CHAPTER 7. Plane Electromagnetic Waves

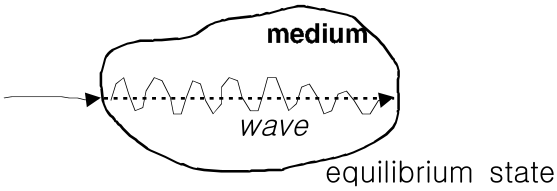
Reading assignments: Cheng Ch.7, Ulaby Ch.6,
Hayt Chs.12, 13, Halliday Chs.33, 34

1. Plane Waves in Lossless Media

A. Representation of Waves

1) General definitions of waves

a) Waves = Externally-excited oscillating or propagating perturbations about an equilibrium state



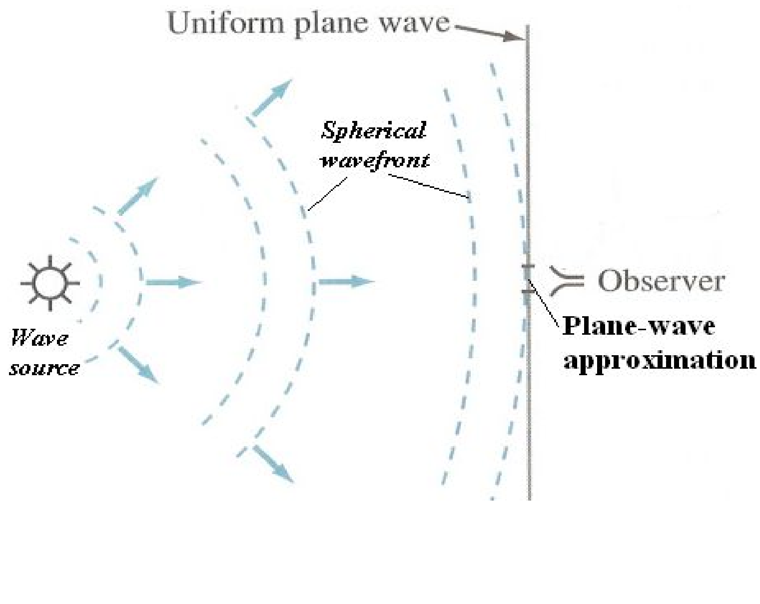
(e.g.) Sound wave, Elastic wave, Spring wave, Water wave, EM wave

Any physical field perturbed by wave can be expressed as

$$g(R, t) = g_o(R) + g_1(R, t) \quad , \qquad |g_1| \ll |g_o| \tag{1}$$

b) Uniform plane wave = Wave with uniform properties (same direction, same magnitude, same phase, ...) at all points in the plane tangent to the wavefront (surface of constant phase)

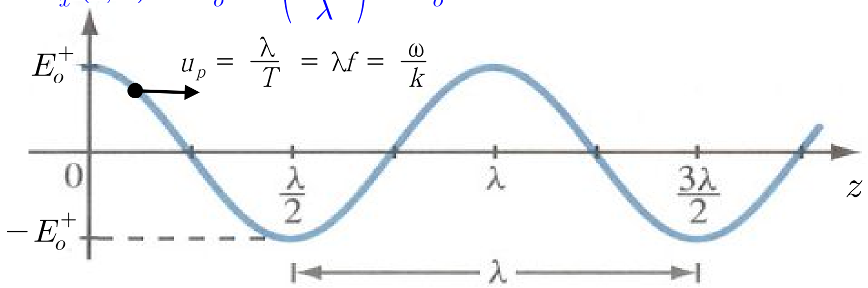
Note) A uniform plane wave ideally exists in a source infinite in extent, but in practice it can be approximated by a spherical wave far away from a source



2) Mathematical representation of waves

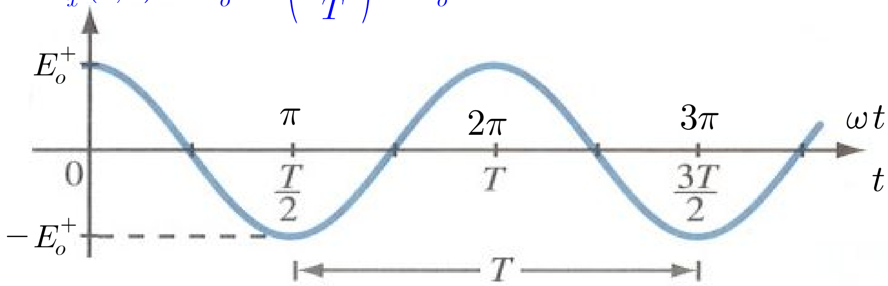
a) A 1-D uniform plane wave E_x travelling in positive z-direction

$$E_x^+(z,0) = E_o^+\cos\left(\frac{2\pi z}{\lambda}\right) = E_o^+\cos kz$$



(a) $E_x^+(z,t)$ versus z at $t=0$

$$E_x^+(0,t) = E_o^+\cos\left(\frac{2\pi t}{T}\right) = E_o^+\cos\omega t$$



(b) $E_x^+(z,t)$ versus t at $z=0$

$$E_x^+(z,t) = E_o^+\cos\left(\frac{2\pi t}{T} - \frac{2\pi z}{\lambda}\right) = E_o^+\cos(\omega t - kz)$$

$$= Re[E_o^+ e^{j(\omega t - kz)}] = Re[E_x^+(z)e^{j\omega t}] \qquad (7\text{-}9)^*$$

where $k \equiv 2\pi/\lambda$ (rad/m) : *wavenumber (propagation constant)* (7-11)

$\omega \equiv 2\pi/T = 2\pi f$ (rad/s) : *angular frequency*

$\phi(z,t) \equiv \frac{2\pi t}{T} - \frac{2\pi z}{\lambda} = \omega t - kz$ (rad) : *phase of the wave*

Phase velocity = velocity of a constant phase point

$$\omega t - kz = const \Rightarrow \frac{d}{dt}(\omega t - kz) = 0 \Rightarrow u_p = \frac{dz}{dt} = \frac{\omega}{k} = \lambda f = \frac{\lambda}{T} \text{ (m/s)} \qquad (7\text{-}10)$$

