2. Plane Waves in Lossy Media

A. EM Waves in Unbounded Lossy Media

1) Complex permittivity of lossy media

In unbounded lossy ($\sigma \neq 0$: conducting) media, time-harmonic Maxwell's equations for lossless media are still applicable (the subscript s will be omitted hereafter) :

$$\nabla \times E = -j\omega\mu H, \quad \nabla \cdot E = \rho_v/\varepsilon, \quad \nabla \cdot B = 0 \qquad \text{(6-80a, c, d)}$$

except for Ampere's law, which are changed with the help of Ohm's law $J = \sigma E$, into

$$\nabla \times H = (\sigma + j\omega\varepsilon)E = j\omega\varepsilon_c E \qquad \text{(7-35)(6-80b)*}$$

where ϵ_c is the complex permittivity of the medium:

$$\epsilon_c \triangleq \epsilon - j\frac{\sigma}{\omega} \triangleq \epsilon' - j\epsilon'' \quad \text{(F/m)} \qquad \text{(7-36, 37)}$$

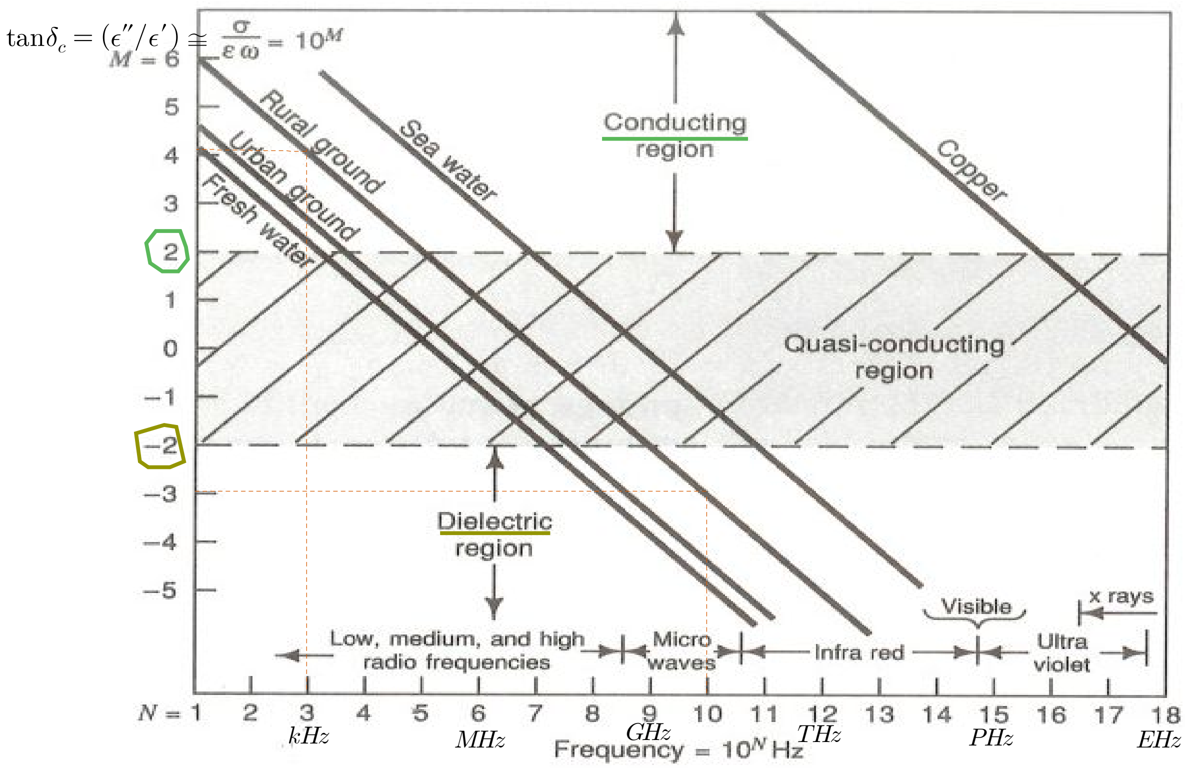
Notes) i) $\epsilon' \equiv \epsilon$ and $\epsilon'' \equiv \sigma/\omega$ (including damping and ohmic losses)

