Chap. 8 Similitude, Dimensional Analysis, and Normalization of Equation of Motion

Approaches to study flow problems:

- Analytical solutions → available only for special cases
- Physical experiments
  - prototype (field experiment)
  - model (smaller or larger than prototype)
- Numerical experiments → solve governing equations + boundary conditions using a computer

8.1 Similitude (相似法則) and Physical Models

Geometric similitude (幾何學的 相似)

$$\frac{L_p}{L_m} = \text{constant in all directions}$$

$$\frac{A_p}{A_m} = \left(\frac{L_p}{L_m}\right)^2, \qquad \frac{\forall_p}{\forall_m} = \left(\frac{L_p}{L_m}\right)^3$$

Kinematic similitude (運動學的 相似)

$$\frac{V_p}{V_m} = \text{const.}, \quad \frac{a_p}{a_m} = \text{const. everywhere in the flow}$$

Dynamic similitude (力學的 相似)

$$\frac{\vec{F}_p}{\vec{F}_m} = \text{const. for any corresponding fluid masses}$$

where $\vec{F}$ can be any type of force (e.g., pressure, inertia, gravity, viscous, etc.)

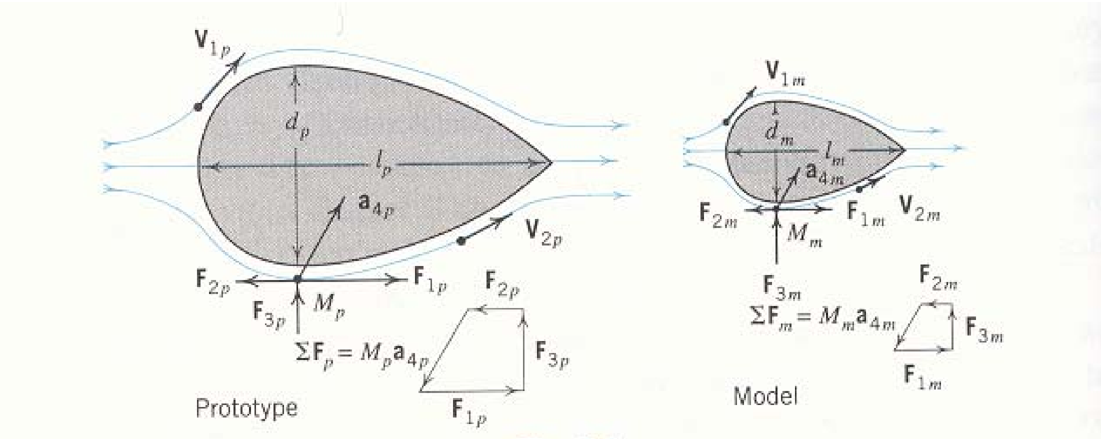


Fig. 8.1

$$\sum \vec{F}_p = \vec{F}_{1_p} + \vec{F}_{2_p} + \vec{F}_{3_p} = M_p \vec{a}_{4_p}, \quad \sum \vec{F}_m = \vec{F}_{1_m} + \vec{F}_{2_m} + \vec{F}_{3_m} = M_m \vec{a}_{4_m}$$

Considering only the magnitudes of the forces under the assumption that the direction of each force is matched between prototype and model, the dynamic similitude gives:

$$\frac{F_{1_p}}{F_{1_m}} = \frac{F_{2_p}}{F_{2_m}} = \frac{F_{3_p}}{F_{3_m}} = \frac{M_p a_{4_p}}{M_m a_{4_m}} = \text{const.}$$

$$\therefore \frac{M_p a_{4_p}}{F_{2_p}} = \frac{M_m a_{4_m}}{F_{2_m}}, \quad \frac{M_p a_{4_p}}{F_{3_p}} = \frac{M_m a_{4_m}}{F_{3_m}}$$

For dynamic similarity among 4 forces, we need 2 equations of force ratios among 3 forces (i.e. F_2, F_3 and $F_I = Ma_4$)

Scalar magnitude of forces:

pressure: $F_P = (\Delta p)A = (\Delta p)l^2$

inertia: $F_I = Ma = \rho l^3 \left(\frac{V^2}{l} \right) = \rho V^2 l^2$

gravity: $F_G = Mg = \rho l^3 g$

viscosity: $F_V = \mu \left(\frac{dv}{dy} \right) A = \mu \left(\frac{V}{l} \right) l^2 = \mu V l$

elasticity: $F_E = EA = El^2$

surface tension: $F_T = \sigma l$

Dimensionless numbers: relative magnitude (or importance) of inertia force w.r.t. other forces