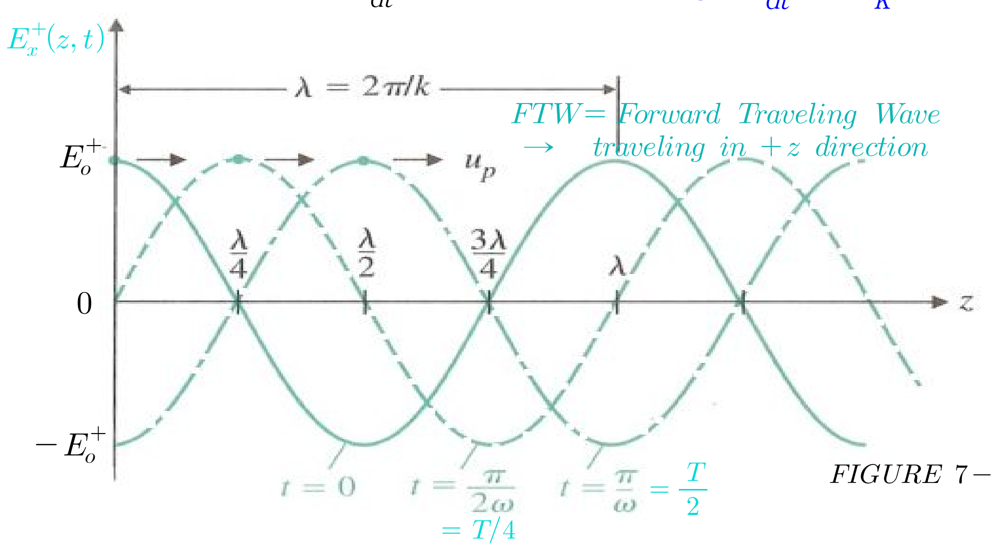
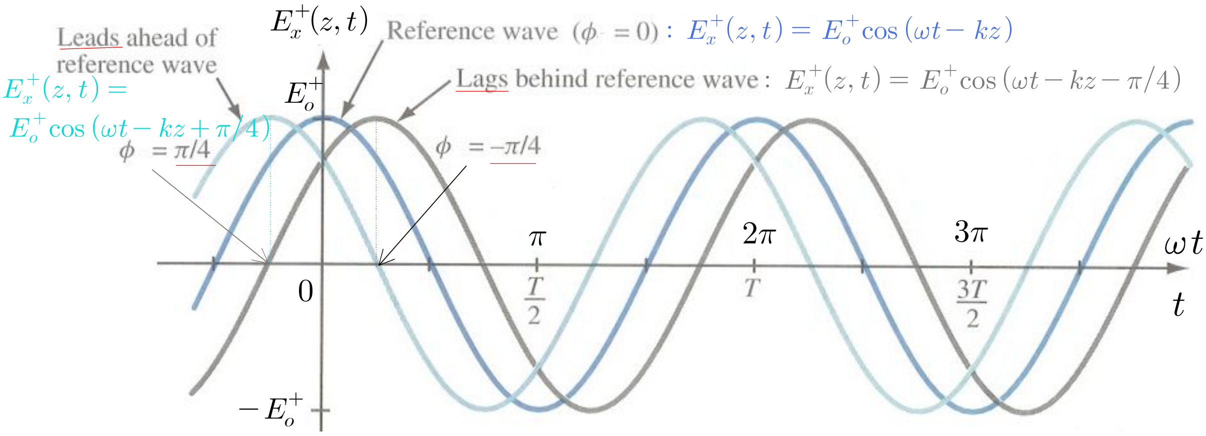


FIGURE 7-1

b) A 1-D uniform plane wave E_x with a reference phase ϕ at $t = 0$



$$E_x^+(z,t) = E_o^+\cos\left(\frac{2\pi t}{T} - \frac{2\pi z}{\lambda} + \phi\right) = E_o^+\cos(\omega t - kz + \phi)$$

$$= Re[E_o^+ e^{j(\omega t - kz + \phi)}] = Re[(E_o^+ e^{j\phi}) e^{j(\omega t - kz)}] = Re[E_{xs}^+(z) e^{j\omega t}]$$

$= E_o^+ \angle \phi$: *complex amplitude* (7-9)**

c) General representation of a monochromatic harmonic sinusoidal wave (Fourier component)

$$g_1 = \overline{g_1} e^{j(\omega t \mp k \cdot R)} \quad \begin{matrix} - & FTW & (u_p = \omega/k > 0) \\ + & BTW & (u_p = \omega/k < 0) \end{matrix} \qquad (2)$$

where $\overline{g_1}$ is a complex amplitude with a phase $\phi = \tan^{-1}(\overline{g_{1i}}/\overline{g_{1r}})$

Any waves can be represented by Fourier superposition principle :

i) Fourier series representation for a periodic wave over an infinite range (-∞,∞) or a non-periodic wave over a limited range [a, b]

$$g_1 = \sum \overline{g_{1k}} e^{j(\omega t - k \cdot R)} \qquad (3)$$

ii) Fourier integral representation for a non-periodic wave over an infinite range (-∞, ∞)

$$g_1 = \frac{1}{\sqrt{2\pi}} \int \overline{g_1}(k) e^{j(\omega t - k \cdot R)} dk \qquad (4)$$

Notes) <u>Type of waves</u>

Longitudinal wave : $g_1 \parallel k \Rightarrow k \times g_1 = 0$

Transverse wave : $g_1 \perp k \Rightarrow k \cdot g_1 = 0$

Electrostatic wave : $E_1 \neq 0$, $B_1 = 0$ (5)

Electromagnetic wave : $E_1 \neq 0$, $B_1 \neq 0$

Acoustic (Sound) wave : $\nabla p_1 \neq 0$

External B fields can couple these waves (O, X, R, L, ...)

B. Plane Waves in Lossless Media

1) Electromagnetic Waves in Unbounded Lossless Media

a) Transverse electromagnetic (TEM) wave along +z direction

In source-free $(\rho_{vs}=0, \boldsymbol{J}=0)$ lossless $(\sigma=0)$ unbounded simple media,

Helmholtz's equation (6-98) by omitting a subscript s :

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \qquad (7\text{-}3)$$

where $k=\omega\sqrt{\mu\epsilon} = \dfrac{\omega}{u_p}$ (7-4)

In Cartesian coordinates (x,y,z), consider a 1-D uniform plane wave traveling along +z direction $\left(\partial^2 E_x/\partial x^2=0 \quad \& \quad \partial^2 E_x/\partial y^2=0\right)$