ii) Loss tangent $\tan\delta_c \triangleq \dfrac{\epsilon''}{\epsilon'} \cong \dfrac{\sigma}{\omega\epsilon}$: measure of power loss (7-39)

iii) $\sigma \gg \omega\epsilon$ (good conductor), $\sigma \ll \omega\epsilon$ (good insulator), $\sigma = 0$ (lossless)

iV)

Medium	Relative permittivity ϵ_r, dimensionless	Conductivity σ, (S/m)
Copper	1	5.8×10^7
Seawater	80	4
Rural ground (*moist*)	14	10^{-2}
Urban ground	3	10^{-4}
Fresh water	80	10^{-3}



2) TEM wave propagation in lossy media

a) Homogeneous Helmholtz's equation and its solution

$$\nabla^2 \boldsymbol{E} + k_c^2 \boldsymbol{E} = 0 \qquad \text{(7-41)}$$

where $k_c = \omega\sqrt{\mu\epsilon_c}$ (7-40)

Rewriting (7-41) as

$$\nabla^2 \boldsymbol{E} - \gamma^2 \boldsymbol{E} = 0 \qquad \text{(7-45a)}$$

with a propagation constant

$$\gamma = jk_c = j\omega\sqrt{\mu\epsilon_c} \quad (\mathrm{m}^{-1}) \qquad \text{(7-42)}$$

$$= \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{1/2} = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} \qquad \text{(7-43, 44)}$$

Note) $(\alpha + j\beta)^2 = (\alpha^2 - \beta^2) + j2\alpha\beta = -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon''$

$$\Rightarrow \left(\begin{array}{l} \alpha^2 - \beta^2 = -\omega^2\mu\epsilon' \\ 2\alpha\beta = \omega^2\mu\epsilon'' \end{array} \right.$$

Solving these two equations,

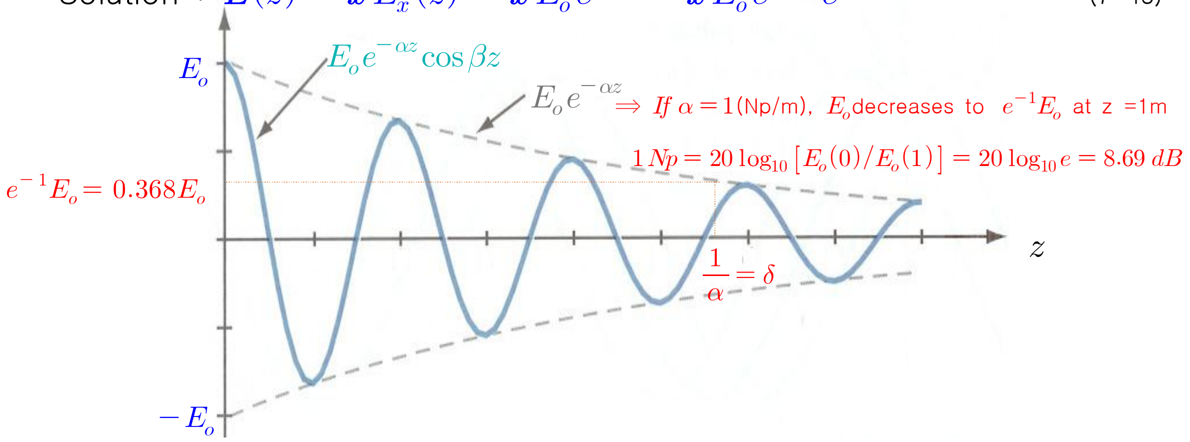
$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]\right\}^{1/2} \quad (\mathrm{Np/m}) : \text{attenuation constant}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]\right\}^{1/2} \quad (\mathrm{rad/m}) : \text{phase constant}$$

