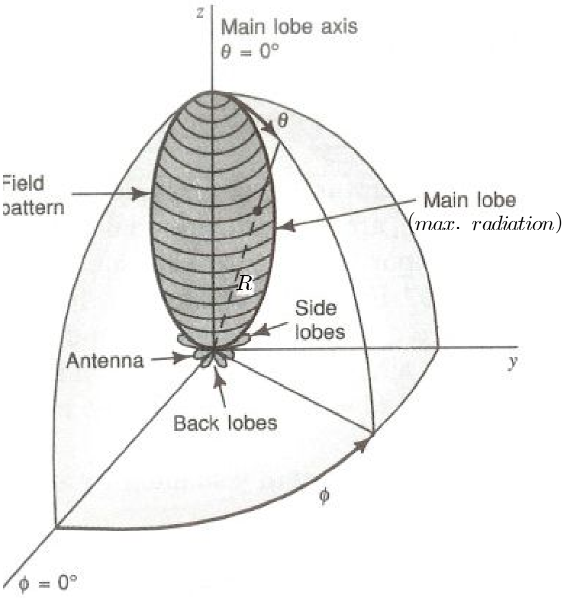
2. Antenna Arrays

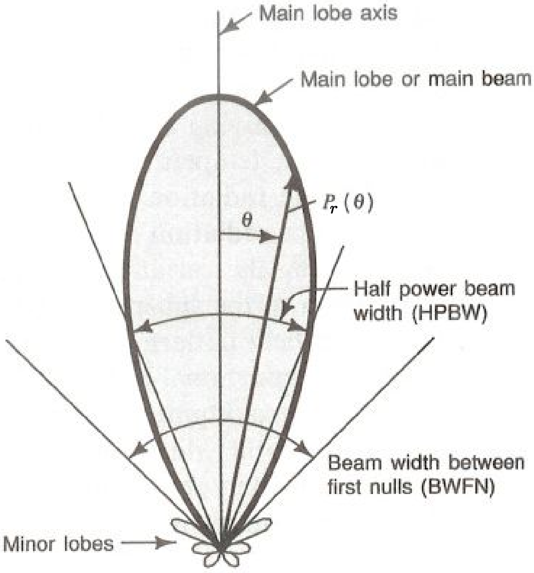
= Arrangements of several antenna elements in various configurations (straight lines, circles, triangles, ...) to have high directivity.

In the far-field condition, the radiation patterns are 3-dim. quantities involving the variation of field or power (proportional to the field square) as a function of the spherical coordinates θ and ϕ.

Antenna field pattern with coordinate system



Antenna power pattern in polar coordinates



A. Two-Element Arrays

1) Principle of pattern multiplication

Consider two identical linear antennas separated by a distance *d*, which are excited with currents of the same amplitude and a phase difference ξ between them.

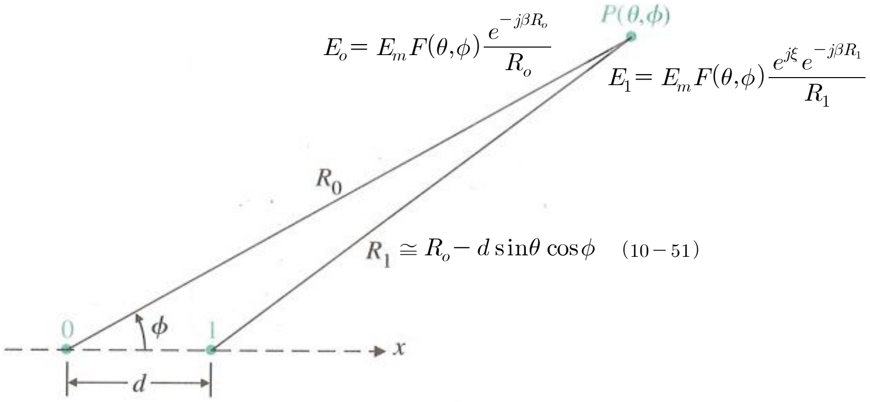


FIGURE 10-6 A two-element array.

