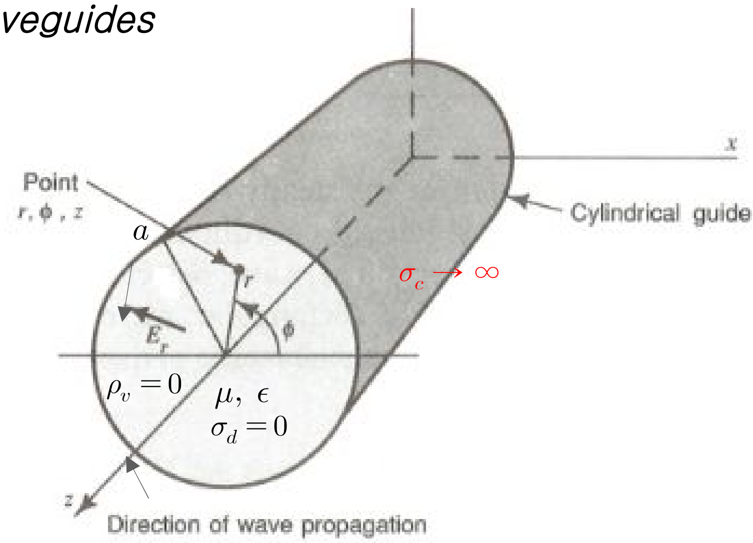
3. Circular Waveguides and Cavity Resonators

A. Cylindrical Waveguides



1) TE wave fields in the cylindrical waveguide

Longitudinal fields: $E_z(r,\phi,z) = 0$, $H_z(r,\phi,z) = H_z^o(r,\phi)\, e^{-\gamma z}$

BVP: $\left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \phi^2} + h^2\right) H_z^o(r,\phi) = 0$: Wave equation

$$\left.\frac{\partial H_z^o(x,y)}{\partial r}\right|_{r=a} = 0 \ , \quad 0 \le \phi \le 2\pi \qquad : \text{BC}$$

Separation of variables: $H_z^o(r,\phi) = R(r)\,\Phi(\phi)$

$$\frac{r^2}{R}\left[\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + h^2 R(r)\right] = -\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} \equiv n^2 = constant$$

$$\Rightarrow \begin{cases} \dfrac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0 \\ \dfrac{d^2R(r)}{dr^2} + \dfrac{1}{r}\dfrac{dR(r)}{dr} + \left(h^2 - \dfrac{n^2}{r^2}\right)R(r) = 0 \ : \text{Bessel diff. eqn.} \end{cases}$$

General solutions: $\begin{cases} \Phi(\phi) = A_1 \sin n\phi + A_2 \cos n\phi \\ R(r) = B_1 J_n(hr) + B_2 N_n(hr) \end{cases}$

Choose $A_1 = 0$ so that $H_z^o(\phi)$ has maximum values at $\phi = 0, \pi$.

$$\Rightarrow \quad \Phi(\phi) = A_2 \cos n\phi$$

At $r = 0$, H_z^o=finite, i.e., $R(r=0)$=finite $\Rightarrow$ $B_2 = 0$ since $N_n(r=0) \to \infty$.

$$\Rightarrow \quad R(r) = B_1 J_n(hr)$$

By putting $H_o \equiv A_2 B_1$ to be determined by IC:

$$H_z^o(r,\phi) = H_o \cos n\phi \ J_n(hr) \quad \text{(A/m)}$$

Applying BC to the general solution:

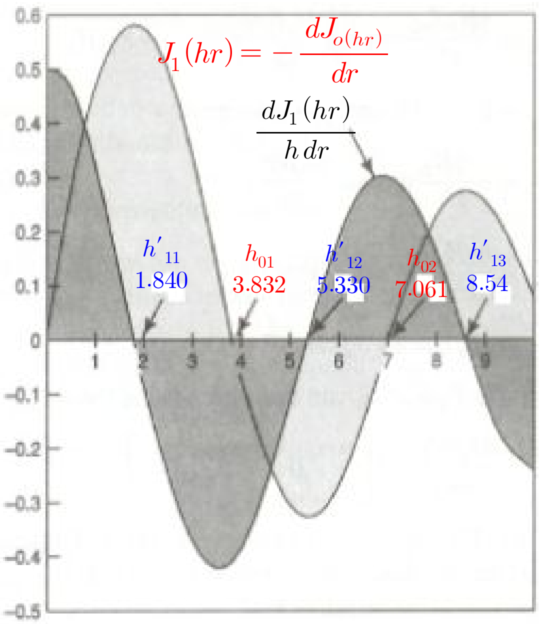
$$\left.\frac{dJ_n(hr)}{dr}\right|_{r=a} = 0$$

