Lecture Note of Topics in Ship Design Automation

Optimum Design

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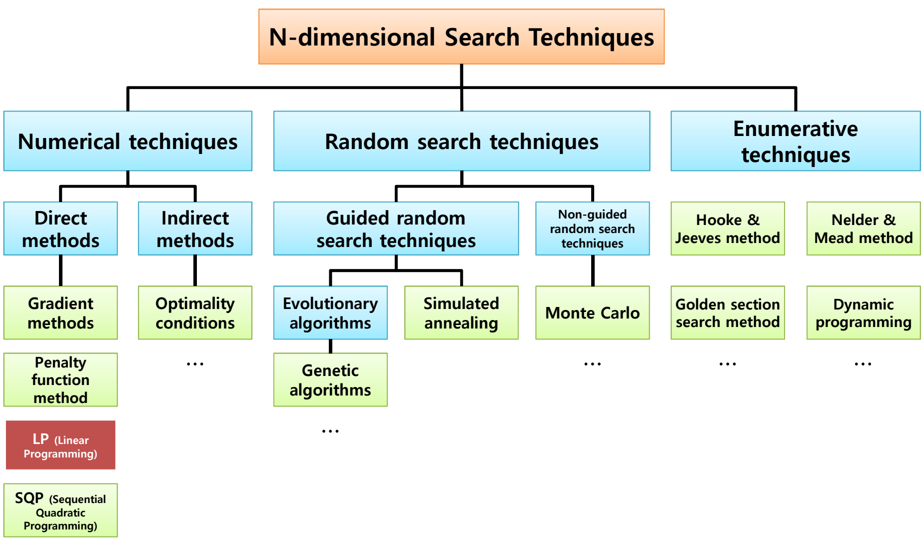
Ch. 5 Constrained Optimization Method: LP (Linear Programming)

Classes of Search Techniques (1/4)



5.1 Linear Programming Problem

Linear Programming Problem

☑ Linear Programming (LP) Problem

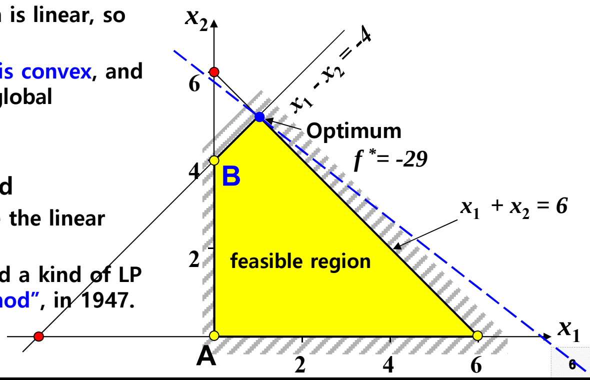
- This problem has linear objective function and linear constraint functions in the design variables.
- Since all functions are linear in an LP problem, the feasible set or feasible region defined by linear equalities or inequalities is convex.
- Also, the objective function is linear, so it is convex.
- Therefore, the LP problem is convex, and if an optimum exists, it is global optimum.

☑ Linear Programming Method

- This is the method to solve the linear programming problem.
- George B. Dantzig proposed a kind of LP method, "the Simplex method", in 1947.

Objective function: *Minimize* $f = -4x_1 - 5x_2$

Constraints: *Subject to*
$$x_1 - x_2 \ge -4$$
$$x_1 + x_2 \le 6$$
$$x_1, x_2 \ge 0$$



Property of the Linear Programming Problem

- ☑ The objective function and constraints represent the linear relationship among the design variables.
  - ■ This problem has one objective function and constraints.
  - ■ The objective function is to minimize or maximize.
- ☑ The constraints are represented as the equality constraints (=) or inequality constraints (≥, ≤).
- ☑ To use the Simplex method, the design variables have to be nonnegative in the LP problem.
  - ■ If a variable is negative, it should be transformed to nonnegative.
    - ● Ex) x = -y (x is negative, y is positive)
  - ■ If a variable is unrestricted in sign, it can always be written as the difference of two nonnegative variables.
    - ● Ex) x = y – z (x is unrestricted in sign and y and z are nonnegative.)

Objective function: *Minimize* $f = -4x_1 - 5x_2$

Constraints: *Subject to* $x_1 - x_2 \ge -4$

$x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

✓ Example of problem which has nonnegative variables
+ Distribution of the feed for animal: the amount of the feed can not be negative.
+ Distribution of the material for products: the amount of the material can not be negative.

✓ Example of variable which is unrestricted in sign
+ Profit of the shipyard = Price of a ship – Shipbuilding cost

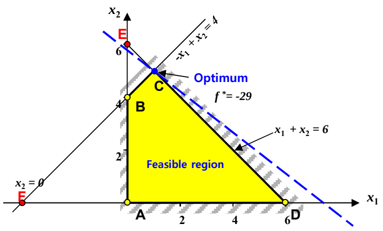
Example of the Linear Programming Problem: Problem with Two Variables and Inequality Constraint(“≤”)

Objective function: *Maximize* $f = 4x_1 + 5x_2$

Constraints: *Subject to* $x_1 - x_2 \ge -4$

$x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$



Minimize $f = -4x_1 - 5x_2$

Subject to $-x_1 + x_2 \le 4$

$x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

Maximization problem can be transformed to a minimization problem.

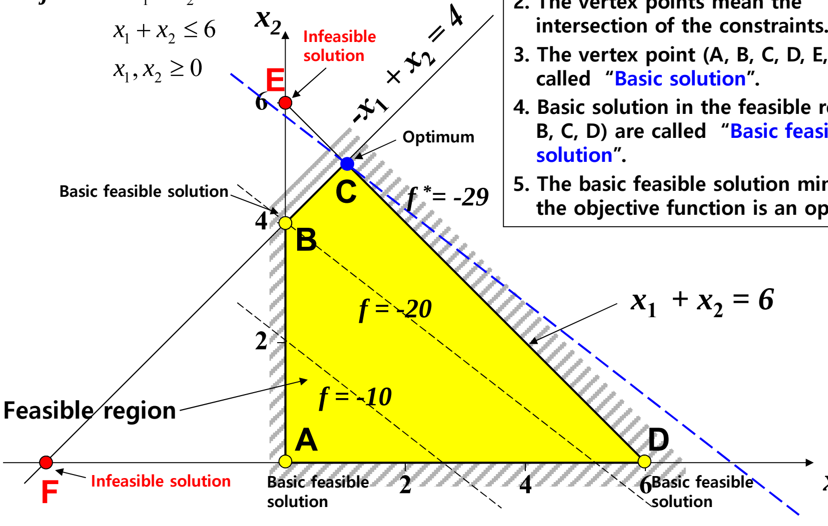
The right hand side of the constraints can always be made nonnegative by multiplying both side of the constraints by -1, if necessary.

Why should we transform the maximization problem to a minimization problem?
If the problem is not transformed to a minimization problem, we also have to find the method which can solve the maximization problem and minimization problem.
➡ For the simplification of the problem

5.2 Geometric Solution of Linear Programming Problem

Geometric Solution of the Linear Programming Problem

$$\textit{Minimize} \quad f = -4x_1 - 5x_2$$
$$\textit{Subject to} \quad -x_1 + x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1, x_2 \ge 0$$



1. The solution of a LP problem lies on a vertex point of the polygon.
2. The vertex points mean the intersection of the constraints.
3. The vertex point (A, B, C, D, E, F) are called "Basic solution".
4. Basic solution in the feasible region (A, B, C, D) are called "Basic feasible solution".
5. The basic feasible solution minimizing the objective function is an optimum.

5.3 Solution of Linear Programming Problem Using Simplex Method

Solution of Linear Programming Problem (1/3)
- Transformation of "≤" Type Inequality Constraint

Minimize $f = -4x_1 - 5x_2$

Subject to $\boxed{-x_1 + x_2 \le 4}$

$x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

For "≤" type inequality constraint, we introduce a nonnegative slack variable.

$$-x_1 + x_2 \le 4 \quad \Rightarrow \quad -x_1 + x_2 + \underbrace{x_3}_{\text{Slack variable(nonnegative)}} = 4$$

Standard form of the Linear Programming Problem

1. Right hand side of the constraints should always be nonnegative.
2. Inequality constraint should be transformed to an equality constraint.

Solution of Linear Programming Problem (2/3)

To transform "≤" type inequality constraints to the equality constraints, we introduce a nonnegative slack variable.

$$\begin{aligned} \textit{Minimize} \quad & f = -4x_1 - 5x_2 \\ \textit{Subject to} \quad & -x_1 + x_2 \le 4 \\ & x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned} \xrightarrow{\text{Transforming the inequality constraints to the equality constraints}} \begin{aligned} \textit{Minimize} \quad & f = -4x_1 - 5x_2 \\ \textit{Subject to} \quad & -x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$$\begin{aligned} -x_1 + x_2 + x_3 &= 4 \\ x_1 + x_2 + x_4 &= 6 \end{aligned}$$

Because the number of variables (4) is larger than the number of equation (2), there are many sets of solution.

➡ If we assume the value of two (=4-2) unknown variables, we can obtain the solution.

➡ When we use the "Simplex method", the two unknown variables are assumed to be zero.

At this time, the variables set to zero are called "nonbasic variables", the remaining ones are called "basic variables".

When the number of unknown variables is n and the number of linearly independent equations (equality constraints) is m (n≥m),
- The degree of freedom is (n-m).
- If we assume the value of (n-m) unknown variables (degree of freedom), we can obtain the solution.
- In the "Simplex method", the (n-m) unknown variables are assumed to be zero.