For a uniform plane wave propagating in +z direction,

$$\frac{d^2E_x}{dz^2} - \gamma^2 E_x = 0 \qquad \text{(7-45b)}$$

Solution : $\boldsymbol{E}(z) = \hat{\boldsymbol{x}}E_x(z) = \hat{\boldsymbol{x}}E_o e^{-\gamma z} = \hat{\boldsymbol{x}}E_o e^{-\alpha z}e^{-j\beta z}$ (7-46)



The associated magnetic wave can be found from $\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H}$

$$\Rightarrow \boldsymbol{H}(z) = \hat{\boldsymbol{y}}H_y(z) = \hat{\boldsymbol{y}}\frac{E_x(z)}{\eta_c} = \hat{\boldsymbol{y}}\frac{E_o}{\eta_c}e^{-\alpha z}e^{-j\beta z}$$: not in phase with $E(z)$ (7-13)*

where $\eta_c = \sqrt{\dfrac{\mu}{\epsilon_c}} = \sqrt{\dfrac{\mu}{\epsilon'}}\left(1 - j\dfrac{\epsilon''}{\epsilon'}\right)^{-1/2}$ (Ω) : complex value

= Intrinsic impedance of lossy medium (7-14)*

b) Wave propagation in low-loss dielectrics $\left(\tan\delta_c \triangleq \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon} \ll 10^{-2}\right)$

For a low-loss dielectric (like ordinary imperfect insulators), $\sigma/\omega\epsilon \ll 1$ $(\epsilon'' \ll \epsilon')$ in (7-44), using the binomial expansion $(a+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}x^k$, becomes

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} \cong j\omega\sqrt{\mu\epsilon'}\left[1 - j\frac{\epsilon''}{2\epsilon'} + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right]$$

from which

$$\alpha \cong \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \quad \text{(Np/m)} \qquad (7\text{-}47)$$

$$\beta \cong \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] = \omega\sqrt{\mu\epsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon}\right)^2\right] \quad \text{(rad/m)} \qquad (7\text{-}48)$$

Intrinsic impedance : (7-14)* $\Rightarrow$

$$\eta_c \cong \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) = \sqrt{\frac{\mu}{\epsilon}}\left(1 + j\frac{\sigma}{2\omega\epsilon}\right) \quad (\Omega) \qquad (7\text{-}49)$$

Phase velocity :

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\epsilon'}}\left[1 - \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] \quad \text{(m/s)} \qquad (7\text{-}50)$$

c) Wave propagation in good conductors $\left(\tan\delta_c \triangleq \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon} \gg 10^{2}\right)$

For a good conductor, $\sigma/\omega\epsilon \gg 1$ $(\epsilon'' \gg \epsilon')$ in (7-43),

$$\gamma = \alpha + j\beta = \omega\sqrt{\mu\epsilon}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{1/2} \cong \sqrt{j}\sqrt{\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\mu\sigma}$$

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = e^{j\pi/4} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{1+j}{\sqrt{2}} = 1\angle\pi/4 \qquad (7\text{-}51)$$

from which

$$\alpha = \beta \cong \sqrt{\pi f\mu\sigma} \qquad (7\text{-}52)$$

Intrinsic impedance : (7-14)* $\Rightarrow$ (7-53)

$$\eta_c \cong \sqrt{\frac{\mu}{\epsilon'}}\left(-j\frac{\epsilon''}{\epsilon'}\right)^{-1/2} = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma} = \frac{\sqrt{2}\,\alpha}{\sigma}\angle\pi/4$$