$$\Rightarrow \quad \frac{dJ_n(ha)}{dr} = \frac{dJ_n(h'_{nr})}{dr} = 0$$

where $h'_{nr} = ha$ = rth root of $\frac{dJ_n(hr)}{dr} = 0$ $(r=1,2,3,....)$: eigenvalues

Then, the eigenmodes for H_z^o becomes

$$H_z^o(r,\phi) = H_o \cos n\phi \; J_n\left(\frac{h'_{nr}}{a}r\right) \qquad \text{for } TE_{nr} \text{ mode}$$



Mode designation†	Eigenvalues h'_{nr}	h_{nr}	$(\lambda_c)_{nr}$
TM_{01}		2.405	$2.61r_0$
TE_{01} (low loss)	3.832		$1.64r_0$
TM_{02}		5.520	$1.14r_0$
TE_{02}	7.016		$0.89r_0$
TE_{11} (dominant)	1.840		$3.41r_0$
TM_{11}		3.832	$1.64r_0$
TE_{12}	5.330		$1.18r_0$
TM_{12}		7.016	$0.89r_0$
TE_{21}	3.054		$2.06r_0$
TM_{21}		5.135	$1.22r_0$
TE_{22}	6.706		$0.94r_0$
TE_{31}	4.201		$1.49r_0$
TM_{31}		6.379	$0.98r_0$
TE_{41}	5.318		$1.18r_0$
TM_{41}		7.588	$0.83r_0$
TE_{51}	6.416		$0.98r_0$

† The subscripts nr as in TE_{nr} or k_{nr} have the following significance:

n = nth-order Bessel function

r = order of root of nth-order Bessel function

Transverse fields can be determined by using H_z^o:

from $\mathbf{E}_\perp^o = \frac{j\omega\mu}{h^2}(\hat{z} \times \nabla_\perp H_z^o)$ (9-11, 12)* ,

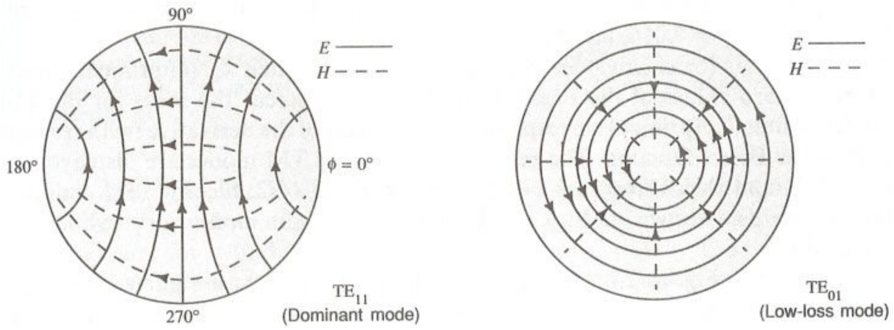
$$\Rightarrow \quad E_r^o = -\frac{j\omega\mu}{h^2}\frac{1}{r}\frac{\partial H_z^o}{\partial\phi}, \qquad E_\phi^o = \frac{j\omega\mu}{h^2}\frac{\partial H_z^o}{\partial r}$$

from $\mathbf{H}_\perp^o = -\frac{\gamma}{h^2}\nabla_\perp H_z^o$ (9-13, 14)* ,

$$\Rightarrow \quad H_r^o = -\frac{\gamma}{h^2}\frac{\partial H_z^o}{\partial r}, \qquad H_\phi^o = -\frac{\gamma}{h^2}\frac{1}{r}\frac{\partial H_z^o}{\partial\phi}$$