$\frac{F_I}{F_P} = \frac{\rho V^2}{\Delta p}$ $\rightarrow$ Euler number, $\mathbf{E} = V\sqrt{\frac{\rho}{2\Delta p}}$

$\frac{F_I}{F_G} = \frac{V^2}{lg}$ $\rightarrow$ Froude number, $\mathbf{F} = \frac{V}{\sqrt{lg}}$

$\frac{F_I}{F_V} = \frac{\rho V l}{\mu}$ $\rightarrow$ Reynolds number, $\mathbf{R} = \frac{Vl}{\nu}$

$\frac{F_I}{F_E} = \frac{\rho V^2}{E}$ $\rightarrow$ Cauchy number, $\mathbf{C} = \mathbf{M}^2 = \frac{\rho V^2}{E}$

$\frac{F_I}{F_T} = \frac{\rho V^2 l}{\sigma}$ $\rightarrow$ Weber number, $\mathbf{W} = \frac{\rho V^2 l}{\sigma}$

For example, if $\mathbf{F} >> 1$, then $F_I >> F_G$, or the gravity force is negligible compared to inertia force.

$$\left(\frac{F_I}{F_P}\right)_p = \left(\frac{F_I}{F_P}\right)_m = \left(\frac{\rho V^2}{\Delta p}\right)_p = \left(\frac{\rho V^2}{\Delta p}\right)_m \text{ : Euler similarity}$$

$$\left(\frac{F_I}{F_G}\right)_p = \left(\frac{F_I}{F_G}\right)_m = \left(\frac{V^2}{lg}\right)_p = \left(\frac{V^2}{lg}\right)_m \text{ : Froude similarity}$$

$$\left(\frac{F_I}{F_V}\right)_p = \left(\frac{F_I}{F_V}\right)_m = \left(\frac{Vl}{\nu}\right)_p = \left(\frac{Vl}{\nu}\right)_m \text{ : Reynolds similarity}$$

$$\left(\frac{F_I}{F_E}\right)_p = \left(\frac{F_I}{F_E}\right)_m = \left(\frac{\rho V^2}{E}\right)_p = \left(\frac{\rho V^2}{E}\right)_m \text{ : Cauchy similarity}$$

$$\left(\frac{F_I}{F_T}\right)_p = \left(\frac{F_I}{F_T}\right)_m = \left(\frac{\rho V^2 l}{\sigma}\right)_p = \left(\frac{\rho V^2 l}{\sigma}\right)_m \text{ : Weber similarity}$$

5 equations for 6 forces ($F_I, F_P, F_G, F_V, F_E, F_T$)

If 4 equations are simultaneously satisfied, the 5th equation is satisfied automatically.

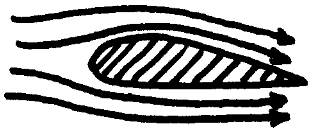
In nature, some of the forces (1) may not act, (2) may be negligible, or (3) may oppose each other so that the effect of both is reduced.

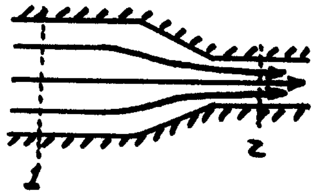
→ do not need to satisfy all the similarity.

→ need experience or good understanding of fluid phenomena to judge which forces are important.

Examples:

(1) Aerofoil (external flow) or pipe flow (internal flow)





F_G, F_E, F_T are not important $\rightarrow$ 3 forces are important (F_P, F_I, F_V)

$\rightarrow$ 1 equation between 2 forces (Reynolds or Euler)

$\rightarrow$ If Reynolds similarity is satisfied, Euler similarity is satisfied automatically, or vice versa.