(7-3) $\Rightarrow$ $$\frac{d^2E_x}{dz^2}+k^2E_x=0 \qquad (7\text{-}6)$$

General solution :

$$E_x(z)=E_o^+e^{-jkz}+E_o^-e^{+jkz} \equiv E_x^+(z)+E_x^-(z) \qquad (7\text{-}7)$$

where E_o^+, E_o^- are integration constants to be determined by B.C.s

$\therefore$ Electric phasor: $\boldsymbol{E}(z)=\hat{\boldsymbol{x}}[E_x^+(z)+E_x^-(z)]=\hat{\boldsymbol{x}}[E_o^+e^{-jkz}+E_o^-e^{+jkz}]$ (7-8)

For a cosine reference, the instantaneous electric wave :

$$\boldsymbol{E}(z,t)=\hat{\boldsymbol{x}}[E_x^+(z,t)+E_x^-(z,t)]=\hat{\boldsymbol{x}}e^{j\omega t}[E_x^+(z)+E_x^-(z)]$$

$$=\hat{\boldsymbol{x}}Re[E_o^+e^{j(\omega t-kz)}+E_o^-e^{j(\omega t+kz)}] \;:\; \text{FTW + BTW}$$

$$=\hat{\boldsymbol{x}}[E_o^+\cos(\omega t-kz)+E_o^-\cos(\omega t+kz)] \;:\; \text{FTW + BTW}$$

In unbounded media, only FTW exists.

$$\therefore\quad \boldsymbol{E}(z,t)=\hat{\boldsymbol{x}}E_x^+(z,t)=\hat{\boldsymbol{x}}E_x^+(z)e^{j\omega t}=\hat{\boldsymbol{x}}Re[E_o^+e^{j(\omega t-kz)}]=\hat{\boldsymbol{x}}E_o^+\cos(\omega t-kz)$$

Notes) (7-9)

i) $\nabla \rightarrow -j\boldsymbol{k}, \quad \nabla^2 \rightarrow -k^2$

ii) (6-80c) $\Rightarrow$ $\nabla\cdot\boldsymbol{E}_s = 0$

$\Rightarrow$ $\boldsymbol{k}\cdot\boldsymbol{E}=0$ *i.e.,* $\boldsymbol{k}\perp\boldsymbol{E}$: Transverse E wave

iii) (6-80a) $\Rightarrow$ $\nabla\times\boldsymbol{E}_s = -j\omega\mu\,\boldsymbol{H}_s$

$\Rightarrow$ $\boldsymbol{k}\times\boldsymbol{E}=\omega\mu\boldsymbol{H}$ *i.e.,* $\boldsymbol{H}\perp\boldsymbol{k}\perp\boldsymbol{E}$: Transv. EM (TEM) wave

$\boldsymbol{E}$ and $\boldsymbol{H}$ are in phase

The associated magnetic wave can be found from (6-80a) :

(7-9) in (6-80a) $\nabla\times\boldsymbol{E}_s = -j\omega\mu\,\boldsymbol{H}_s$

$$\Rightarrow \boldsymbol{H}(z,t)=\hat{\boldsymbol{y}}H_y^+(z,t)=\hat{\boldsymbol{y}}Re[H_o^+e^{j(\omega t-kz)}]=\hat{\boldsymbol{y}}\frac{E_o^+}{\eta}\cos(\omega t-kz) \qquad (7\text{-}15)$$

where $\eta=\sqrt{\mu/\epsilon}=|\boldsymbol{E}|/|\boldsymbol{H}|$ (Ω): Intrinsic impedance of medium (7-14)

In free space (or air), $\eta_o=\sqrt{\mu_o/\epsilon_o}=120\pi=377\ (\Omega)$

(e.g. 7-1)

A uniform plane wave $\boldsymbol{E} = \hat{\boldsymbol{x}} E_x(z,t)$ propagating along +z-direction $(\boldsymbol{k} \parallel \hat{\boldsymbol{z}})$ in $\epsilon_r = 4$, $\mu_r = 1$, $\sigma = 0$ with $f = 100$ MHz, $E_o^+(1/8, 0) = 10^{-4}$ (V/m)

a) $\boldsymbol{E}(z,t) = ?$ b) $\boldsymbol{H}(z,t) = ?$ c) $z_m = ?$ where E_x=+max. at $t = 10^{-8}$s

Solutions)

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi 10^8}{3\times 10^8}\sqrt{1\cdot 4} = \frac{4\pi}{3} \quad \text{(rad/m)}$$

$$\lambda = 2\pi/k = 3/2 \quad \text{(m)}$$

a) $\left.\begin{array}{l} \boldsymbol{E}(z,t) = \hat{\boldsymbol{x}} E_x(z,t) = \hat{\boldsymbol{x}} 10^{-4}\cos(2\pi 10^8 t - kz + \phi) \\ E_x(1/8, 0) = 10^{-4} \end{array}\right\}$

$$\Rightarrow \quad 2\pi 10^8 t - \frac{4\pi}{3}z + \phi = 0 \quad \Rightarrow \quad \phi = \pi/6$$

$$\therefore \ \boldsymbol{E}(z,t) = \hat{\boldsymbol{x}} 10^{-4}\cos\left[2\pi 10^8 t - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \quad \text{(V/m)}$$

b) $\eta = \sqrt{\mu/\epsilon} = \eta_o/\sqrt{\epsilon_r} = 120\pi/\sqrt{4} = 60\pi$ (Ω)

$$\boldsymbol{H}(z,t) = \hat{\boldsymbol{y}}\frac{E_x}{\eta} = \hat{\boldsymbol{y}}\frac{10^{-4}}{60\pi}\cos\left[2\pi 10^8 t - \frac{4\pi}{3}\left(z - \frac{1}{8}\right)\right] \quad \text{(A/m)}$$

c) $E_x(z_m)$ = +max. at $t = 10^{-8}$s

$$\Rightarrow \quad 2\pi 10^8(10^{-8}) - \frac{4\pi}{3}\left(z_m - \frac{1}{8}\right) = \pm 2n\pi, \quad \text{n=0, 1, 2, ...}$$

$$\Rightarrow \quad z_m = \frac{13}{8} \pm \frac{3}{2}n = \frac{13}{8} \pm n\lambda \quad \text{(m)}$$

