb_{B2}$, respectively.

Transmission-Line Impedance Matching (7)

Ex.9-20. $R_0 = 50,\ Z_L = 35 - j47.5(\Omega)\ \Rightarrow$ single-stub matching
find d and l.

(sol) $z_L = \dfrac{Z_L}{R_0} = 0.70 - j0.95$

1. Enter z_L point $\Rightarrow P_1$
2. Draw a $|\Gamma|$ circle.
3. Find a $y_L \Rightarrow P_2$(rotation of π radian)
4. Two points of intersection of the $|\Gamma|$ circle with the g=1 circle.

   At $P_3 : y_{B1} = 1 + j1.2 = 1 + jb_{B1}$

   At $P_4 : y_{B2} = 1 - j1.2 = 1 + jb_{B2}$

Transmission-Line Impedance Matching (8)

5. Solutions for the position of the stub.

For P_3 $(from\ P_2'\ to\ P_3')$: $d_1 = (0.168 - 0.109)\lambda = 0.059\lambda$

For P_4 $(from\ P_2'\ to\ P_4')$: $d_2 = (0.332 - 0.109)\lambda = 0.223\lambda$

6. Solutions for the length of short-circuited stub to provide $y_S = -jb_B$

For P_3 $\quad -jb_{B1} = -j1.2,$

$l_{B1} = (0.361 - 0.250)\lambda = 0.111\lambda$

For P_4 $\quad -jb_{B2} = j1.2,$

$l_{B2} = (0.139 + 0.250)\lambda = 0.389\lambda$

cf) from P_{sc} to P_4'' in the clock-wise direction.

Homework

H.W.

9-15, 9-19, 9-23, 9-30, 9-33, 9-42, 9-48