2) Characteristics of TE and TM modes

Field configurations of TE_{nr} modes:



Dispersion relation: $h = h'_{nr}/a$

$$h^2 = \gamma^2 + \omega^2\mu\epsilon \quad \Rightarrow \quad \gamma = \sqrt{(h'_{nr}/a)^2 - \omega^2\mu\epsilon} = \alpha + j\beta$$

Cutoff ($\gamma = 0$): $\omega^2\mu\epsilon = (h'_{nr}/a)^2 \quad \Rightarrow \quad (f_c)_{nr} = \dfrac{u}{2\pi}\dfrac{h'_{nr}}{a}$: cutoff frequency

$$\Rightarrow (\lambda_c)_{nr} = \frac{u}{(f_c)_{nr}} = \frac{2\pi a}{h'_{nr}} \quad \Rightarrow \text{Dominant mode } TE_{11}:\ (\lambda_c)_{11} = \frac{2\pi a}{h'_{11}} = 3.41a$$

h'_{11} = 1.84

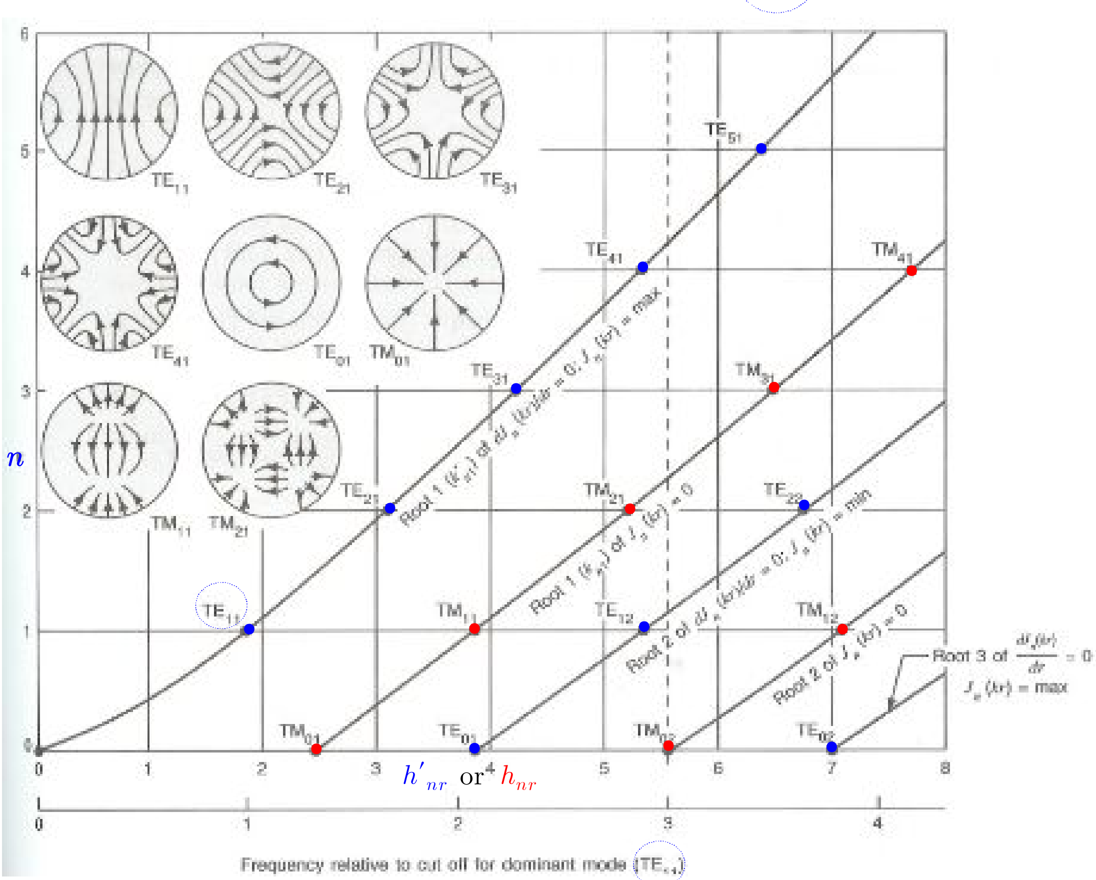
Propagation in the waveguide ($\omega^2\mu\epsilon > (h'_{nr}/a)^2$):