TEM wave : (7-13)* $\Rightarrow$

$$1/\sqrt{j} = (e^{j\pi/2})^{-1/2} = e^{j(-\pi/4)}$$

$$\boldsymbol{H}(z) = \hat{\boldsymbol{y}}\frac{E_o}{\eta_c}e^{-\alpha z}e^{-j\beta z} = \hat{\boldsymbol{y}}\frac{E_o}{\sqrt{j\omega\mu/\sigma}}e^{-\alpha z}e^{-j\beta z} = \hat{\boldsymbol{y}}\frac{E_o}{\sqrt{\omega\mu/\sigma}}e^{-\alpha z}e^{-j(\beta z + \pi/4)}$$

: $H(z)$ lags behind $E(z)$ by $\pi/4$

Phase velocity :

$$u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \ll c \qquad (7\text{-}54)$$

Note) $u_p^{\downarrow}$ as $\sigma^{\uparrow}$

(e.g.) For Cu with $\sigma = 5.8\times10^7$, $u_p = 720\ m/s \ll c$ at $f = 3$ MHz

Wavelength :

$$\lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} \cong 2\sqrt{\frac{\pi}{f\mu\sigma}} \quad \text{(m)} \qquad (7\text{-}55)$$

Note) $\lambda^{\downarrow}$ as $\sigma^{\uparrow}$

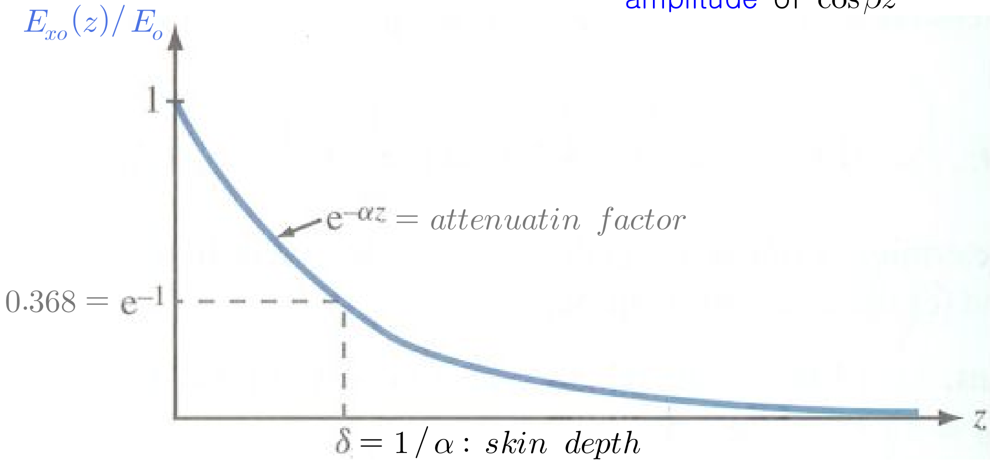
(e.g.) For Cu, $\lambda = 0.24\ mm \ll 100m$ in air at $f = 3$ MHz

Skin depth δ = Depth of penetration of a good conductor

= Distance thru which the wave amplitude decrease by e^{-1}

For $\boldsymbol{E}(z) = \hat{\boldsymbol{x}} E_x(z) = \hat{\boldsymbol{x}} E_o e^{-\gamma z} = \hat{\boldsymbol{x}} E_o e^{-\alpha z} e^{-j\beta z}$ (7-46) ,

amplitude of $\cos\beta z$



(7-52)

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \text{or} \quad \delta = \frac{1}{\beta} = \frac{\lambda}{2\pi} \quad \text{(m)} \qquad (7\text{-}56,\ 57)$$

Note) $\delta^{\downarrow}$ as $\sigma^{\uparrow}$ and/or $f^{\uparrow}$

(e.g.) For Cu, $\delta = 0.038\ mm$ at $f = 3$ MHz

$\delta = 0.66\ \mu m$ at $f = 10$ GHz

(e.g. 7-4)

A LP plane wave $\boldsymbol{E} = \hat{\boldsymbol{x}} E(z,t)$ propagating along +z-direction ($\boldsymbol{k} \parallel \hat{\boldsymbol{z}}$) in seawater ($\epsilon_r = 72$, $\mu_r = 1$, $\sigma = 4$ S/m) with $\boldsymbol{E}(0,t) = \hat{\boldsymbol{x}} 100\cos(10^7 \pi t)$ (V/m) at $z = 0$.

a) α, β, η_c, u_p, λ, δ = ? b) z_1 = ? where $E(z_1) = 0.01\, E(z=0)$,

c) $\boldsymbol{E}(z=0.8, t)$, $\boldsymbol{H}(z=0.8, t)$ = ?