For drag force, $D = \rho C_D A V^2$, on aerofoil

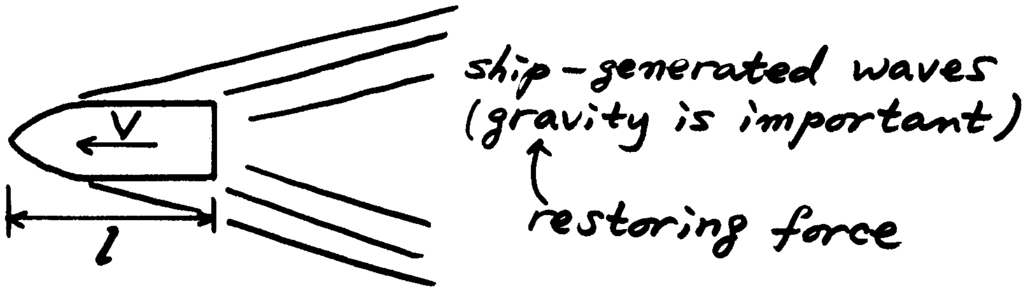
$$\left(\frac{Vl}{\nu}\right)_p = \mathbf{R}_p = \mathbf{R}_m = \left(\frac{Vl}{\nu}\right)_m$$

$$\frac{D_p}{D_m} = \frac{(\rho V^2 l^2)_p}{(\rho V^2 l^2)_m} \quad \rightarrow \quad \left(\frac{D}{\rho V^2 l^2}\right)_p = \left(\frac{D}{\rho V^2 l^2}\right)_m$$

For pressure drop between 1 and 2 in pipe flow

$$\left(\frac{p_1 - p_2}{\rho V^2}\right)_p = \left(\frac{p_1 - p_2}{\rho V^2}\right)_m$$

(2) Ship movement



F_P, F_E, F_T are not important, but F_I, F_G, F_V are important.

Froude similarity: $\left(\dfrac{V}{\sqrt{lg}}\right)_p = \mathbf{F}_p = \mathbf{F}_m = \left(\dfrac{V}{\sqrt{lg}}\right)_m$

Reynolds similarity: $\left(\dfrac{Vl}{\nu}\right)_p = \mathbf{R}_p = \mathbf{R}_m = \left(\dfrac{Vl}{\nu}\right)_m$

8.2 Dimensional Analysis (次元解析)

→ gives relationship among variables of fluid properties and flow characteristics.

Basic dimensions in fluid mechanics:
length (L), mass (M), time (t)

유체역학에 사용되는 변수들의 차원은 basic dimension 들의 곱의 형태로 표시 가능 (Appendix 1 참조)

Ex) Velocity: $V = \frac{L}{t}$

Acceleration: $a = \frac{L}{t^2}$

Force: $F = ma = \frac{ML}{t^2}$

Principle of dimensional homogeneity: 방정식의 각 항들은 같은 차원을 갖는다.

Ex) $P = f(Q, \gamma, E_t)$

where P = power of turbine $\left(\frac{ML^2}{t^3}\right)$

Q = flow rate $\left(\frac{L^3}{t}\right)$

γ = unit weight of fluid $\left(\frac{M}{L^2 t^2}\right)$

E_t = unit mechanical energy of turbine (L)

터빈의 power, P는 Q, γ, E_t의 함수임을 경험적으로 안다고 가정하고 이들 사이에 어떠한 관계가 있는지 dimensional analysis(차원해석)을 통해서 알아내고자 한다. 각 변수의 차원이 서로 다르므로, 더하기나 빼기는 성립하지 않고 각 변수의 거듭제곱의 형태로 나타내야 한다.

$P = CQ^a \gamma^b E_t^c$ 여기서 C = 무차원 경험상수

$$\frac{ML^2}{t^3} = \left(\frac{L^3}{t}\right)^a \left(\frac{M}{L^2 t^2}\right)^b (L)^c$$

$$\left.\begin{aligned} M&: \quad 1 = b \\ L&: \quad 2 = 3a - 2b + c \\ t&: \quad -3 = -a - 2b \end{aligned}\right\} \rightarrow \quad a = 1,\ b = 1,\ c = 1$$