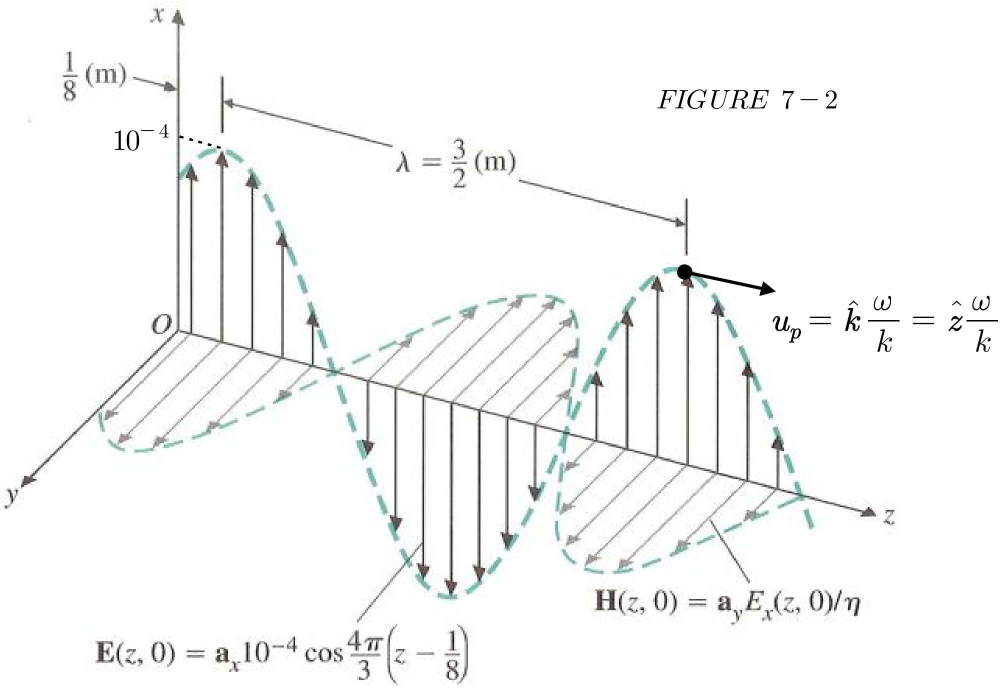
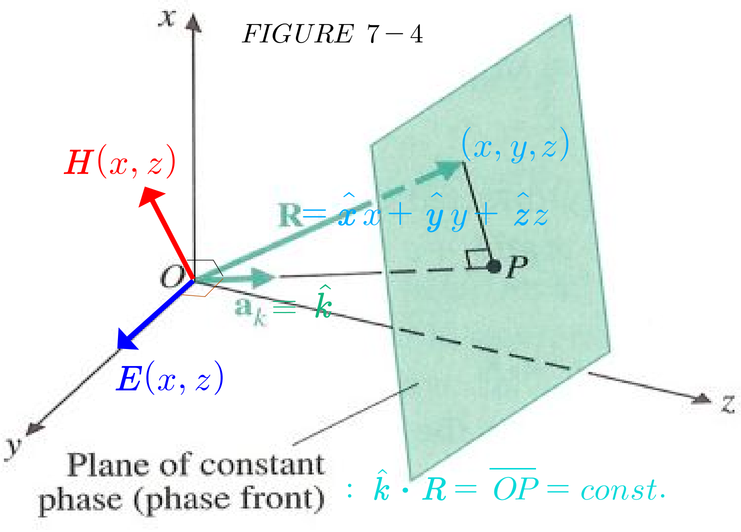


FIGURE 7−2

b) Transverse electromagnetic (TEM) wave along an arbitrary direction



Consider a uniform plane E-wave propagating in k-direction.

$$E(x,z)=\hat{y}E_o e^{-j(k_x x+k_z z)}=\hat{y}E_o e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}=\hat{y}E_o e^{-jk\hat{k}\cdot\boldsymbol{R}} \qquad (7\text{-}20,\ 23)$$

where $\boldsymbol{k}=\hat{x}k_x+\hat{z}k_z=\hat{k}k$: wavenumber vector (7-21)

The associated H-wave can be found from (6-80a) :

(7-20) in (6-80a) $H = -\dfrac{1}{j\omega\mu}\nabla\times E$

$$\Rightarrow \quad \boldsymbol{H}(x,z)=\frac{E_o}{\omega\mu}(-\hat{x}k_z+\hat{z}k_x)e^{-j(k_x x+k_z z)} \qquad (7\text{-}24)$$

or (7-23) in (6-80a) with $\nabla \rightarrow -j\boldsymbol{k}$

$$\Rightarrow \quad \boldsymbol{H}=\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega\mu}=\frac{\hat{k}\times\boldsymbol{E}}{\eta} \qquad (\boldsymbol{H}\text{ and }\boldsymbol{E}\text{ are in phase}) \qquad (7\text{-}25)$$

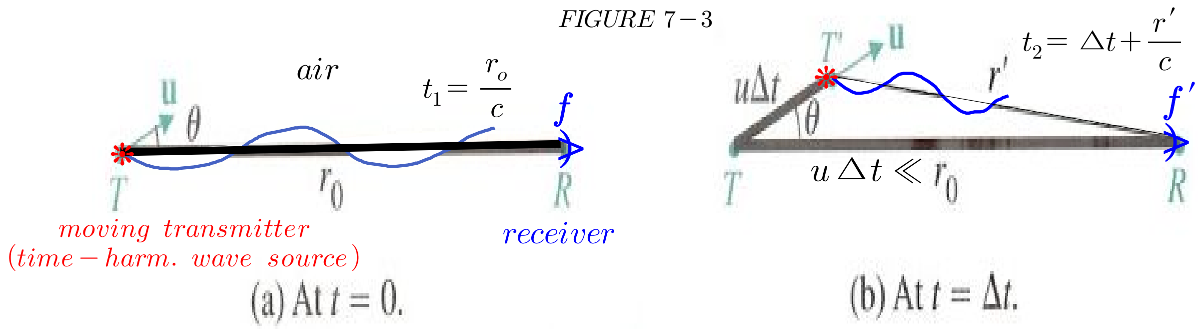
$$\frac{k}{\omega\mu}=\frac{\sqrt{\mu\epsilon}}{\mu}=\sqrt{\frac{\epsilon}{\mu}}=\frac{1}{\eta}$$

$\Rightarrow \quad \boldsymbol{H}\perp\boldsymbol{k}\perp\boldsymbol{E}$: Transverse EM (TEM) wave

$\boldsymbol{E}$ and $\boldsymbol{H}$ are in phase.

C. Doppler Effect

= Frequency shift detected by a receiver from a transmitter when there is relative motion between them.