Solution of Linear Programming Problem (3/3)

$$\begin{aligned} \textit{Minimize} \quad & f = -4x_1 - 5x_2 \\ \textit{Subject to} \quad & -x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$-x_1 + x_2 + x_3 = 4$ ---- ①
$x_1 + x_2 + x_4 = 6$ ---- ②
$x_1, x_2, x_3, x_4 \ge 0$

Convert the inequality constraints to the equality constraint

Nonbasic variables (assumed to be zero)	Basic variables	Solution (x_1, x_2, x_3, x_4)	Location of the solution ("Vertex point")	Objective function
(x_2, x_3)	(x_1, x_4)	(-4, 0, 0, 10)	F	16
(x_1, x_4)	(x_2, x_3)	(0, 6, -2, 0)	E	-30
(x_1, x_2)	(x_3, x_4)	(0, 0, 4, 6)	A	0
(x_2, x_4)	(x_1, x_3)	(6, 0, 10, 0)	D	-24
(x_1, x_3)	(x_2, x_4)	(0, 4, 0, 2)	B	-20
(x_3, x_4)	(x_1, x_2)	(1, 5, 0, 0)	C	-29

(1) Select the two variables assumed to be zero (Total 6 sets).

(2) Substitute the 6 sets into the equations ①, ② and calculate the value of the basic variables (vertex point).

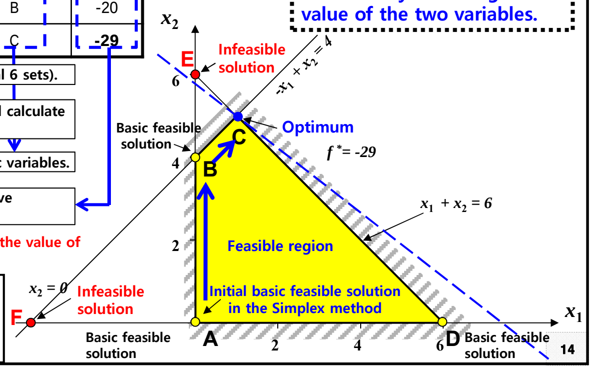
(3) Find the basic feasible solution among the 6 basic variables.

(4) The basic feasible solution minimizing the objective function is the optimum.

Q: Do we have to find all vertex points and calculate the value of the objective function? It's inefficient!

General solution of a LP problem:
"Simplex Method" starts at the initial basic feasible solution and finds the optimum by improving the objective function through iteration. ➡ We can minimize the number of calculating the vertex points.

Each vertex point is obtained by assuming the value of the two variables.



Solution of Linear Programming Problem by Using Simplex Method (1/7) - Classification between Basic Variables and Nonbasic Variables

- In this example, we can solve this problem by assuming the two variables as the nonbasic variables (=0).

Transform the inequality constraints to the equality constraints.

Minimize $f = -4x_1 - 5x_2$

Subject to $-x_1 + x_2 \le 4$

$x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

Mark the basic variable included in each row

Nonbasic variable | Basic variable

$\square$: Nonbasic variable (=0)

$\bigcirc$: Basic variable

Row 1: x_3 $\quad \boxed{-x_1} \boxed{+x_2} + x_3 = 4$

Row 2: x_4 $\quad \boxed{x_1} \boxed{+x_2} + x_4 = 6$

Row 3: $\quad \boxed{-4x_1} \boxed{-5x_2} = f - 0$

$x_1, x_2, x_3, x_4 \ge 0$

Type of variables	Explanation	Method to classify
Nonbasic variables	A variable set to zero in variables	Objective function is only composed of the nonbasic variables.
Basic variables	A variable obtained by setting the nonbasic variable and solving the equations simultaneously	Each basic variable appears in only one row.

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Solution of Linear Programming Problem by Using Simplex Method (2/7) - Interchange of Basic and Nonbasic Variables

Row 1: x_3 $\quad -x_1 + x_2 + x_3 = 4$ $\quad \leftarrow 4/1 = 4$

Row 2: x_4 $\quad x_1 + x_2 + x_4 = 6$ $\quad \leftarrow 6/1 = 6$

Row 3: $\quad -4x_1 - 5x_2 = f - 0$

$x_1, x_2, x_3, x_4 \ge 0$

$\square$: Nonbasic variable (=0)

$\bigcirc$: Basic variable

Interchange the basic variable included in the Row 1, i.e., x3 and the nonbasic variable, i.e., x2.

Nonbasic variable: x1, x2, x3

Basic variable: x3, x4, x2

The greatest reduction in the objective function can be achieved by increasing x2, because its coefficient is most negative. ➨ The nonbasic variable x2 should be replaced by a basic variable.

Because two variables should be the nonbasic variables (=0), x3 or x4 should be a nonbasic variable.

$$\frac{\text{Right hand side parameter in each row}}{\text{Positive coefficient of the element in the selected column}}$$

Select the variable whose coefficient is positive and the row having the smallest positive ratio in the constraints ➨ x3 is selected as the nonbasic variable.

<Ref.> What would be done if we do not select the row having the smallest positive ratio?

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[Reference] The reason why the column which has the minimum coefficient of the objective function is selected for pivot.

Row 1: x_3	$-x_1 + x_2 + x_3 \qquad = 4$
Row 2: x_4	$x_1 + x_2 \qquad + x_4 = 6$
Row 3:	$-4x_1 - 5x_2 \qquad = f - 0$
	$x_1, x_2, x_3, x_4 \geq 0$

□: Nonbasic variable (=0)
○: Basic variable

At first, the nonbasic variables (x_1 and x_2) are equal to zero. $(x_3 = 4,\ x_4 = 6,\ f = 0)$

If there are some variables whose coefficients are negative in the objective function, the variables (x_1 and x_2) can be increased for decreasing the value of the objective function.

The greatest reduction in the value of the objective function can be achieved by increasing x_2, because its coefficient is most negative.

[Reference] The reason why the row having the smallest positive ratio in the constraints is selected.

Select the variable whose coefficient is positive and the row having the smallest positive ratio in the constraints. ➡ x3 will be selected as the nonbasic variable.

Row 1: x_3	$-x_1 + x_2 + x_3 \qquad = 4$	← $4/1 = 4$ ← The row having the smallest positive ratio (Row 1)
Row 2: x_4	$x_1 + x_2 \qquad + x_4 = 6$	← $6/1 = 6$
Row 3:	$-4x_1 - 5x_2 \qquad = f - 0$	
	$x_1, x_2, x_3, x_4 \geq 0$	

□: Nonbasic variable (=0)
○: Basic variable

The Row 1 and 2 are rearranged as follows.

$$-x_1 + x_3 = 4 - x_2$$
$$x_1 + x_4 = 6 - x_2$$

(1) If the Row 1 is selected, then x_3 becomes nonbasic variable.

Row 1: $x_1 = x_3 = 0, x_2 = 4 \quad (\because x_1, x_3 \text{ are nonbasic variables})$

Row 2: $x_1 = 0, x_2 = 4, x_4 = 2$

(2) If the Row 2 is selected, then x_4 becomes nonbasic variable.

Row 2: $x_1 = x_4 = 0, x_2 = 6 \quad (\because x_1, x_4 \text{ are nonbasic variables})$

Row 1: $x_1 = 0, x_2 = 6, x_3 = -2$ ➡ **The constraint, the variables have to be nonnegative, is violated.**

Solution of Linear Programming Problem by Using Simplex Method (3/7) - Pivot Operation

Type of variables	Explanation	Method to classify
Nonbasic variables	A variable set to zero in variables	Objective function is only composed of the nonbasic variables.
Basic variables	A variable obtained by setting the nonbasic variable and solving the equations simultaneously	Each basic variables appears in only one row.

Row 1: x_3 | $-x_1 + x_2 + x_3 = 4$ ← $4/1=4$

Row 2: x_4 | $x_1 + x_2 + x_4 = 6$ ← $6/1=6$

Row 3: | $-4x_1 - 5x_2 = f - 0$

$$x_1, x_2, x_3, x_4 \geq 0$$

□: Nonbasic variable (=0)
○: Basic variable

Interchange the basic variable included in the Row 1, i.e., x3 and the nonbasic variable, i.e., x2.

Rearrange the Row 1 as: $x_2 = 4 + x_1 - x_3$

and substitute this into the Row 2 and 3.

$$x_1 + (4 + x_1 - x_3) + x_4 = 6$$
$$\Rightarrow 2x_1 - x_3 + x_4 = 2$$
$$-4x_1 - 5(4 + x_1 - x_3) = f$$
$$\Rightarrow -9x_1 + 5x_3 = f + 20$$

Nonbasic variable: x1, x2, x3

Basic variable: x3, x4, x2

Pivot on the selected variable (x_2: 1st Row, 2nd Column)

Row 1: x_2 | $-x_1 + x_2 + x_3 = 4$

Row 2: x_4 | $2x_1 - x_3 + x_4 = 2$

Row 3: | $-9x_1 + 5x_3 = f + 20$

$$x_1, x_2, x_3, x_4 \geq 0$$

□: Nonbasic variable (=0)
○: Basic variable

Pivot: It is the same concept with Gauss-Jordan elimination. This eliminates the selected variables from all the equations except one equation.

Solution of Linear Programming Problem by Using Simplex Method (4/7) - New Basic Variable ("Vertex Point") after Pivot Operation

Type of variables	Explanation	Method to classify
Nonbasic variables	A variable set to zero in variables	Objective function is only composed of the nonbasic variables.
Basic variables	A variable obtained by setting the nonbasic variable and solving the equations simultaneously	Each basic variables appears in only one row.