Phase constant: $(\beta)_{nr} = \sqrt{\omega^2\mu\epsilon - (h'_{nr}/a)^2} = \dfrac{\omega}{u}\sqrt{1 - [(f_c)_{nr}/f]^2}$

Wavelength in the guide: $(\lambda_g)_{nr} = \dfrac{2\pi}{(\beta)_{nr}} = \dfrac{u}{f}\dfrac{1}{\sqrt{1 - [(f_c)_{nr}/f]^2}}$

Phase velocity: $(u_p)_{nr} = \dfrac{\omega}{(\beta)_{nr}} = \dfrac{u}{\sqrt{1 - [(f_c)_{nr}/f]^2}}$

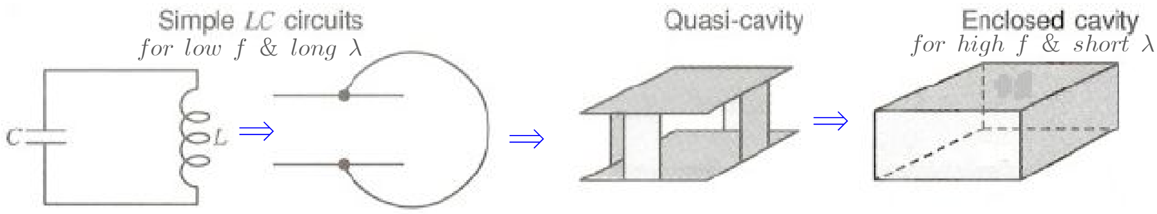
Cutoff frequencies relative to the dominant mode TE_{11}



B. Cavity Resonators

= EM energy storage devices in the form of enclosed metal boxes confining EM fields inside and eliminating radiation and high-resistant effects by large-area current flow on the metal surfaces. $\Rightarrow$ $\exists$ resonant f_{mnp} and high Q value

Types of resonators



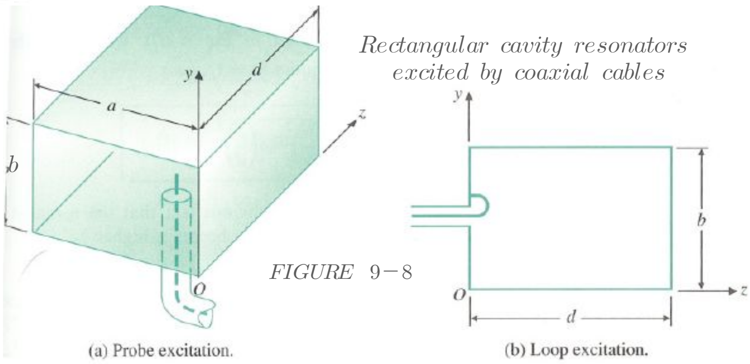
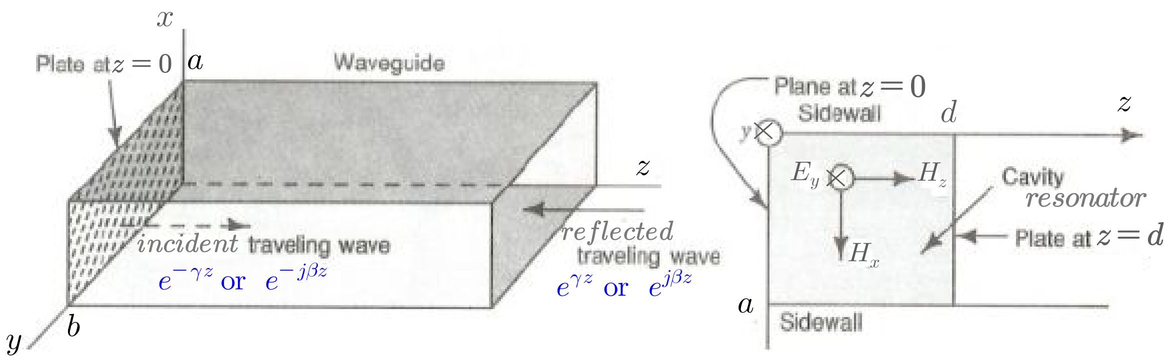