$$\therefore P = CQ\gamma E_t$$

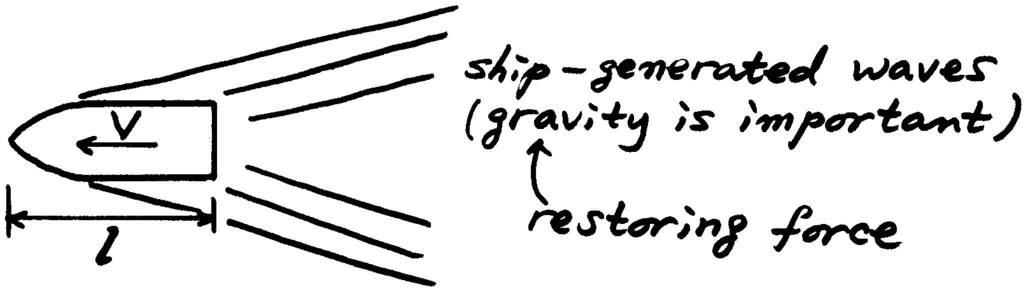
Note: 3 variables (Q, γ, E_t) and 3 basic dimensions (M, L, t) are OK. How about # of variables > 3 ?

Buckingham Π theorem:

1. # of variables = n $\rightarrow$ $f(a_1, a_2, \cdots, a_n) = 0$
2. # of basic dimensions involved = $k \le 3$
3. # of dimensionless groups (Π's) = $n - k$
   $\rightarrow$ $f'(\Pi_1, \Pi_2, \cdots, \Pi_{n-k}) = 0$
4. Dimensional analysis $\rightarrow$ expression of Π's in terms of variables

각각의 variable 간의 관계를 구할 수는 없지만, variable들의 조합으로 이루어지는 dimensionless group (Π's)들을 구할 수 있다.

Ex) Ship movement



ρ, μ, and drag force, D, are important.

$$f(D, \rho, \mu, V, l, g) = 0$$

$$n = 6,\ k = 3,\ \therefore n - k = 3$$

$$f'(\Pi_1, \Pi_2, \Pi_3) = 0$$

$\Pi_1, \Pi_2, \Pi_3 = f(D, \rho, \mu, V, l, g)$ ← 수많은 조합이 가능함

Introduce 'repeating variables', ρ, V, l.

→ 8.1의 무차원수들에 거의 공통적으로 포함되는 변수들

$$\Pi_1 = f(D, \rho, V, l)$$

$$\Pi_2 = f(\mu, \rho, V, l)$$

$$\Pi_3 = f(g, \rho, V, l)$$

Dimensional analysis for Π_1, Π_2, Π_3

$$\Pi_1: \ M^0L^0t^0 = D^a\rho^bV^cl^d = \left(\frac{ML}{t^2}\right)^a\left(\frac{M}{L^3}\right)^b\left(\frac{L}{t}\right)^c(L)^d$$

$$\left.\begin{array}{l} M: a+b=0 \\ L: a-3b+c+d=0 \\ t: -2a-c=0 \end{array}\right\} \rightarrow \left\{\begin{array}{l} b=-a \\ c=-2a \\ d=-2a \end{array}\right.$$

$$\therefore \Pi_1 = D^a\rho^{-a}V^{-2a}l^{-2a} = \left(\frac{D}{\rho V^2 l^2}\right)^a$$

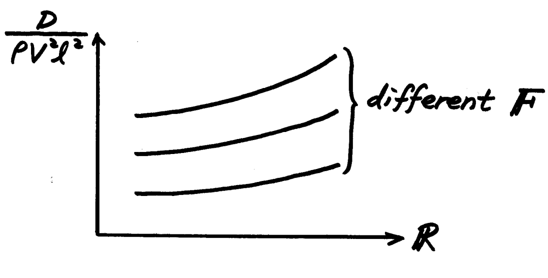
Since Π_1 is dimensionless, the power 'a' is meaningless.

$$\therefore \Pi_1 = \frac{D}{\rho V^2 l^2}$$

Similarly, $\Pi_2 = \frac{Vl\rho}{\mu} = \frac{Vl}{\nu} = \mathbf{R}$, $\Pi_3 = \frac{V}{\sqrt{lg}} = \mathbf{F}$

$$\therefore f'\left(\frac{D}{\rho V^2 l^2}, \mathbf{R}, \mathbf{F}\right) = 0 \rightarrow \frac{D}{\rho V^2 l^2} = f''(\mathbf{R}, \mathbf{F})$$

Carry out experiment and plot experimental results:



Calculate the drag force on prototype using similitude laws.

$$R_p = R_m, F_p = F_m \rightarrow \left(\frac{D}{\rho V^2 l^2}\right)_p = \left(\frac{D}{\rho V^2 l^2}\right)_m$$