Row 1: x_2 | $-x_1 + x_2 + x_3 = 4$

Row 2: x_4 | $2x_1 - x_3 + x_4 = 2$

Row 3: | $-9x_1 + 5x_3 = f + 20$

$$x_1, x_2, x_3, x_4 \geq 0$$

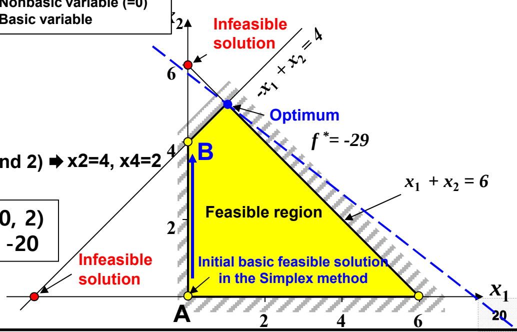
□: Nonbasic variable (=0)
○: Basic variable

Nonbasic variable: x1, x3

Basic variable: x2, x4

Substitute x1=x3=0 into the equations (Row 1 and 2) ➡ x2=4, x4=2

➡ New solution B (x_1, x_2, x_3, x_4) = (0, 4, 0, 2)
Value of the objective function at B = -20



Solution of Linear Programming Problem by Using Simplex Method (5/7) - Interchange of Basic and Nonbasic Variables

Type of variables	Explanation	Method to classify
Nonbasic variables	A variable set to zero in variables	Objective function is only composed of the nonbasic variables.
Basic variables	A variable obtained by setting the nonbasic variable and solving the equations simultaneously	Each basic variables appears in only one row.

Row 1: x_2 $\quad -x_1 + x_2 + x_3 \quad = 4$

Row 2: x_4 $\quad 2x_1 \quad - x_3 + x_4 = 2 \quad \longleftarrow 2/2 = 1$

Row 3: $\quad -9x_1 \quad + 5x_3 \quad = f + 20$

$$x_1, x_2, x_3, x_4 \geq 0$$

□: Nonbasic variable (=0)
○: Basic variable

→ Interchange the basic variable included in the Row 2, i.e., x4 and the nonbasic variable, i.e., x1.

Nonbasic variable: x1, x3, x4

Basic variable: x2, x4, x1

The greatest reduction in the objective function can be achieved by increasing x1, because its coefficient is most negative.
➡ The nonbasic variable x1 should be replaced by a basic variable.

Because two variables should be the nonbasic variables (=0),
➡ x2 or x4 should be the nonbasic variable.

$$\frac{\text{Right hand side parameter in each row}}{\text{Positive coefficient of the element in the selected column}} =$$

Select the variable whose coefficient is positive and row and the row having the smallest positive ratio in the constraints.
➡ x4 is selected as the nonbasic variable.
<Ref.> What would be done if we do the row having the negative coefficient?

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[Reference] The reason why the row having the negative coefficient in the selected column is not selected. (1/2)

Row 1: x_2 $\quad -x_1 + x_2 + x_3 \quad = 4 \quad \longleftarrow$ The row having the negative coefficient in the selected column is not selected.

Row 2: x_4 $\quad 2x_1 \quad - x_3 + x_4 = 2 \quad \longleftarrow 2/2 = 1$

Row 3: $\quad -9x_1 \quad + 5x_3 \quad = f + 20$

$$x_1, x_2, x_3, x_4 \geq 0$$

□: Nonbasic variable (=0)
○: Basic variable

Nonbasic variable: x1, x3, x4

Basic variable: x2, x4, x1

1. The Row 1 and 2 are rearranged as follows.

$$x_2 + x_3 = 4 + x_1 \quad \text{------ ①}$$

$$-x_3 + x_4 = 2 - 2x_1 \quad \text{------ ②}$$

2. x2 or x4 will become a nonbasic variable.

3. If x4 becomes a nonbasic variable,

3-1. Equation ② is changed as follows. (nonbasic variable x_3=0, x_4=0)

$$0 = 2 - 2x_1 \;\rightarrow\; 2 = 2x_1 \;\rightarrow\; 1 = x_1$$

3-2. Equation ① is changed as follows. (nonbasic variable x_3=0, x_4=0)

$$x_2 = 4 + x_1 \geq 0$$

In 3-1, any value of x_1 satisfies the equation ①. ➡ If the row having the positive coefficient in the selected column is selected, the row having the negative coefficient in the selected column is always satisfied.

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[Reference] The reason why the row having the negative coefficient in the selected column is not selected. (2/2)

Row 1: x_2 | $-x_1 + x_2 + x_3 = 4$ ← The row having the negative coefficient in the selected column is not selected.

Row 2: x_4 | $2x_1 - x_3 + x_4 = 2$ ← 2/2 = 1

Row 3: | $-9x_1 + 5x_3 = f + 20$

$x_1, x_2, x_3, x_4 \geq 0$

□: Nonbasic variable (=0)
○: Basic variable

Nonbasic variable: $x1$, x3, x2
Basic variable: $x2$, x4, x1

1. The Row 1 and 2 are rearranged as follows.

$x_2 + x_3 = 4 + x_1$ ------ ①

$-x_3 + x_4 = 2 - 2x_1$ ------ ②

2. x2 or x4 will become a nonbasic variable.

3. If x2 becomes a nonbasic variable,

3-1. Equation ① is changed as follows. (nonbasic variable x_2=0, x_3=0)

$0 = 4 + x_1 \rightarrow x_1 = -4$ ➡ The constraint, the variables have to be nonnegative, is violated.

Solution of Linear Programming Problem by Using Simplex Method (6/7) - Pivot Operation

Type of variables	Explanation	Method to classify
Nonbasic variables	A variable set to zero in variables	Objective function is only composed of the nonbasic variables.
Basic variables	A variable obtained by setting the nonbasic variable and solving the equations simultaneously	Each basic variables appears in only one row.

Row 1: x_2 | $-x_1 + x_2 + x_3 = 4$

Row 2: x_4 | $2x_1 - x_3 + x_4 = 2$ ← 2/2 = 1 → Interchange the basic variable included in the Row 2, i.e., x4 and the nonbasic variable, i.e., x1.

Row 3: | $-9x_1 + 5x_3 = f + 20$

$x_1, x_2, x_3, x_4 \geq 0$

□: Nonbasic variable (=0)
○: Basic variable

Nonbasic variable: x1, x3, x4
Basic variable: x2, x4, x1

Pivot on the selected variable (x_1: 2nd Row, 1st Column)

(Row 1 + 0.5×Row 2) → Row 1: x_2 | $x_2 + 0.5x_3 + 0.5x_4 = 5$

(0.5×Row 2) → Row 2: x_1 | $x_1 - 0.5x_3 + 0.5x_4 = 1$

(Row 3 + 4.5×Row 2) → Row 3: | $+0.5x_3 + 4.5x_4 = f + 29$

$x_1, x_2, x_3, x_4 \geq 0$

□: Nonbasic variable(=0)
○: Basic variable

Solution of Linear Programming Problem by Using Simplex Method (7/7)
- New Basic Variable ("Vertex Point") after Pivot Operation / Stop to Simplex

Type of variables	Explanation	Method to classify
Nonbasic variables	A variable set to zero in variables	Objective function is only composed of the nonbasic variables.
Basic variables	A variable obtained by setting the nonbasic variable and solving the equations simultaneously	Each basic variables appears in only one row.

$$\begin{array}{lll} \text{Row 1: } x_2 & & x_2 + 0.5x_3 + 0.5x_4 = 5 \\ \text{Row 2: } x_1 & x_1 & \quad - 0.5x_3 + 0.5x_4 = 1 \\ \text{Row 3: } & & \quad + 0.5x_3 + 4.5x_4 = f + 29 \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

□: Nonbasic variable(=0)
○: Basic variable

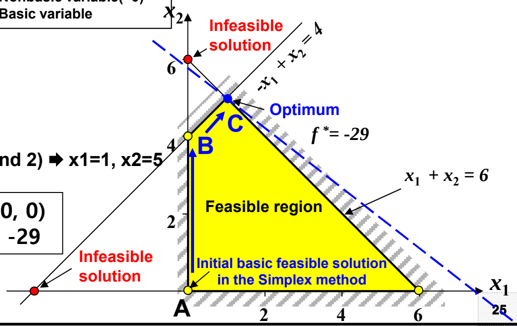
Because the coefficients of the objective function are nonnegative, the current solution is the optimum.
➡ Stop the Simplex method

Nonbasic variable: **x3, x4**

Basic variable: **x1, x2**

Substitute x3=x4=0 into the equations (Row 1 and 2) ➡ x1=1, x2=5

➡ New solution C (x_1, x_2, x_3, x_4) = (1, 5, 0, 0)
Value of the objective function at B = -29



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Solution of Linear Programming Problem by Using Simplex Tableau

Pivot: It is the same concept with Gauss-Jordan elimination. This eliminates the selected variables from all the equations except one equation.

Basic variable / Nonbasic variable (=0) / Basic variable

$$\begin{array}{ll} \text{Row 1: } x_3 & -x_1 + x_2 + x_3 = 4 \quad \longleftarrow 4/1 = 4 \\ \text{Row 2: } x_4 & x_1 + x_2 + x_4 = 6 \quad \longleftarrow 6/1 = 6 \\ \text{Row 3: } & -4x_1 - 5x_2 = f - 0 \end{array}$$

Basic variable

	↓	x1	x2	x3	x4	bi	bi/ai
Row 1:	x3	-1	1	1	0	4	4
Row 2:	x4	1	1	0	1	6	6
Row 3:	Obj.	-4	-5	0	0	f-0	-

Pivot on x2(1st Row and 2nd Column)

New Row 2 = (Row 2 - Row 1)
New Row 3 = (Row 3 + 5×Row 1)

Basic variable / Nonbasic variable (=0)

$$\begin{array}{ll} \text{Row 1: } x_2 & -x_1 + x_2 + x_3 = 4 \quad \longleftarrow 4/-1 = -4 \\ \text{Row 2: } x_4 & 2x_1 - x_3 + x_4 = 2 \quad \longleftarrow 2/2 = 1 \\ \text{Row 3: } & -9x_1 + 5x_3 = f + 20 \end{array}$$

(If the coefficient of the variable is negative, the variable is not selected.)