Time elapsed at R while T is moving during Δt :

$$\Delta t' = t_2 - t_1 = \left(\Delta t + \frac{r'}{c}\right) - \frac{r_o}{c} \cong \Delta t\left(1 - \frac{u}{c}cos\theta\right) \qquad (7\text{-}18)$$

($u\Delta t \ll r_o$)

For the time-harmonic source of f, let $\Delta t = 1/f$,

then, the frequency received at R from the moving transmitter is

$$f' = \frac{1}{\Delta t'} = \frac{f}{\left(1 - \frac{u}{c}cos\theta\right)} \cong f\left(1 + \frac{u}{c}cos\theta\right) \qquad (7\text{-}19)$$

Doppler shift of the received frequency at R :

$$\Delta f \equiv f' - f \cong f\frac{u}{c}cos\theta \qquad (7\text{-}19)*$$

Notes)

i) For approaching T and/or R ($0 \le \theta < \pi/2$), $\Delta f > 0$: Increasing f.
For receding T and/or R ($\pi/2 \le \theta \le \pi$), $\Delta f < 0$: Decreasing f.

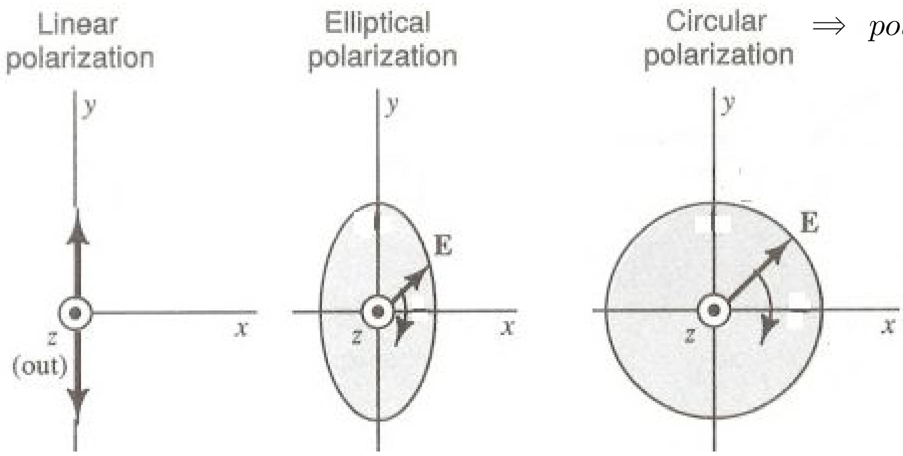
ii) Application ⇒ Doppler radar speed gun : $u \propto \Delta f$

iii) Red shift = Lower-frequency light spectrum (red) emitted by a receding star

D. Polarization of Plane Waves

1) Polarization State

Polarization of wave = Shape and locus of the tip of the wave vector moved at a fixed location in space during a period $T = 2\pi/\omega$



$\Rightarrow$ *polarization state*

Consider transverse electric wave traveling in the z-direction

$$\boldsymbol{E}(z,t) = Re[\boldsymbol{E}(z)\,e^{j\omega t}] = Re\{[\hat{\boldsymbol{x}}E_x(z) + \hat{\boldsymbol{y}}E_y(z)]\,e^{j\omega t}\}$$

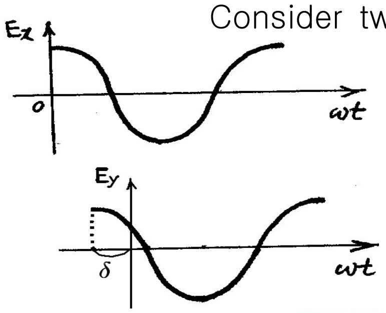
$$= Re\{[\hat{\boldsymbol{x}}E_{xo}e^{j\phi_x} + \hat{\boldsymbol{y}}E_{yo}e^{j\phi_y}]\,e^{j(\omega t - kz)}\}$$

$$= \hat{\boldsymbol{x}}\,E_{xo}\cos(\omega t - kz + \phi_x) + \hat{\boldsymbol{y}}\,E_{yo}\cos(\omega t - kz + \phi_y) \qquad (6)$$

Real wave at a fixed plane z=0

$$\boldsymbol{E}(0,t) = \hat{\boldsymbol{x}}\,E_{xo}\cos(\omega t + \phi_x) + \hat{\boldsymbol{y}}\,E_{yo}\cos(\omega t + \phi_y) \qquad (7),(7\text{-}27)^*$$

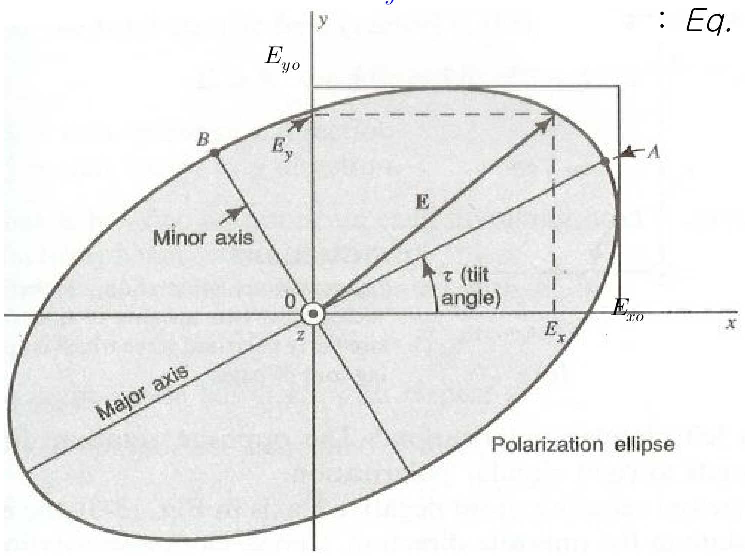
Consider two components of $\boldsymbol{E}(0,t)$ for $\phi_x = 0,\ \phi_y = \delta$



$$\begin{cases} E_x = E_{xo}\cos\omega t \\ E_y = E_{yo}\cos(\omega t + \delta) = E_{yo}(\cos\omega t\,\cos\delta - \sin\omega t\,\sin\delta) \end{cases} \qquad (8)$$

$$\Rightarrow \frac{E_x^2}{E_{xo}^2} + \frac{E_y^2}{E_{yo}^2} - \frac{2E_xE_y}{E_{xo}E_{yo}}cos\delta = \sin^2\delta$$

: *Eq. of ellipse* (9)(7-28)*



Axial ratio :

$$AR = \frac{major\ axis}{minor\ axis}$$

$$1 \leq AR < \infty$$

circle　　linear line

2) Linear Polarization

a) Linear polarization with a positive slope

For $\delta = 0$ ($\phi_x = \phi_y$: same phase) (or $\pm 2n\pi$, $n = 0, 1, \cdots$)