Solutions)

$$\omega = 10^7 \pi,\ f = \frac{\omega}{2\pi} = 5 \times 10^6,\ \tan\delta_c = \frac{\sigma}{\omega\epsilon} = 200 \gg 1 \ :\ \text{good conductor}$$

a) $\alpha = \beta \cong \sqrt{\pi f \mu \sigma}$ = 8.89 (rad/m)

$$\eta_c = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = \pi e^{j\pi/4} \quad (\Omega)$$

$$u_p = \frac{\omega}{\beta} = 3.53\times 10^6 \text{ (m/s)}, \ \lambda = \frac{2\pi}{\beta} = 0.707 \text{ (m)}, \ \delta = \frac{1}{\alpha} = 0.112 \text{ (m)}$$

b) $E(z_1) = 0.01\,E(z=0) \quad \Rightarrow \quad E_0 e^{-\alpha z_1} = 0.01\,E_0 e^{-\alpha 0}$

$$\Rightarrow \quad z_1 = -\frac{1}{\alpha}\ln 0.01 = \frac{1}{\alpha}\ln 100 = 0.518 \quad \text{(m)}$$

c) $\boldsymbol{E}(z) = \hat{\boldsymbol{x}}\,100\,e^{-\alpha z}\,e^{-j\beta z}$ in the phasor domain

$\boldsymbol{E}(z,t) = Re[\boldsymbol{E}(z)\,e^{j\omega t}] = \hat{\boldsymbol{x}}\,100\,e^{-\alpha z}\cos(\omega t - \beta z)$ in the time domain

$\therefore \ \boldsymbol{E}(z=0.8,\ t) = \hat{\boldsymbol{x}}\,0.082\cos(10^7\pi t - 7.11)$ (V/m)

(7-15) $\Rightarrow \quad \boldsymbol{H}(z,\ t) = Re\left[\hat{\boldsymbol{y}}\,\frac{E_x(z)}{\eta_c}\,e^{j\omega t}\right] = Re\left[\hat{\boldsymbol{y}}\,\frac{100\,e^{-\alpha z}\,e^{-j\beta z}}{\pi\,e^{j\pi/4}}\,e^{j\omega t}\right]$

$$= Re\left[\hat{\boldsymbol{y}}\,(100/\pi)\,e^{-\alpha z}\,e^{j(\omega t - \beta z - \pi/4)}\right]$$

$$= \hat{\boldsymbol{y}}\,(100/\pi)\,e^{-\alpha z}\cos(\omega t - \beta z - \pi/4)$$

$$\Rightarrow \quad \boldsymbol{H}(0.8,t) = \hat{\boldsymbol{y}}\,(100/\pi)\,e^{-0.8\alpha}\cos(10^7\pi t - 0.8\beta - \pi/4)$$

$$\cong \hat{\boldsymbol{y}}\,0.026\cos(10^7\pi t - 7.89)$$

$$\cong \hat{\boldsymbol{y}}\,0.026\cos(10^7\pi t - 2\pi - 1.61)$$

$$= \hat{\boldsymbol{y}}\,0.026\cos(10^7\pi t - 1.61) \quad \text{(A/m)}$$

Summary of EM Plane Wave in Media

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu \sigma}$	(Np/m)
$\beta =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu \sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1+j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