FIGURE 9−8

1) Fields and resonant frequencies of rectangular cavity resonators



Choose the z-axis as the reference direction of propagation, then there exist standing waves in $0 \le z \le d$ by incident and reflected TE or TM waves in the cavity.

a) TM_{mnp} modes

TM wave fields in the rectangular guide:

$$H_z(x,y,z) = 0\ , \quad E_z(x,y,z) = E_z^o(x,y)\, e^{-j\beta z} \ : \text{ incident wave} \qquad \text{(9-52)}$$

$$= E_z^o(x,y)\, e^{j\beta z} \ : \text{ reflected wave} \qquad \text{(9-52)*}$$

where $\quad E_z^o(x,y) = E_o \sin\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right)$ (9−65)

(9−52) + (9−52)* = standing wave $\propto$ $\cos\beta z$ or $\sin\beta z$

Application of BCs: $E_x^o(x,y,z)|_{z=0,\,d} = 0$ and $E_y^o(x,y,z)|_{z=0,\,d} = 0$

$$\Rightarrow \quad E_x^o,\ E_y^o \propto \sin\beta z \ \ \&\ \ \beta = p\pi/d$$

(9-13), (9-14) for E_x^o, E_y^o with $H_z(x,y,z)=0$ and $-\gamma=\partial/\partial z$

$$\Rightarrow \quad E_z^o \propto \cos\beta z \ \& \ \beta = p\pi/d$$

$$\therefore \ E_z(x,y,z) = E_z^o(x,y)\cos\left(\frac{p\pi}{d}z\right) = E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right) \quad \text{(9-102)}$$

$m,n = 1,2,....$ and $p = 0,1,2,....$

Other transverse fields are obtained from (9-13), (9-14) with $H_z(x,y,z)=0$ and $-\gamma=\partial/\partial z$.

Resonant frequency of TM_{mnp} modes from (9-68) :

$$f_{mnp} = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2} \quad \text{(9-104)}$$

$m,n = 1,2,....$ and $p = 0,1,2,....$

b) TE_{mnp} modes

TE wave fields in the rectangular guide:

$E_z(x,y,z)=0$, $\quad H_z(x,y,z) = H_z^o(x,y)e^{-j\beta z}$: incident wave (9-70)

$= H_z^o(x,y)e^{j\beta z}$: reflected wave (9-70)*

where $\quad H_z^o(x,y) = H_o\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$ (9-76)

In a similar manner as TM_{mnp}, we can get

$$H_z(x,y,z) = H_z^o(x,y)\sin\left(\frac{p\pi}{d}z\right) = H_o\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right) \quad \text{(9-103)}$$

$m,n = (either\ m \text{ or } n=0),1,2,....$ and $p = 1,2,....$

Note) If m=n=0, H_z is ind. of x and y

$\Rightarrow$ all transv. fields = 0 by (9-11)~(9-14)

$\Rightarrow$ ∄ no TE modes

Other transverse fields are obtained from (9-13), (9-14) with $E_z(x,y,z)=0$ and $-\gamma=\partial/\partial z$.

Resonant frequency of TE_{mnp} modes are the same as that of TM_{mnp}

$$f_{mnp} = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2} \quad \text{(9-103)}$$

but, $m,n = (either\ m \text{ or } n=0),1,2,....$ and $p = 1,2,....$

Therefore, TM_{mnp} and TE_{mnp} are always **degenerate** with the same f_{mnp} excluding the cases for none of m, n, p = 0 (TM_{mn0}, TE_{0np}, TE_{m0p}).

(eg.9-8)

Dominant modes in an air-filled rectangular cavity with $a \times b \times d$.