(9) $\Rightarrow \left(\dfrac{E_x}{E_{xo}} - \dfrac{E_y}{E_{yo}}\right)^2 = 0 \;\Rightarrow\; \dfrac{E_x}{E_{xo}} = \dfrac{E_y}{E_{yo}}$: *Eq. of straight line* (9L+)

(7) $\Rightarrow$ $\boldsymbol{E}(0,t) = (\hat{\boldsymbol{x}}\, E_{xo} + \hat{\boldsymbol{y}}\, E_{yo})\cos\omega t$ (7L+)(7-32)

(8) $\Rightarrow$ $E_x = E_{xo}\cos\omega t$ and $E_y = E_{yo}\cos\omega t$: in-phase (8L+)

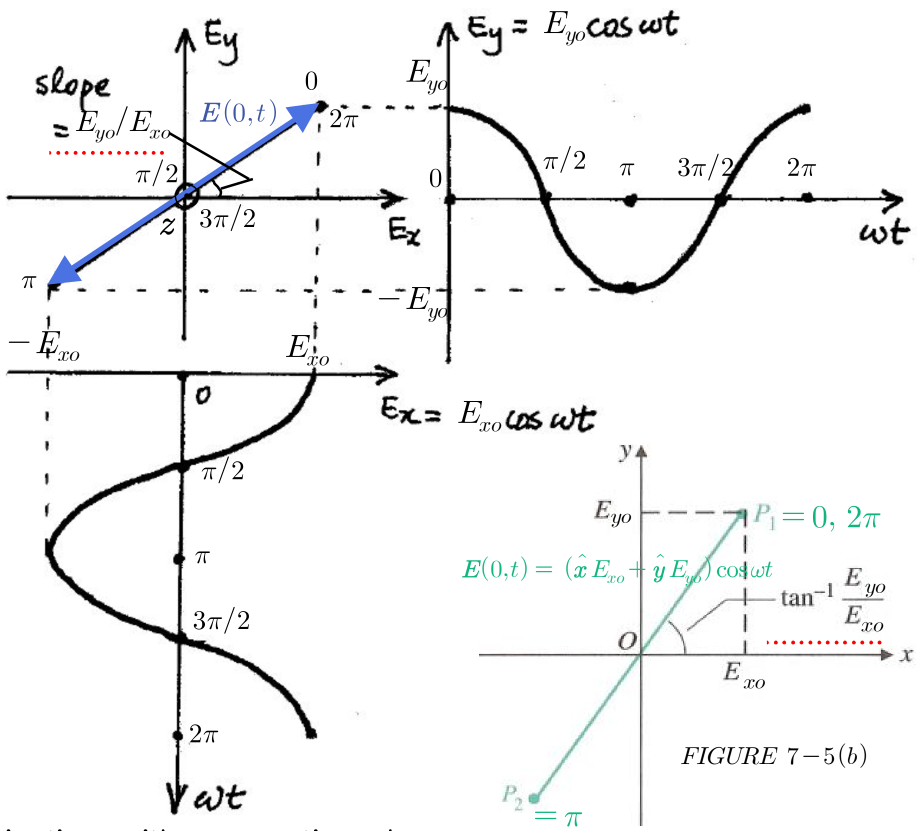
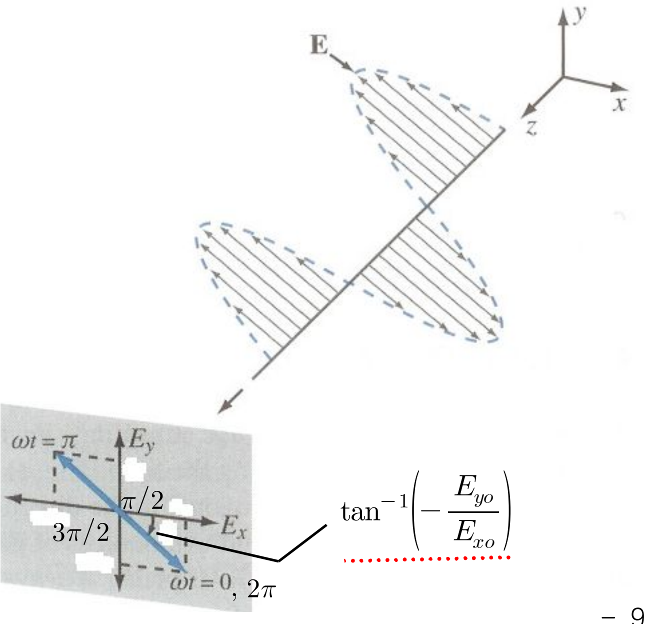


FIGURE 7-5(b)

b) Linear polarization with a negative slope

For $\delta = \pi$ [or $\pm(2n-1)\pi$, $n = 1, 2, \cdots$]



$\dfrac{E_x}{E_{xo}} = -\dfrac{E_y}{E_{yo}}$ (9L−)

$\boldsymbol{E}(0,t) = \hat{\boldsymbol{x}}\,E_{xo}\cos\omega t + \hat{\boldsymbol{y}}\,E_{yo}\cos(\omega t + \pi)$ (7L−)

$E_x = E_{xo}\cos\omega t$ and $E_y = -E_{yo}\cos\omega t$ (8L−)

: out-of-phase

3) Circular polarization

a) Right-hand (or positive) circularly polarized wave

For phase difference $\delta = -\frac{\pi}{2}$ $\left[\text{or} \quad \delta = -\left(n-\frac{1}{2}\right)\pi, \quad n = 1, 2, \cdots\right]$

and $E_{xo} = E_{yo} \equiv E_o$ (same amplitude)

(9) $\Rightarrow$ $\frac{E_x^2}{E_{xo}^2} + \frac{E_y^2}{E_{yo}^2} = 1 \quad \Rightarrow \quad E_x^2 + E_y^2 = E_o^2$: *Eq. of circle* (9RCP)(7-28)

(7) $\Rightarrow$ $\boldsymbol{E}(0,t) = \hat{x} E_o \cos\omega t + \hat{y} E_o \cos(\omega t - \frac{\pi}{2}) = \hat{x} E_o \cos\omega t + \hat{y} E_o \sin\omega t$ (7RCP)(7-27)

(8) $\Rightarrow$ $E_x = E_o \cos\omega t$ and $E_y = E_o \sin\omega t$: lags by $\pi/2$ (8RCP)