Basic variable

	↓	x1	x2	x3	x4	bi	bi/ai
Row 1:	x2	-1	1	1	0	4	-4
Row 2:	x4	2	0	-1	1	2	1
Row 3:	Obj.	-9	0	5	0	f+20	-

Pivot on x1(2nd Row and 1st Column)

New Row 1 = (Row 1 + 0.5×Row 2)
New Row 2 = (0.5×Row 2)
New Row 3 = (Row 3 + 4.5×Row 2)

Basic variable / Nonbasic variable (=0)

$$\begin{array}{ll} \text{Row 1: } x_2 & x_2 + 0.5x_3 + 0.5x_4 = 5 \\ \text{Row 2: } x_1 & x_1 \quad - 0.5x_3 + 0.5x_4 = 1 \\ \text{Row 3: } & \boxed{+0.5x_3 + 4.5x_4 = f + 29} \end{array}$$

Basic variable

	↓	x1	x2	x3	x4	bi	bi/ai
Row 1:	x2	0	1	0.5	0.5	5	-
Row 2:	x1	1	0	-0.5	0.5	1	-
Row 3:	Obj.	0	0	0.5	4.5	f+29	-

Because all the coefficients of the objective function are nonnegative, the current solution is the optimum. (x_1=1, x_2=5, x_3=x_4=0, f=-29)

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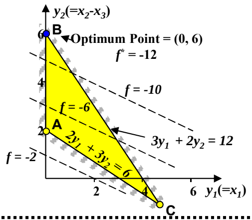
Solution of Linear Programming Problem (1/2)
- Problem with "≥" Type Inequality Constraint and Two Design Variable

$$\textit{Maximize} \quad z = y_1 + 2y_2$$
$$\textit{Subject to} \quad 3y_1 + 2y_2 \le 12$$
$$2y_1 + 3y_2 \ge 6$$
$$y_1 \ge 0$$
y_2 is unrestricted in sign.



$$\textit{Minimize} \quad F = -y_1 - 2y_2$$
$$\textit{Subject to} \quad 3y_1 + 2y_2 \le 12$$
$$2y_1 + 3y_2 \ge 6$$
$$y_1 \ge 0$$
y_2 is unrestricted in sign.

Maximization problem can be transformed to a minimization problem.

The variable unrestricted in sign is expressed with two nonnegative variables.
($y_2 = y_2^+ - y_2^-$)
Let be $x_1 = y_1, x_2 = y_2^+, x_3 = y_2^-$.

$$\textit{Minimize} \quad f = -x_1 - 2x_2 + 2x_3$$
$$\textit{Subject to} \quad 3x_1 + 2x_2 - 2x_3 \le 12$$
$$2x_1 + 3x_2 - 3x_3 \ge 6$$
$$x_1, x_2, x_3 \ge 0$$

Solution of Linear Programming Problem (2/2)
- Transformation of "≥" Type Inequality Constraint

$$\textit{Minimize} \quad f = -x_1 - 2x_2 + 2x_3$$
$$\textit{Subject to} \quad 3x_1 + 2x_2 - 2x_3 \le 12$$
$$\boxed{2x_1 + 3x_2 - 3x_3 \ge 6}$$
$$x_1, x_2, x_3 \ge 0$$

[Review] For "≤" type inequality constraint: we introduce a nonnegative slack variable.

$$3x_1 + 2x_2 - 2x_3 + \underline{x_4} = 12$$

For "≥" type inequality constraint, we introduce a surplus variable and artificial variable.

$$2x_1 + 3x_2 - 3x_3 \ge 6 \quad \Rightarrow \quad 2x_1 + 3x_2 - 3x_3 - \underline{x_5} + \underline{x_6} = 6$$

Surplus variable (nonnegative) — Artificial variable (nonnegative)

"The reason why we introduce the artificial variable"

At starting the Simplex method, we assume the original design variables (x_1, x_2, x_3) as "nonbasic variables" ($x_1=x_2=x_3=0$), $-x_5 = 6$.

➡ This violates the nonnegativity requirement.
For satisfying the requirement, we introduce the variable x_6 artificially.
However, the artificial variable should be equal to zero in the feasible region, because x_6 is augmented artificially.

Solution of Linear Programming Problem Using Simplex Method (Simplex Tableau) - Simplex Method for the Problem with "≥" Type Inequality Constraint (1/4)

①
Maximize $z = y_1 + 2y_2$
Subject to $3y_1 + 2y_2 \le 12$
$2y_1 + 3y_2 \ge 6$
$y_1 \ge 0$
y_2 is unrestricted in sign.

1. Transform to a minimization problem.
2. Since y_2 is unrestricted in sign, transform as $y_2 = y_2^+ - y_2^-$.
3. Let be $x_1 = y_1, x_2 = y_2^+, x_3 = y_2^-$.
4. Transform the inequality constraints to the equality constraints (Introduce the slack and surplus variable).

②
Minimize $f = -x_1 - 2x_2 + 2x_3$
Subject to $3x_1 + 2x_2 - 2x_3 + x_4 = 12$
$2x_1 + 3x_2 - 3x_3 - x_5 = 6$
$x_i \ge 0; i = 1\ to\ 5$

Slack variable

Surplus variable

Assume the original variables (x_1, x_2, x_3) as nonbasic variables (=0) and calculate the basic variable (x_4, x_5).

$x_4 = 12, x_5 = -6$ ➡ This violates the nonnegativity requirement

Introduce an artificial variable x_6 in the "≥" type inequality constraints.

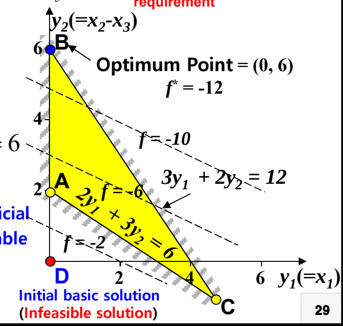
③
Minimize $f = -x_1 - 2x_2 + 2x_3$
Subject to $3x_1 + 2x_2 - 2x_3 + x_4 = 12$
$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$
$x_i \ge 0; i = 1\ to\ 6$

Slack variable

Surplus variable

Artificial variable

Assume the original design variables (x_1, x_2, x_3) and the surplus variable (x_5) as nonbasic variables (=0) and calculate the basic variable (x_4, x_6).
The result is x_4=12, x_6=6. ➡ Initial basic solution (Infeasible solution)
However, the artificial variable should be equal to zero in the feasible region, because x_6 is augmented artificially.



Solution of Linear Programming Problem Using Simplex Method (Simplex Tableau) - Simplex Method for the Problem with "≥" Type Inequality Constraint (2/4)

③
Minimize $f = -x_1 - 2x_2 + 2x_3$
Subject to $3x_1 + 2x_2 - 2x_3 + x_4 = 12$
$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$
$x_i \ge 0; i = 1\ to\ 6$

Slack variable

Surplus variable

Artificial variable

Define an artificial objective function which is a sum of all the artificial variables ($w = x_6$).

④
$3x_1 + 2x_2 - 2x_3 + x_4 = 12$
$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$
$-x_1 - 2x_2 + 2x_3 = f$
$\boxed{-2x_1 - 3x_2 + 3x_3 + x_5 = w - 6}$

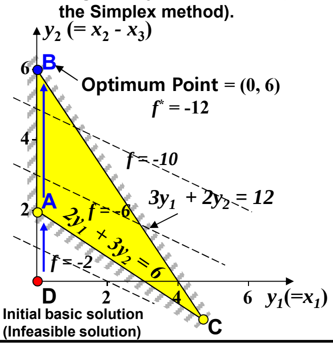
Artificial objective function

Designate $x_6 = w$ and rearrange $2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$.

⑤ Find the basic feasible solution(minimize the artificial objective function, $w = x_6$ ("w = 0"). (Phase 1 of the Simplex method)

Since x_6 is augmented artificially, the artificial variable should be equal to zero in the feasible region.

⑥ Find the optimum to minimize the original objective function (Phase 2 of the Simplex method).



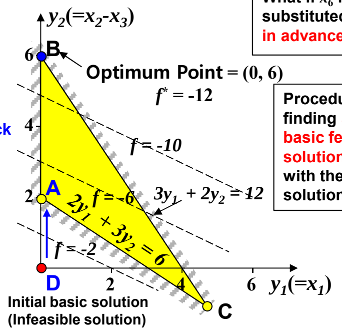
Solution of Linear Programming Problem Using Simplex Method (Simplex Tableau) - Simplex Method for the Problem with "≥" Type Inequality Constraint (3/4)

④

$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$
$$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$
$$-x_1 - 2x_2 + 2x_3 = f$$
$$-2x_1 - 3x_2 + 3x_3 + x_5 = w - 6$$

At first, we assume the original design variables (x_1, …, x_3) and surplus variable (x_5) as nonbasic variables (=0), whereas the slack variable (x_4) and artificial variable(x_6) as basic variables. Then solve the equation. ("Starting with the initial basic solution")

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	3	2	-2	1	0	0	12	-
x6	2	3	-3	0	-1	1	6	-
Obj.	-1	-2	2	0	0	0	f-0	-
A. Obj.	-2	-3	3	0	1	0	w-6	-



What if x_6 is substituted for zero in advance?