Lowest-oder modes: TM_{110}, TE_{011}, TE_{101}

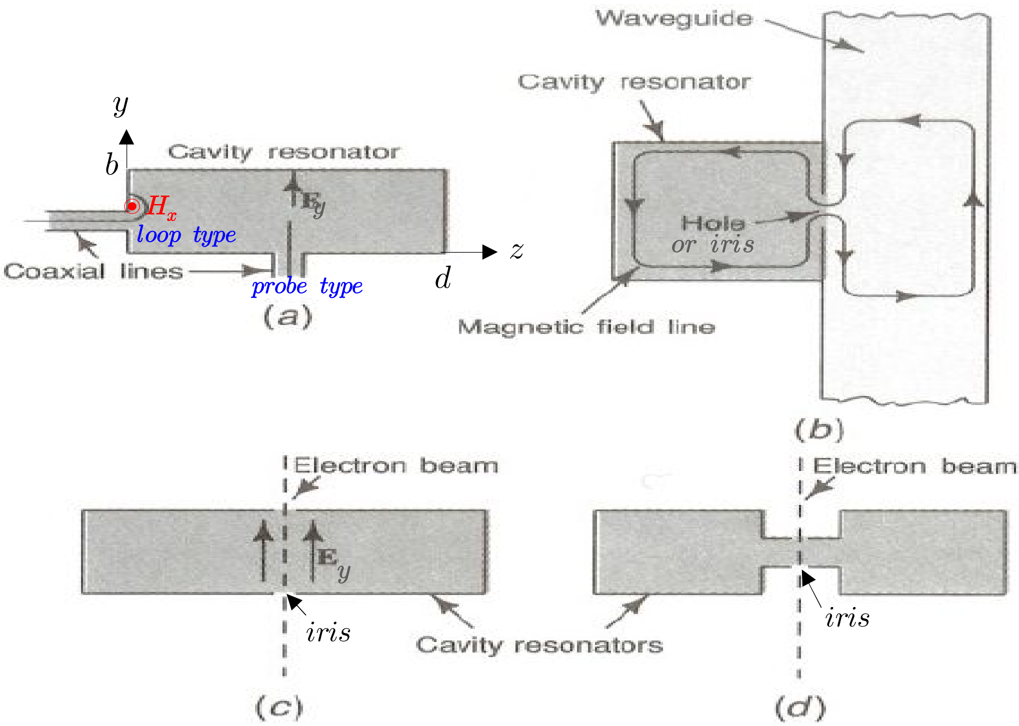
(a) For $a>b>d$, $\quad (f_{mnp})_{\min} = f_{110} = (c/2)\sqrt{a^{-2}+b^{-2}} \quad \Rightarrow \quad TM_{110}$ (9-108)

(b) For $a>d>b$, $\quad (f_{mnp})_{\min} = f_{101} = (c/2)\sqrt{a^{-2}+d^{-2}} \quad \Rightarrow \quad TE_{101}$ (9-109)

(c) For $a=b=d$, $\quad (f_{mnp})_{\min} = f_{110} = f_{011} = f_{101} = (c/2)\sqrt{2a^{-2}} = c/\sqrt{2}a$ (9-110)