$\Longrightarrow$ *Right-hand Circular Polarization (RCP)*

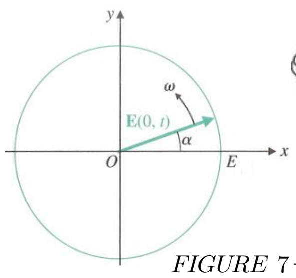
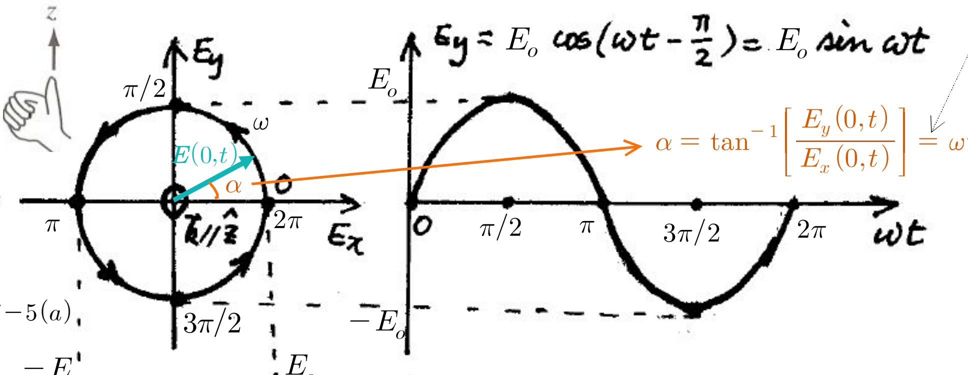
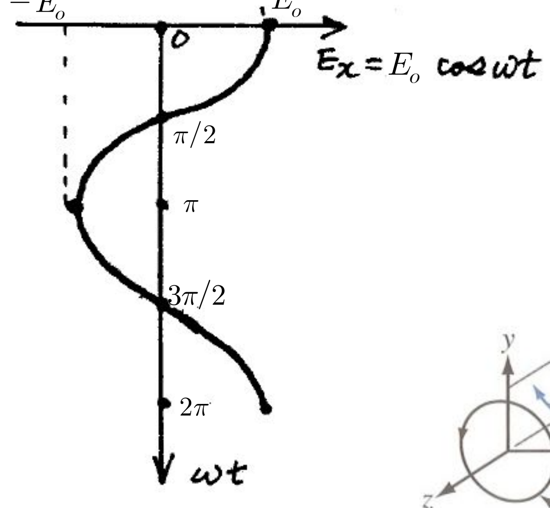
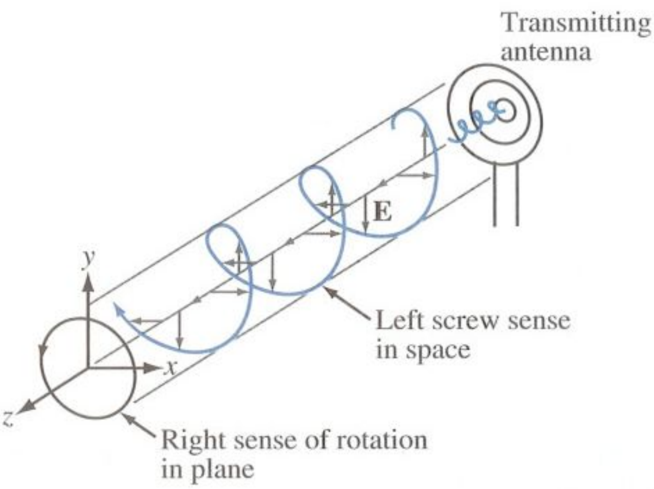


FIGURE 7−5(a)







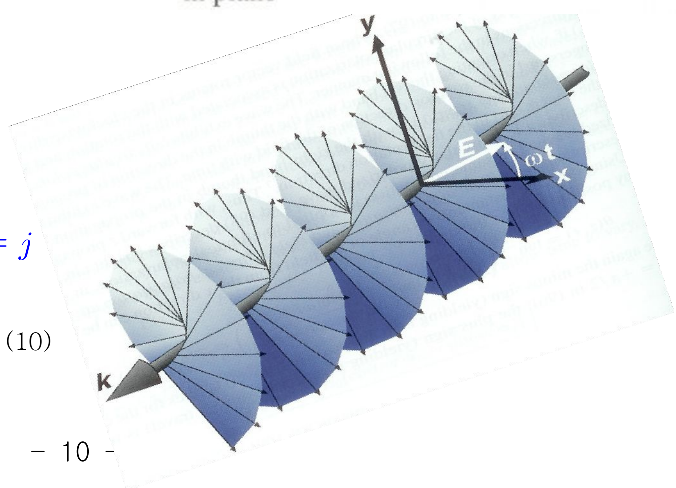
Note)

$\boldsymbol{E}(0,t) = \hat{x} E_o \cos\omega t + \hat{y} E_o \cos(\omega t - \frac{\pi}{2})$

$= Re[(\hat{x} E_o + \hat{y} E_o e^{-j\frac{\pi}{2}}) e^{j\omega t}]$

$\Rightarrow \frac{E_x}{E_y} = \frac{E_{xo}}{E_{yo} e^{-j(\pi/2)}} = \frac{E_o}{-j E_o} = j$

or $\frac{jE_x}{E_y} = -1$ for RCP (10)



b) Left-hand (or negative) circularly polarized wave

For phase difference $\delta = +\frac{\pi}{2}$ $\left[\text{or} \quad \delta = -\left(n+\frac{1}{2}\right)\pi \;, \quad n = 1, 2, \cdots\right]$

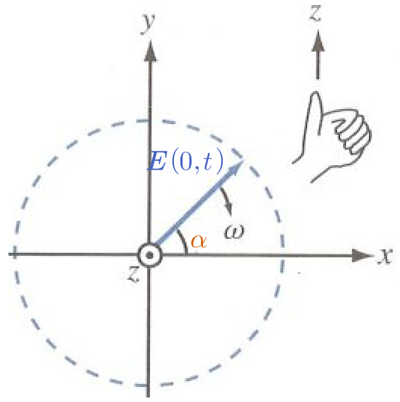
and $E_{xo} = E_{yo} \equiv E_o$ (same amplitude)

(9) $\Rightarrow \dfrac{E_x^2}{E_{xo}^2} + \dfrac{E_y^2}{E_{yo}^2} = 1 \quad \Rightarrow \quad E_x^2 + E_y^2 = E_o^2$: *Eq. of circle* (9LCP)(7-28)