Total electric field:

$$E = E_o + E_1 = \underset{amplitude\ ftn.}{E_m}\underset{pattern\ ftn.}{\boxed{F(\theta,\phi)}}\left[\frac{e^{-j\beta R_o}}{R_o} + \frac{e^{j\xi}e^{-j\beta R_1}}{R_1}\right] \qquad (10\text{-}50)$$

(10-51) in (10-50):

$$E = E_m \frac{F(\theta,\phi)}{R_0} e^{-j\beta R_0} [\underbrace{1 + e^{j\beta d \sin\theta\cos\phi} e^{j\xi}}_{1+e^{j\psi}\,(array\ factor)}]$$

$$= E_m \frac{F(\theta,\phi)}{R_o} e^{-j\beta R_o} e^{j\psi/2} \left(2\cos\frac{\psi}{2} \right) \qquad (10\text{-}52)$$

where $\psi = \beta d \sin\theta \cos\phi + \xi$ (10-53)

$$\Rightarrow \quad |E| = \frac{2E_m}{R_o} |F(\theta,\phi)| \left| \cos\frac{\psi}{2} \right| \qquad (10\text{-}54)$$

⇒ Principle of pattern multiplication:

Total array pattern function of identical elements

= *Element factor* (individual source pattern function)

× *Array factor* (depending on array geometry, ampl. and phase)

2) Broadside array and Endfire array

Consider H-plane $(\theta = \pi/2)$ radiation patterns of two-element parallel dipole array directed in z and placed along the x-axis *(e.g. 10-6)*.

a) Broadside array

For $d = \lambda/2\ (\beta d = \pi)$, $\xi = 0$ (in phase),

Normalized array factor is

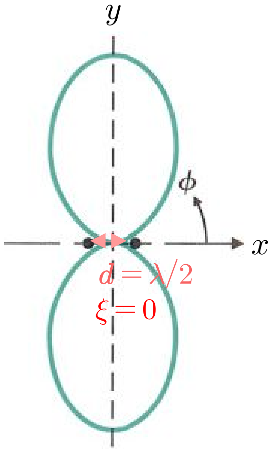
$$|A(\phi)|_n = \left|\cos\frac{\psi}{2}\right| = \left|\cos\frac{1}{2}(\beta d\cos\phi + \xi)\right|$$

$$= \left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right| \qquad (10\text{-}55)$$

At $\phi = \pm\pi/2$, $\exists\ |E|_{\max}$.

At $\phi = 0,\ \pi$, $\exists\ |E|_{\min} = 0$.

Main beams only, no side lobes



b) Endfire array

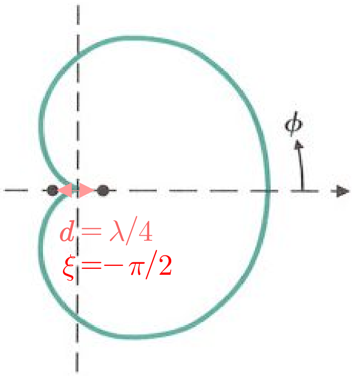
For $d = \lambda/4\ (\beta d = \pi/2)$, $\xi = -\pi/2$ (lags),

$$|A(\phi)|_n = \left|\cos\left(\frac{\pi}{4}(\cos\phi - 1)\right)\right| \qquad (10\text{-}56)$$

At $\phi = 0$, $\exists\ |E|_{\max}$.

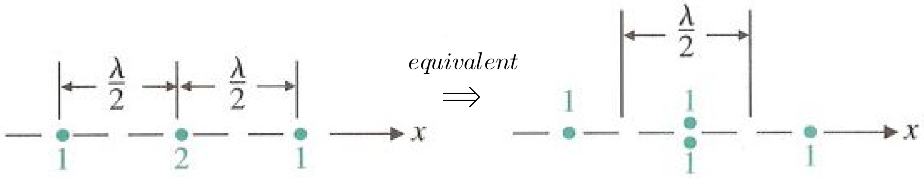
At $\phi = \pi$, $\exists\ |E|_{\min} = 0$

No minor lobes



3) Binomial arrays

Consider a linear array of three isotropic sources in the $\theta=\pi/2$ plane: $d=\lambda/2$ $(\beta d=\pi)$, $\xi=0$ (in phase), amplitude ratio = $1:2:1$ *(e.g. 10-7)*