$\Rightarrow \quad TM_{110}$, TE_{011}, TE_{101}

2) Excitation of TM_{mnp} and TE_{mnp} modes in a cavity resonator

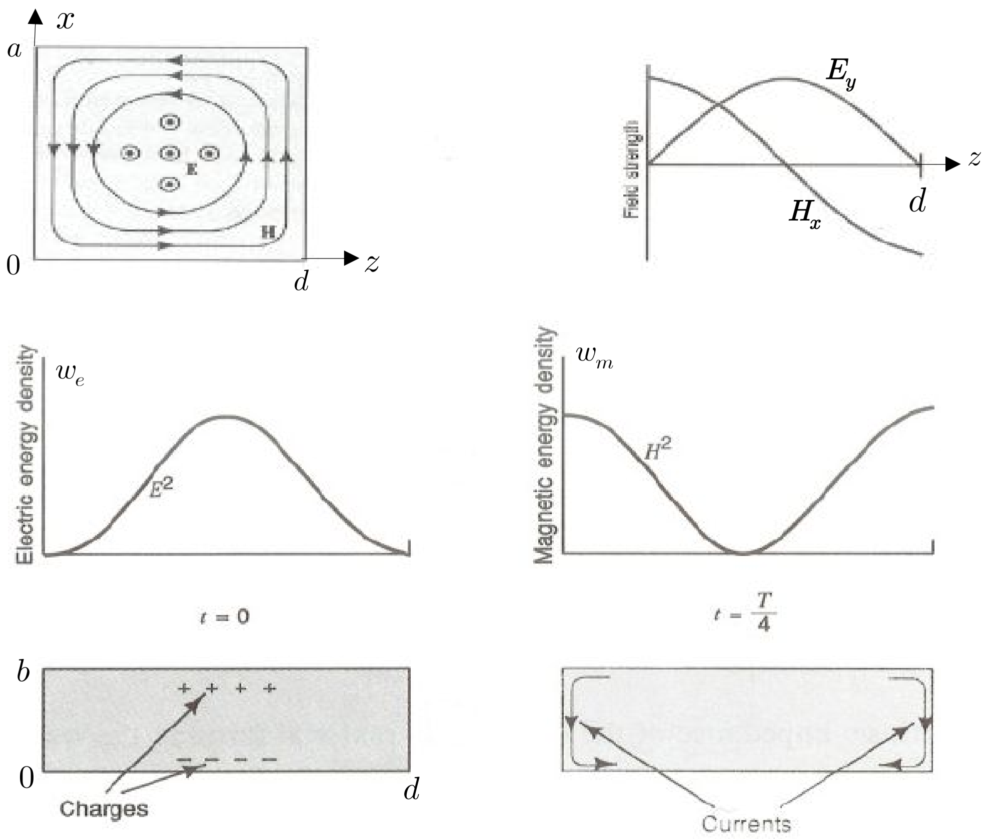


(e.g.) TE_{101} mode in an $a \times b \times d$ rectangular cavity :

$$E_y = -\frac{j\omega\mu a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{d}z\right) \tag{9-105}$$

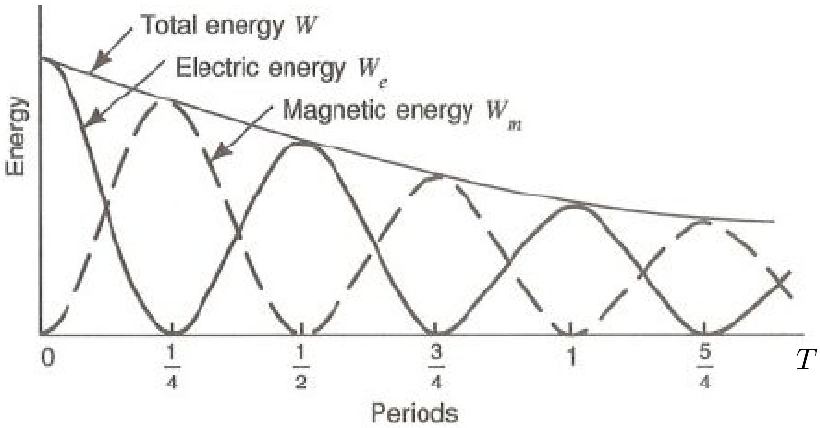
$$H_x = -\frac{a}{d} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{d}z\right) \tag{9-106}$$

$$H_z = H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{d}z\right) \tag{9-107}$$



3) Quality factor Q of cavity resonator

Dissipation of stored EM energy into metal walls of finite conductivity:



Quality factor Q : a measure of the bandwidth of a resonator

$$Q \equiv 2\pi \frac{total\ time-average\ energy\ stored\ at\ f_{mnp}}{dissipated\ energy\ in\ a\ period} \tag{9-111}$$

$$\Rightarrow \quad Q = \frac{\omega W}{P_L} \quad (\gg 1 \text{ at } f_{mnp} : \text{narrow bandwidth}) \tag{9-113}$$

where $$W = W_e + W_m = \frac{1}{2}\left[\int_V \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv\right] \tag{9-112}$$

$$P_L = -\frac{dW}{dt} = \oint_S \mathscr{P}_{av} \cdot d\mathbf{s} \tag{9-112)*$$