(7) $\Rightarrow \boldsymbol{E}(0,t) = \hat{\boldsymbol{x}} E_o \cos\omega t + \hat{\boldsymbol{y}} E_o \cos(\omega t + \frac{\pi}{2}) = \hat{\boldsymbol{x}} E_o \cos\omega t - \hat{\boldsymbol{y}} E_o \sin\omega t$ (7LCP)(7-31)

(8) $\Rightarrow E_x = E_o \cos\omega t$ and $E_y = -E_o \sin\omega t$: leads by $\pi/2$ (8LCP)

$\Longrightarrow$ *Left-hand Circular Polarization (LCP)*



Note)

$$\boldsymbol{E}(0,t) = \hat{\boldsymbol{x}} E_o \cos\omega t + \hat{\boldsymbol{y}} E_o \cos(\omega t + \frac{\pi}{2})$$

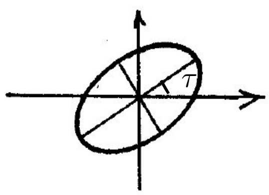
$$= Re[(\hat{\boldsymbol{x}} E_o + \hat{\boldsymbol{y}} E_o e^{+j\frac{\pi}{2}}) e^{j\omega t}]$$

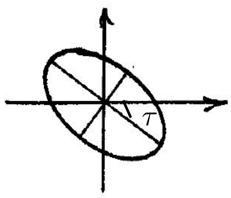
$$\Rightarrow \frac{E_x}{E_y} = \frac{E_{xo}}{E_{yo} e^{j(\pi/2)}} = \frac{E_o}{jE_o} = -j$$

or $\dfrac{jE_x}{E_y} = +1$ for LCP (11)

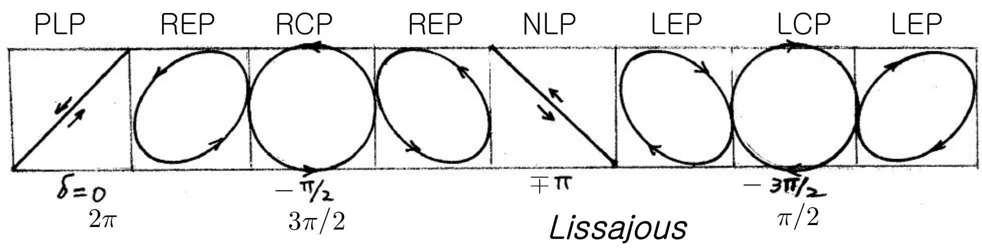
4) Elliptical polarization

For $\delta \neq 0$ and $E_{xo} \neq E_{yo}$ $[\delta \rightarrow \textit{shape}, \quad E_{yo}/E_{xo} \rightarrow \textit{tilt angle}\,(\tau)]$

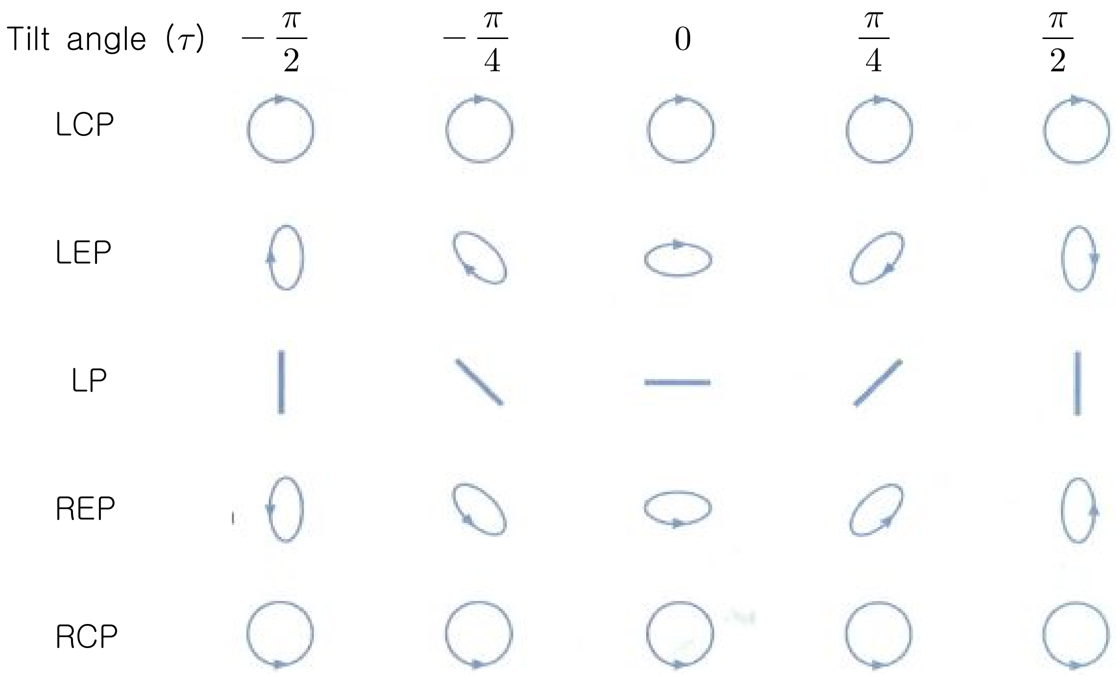




Notes) For same amplitudes $(E_{xo} = E_{yo})$



phase, amplitude, frequency



(e.g. 7-2)

LP plane wave = (RCP + LCP) waves of equal amplitude

(Proof)

From (10) $\frac{jE_x}{E_y} = -1$ for RCP and (11) $\frac{jE_x}{E_y} = +1$ for LCP,

$$E_{RCP}(z) = (\hat{\boldsymbol{x}}\,E_x + \hat{\boldsymbol{y}}\,E_y)\,e^{-jkz} = \frac{E_o}{2}(\hat{\boldsymbol{x}} - j\,\hat{\boldsymbol{y}})\,e^{-jkz} \qquad ①$$

$$E_{LCP}(z) = (\hat{\boldsymbol{x}}\,E_x + \hat{\boldsymbol{y}}\,E_y)\,e^{-jkz} = \frac{E_o}{2}(\hat{\boldsymbol{x}} + j\,\hat{\boldsymbol{y}})\,e^{-jkz} \qquad ②$$

① + ② $\Rightarrow$ $E_{RCP}(z) + E_{LCP}(z) = \hat{\boldsymbol{x}}\,E_o e^{-jkz} = E_{LP}(z)_{///}$