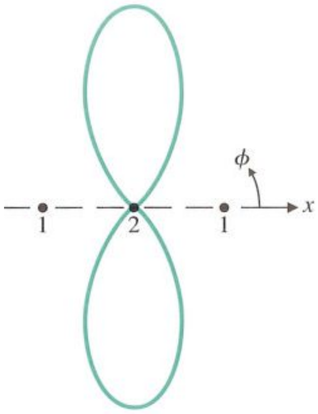


(a) Three-element binomial array.

(b) Two displaced two-element arrays.

Broadside pattern by the principle of pattern multiplication using (10-54, 55),

$$|E|=\frac{4E_m}{R_o}\left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right|^2 \qquad (10\text{-}57)$$



More directive than the two-element array

Main beams only, no sidelobes

Generalization for the binomial array of N elements:

Array factor by the principle of pattern multiplication using (10-52),

$$(1+e^{j\psi})^{N-1}$$

(cf) Binomial series:

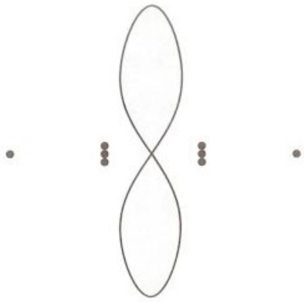
$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \;=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ \ldots$$

where $\binom{n}{k}=\dfrac{n(n-1)(n-2)....(n-k+1)}{k!}=\dfrac{n!}{k!\,(n-k)!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
....

Broadside patterns:

$\left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right|^2$ for N=3



$\left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right|^3$ for N=4

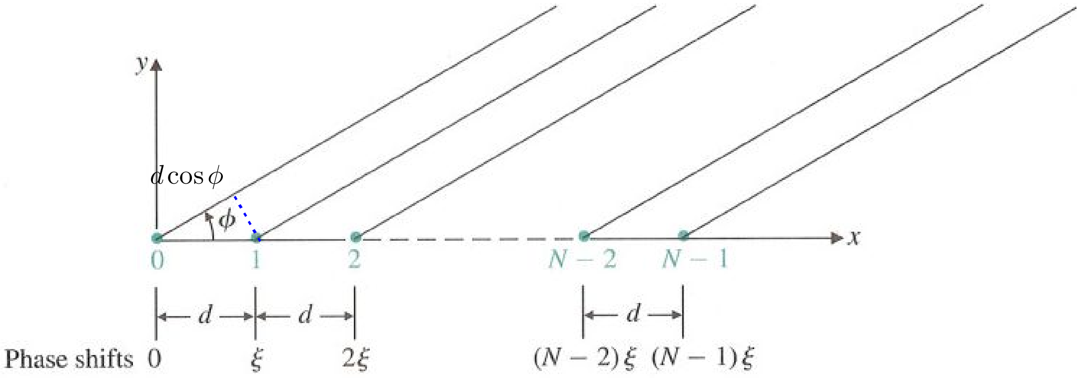
$\Rightarrow$ A directive pattern w/o sidelobes if $d=\lambda/2$

B. General Uniform Linear Array

1) Array factor

Consider a uniform linear array (equally-spaced identical antennas fed with currents of equal magnitude and uniform progressive phase shift),

FIGURE 10-10 A general uniform linear array.



Total phase difference from adjacent sources:

$$\psi = \beta d\cos\phi + \xi \tag{10-59}$$

Array factor of an N-element uniform linear array using (10-52):

$$A(\psi) = 1 + e^{j\psi} + e^{j2\psi} + \,.....\, + e^{j(N-1)\psi} \qquad ①$$

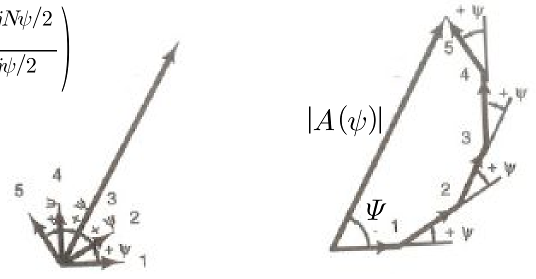
[① - ①× $e^{j\psi}$]/$(1-e^{j\psi})$:

$$A(\psi) = \frac{1-e^{jN\psi}}{1-e^{j\psi}} = \frac{e^{jN\psi/2}}{e^{j\psi/2}}\left(\frac{e^{jN\psi/2}-e^{-jN\psi/2}}{e^{j\psi/2}-e^{-j\psi/2}}\right)$$

$$= e^{j\Psi}\frac{\sin(N\psi/2)}{\sin(\psi/2)} \qquad ②$$

where $\Psi = \frac{N-1}{2}\psi$

$$\Rightarrow \quad |A(\psi)| = \left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| \qquad ③$$



For $\psi = 0$, $|A(\psi)|_{\max} = N$

Therefore, the normalized array factor becomes

$$|A(\psi)|_n \equiv \frac{|A(\psi)|}{|A(\psi)|_{\max}} = \frac{1}{N}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| \tag{10-60}$$

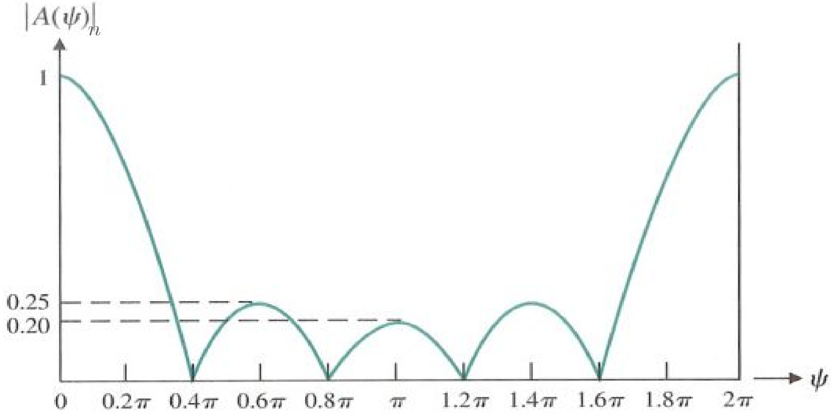
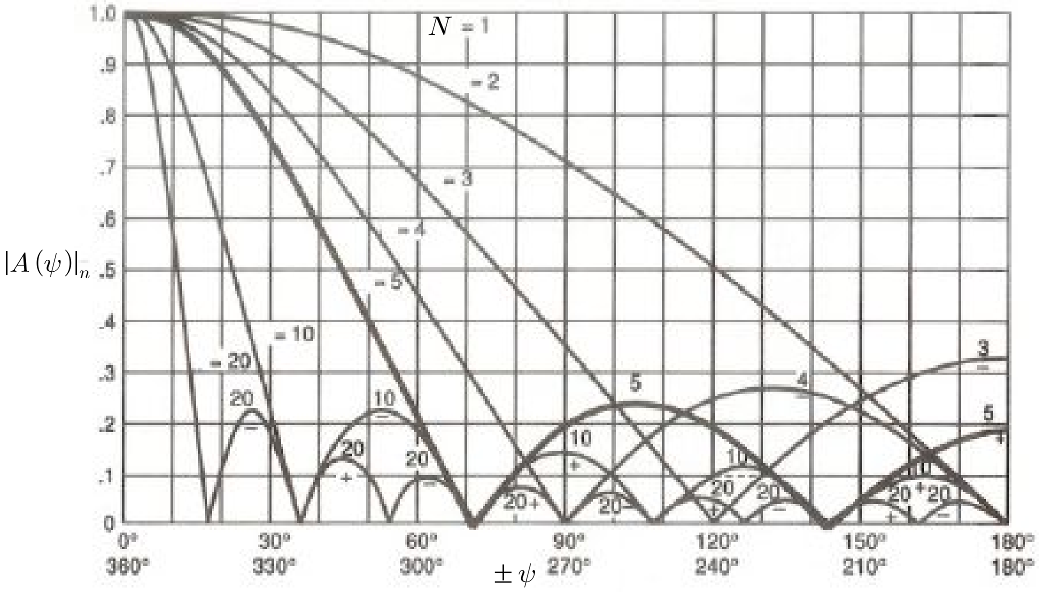


FIGURE 10-11 Normalized array factor of a five-element uniform linear array.