For T_{101} in an $a \times b \times d$ cavity by using (9-105, 106, 107):

$$\begin{aligned} W_e &= \frac{\epsilon_0}{4}\int |E_y|^2 \, dv \\ &= \frac{\epsilon_0 \omega_{101}^2 \mu_0^2 a^2}{4\pi^2} H_0^2 \int_0^d \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{d}z\right) dx\,dy\,dz \\ &= \frac{\epsilon_0 \omega_{101}^2 \mu_0^2 a^2}{4\pi^2} H_0^2 \left(\frac{a}{2}\right) b \left(\frac{d}{2}\right) = \frac{1}{4}\epsilon_0 \mu_0^2 a^3 b d f_{101}^2 H_0^2 \end{aligned} \tag{9-114}$$

$$\begin{aligned} W_m &= \frac{\mu_0}{4}\int \{|H_x|^2 + |H_z|^2\} \, dv \\ &= \frac{\mu_0}{4} H_0^2 \int_0^d \int_0^b \int_0^a \left\{\frac{a^2}{d^2}\sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{d}z\right) \right. \\ &\quad \left. + \cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{d}z\right)\right\} dx\,dy\,dz \\ &= \frac{\mu_0}{4} H_0^2 \left\{\frac{a^2}{d^2}\left(\frac{a}{2}\right) b \left(\frac{d}{2}\right) + \left(\frac{a}{2}\right) b \left(\frac{d}{2}\right)\right\} = \frac{\mu_0}{16} abd \left(\frac{a^2}{d^2} + 1\right) H_0^2 \end{aligned} \tag{9-115}$$

$f_{101} = (1/2\sqrt{\mu_o \epsilon_o})\sqrt{a^{-2} + d^{-2}}$ in (9-114) $\Rightarrow$ $W_e = W_m$

$$\Rightarrow \quad W = 2W_e = 2W_m = \frac{\mu_0 H_0^2}{8} abd \left(\frac{a^2}{d^2} + 1\right) \tag{9-117}$$

$\mathscr{P}_{av} = \frac{1}{2}|J_s|^2 R_s = \frac{1}{2}|H_t|^2 R_s$ in (9-112)* :

$$P_L = \oint \mathscr{P}_{av}\, ds = R_s \left\{ \int_0^b \int_0^a |H_x(z=0)|^2\, dx\, dy + \int_0^d \int_0^b |H_z(x=0)|^2\, dy\, dz + \int_0^d \int_0^a |H_x|^2\, dx\, dz + \int_0^d \int_0^a |H_z|^2\, dx\, dz \right\}$$

$$= \frac{R_s H_0^2}{2} \left\{ \frac{a^2}{d}\left(\frac{b}{d} + \frac{1}{2}\right) + d\left(\frac{b}{a} + \frac{1}{2}\right) \right\} \qquad (9-119)$$

(9-117, 119) in (9-113) :

$$Q_{101} = \frac{\pi f_{101} \mu_0 abd(a^2 + d^2)}{R_s[2b(a^3 + d^3) + ad(a^2 + d^2)]} \qquad (9-120)$$

(e.g. 9-9)

Given: a hollow cubic cavity ($a = b = d$) of Cu ($\sigma = 5{,}80 \times 10^7$ S/m) having a dominant freq. = 10 GHz

Find (a) a = ? (b) Q = ?

(a) For $a = b = d$, dominant modes = TM_{110}, TE_{011}, TE_{101}

$$\Rightarrow \quad f_{110} = f_{011} = f_{101} = \frac{c}{\sqrt{2}\, a} = \frac{3 \times 10^8}{\sqrt{2}\, a} = 10^{10}$$

$$\Rightarrow \quad a = 2.12 \times 10^{-2} \quad \text{(m)}$$

(b) $a = b = d$, $\sigma = 5{,}80 \times 10^7$, $R_s = \sqrt{\pi f_{101} \mu_o / \sigma}$ in (9-120) :

$$Q_{101} = \frac{\pi f_{101} \mu_o a}{3 R_s} = \frac{a}{3} \sqrt{\pi f_{101} \mu_o \sigma} \qquad (9-121)$$

$$= 10{,}693 \gg 1$$

Homework Set 7

1) P.9-17 2) P.9-20