B. *Wave Velocities and Dispersive Medium*

1) **Phase velocity** = Propagation velocity of an equiphase wavefront

For plane waves in a lossless medium $[\mathbf{E}(z,t) = \hat{\mathbf{x}}E_o \cos(\omega t - kz + \phi_z)]$,

$$u_p = \frac{\omega}{k} = \lambda f = \frac{1}{\sqrt{\mu\epsilon}} = \text{const. (m/s) : indep. of frequency} \qquad (7\text{-}10)$$

For plane waves in a lossy medium $[\mathbf{E}(z,t) = \hat{\mathbf{x}}E_o e^{-\alpha z} \cos(\omega t - \beta z + \phi_z)]$,

$$u_p = \frac{\omega}{\beta} = \lambda f \quad \text{(m/s) : dep. on frequency} \qquad (7\text{-}50,\ 58)$$

$$\text{where} \quad \beta = \omega \left\{ \frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right] \right\}^{1/2} \qquad (7\text{-}43,\ 44)$$

2) **Group velocity** = Propagation velocity of the wave-packet envelope of a group of frequencies

Consider two plane waves with slightly different $\omega(f)$ and $\beta(\lambda)$,

$$\begin{cases} E_1 = E_o \cos[(\omega + \Delta\omega)t - (\beta + \Delta\beta)z] \\ E_2 = E_o \cos[(\omega - \Delta\omega)t - (\beta - \Delta\beta)z] \end{cases}$$

Addition of two waves $\Rightarrow$ Wave packet *(cf)* Beat wave

$$E(z,\ t) = E_1 + E_2 = 2E_o \cos(t\Delta\omega - z\Delta\beta)\cos(\omega t - \beta z) \qquad (7\text{-}59)$$

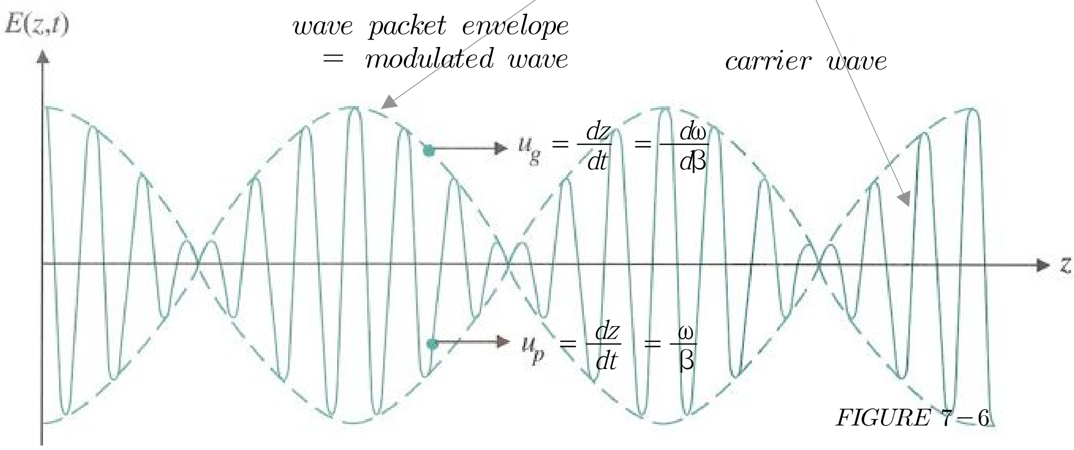


FIGURE 7-6

group vel. = vel. of modulated wave carrying information

Constant phase of modulated wave : $t\Delta\omega - z\Delta\beta = constant$

$$\Rightarrow \quad \text{Group velocity :} \quad u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega}{d\beta} \quad \text{(m/s)} \qquad (7\text{-}60)$$

3) **Index of refraction and dispersive medium**

Index of refraction of the medium : $n_r = c/u_p$ (10), (7-117)

$\Rightarrow$ If u_p depends on $\omega(f)$ and $\beta(\lambda)$, the information-bearing waves consisting of different f and λ will be dispersed (distorted).