Procedure of finding another basic feasible solution starting with the initial basic solution

⑤ Phase 1: Repeat Pivot operation until the artificial objective function w becomes zero.

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	3	2	-2	1	0	0	12	6
x6	2	3	-3	0	-1	1	6	2
Obj.	-1	-2	2	0	0	0	f-0	-
A. Obj.	-2	-3	3	0	1	0	w-6	-

➡

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	5/3	0	0	1	2/3	-2/3	8	-
x2	2/3	1	-1	0	-1/3	1/3	2	-
Obj.	1/3	0	0	0	-2/3	2/3	f+4	-
A. Obj.	0	0	0	0	0	1	w-0	-

Since the artificial variable (x_6) is augmented artificially, the variable should be equal to zero in the feasible region.

New Row 1 = Row 1 - (2/3)×Row 2
New Row 2 = (1/3)×Row 2
New Row 3 = Row 3 - (2/3)×Row 2
New Row 4 = Row 4 + Row 2

Since the value of the artificial objective function becomes zero, the Phase 1 is completed.
Point A(x_1=x_3=x_5=x_6=0, x_2=2, x_4=8)

[Reference] What if x_6 is substituted for zero in advance?

$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$
$$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$
$$-x_1 - 2x_2 + 2x_3 = f$$

When x_6 is substituted for zero,

the other variables (x_1, x_2, x_3, x_5) in the same equation should not be negative.

The procedure of the calculating the values of x_1, x_2, x_3, x_5 is identical with that of reducing the artificial objective function (x_6) to zero in the Simplex method.

Solution of Linear Programming Problem Using Simplex Method(Simplex Tableau)
- Simplex Method for the Problem with "≥" Type Inequality Constraint (4/4)

5 Phase 1: Repeat Pivot operation until the artificial objective function w becomes zero.

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	3	2	-2	1	0	0	12	6
x6	2	3	-3	0	-1	1	6	2
Obj.	-1	-2	2	0	0	0	f-0	-
A. Obj.	-2	-3	3	0	1	0	w-6	-

➡

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	5/3	0	0	1	2/3	-2/3	8	-
x2	2/3	1	-1	0	-1/3	1/3	2	-
Obj.	1/3	0	0	0	-2/3	2/3	f+4	-
A. Obj.	0	0	0	0	0	1	w-0	-

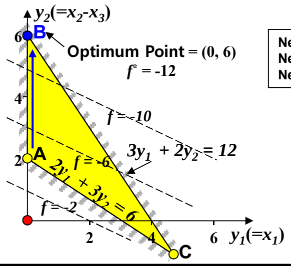
6 Phase 2: Repeat Pivot operation until all the coefficients of the original objective function f are nonnegative.

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	5/3	0	0	1	2/3	-2/3	8	12
x2	2/3	1	-1	0	-1/3	1/3	2	-6
Obj.	1/3	0	0	0	-2/3	2/3	f+4	-

➡

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x5	5/2	0	0	3/2	1	-1	12	-
x2	3/2	1	-1	1/2	0	0	6	-
Obj.	2	0	0	1	0	0	f+12	-

New Row 1 = Row 1×(2/3)
New Row 2 = Row 2 + (1/2)×Row 1
New Row 3 = Row 3 + Row 1



Since all the coefficients of the objective function are nonnegative, the current solution is the optimum.
$(x_1=x_3=x_4=0, x_2=6, x_5=12, f=-12)$

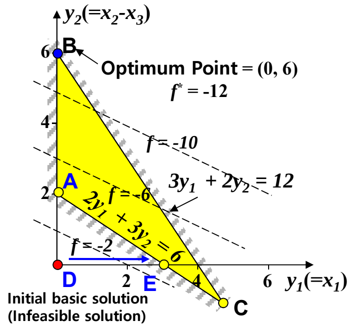
[Reference] Procedure of finding another basic feasible solution starting with the initial basic solution (1/2)

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	3	2	-2	1	0	0	12	4
x6	2	3	-3	0	-1	1	6	3
Obj.	-1	-2	2	0	0	0	f-0	-
A. Obj.	-2	-3	3	0	1	0	w-6	-

Select the first column and perform the Pivot.
(In the general Simplex method, the second column is selected.)

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	0	-5/2	5/2	1	3/2	-3/2	3	-
x1	1	3/2	-3/2	0	-1/2	1/2	3	-
Obj.	0	-1/2	1/2	0	-1/2	1/2	f+3	-
A. Obj.	0	0	0	0	0	1	w-0	-

Since the value of the artificial objective function becomes zero, the Phase 1 is completed. Point E($x_2=x_3=x_5=x_6=0$, $x_1=3$, $x_4=3$)



The basic feasible solution can be found from the initial basic solution through the near corner.

➡ It is similar with the procedure of finding the optimum from the initial basic feasible solution.

- Since Phase1 is completed, Phase 2 is performed.
- Phase2: Pivot operation for the original objective function f

[Reference] Procedure of finding another basic feasible solution starting with the initial basic solution (2/2)

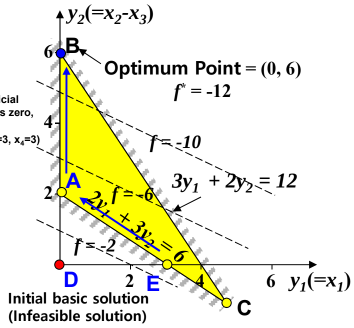
	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	0	-5/2	5/2	1	3/2	-3/2	3	-6/5
x1	1	3/2	-3/2	0	-1/2	1/2	3	2
Obj.	0	-1/2	1/2	0	-1/2	1/2	f+3	-
A. Obj.	0	0	0	0	0	1	w-0	-

Since the value of the artificial objective function becomes zero, the Phase 1 is completed.
Point E ($x_2=x_3=x_5=x_6=0$, $x_1=3$, $x_4=3$)

New Row 1 = Row 1 + Row 2×(5/3)
New Row 2 = Row 2×(2/3)
New Row 3 = Row 3 + Row 2×(1/3)

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x4	5/3	0	0	1	**2/3**	-2/3	8	**12**
x2	2/3	1	-1	0	-1/3	1/3	2	-6
Obj.	1/3	0	0	0	**-2/3**	2/3	f+4	-

Point A ($x_1=x_3=x_5=x_6=0, x_2=2, x_4=8$)



New Row 1 = Row 1×(2/3)
New Row 2 = Row 2 + (1/2)×Row 1
New Row 3 = Row 3 + Row 1

	x1	x2	x3	x4	x5	x6	bi	bi/ai
x5	5/2	0	0	3/2	1	-1	**12**	-
x2	3/2	1	-1	1/2	0	0	**6**	-
Obj.	2	0	0	1	0	0	f+12	-

Since all the coefficients of the objective function are nonnegative, the current solution is the optimum.

Point B ($x_1=x_3=x_4=x_6=0, x_2=6, x_5=12, f=-12$)

Solution of Linear Programming Problem - Transformation of Equality("=") Constraint

Minimize $f = -x_1 - 2x_2 + 2x_3$

Subject to $3x_1 + 2x_2 - 2x_3 \le 12$ ⇒ $3x_1 + 2x_2 - 2x_3 + \underline{x_4} = 12$

$2x_1 + 3x_2 - 3x_3 \ge 6$ ⇒ $2x_1 + 3x_2 - 3x_3 - \underline{x_5} + \underline{x_6} = 6$

$\boxed{x_1 + x_2 + x_3 = 6}$

$x_1, x_2, x_3 \ge 0$

[Review] For "≤" type inequality constraint, we introduce a nonnegative slack variable.

[Review] For "≥" type inequality constraint, we introduce a surplus variable and artificial variable.

For "=" type equality constraint, we introduce an artificial variable.

$$x_1 + x_2 + x_3 = 6 \quad \Rightarrow \quad x_1 + x_2 + x_3 + \underline{x_7} = 6$$

Artificial variable (nonnegative)

"The reason why we introduce the artificial variable"

At starting the Simplex method, we assume the original design variables (x_1, x_2, x_3) as "nonbasic variables" ($x_1 = x_2 = x_3 = 0$). Then the equality constraint is violated ($0 = 6$).

➡ To satisfy the equality constraint, we introduce the variable x_7 artificially.
However, because x_7 is augmented artificially, the artificial variable should be equal to zero in the feasible region.

Solution of Linear Programming Problem Using Simplex Method - Method for Formulating the Artificial Objective Function

①

Minimize $f = -x_1 - 2x_2 + 2x_3$

Subject to
$$3x_1 + 2x_2 - 2x_3 \le 12$$
$$2x_1 + 3x_2 - 3x_3 \ge 6$$
$$x_1 + x_2 + x_3 = 6$$
$$x_1, x_2, x_3 \ge 0$$

Transform the inequality constraints to equality constraints

②

Minimize $f = -x_1 - 2x_2 + 2x_3$

Subject to
$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$
$$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$
$$x_1 + x_2 + x_3 + x_7 = 6$$
$$x_i \ge 0;\ i = 1\ to\ 7$$

$$x_6 - 6 = -2x_1 - 3x_2 + 3x_3 + x_5$$
$$x_7 - 6 = -x_1 - x_2 - x_3$$
$$w(= x_6 + x_7) - 12 = -3x_1 - 4x_2 + 2x_3 + x_5$$

Define an artificial objective function which is a sum of all the artificial variables ($w = x_6 + x_7$).