Normalized array factor of various N-element uniform linear arrays

2) Main beam direction

When $\psi = 0$, $\exists\ |A(\psi)|_{\max}$.

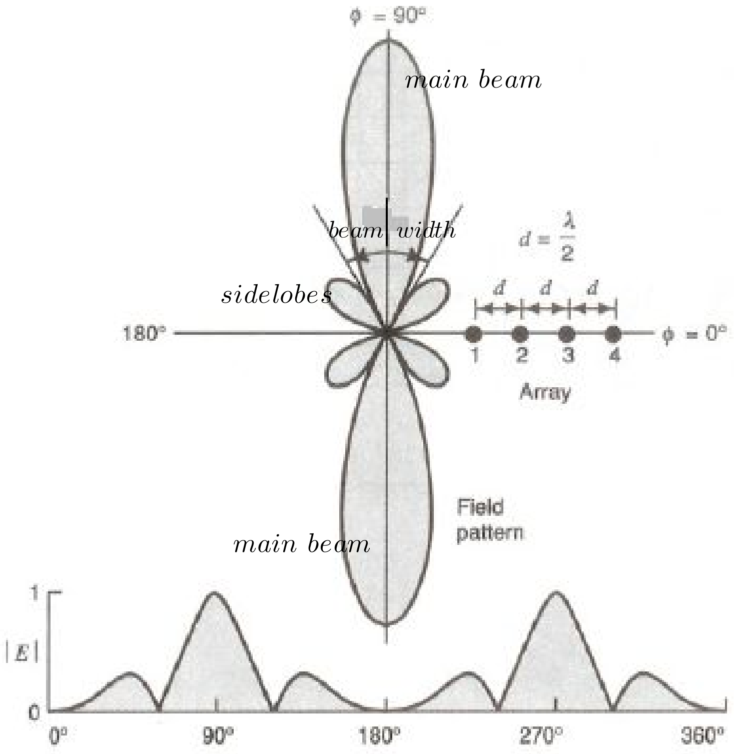
$$\Rightarrow \quad \beta d \cos\phi + \xi = 0 \quad \Rightarrow \quad \cos\phi_o = -\xi/\beta d \qquad (10\text{-}61)$$

a) Broadside array

For sources in phase $(\xi = 0)$, $\cos\phi_o = 0 \ \Rightarrow \ \phi_o = \pm\pi/2$.

i.e., Maximum radiation at perpendicular directions to the array line

(e.g.) For N=4,



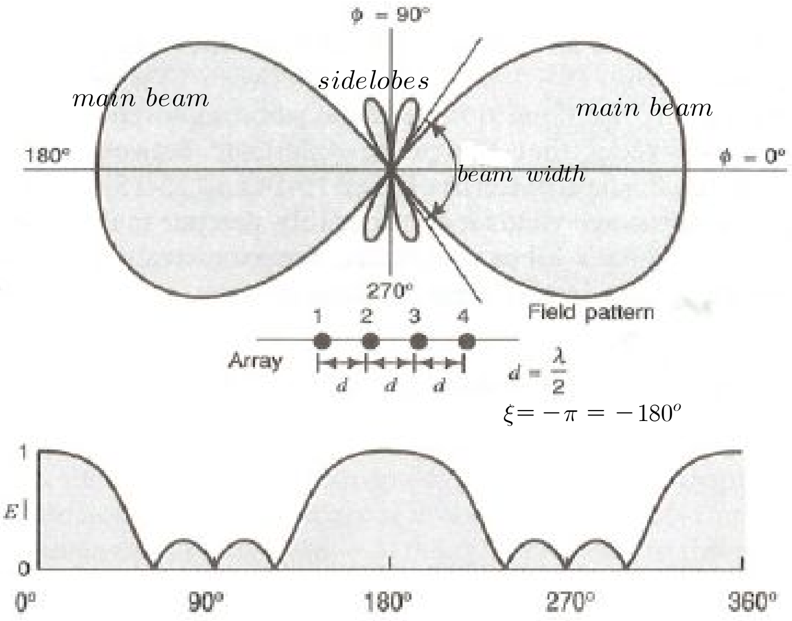
b) Endfire array

In order to make the radiation field maximum in the direction of the array ($\phi_o = 0$), $\psi = 0$ and $\phi_o = 0$ in (10-59):

$$\xi = -\beta d \cos\phi_o = -2\pi d/\lambda$$

⇒ **Phased array** = antenna array with phase shifter to scan the main beam direction