(e.g.) waves in dispersive medium, such as lossy dielectrics, transmission lines, waveguides, ...

$$u_p = \frac{\omega}{\beta} \qquad \beta = \frac{2\pi}{\lambda}, \quad d\beta = -\frac{2\pi}{\lambda^2}d\lambda \qquad (7-60)$$

$$u_g = \frac{d\omega}{d\beta} = u_p + \beta\frac{du_p}{d\beta} = u_p - \lambda\frac{du_p}{d\lambda} \qquad \frac{dn_r}{d\lambda} = -\frac{c}{u_p^2}\left(\frac{du_p}{d\lambda}\right) \quad (7\text{-}60)^*$$

$$\Rightarrow \begin{cases} u_g < u_p : & \text{normal dispersion } [du_p/d\lambda > 0,\ dn_r/d\lambda < 0] \\ u_g > u_p : & \text{anomalous dispersion } [du_p/d\lambda < 0,\ dn_r/d\lambda > 0] \\ u_g = u_p : & \text{no dispersion } [du_p/d\lambda = 0,\ n_r = ind.\ of\ \lambda] \end{cases}$$

C. EM Power Flow and Poynting's Theorem

1) Poynting vector = EM power flow per unit area

$\boldsymbol{H}\cdot$[(6-45a) $\nabla\times E = -\frac{\partial B}{\partial t}$] − $\boldsymbol{E}\cdot$[(6-45b) $\nabla\times H = J + \frac{\partial D}{\partial t}$] by using a vector identity $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{H})$

$$\Rightarrow \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = -\frac{\partial}{\partial t}\left(\frac{\boldsymbol{E}\cdot\boldsymbol{D}}{2} + \frac{\boldsymbol{H}\cdot\boldsymbol{B}}{2}\right) - \boldsymbol{E}\cdot\boldsymbol{J} \qquad (7\text{-}64)$$

In a simple medium, substitution of constitutive relations in (7-64) yields

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2 \qquad (7\text{-}65)$$

Integral form : $\int_V (7-65)dv$ using Gauss' theorem

$$\oint_S (\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{s} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)dv - \int_V \sigma E^2 dv \qquad (7\text{-}66)$$

EM power outflow = EM energy decreasing rate − Ohmic power dissipation

⟹ instantaneous EM energy conservation

Definition of Poynting vector $\mathscr{P}$:

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} \quad (\mathrm{W/m^2}) \qquad (7\text{-}67)$$

2) Poynting's theorem = Instantaneous EM energy conservation

Rewriting of (7-66) :

$$-\oint_S \mathscr{P}\cdot d\boldsymbol{s} = \frac{\partial}{\partial t}\int_V (w_e + w_m)\,dv + \int_V P_\sigma\,dv \qquad (7\text{-}68)$$

EM power inflow = EM energy increasing rate + Ohmic power dissipation

$w_e = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\epsilon\boldsymbol{E}\cdot\boldsymbol{E}^*$ = electric energy density (7-69)

$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu\boldsymbol{H}\cdot\boldsymbol{H}^*$ = magnetic energy density (7-70)

$p_\sigma = \sigma E^2 = J^2/\sigma = \sigma\boldsymbol{E}\cdot\boldsymbol{E}^* = \boldsymbol{J}\cdot\boldsymbol{J}^*/\sigma$ = Ohmic power density (7-71) (e.g. 7-5) Illustrating Poynting's theorem:

$$\mathbf{J} = \hat{z}(I/\pi b^2)$$

$$\mathbf{E} = \mathbf{J}/\sigma = \hat{z}(I/\sigma\pi b^2)$$

$$\mathbf{H} = \hat{\phi}(I/2\pi b)$$

$$\Rightarrow \quad \mathscr{P} = \mathbf{E}\times\mathbf{H} = -\hat{\mathbf{r}}(I^2/2\sigma\pi^2 b^3)$$