③
$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$
$$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$
$$x_1 + x_2 + x_3 + x_7 = 6$$
$$-x_1 - 2x_2 + 2x_3 = f$$
$$-3x_1 - 4x_2 + 2x_3 + x_5 = w - 12$$

Find the basic feasible solution(minimize the artificial objective function, $w = x_6 + x_7$ ("$w = 0$"; $x_6 = x_7 = 0$)

<Ref.> If we define the artificial objective functions for each artificial variable,
$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$
$$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$
$$x_1 + x_2 + x_3 + x_7 = 6$$
$$-x_1 - 2x_2 + 2x_3 = f$$
$$\left.\begin{array}{l} -2x_1 - 3x_2 + 3x_3 + x_5 = w_1 - 6 \\ -x_1 - x_2 - x_3 = w_2 - 6 \end{array}\right\}$$

➡ We have to minimize w_1 ($x_6 = 0$) and w_2 ($x_7 = 0$).

Since the artificial variables are nonnegative, solutions of minimizing the sum of all the artificial objective functions are the same as those of minimizing of each artificial objective function. Therefore, it is convenient to define the artificial objective function as a sum of all the artificial variables.

[Reference] Time to transform the variables unrestricted in sign to the nonnegative variables

Mathematical Model

Minimize $z = -y_1 - 2y_2$

Subject to
$$3y_1 + 2y_2 \le 12$$
$$2y_1 + 3y_2 \ge 6$$
$$y_1 \ge 0$$
y_2 is unrestricted in sign.

$$y_2 = y_2^+ - y_2^-$$
$$y_2^+, y_2^- \ge 0$$

Transform the variable unrestricted in sign to nonnegative variable

Minimize $f = -y_1 - 2y_2^+ + 2y_2^-$

Subject to
$$3y_1 + 2y_2^+ - 2y_2^- \le 12$$
$$2y_1 + 3y_2^+ - 3y_2^- \ge 6$$
$$y_1, y_2^+, y_2^- \ge 0$$

Order (1)

"≤" type inequality constraint: Introduce the slack variable.

"≥" type inequality constraint: Introduce the surplus variable and the artificial variable.

"≤" type inequality constraint: Introduce the slack variable.

"≥" type inequality constraint: Introduce the surplus variable and the artificial variable.

Order (2)

Minimize $f = -y_1 - 2y_2$

Subject to
$$3y_1 + 2y_2 + x_1 = 12$$
$$2y_1 + 3y_2 - x_2 + x_3 = 6$$
$$y_1, x_i \ge 0; i = 1\ to\ 3$$
y_2 is unrestricted in sign.

$$y_2 = y_2^+ - y_2^-$$
$$y_2^+, y_2^- \ge 0$$

Transform the variable unrestricted in sign to nonnegative variable

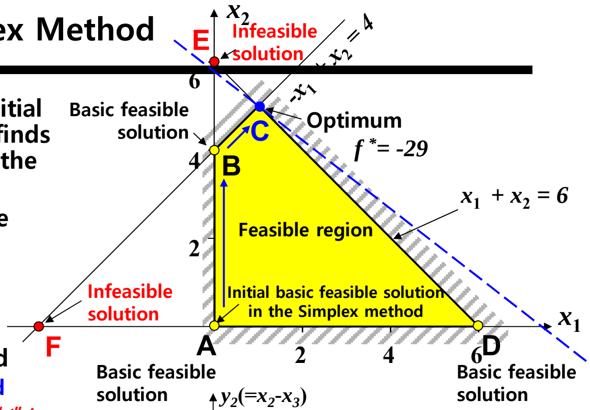
Minimize $f = -y_1 - 2y_2^+ + 2y_2^-$

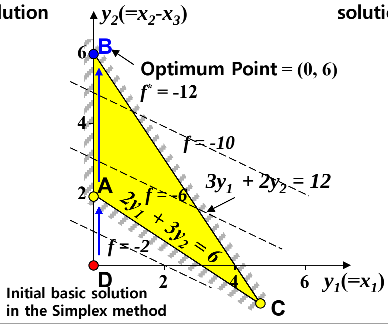
Subject to
$$3y_1 + 2y_2^+ - 2y_2^- + x_1 = 12$$
$$2y_1 + 3y_2^+ - 3y_2^- - x_2 + x_3 = 6$$
$$y_1, y_2^+, y_2^- \ge 0,\ x_i \ge 0; i = 1\ to\ 3$$

After formulating the mathematical model, there is no restriction in order between transforming the variables unrestricted in sign to the nonnegative variables and introducing the slack, surplus and artificial variables.

Summary of the Simplex Method

- ☑ This method starts at the initial basic feasible solution and finds the optimum by improving the objective function.
- ☑ This method is based on the theory of the first-order simultaneous equations.
  - ■ Matrix calculation is used. (Gauss-Jordan elimination)
- ☑ Type of the Simplex method
  - ■ One-phase Simplex method
    - ● The problem only having "≤" type inequality constraints
  - ■ Two-phase Simplex method
    - ● The problem having "≥" type inequality or equality ("=") constraint
    - ● Phase 1: Find the initial basic feasible solution to satisfy the artificial objective function(w) to be zero.
    - ● Phase 2: Find the optimum by starting with the initial basic feasible solution.





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Summary of the Simplex Algorithm

- ☑ Step 1: initial basic feasible solution
  - ■ "≤" type inequality constraints: Find the initial basic feasible variables by assuming the slack variables as basic and the original variables as nonbasic variables(=0).
  - ■ "≥" type inequality constraints: By using the Two-phase Simplex method, find the initial basic feasible variables to satisfy the artificial objective function to be zero in the Phase 1.
- ☑ Step 2: The objective function must be expressed with the nonbasic variables.
- ☑ Step 3: If all the reduced coefficient of the objective function for nonbasic variables are nonnegative, the current basic solution is the optimum. Otherwise, continue.
- ☑ Step 4: Determine the Pivot column and row. At this time, the nonbasic variable in the selected Pivot column should become the new basic variable and the basic variable in the selected Pivot row should become the new nonbasic variable.
- ☑ Step 5: Pivot operation by using the Gauss-Jordan elimination
- ☑ Step 6: Calculate the value of the basic and nonbasic variable and go to Step 3.

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5.4 Examples for Linear Programming

[Example 1] Optimal Transportation of Cargo

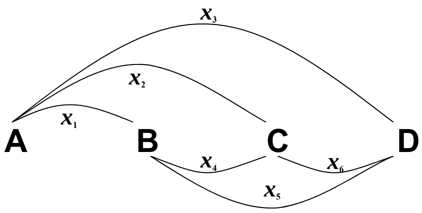
Consider a cargo ship departing from the port A to E via the ports B, C, and D. The maximum cargo loading capacity of the ship is 50,000ton and the loadable cargo at each port is as follows. Formulate and find the optimum cargo transportation that maximizes the freight income.

Type of cargo	Port of departure	Port of arrival	Loadable cargo at each port of departure (1,000ton)	Freight income ($/ton)
1	A	B	100	5
2	A	C	40	10
3	A	D	25	20
4	B	C	50	8
5	B	D	100	12
6	C	D	50	6

[Example 1] Optimal Transportation of Cargo - Solution (1/7)

Type of cargo	Port of departure	Port of arrival	Loadable cargo at the each ports of departure (1,000ton)	Freight income ($/ton)
1	A	B	100	5
2	A	C	40	10
3	A	D	25	20
4	B	C	50	8
5	B	D	100	12
6	C	D	50	6

The loadable cargo at each port (x_i, *i* type of cargo) by 1,000 ton is as follows.



Design variables: $x_1, x_2 x_3, x_4, x_5, x_6$

Objective function: Maximization of the freight income

$$\textit{Maximize } Z = 5x_1 + 10x_2 + 20x_3 + 8x_4 + 12x_5 + 6x_6$$

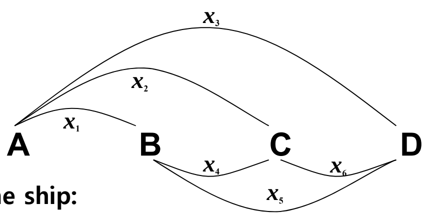
➡ **The maximization problem should be converted to a minimization problem by assuming *f* = -*Z***

$$\textit{Minimize } f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$$

[Example 1] Optimal Transportation of Cargo - Solution (2/7)

Type of cargo	Port of departure	Port of arrival	Loadable cargo at the each ports of departure (1,000ton)	Freight income ($/ton)
1	A	B	100	5
2	A	C	40	10
3	A	D	25	20
4	B	C	50	8
5	B	D	100	12
6	C	D	50	6

The loadable cargo at each port (x_i, *i* type of cargo) by 1,000 ton is as follows.



Constraints:

The maximum cargo to be loaded in the ship:

$$A \Rightarrow B: x_1 + x_2 + x_3 \le 50 \quad B \Rightarrow C: x_2 + x_3 + x_4 + x_5 \le 50$$
$$C \Rightarrow D: x_3 + x_5 + x_6 \le 50$$

The maximum cargo according to the type:

$$0 \le x_2 \le 40,\ 0 \le x_3 \le 25,\ 0 \le x_4 \le 50,\ 0 \le x_6 \le 50$$

The maximum loadable cargoes x_1, x_5 are larger than 50,000 ton, there are no upper limit related with x_1, x_5.

The maximum loadable cargoes x_4, x_6 are 50,000 ton, there are no upper limit related with x_4, x_6.