(e.g.) For N=4,



3) Sidelobe locations and first sidelobe level

When $|\sin(N\psi/2)| = 1$ in (10-60), sidelobes (minor maxima) occur:

$$\frac{N\psi}{2} = \pm(2m+1)\frac{\pi}{2,} \qquad m = 1, 2, 3, \ldots.$$

The locations of the first sidelobes :

$$\frac{N\psi}{2} = \pm\frac{3\pi}{2}, \quad (m=1) \tag{10-62}$$

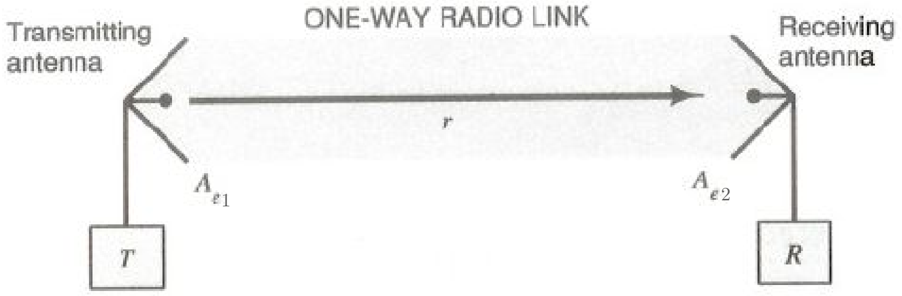
Amplitude of the first sidelobes from (10-62) in (10-60):

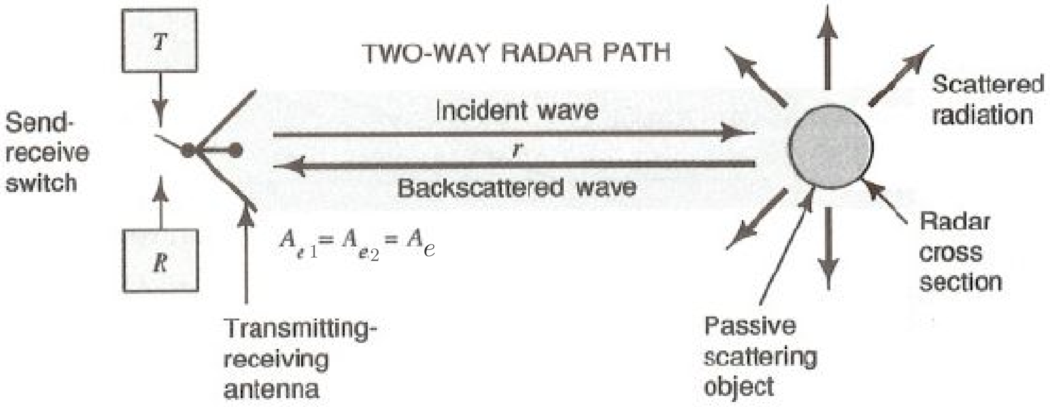
$$\frac{1}{N}\left|\frac{1}{\sin(3\pi/2N)}\right| \cong \frac{1}{N}\left|\frac{1}{3\pi/2N}\right| = \frac{2}{3\pi} \cong 0.212 \quad \text{for } N \gg 1$$

⇒ 13.5 (dB)