$$-\oint_S \mathscr{P}\cdot d\mathbf{s} = -\int_0^l\left(-\hat{\mathbf{r}}\frac{I^2}{2\sigma\pi^2 b^3}\right)\cdot(\hat{\mathbf{r}}2\pi b\,dz)$$

$$= I^2\left(\frac{l}{\sigma\pi b^2}\right) = I^2 R$$

$\Rightarrow$ EM power inflow = Ohmic power loss

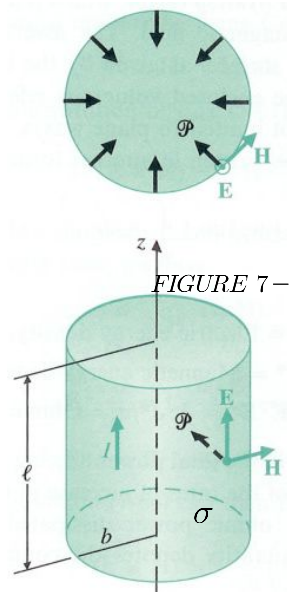


FIGURE 7−7

3) Time-Average Poynting vector

Instantaneous time-harmonic EM waves :

$$\mathbf{E}(z,t) = Re[\mathbf{E}(z)e^{j\omega t}] = \hat{\mathbf{x}}E_o e^{-\alpha z}Re[e^{i(\omega t-\beta z)}]$$

$$= \hat{\mathbf{x}}E_o e^{-\alpha z}\cos(\omega t-\beta z) \qquad \text{(7-73)}$$

$$\eta_c = |\eta_c|e^{j\phi_\eta}$$

$$\mathbf{H}(z,t) = Re[\mathbf{H}(z)e^{j\omega t}] = Re\left[\hat{\mathbf{y}}\frac{E(z)}{\eta_c}e^{j\omega t}\right] = \hat{\mathbf{y}}\frac{E_o}{|\eta_c|}e^{-\alpha z}Re[e^{i(\omega t-\beta z-\phi_\eta)}]$$

$$= \hat{\mathbf{y}}\frac{E_o}{|\eta_c|}e^{-\alpha z}\cos(\omega t-\beta z-\phi_\eta) \qquad \text{(7-75)}$$

Instantaneous Poynting vector :

$$\mathscr{P}(z,t) = \mathbf{E}(z,t)\times\mathbf{H}(z,t)$$

(7 − 73), (7 − 75)

$$= \hat{\mathbf{z}}\frac{E_o^2}{2|\eta_c|}e^{-2\alpha z}[\cos\phi_\eta + \cos(2\omega t-2\beta z-\phi_\eta)] \qquad \text{(7-76)}$$

Time-average Poynting vector :

$$\mathscr{P}_{av}(z) = \frac{1}{T}\int_0^T \mathscr{P}(z,t)\,dt$$

(7 − 76)

$$= \hat{\mathbf{z}}\frac{E_o^2}{2|\eta_c|}e^{-2\alpha z}\left[\cos\phi_\eta + \underbrace{\frac{1}{T}\int_0^T \cos(2\omega t-2\beta z-\phi_\eta)\,dt}_{=0}\right]$$

$$= \hat{\mathbf{z}}\frac{E_o^2}{2|\eta_c|}e^{-2\alpha z}\cos\phi_\eta \quad (\text{W/m}^2) \qquad \text{(7-77)}$$

In lossless media ($\alpha = 0,\ \phi_\eta = 0,\ \eta_c = \eta$), $\mathscr{P}_{av}(z) = \hat{\mathbf{z}}E_o^2/2\eta$ (7-78)

Generalization : $\mathscr{P}_{av}(z) = \frac{1}{2}Re[\mathbf{E}\times\mathbf{H}^*]$ (W/m^2) (7-79)

Total average power : $P_{av} = \oint_S \mathscr{P}_{av}(z)\cdot d\mathbf{s}$ (W) (7-79)*

Homework Set 2

1) P.7-1 2) P.7-3 3) P.7-5

4) P.7-7 5) P.7-10 6) P.7-11