[Example 1] Optimal Transportation of Cargo - Solution (3/7)

Find $x_1, x_2, x_3, x_4, x_5, x_6$

Minimize $f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$

Subject to

$$\left.\begin{array}{l} x_1 + x_2 + x_3 \le 50 \\ x_2 + x_3 + x_4 + x_5 \le 50 \\ x_3 + x_5 + x_6 \le 50 \end{array}\right\}$$

: Constraints related with the maximum cargo to be loaded in the ship

$$\left.\begin{array}{l} 0 \le x_2 \le 40,\ 0 \le x_3 \le 25, \\ 0 \le x_4 \le 50,\ 0 \le x_6 \le 50 \end{array}\right\}$$

: Constraints related with the maximum cargo according to the type

➡ Optimization problem having the 6 unknown variables and 7 inequality constraints

[Example 1] Optimal Transportation of Cargo - Solution (4/7)

① Convert to the standard form. → ② Solve this problem by using the Simplex method.

Constraints

$x_1 + x_2 + x_3 \le 50$

$x_2 + x_3 + x_4 + x_5 \le 50$

$x_3 + x_5 + x_6 \le 50$

$0 \le x_2 \le 40,\ 0 \le x_3 \le 25,$

$0 \le x_4 \le 50,\ 0 \le x_6 \le 50$

Objective function

$f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$

Standard form:

$x_1 + x_2 + x_3 + x_7 = 50$

$x_2 + x_3 + x_4 + x_5 + x_8 = 50$

$x_3 + x_5 + x_6 + x_9 = 50$

$x_2 + x_{10} = 40,\ x_3 + x_{11} = 25,$

$x_4 + x_{12} = 50,\ x_6 + x_{13} = 50$

Where, $x_7, x_8, x_9, x_{10}, x_{11}, x_{12} x_{13}$: slack variables[1]

$f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$

→ ③

Perform the Simplex method.

starts at the initial basic feasible solution and finds the optimum by improving the objective function

1: Slack variable – The variables introduced for converting “≤” type inequality constraints.

[Example 1] Optimal Transportation of Cargo - Solution (5/7)

positive ratio = $\dfrac{\text{Right hand side parameter in each column}}{\text{Positive coefficient of the element in the selected column}}$

1

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	bi	bi/ai
x7	1	1	1	0	0	0	1	0	0	0	0	0	0	50	50
x8	0	1	1	1	1	0	0	1	0	0	0	0	0	50	50
x9	0	0	1	0	1	1	0	0	1	0	0	0	0	50	50
x10	0	1	0	0	0	0	0	0	0	1	0	0	0	40	-
x11	0	0	**1**	0	0	0	0	0	0	0	1	0	0	25	**25**
x12	0	0	0	1	0	0	0	0	0	0	0	1	0	50	-
x13	0	0	0	0	0	1	0	0	0	0	0	0	1	50	-
Obj.	-5	-10	**-20**	-8	-12	-6	0	0	0	0	0	0	0	f+0	-

Select the variable whose coefficient is positive and row has the smallest positive ratio in the constraints.

(1) Select the column which has the minimum coefficient of the objective function. (3) Pivot on the selected variable(x_3 / 5th row, 3rd column).

2

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	bi	bi/ai
x7	1	1	0	0	0	0	1	0	0	0	-1	0	0	25	-
x8	0	1	0	1	**1**	0	0	1	0	0	-1	0	0	25	**25**
x9	0	0	0	0	1	1	0	0	1	0	-1	0	0	25	25
x10	0	1	0	0	0	0	0	0	0	1	0	0	0	40	-
x3	0	0	1	0	0	0	0	0	0	0	1	0	0	25	-
x12	0	0	0	1	0	0	0	0	0	0	0	1	0	50	-
x13	0	0	0	0	0	1	0	0	0	0	0	0	1	50	-
Obj.	-5	-10	0	-8	**-12**	-6	0	0	0	0	20	0	0	f+500	-

[Example 1] Optimal Transportation of Cargo - Solution (6/7)

3

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	bi	bi/ai
x7	1	1	0	0	0	0	1	0	0	0	-1	0	0	25	-
x5	0	1	0	1	1	0	0	1	0	0	-1	0	0	25	-
x9	0	-1	0	-1	0	**1**	0	-1	1	0	0	0	0	0	**0**
x10	0	1	0	0	0	0	0	0	0	1	0	0	0	40	-
x3	0	0	1	0	0	0	0	0	0	0	1	0	0	25	-
x12	0	0	0	1	0	0	0	0	0	0	0	1	0	50	-
x13	0	0	0	0	0	1	0	0	0	0	0	0	1	50	50
Obj.	-5	2	0	4	0	**-6**	0	12	0	0	8	0	0	f+800	-

4

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	bi	bi/ai
x7	**1**	1	0	0	0	0	1	0	0	0	-1	0	0	25	**25**
x5	0	1	0	1	1	0	0	1	0	0	-1	0	0	25	-
x6	0	-1	0	-1	0	1	0	-1	1	0	0	0	0	0	-
x10	0	1	0	0	0	0	0	0	0	1	0	0	0	40	-
x3	0	0	1	0	0	0	0	0	0	0	1	0	0	25	-
x12	0	0	0	1	0	0	0	0	0	0	0	1	0	50	-
x13	0	1	0	1	0	0	0	1	-1	0	0	0	1	50	-
Obj.	**-5**	-4	0	-2	0	0	0	6	6	0	8	0	0	800	-

[Example 1] Optimal Transportation of Cargo - Solution (7/7)

5

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	bi	bi/ai
x1	1	1	0	0	0	0	1	0	0	0	-1	0	0	25	
x5	0	1	0	**1**	1	0	0	1	0	0	-1	0	0	25	**25**
x6	0	-1	0	-1	0	1	0	-1	1	0	0	0	0	0	
x10	0	1	0	0	0	0	0	0	0	1	0	0	0	40	
x3	0	0	1	0	0	0	0	0	0	0	1	0	0	25	
x12	0	0	0	1	0	0	0	0	0	0	0	1	0	50	50
x13	0	1	0	1	0	0	0	1	-1	0	0	0	1	50	50
Obj.	0	1	0	**-2**	0	0	5	6	6	0	3	0	0	f+925	

The row having the negative coefficient (-1) in the selected column is not selected.

6

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	bi	bi/ai
x1	1	1	0	0	0	0	1	0	0	0	-1	0	0	25	
x4	0	1	0	1	1	0	0	1	0	0	-1	0	0	25	
x6	0	0	0	0	1	1	0	0	1	0	-1	0	0	25	
x10	0	1	0	0	0	0	0	0	0	1	0	0	0	40	
x3	0	0	1	0	0	0	0	0	0	0	1	0	0	25	
x12	0	-1	0	0	-1	0	0	-1	0	0	1	1	0	25	
x13	0	0	0	0	-1	0	0	0	-1	0	1	0	1	25	
Obj.	0	3	0	0	2	0	5	8	6	0	1	0	0	f+975	

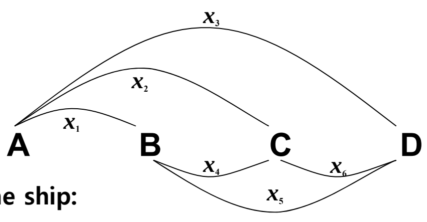
Because all the coefficients of the objective function are nonnegative, the current solution is the optimum. ($x_2=x_5=0, x_1=x_3=x_4=x_6=25, f=-975$)

Therefore, the maximum freight income (975,000$) can be achieved by loading 25,000 tons per the cargo type(1, 3, 4, 6).

[Example 1] Optimal Transportation of Cargo - Solution: Deletion of Duplicated Constraints (1/6)

Type of cargo	Port of departure	Port of arrival	Loadable cargo at the each ports of departure (1,000ton)	Freight income ($/ton)
1	A	B	100	5
2	A	C	40	10
3	A	D	25	20
4	B	C	50	8
5	B	D	100	12
6	C	D	50	6

The loadable cargo at each port (x_i, i type of cargo) by 1,000ton is as follows.



Constraints:

The maximum cargo to be loaded in the ship:

$$A \Rightarrow B : x_1 + x_2 + x_3 \le 50 \quad B \Rightarrow C : x_2 + x_3 + x_4 + x_5 \le 50$$

$$C \Rightarrow D : x_3 + x_5 + x_6 \le 50$$

The maximum cargo according to the type:

$$0 \le x_2 \le 40,\ 0 \le x_3 \le 25,\ \cancel{0 \le x_4 \le 50,\ 0 \le x_6 \le 50}$$

The maximum loadable cargoes x_1, x_5 are larger than 50,000 ton, there are no upper limit related with x_1, x_5.

The maximum loadable cargoes x_4, x_6 are 50,000 ton, there are no upper limit related with x_4, x_6.

[Example 1] Optimal Transportation of Cargo - Solution: Deletion of Duplicated Constraints (2/6)

Find $x_1, x_2, x_3, x_4, x_5, x_6$

Minimize $f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$

Subject to

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\le 50 \\ x_2 + x_3 + x_4 + x_5 &\le 50 \\ x_3 + x_5 + x_6 &\le 50 \end{aligned}\right\}$$

: Constraints related with the maximum cargo to be loaded in the ship

$0 \le x_2 \le 40, \quad 0 \le x_3 \le 25$: Constraints related with the maximum cargo according to the type

➡ Optimization problem having the 6 unknown variables and 5 inequality constraints

[Example 1] Optimal Transportation of Cargo - Solution: Deletion of Duplicated Constraints (3/6)

① Convert to the standard form.

Constraints

$x_1 + x_2 + x_3 \le 50$

$x_2 + x_3 + x_4 + x_5 \le 50$

$x_3 + x_5 + x_6 \le 50$

$0 \le x_2 \le 40, \ 0 \le x_3 \le 25,$

$0 \le x_4 \le 50, \ 0 \le x_6 \le 50$

Objective function

$f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$

② Solve this problem by using the Simplex method.