3. Transmitting and Receiving Antennas

A. Effective Area and Backscatter Cross Section

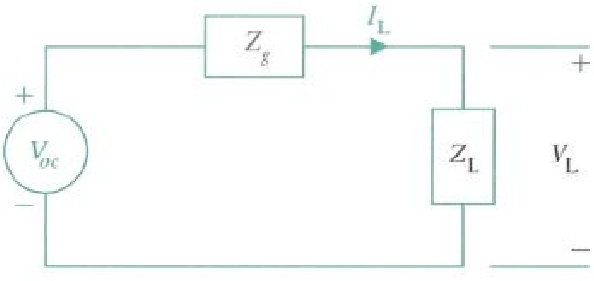


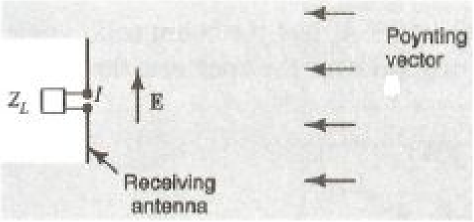


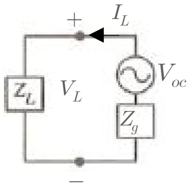
1) Effective area of an antenna

Reciprocity relations for transmitting and receiving antennas:

- ◦ Z_g of receiving antenna = Z_i of transmitting antenna
- ◦ directional patterns are identical for reception and transmission







Equivalent circuit at the receiving end

For matching condition,

$$Z_L = Z_g = Z_i = R_r + jX_i \qquad (10\text{-}66,\ 67)$$

Effective area of an antenna

$$= \frac{Ave.\ power\ delivered\ to\ a\ matched\ load}{Time - ave.\ incident\ EM\ wave\ power}$$

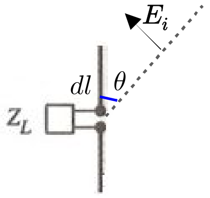
$$\Rightarrow \quad A_e = \frac{P_L}{\mathscr{P}_{av}} \quad (\mathrm{m}^2) \qquad (10\text{-}65)$$

where $$P_L = \frac{1}{2}\left[\frac{|V_{oc}|}{2R_r}\right]^2 R_r = \frac{|V_{oc}|^2}{8R_r} \qquad (10\text{-}68)$$

$$\mathscr{P}_{av} = \frac{E_i^2}{2\eta_o} = \frac{E_i^2}{240\pi} \qquad (10\text{-}69)$$

(e.g. 10-9)

For a Hertzian dipole antenna of $dl\ (\ll \lambda)$,



$V_{oc} = E_i dl\sin\theta$ and $R_r = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$ from (10-25) in (10-68):

$$P_L = \frac{E_i^2}{640\pi^2}(\lambda\sin\theta)^2 \qquad (10\text{-}72)$$

(10-69) and (10-72) in (10-65): $A_e = \frac{3}{8\pi}(\lambda\sin\theta)^2$ (10-73)

$G_D(\theta,\phi) = (3/2)\sin^2\theta$ (10-19) in (10-73) :

$$A_e(\theta,\phi) = \frac{\lambda^2}{4\pi}G_D(\theta,\phi) \quad (\text{m}^2) \qquad (10\text{-}75)$$

2) Backscatter (or Radar) cross section of an object

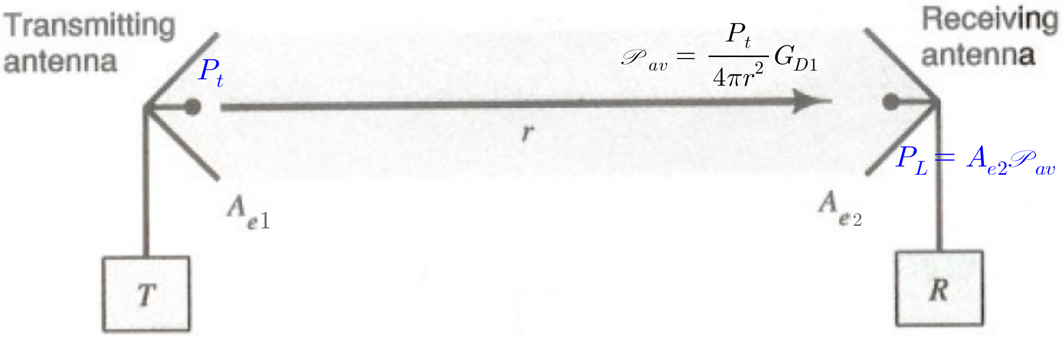
: Equivalent area intercepting the incident power

$$\frac{\mathscr{P}_i\ at\ \ the\ object}{4\pi r^2} = \frac{\mathscr{P}_s\ at\ the\ receiver}{\sigma_{bs}}$$

$$\Rightarrow \quad \sigma_{bs} = 4\pi r^2\frac{\mathscr{P}_s}{\mathscr{P}_i} \quad (\text{m}^2) \qquad (10\text{-}76)$$