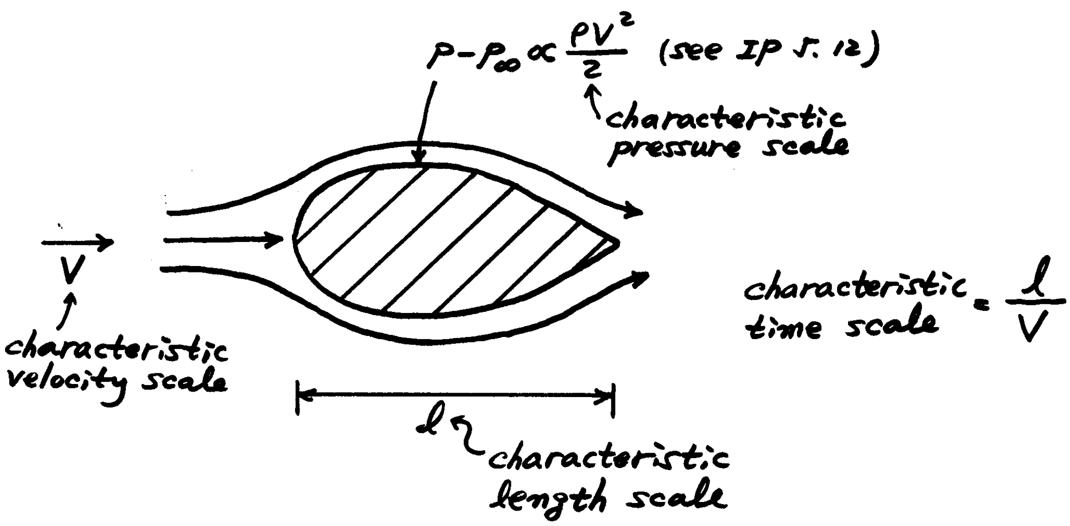
8.3 Normalization of Equations

→ indicates important terms, similarities, Π terms.

Procedure:

(1) Select characteristic scales (for dependent and independent variables).
(2) Define nondimensional variables using the characteristic scales.
(3) Nondimensionalize the equations.

Ex) Flow past a body



G.E.: Navier-Stokes equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-g$$

Define nondimensional variables:

$$X = \frac{x}{l},\ \ Z = \frac{z}{l},\ \ T = \frac{t}{l/V},\ \ U = \frac{u}{V},\ \ W = \frac{w}{V},\ \ P = \frac{p}{\frac{1}{2}\rho V^2}$$

Nondimensionalize the equations:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial (VU)}{\partial T}\frac{\partial T}{\partial t} = \left(\frac{1}{l/V}\right)\frac{\partial (VU)}{\partial T} = \frac{V^2}{l}\frac{\partial U}{\partial T}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial (VU)}{\partial X}\frac{\partial X}{\partial x} = \left(\frac{1}{l}\right)\frac{\partial (VU)}{\partial X} = \frac{V}{l}\frac{\partial U}{\partial X}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{V}{l}\frac{\partial U}{\partial X}\right) = \frac{V}{l}\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial x}\right)$$

$$= \frac{V}{l}\frac{\partial}{\partial X}\left(\frac{1}{V}\frac{\partial u}{\partial x}\right) = \frac{V}{l}\frac{\partial}{\partial X}\left(\frac{1}{l}\frac{\partial U}{\partial X}\right) = \frac{V}{l^2}\frac{\partial^2 U}{\partial X^2}$$

$$\vdots$$

Finally, we get normalized equations:

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z} = -\frac{1}{2}\frac{\partial P}{\partial X} + \frac{\mu}{\rho V l}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2}\right)$$

$$\frac{\partial W}{\partial T} + U\frac{\partial W}{\partial X} + W\frac{\partial W}{\partial Z} = -\frac{1}{2}\frac{\partial P}{\partial Z} + \frac{\mu}{\rho V l}\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Z^2}\right) - \frac{gl}{V^2}$$

1. Important Π terms: $\mathbf{R}, \mathbf{F}$ $\rightarrow$ We can evaluate the importance of viscous force and gravity force w.r.t. inertia force.
2. If $\mathbf{R}_m = \mathbf{R}_p$, $\mathbf{F}_m = \mathbf{F}_p$, then $U_m = U_p, P_m = P_p, \cdots$,

   but $u_m \neq u_p$ because $V_m \neq V_p$.