$x_1 + x_2 + x_3 + x_7 = 50$

$x_2 + x_3 + x_4 + x_5 + x_8 = 50$

$x_3 + x_5 + x_6 + x_9 = 50$

$x_2 + x_{10} = 40, \quad x_3 + x_{11} = 25$

Where, $x_7, x_8, x_9, x_{10}, x_{11}$: slack variables[1]

$f = -5x_1 - 10x_2 - 20x_3 - 8x_4 - 12x_5 - 6x_6$

③

Perform the Simplex method.

starts at the initial basic feasible solution and finds the optimum by improving the objective function

1: Slack variable – The variables introduced for converting "≤" type inequality constraints.

[Example 1] Optimal Transportation of Cargo - Solution: Deletion of Duplicated Constraints (4/6)

positive ratio = Right hand side parameter in each column / Positive coefficient of the element in the selected column

1

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	bi	bi/ai
x7	1	1	1	0	0	0	1	0	0	0	0	50	50
x8	0	1	1	1	1	0	0	1	0	0	0	50	50
x9	0	0	1	0	1	1	0	0	1	0	0	50	50
x10	0	1	0	0	0	0	0	0	0	1	0	40	-
x11	0	0	**1**	0	0	0	0	0	0	0	1	25	**25**
Obj.	-5	-10	**-20**	-8	-12	-6	0	0	0	0	0	f+0	-

(2) Select the variable whose coefficient is positive and row has the smallest positive ratio in the constraints.

(1) Select the column which has the minimum coefficient of the objective function.

(3) Pivot on the selected variable(x_3 / 5th row, 3rd column).

2

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	bi	bi/ai
x7	1	1	0	0	0	0	1	0	0	0	-1	25	-
x8	0	1	0	1	**1**	0	0	1	0	0	-1	25	**25**
x9	0	0	0	0	1	1	0	0	1	0	-1	25	25
x10	0	1	0	0	0	0	0	0	0	1	0	40	-
x3	0	0	1	0	0	0	0	0	0	0	1	25	-
Obj.	-5	-10	0	-8	**-12**	-6	0	0	0	0	20	f+500	-

[Example 1] Optimal Transportation of Cargo - Solution: Deletion of Duplicated Constraints (5/6)

3

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	bi	bi/ai
x7	1	1	0	0	0	0	1	0	0	0	-1	25	-
x5	0	1	0	1	1	0	0	1	0	0	-1	25	-
x9	0	-1	0	-1	0	**1**	0	-1	1	0	0	0	**0**
x10	0	1	0	0	0	0	0	0	0	1	0	40	-
x3	0	0	1	0	0	0	0	0	0	0	1	25	-
Obj.	-5	2	0	4	0	**-6**	0	12	0	0	8	f+800	-

4

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	bi	bi/ai
x7	**1**	1	0	0	0	0	1	0	0	0	-1	25	**25**
x5	0	1	0	1	1	0	0	1	0	0	-1	25	-
x6	0	-1	0	-1	0	1	0	-1	1	0	0	0	-
x10	0	1	0	0	0	0	0	0	0	1	0	40	-
x3	0	0	1	0	0	0	0	0	0	0	1	25	-
Obj.	**-5**	-4	0	-2	0	0	0	6	6	0	8	800	-

[Example 1] Optimal Transportation of Cargo - Solution: Deletion of Duplicated Constraints (6/6)

5

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	bi	bi/ai
x1	1	1	0	0	0	0	1	0	0	0	-1	25	
x5	0	1	0	1	1	0	0	1	0	0	-1	25	25
x6	0	-1	0	-1	0	1	0	-1	1	0	0	0	
x10	0	1	0	0	0	0	0	0	0	1	0	40	
x3	0	0	1	0	0	0	0	0	0	0	1	25	
Obj.	0	1	0	-2	0	0	5	6	6	0	3	f+925	

The row having the negative coefficient (-1) in the selected column is not selected.

6

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	bi	bi/ai
x1	1	1	0	0	0	0	1	0	0	0	-1	25	
x4	0	1	0	1	1	0	0	1	0	0	-1	25	
x6	0	0	0	0	1	1	0	0	1	0	-1	25	
x10	0	1	0	0	0	0	0	0	0	1	0	40	
x3	0	0	1	0	0	0	0	0	0	0	1	25	
Obj.	0	3	0	0	2	0	5	8	6	0	1	f+975	

Because all the coefficients of the objective function are nonnegative, the current solution is the optimum. ($x_2=x_5=0, x_1=x_3=x_4=x_6=25, f=-975$)

Therefore, the maximum freight income (975,000$) can be achieved by loading 25,000 tons per the cargo type (1, 3, 4, 6).

[Example 2] Linear Programming Problem

☑ Solve the linear programming problem only having the equality constraints(linear indeterminate equation).

$$
\begin{aligned}
&2x_1 + y - z - \zeta_1 = 3 \\
&2x_2 + y - z - \zeta_2 = 3 \\
&x_1 + x_2 = 2 \\
&where, x_1, x_2 y, z, \zeta_1, \zeta_2 \geq 0
\end{aligned}
$$

Initial basic feasible solution: $x_1 = x_2 = 1, y = 1, z = 0, \zeta_1 = \zeta_2 = 0$

[Example 2] Linear Programming Problem - Solution (1/3)

1. The problem is the linear programming problem only having the equality constraints(linear indeterminate equation).

2. To solve this problem, we introduce the artificial variables and artificial objective function to find the initial basic feasible solution in the Simplex method.

$$\mathbf{B}_{(3\times6)}\mathbf{X}_{(6\times1)} + \underline{\mathbf{Y}_{(3\times1)}} = \mathbf{D}_{(3\times1)}$$

Artificial variable

3. The artificial objective function is defined as follows.

$$w = \sum_{i=1}^{3} Y_i = \sum_{i=1}^{3} D_i - \sum_{j=1}^{6}\sum_{i=1}^{3} B_{ij}X_j = w_0 + \sum_{j=1}^{6} C_j X_j$$

where, $C_j = -\sum_{i=1}^{3} B_{ij}$: Sum the all the elements at the j column in Matrix B and change the sign. (Relative objective coefficient)

$w_0 = \sum_{i=1}^{3} D_i = 3+3+2=8$: Sum of all the elements in the Matrix D. (Initial basic solution for the artificial objective function)

[Example 2] Linear Programming Problem - Solution (2/3)

$$2x_1 + y - z - \zeta_1 = 3$$
$$2x_2 + y - z - \zeta_2 = 3$$
$$x_1 + x_2 = 2$$
$$where, x_1, x_2 y, z, \zeta_1, \zeta_2 \geq 0$$

$$\mathbf{B}_{(3\times6)}\mathbf{X}_{(6\times1)} + \underline{\mathbf{Y}_{(3\times1)}} = \mathbf{D}_{(3\times1)}$$

Artificial variable

$$\begin{bmatrix} 2 & 0 & 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(=X_1) \\ x_2(=X_2) \\ y(=X_3) \\ z(=X_4) \\ \zeta_1(=X_5) \\ \zeta_2(=X_6) \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

1	X1	X2	X3	X4	X5	X6	Y1	Y2	Y3	bi	bi/ai
Y1	2	0	1	-1	-1	0	1	0	0	3	3/2
Y2	0	2	1	-1	0	-1	0	1	0	3	-
Y3	1	1	0	0	0	0	0	0	1	2	2
A. Obj.	-3	-3	-2	2	1	1	0	0	0	w-8	-

Artificial objective function

Sum the all the elements at the each column in Matrix B and change the sign. (ex. 1st column: -(2+0+1)=-3)

[Example 2] Linear Programming Problem - Solution (3/3)

2

	X1	X2	X3	X4	X5	X6	Y1	Y2	Y3	bi	bi/ai
X1	1	0	1/2	-1/2	-1/2	0	1/2	0	0	3/2	-
Y2	0	2	1	-1	0	-1	0	1	0	3	3/2
Y3	0	**1**	-1/2	1/2	1/2	0	-1/2	0	1	1/2	**1/2**
A. Obj.	0	**-3**	-1/2	1/2	-1/2	1	3/2	0	0	w-7/2	-

3

	X1	X2	X3	X4	X5	X6	Y1	Y2	Y3	bi	bi/ai
X1	1	0	1/2	-1/2	-1/2	0	1/2	0	0	3/2	3
Y2	0	0	**2**	-2	-1	-1	1	1	-2	2	**1**
X2	0	1	-1/2	1/2	1/2	0	-1/2	0	1	1/2	-
A. Obj.	0	0	**-2**	2	1	1	0	0	3	w-2	-

4

	X1	X2	X3	X4	X5	X6	Y1	Y2	Y3	bi	bi/ai
X1	1	0	0	0	-1/4	1/4	1/4	-1/4	1/2	1	
X3	0	0	1	-1	-1/2	-1/2	1/2	1/2	-1	1	
X2	0	1	0	0	1/4	-1/4	-1/4	1/4	1/2	1	-
A. Obj.	0	0	0	0	0	0	1	1	1	**w-0**	-

Since the value of the artificial objective function becomes zero, the initial basic feasible solution is obtained.

$$\mathbf{X}^T{}_{(1\times 5)} = [x_1 \quad x_2 \quad y \quad z \quad \zeta_1 \quad \zeta_2]$$

$\rightarrow X_1=1, X_2=1, X_3=1, X_4=X_5=X_6=0$

Therefore, one of the initial basic feasible solutions is $x_1 = x_2 = 1, v = y - z = 1, \zeta_1 = \zeta_2 = 0.$