Note) Since $\mathscr{P}_s \propto E^2 \propto r^{-2}$, σ_{bs} is indep. of r.

B. Rriis Transmission Formular and Radar Equation

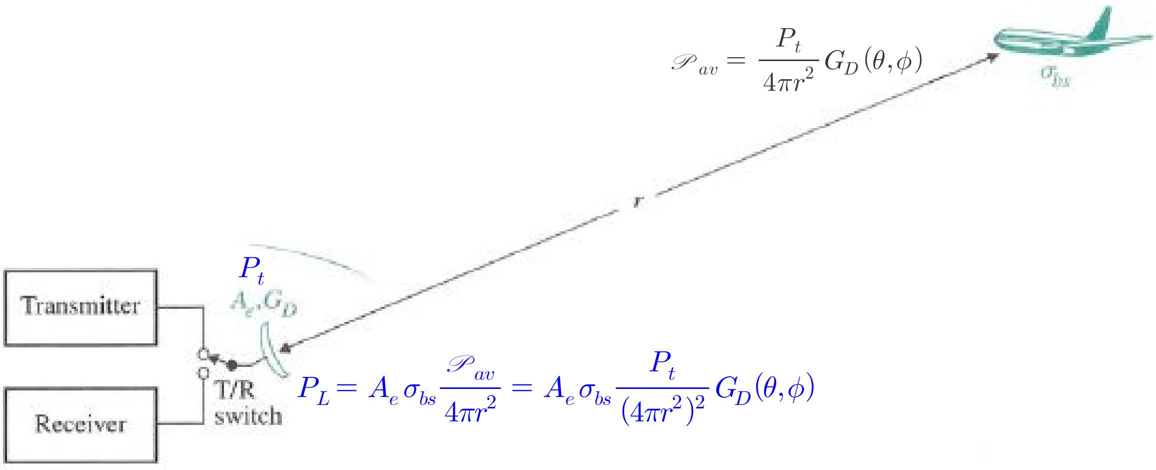


1) Rriis Transmission Formular

$$\frac{P_L}{P_t} = \left(\frac{A_{e2}}{4\pi r^2}\right)G_{D1} = \left(\frac{A_{e2}}{4\pi r^2}\right)\left(\frac{4\pi A_{e1}}{\lambda^2}\right) \quad \Rightarrow \quad \frac{P_L}{P_t} = \frac{A_{e1}A_{e2}}{r^2\lambda^2} \qquad (10\text{-}79)$$

(10-75) in (10-79): $$\frac{P_L}{P_t} = \frac{G_{D1}G_{D2}\lambda^2}{(4\pi r)^2} \qquad (10\text{-}80)$$

2) Radar equation



$$\mathscr{P}_{av} = \frac{P_t}{4\pi r^2} G_D(\theta,\phi)$$

$$P_L = A_e \sigma_{bs} \frac{\mathscr{P}_{av}}{4\pi r^2} = A_e \sigma_{bs} \frac{P_t}{(4\pi r^2)^2} G_D(\theta,\phi)$$

FIGURE 10-13 A monostatic radar system.

$$\frac{P_L}{P_t} = \frac{\sigma_{bs}\lambda^2}{(4\pi)^3 r^4} G_D^2(\theta,\phi) \overset{(10\text{-}75)}{=} \frac{\sigma_{bs}}{4\pi}\left(\frac{A_e}{\lambda r^2}\right)^2 \qquad (10\text{-}83,\ 84)$$

Homework Set 8

1) P.10-2
2) P.10-5
3) P.10-9
4) P.10-13 a), b)
5) P.10-15
6) P.10-20