Digital Modulation and Detection

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Signal Space Analysis

Signal and System Model (1)

- System sends $K = \log_2 M$ bits every T seconds
- Each bit sequence of length K comprises a message $m_i = \{b_1, \ \ldots, b_K\}$
- The message i has the probability p_i of being selected for transmission ($\sum_{i=1}^{M} p_i = 1$)
- Each message is mapped to a unique analog signal $s_i(t)$, is transmitted over the channel during the time interval [0, T], and has energy

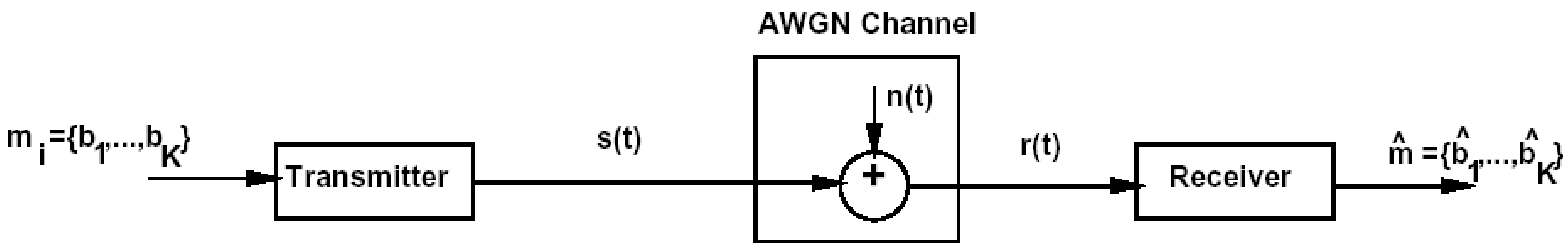
$$E_{S_i} = \int_0^T s_i^2(t)\, dt, \quad i = 1, \cdots, M$$

- The transmitted signal is sent through an AWGN channel (a white Gaussian noise process $n(t)$ of power spectral density $N_0/2$)
- Received signal $r(t)=s(t)+n(t)$
- The receiver should determine the best estimate of the transmitted signal and outputs the best estimate of the transmitted message $\hat{m} = \{\hat{b}_1, \cdots, \hat{b}_K\}$

Signal and System Model (2)

- Goal of the receiver design
  - minimizing the probability of message estimation error

$$P_e = \sum_{i=1}^{M} p(\hat{m} \neq m_i \mid m_i \text{ sent})\, p(m_i \text{ sent})$$



Communication system model

Geometric Representation of Signals (1)

- By representing the signals geometrically, we can solve for the optimal receiver design in AWGN based on a minimum distance criterion
- Basis function representation
  - Any set of M real energy signals $S=\{s_1(t),\cdots,s_M(t)\}$ can be represented as a liner combination of $N\ (\le M)$ real orthonormal basis functions $\{\phi_1(t),\cdots,\phi_N(t)\}$ according to Gram-Schmidt orthogonalization procedure

$$s_i(t)=\sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad 0\le t<T$$

: Modulation

  - Orthonormal basis function: $\int_0^T \phi_i(t)\phi_j(t)dt=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\ne j\end{cases}$
  - Real coefficient representing the projection of $s_i(t)$ onto $\phi_j(t)$

$$s_{ij}=\int_0^T s_i(t)\phi_j(t)dt$$

: Demodulation

Geometric Representation of Signals (2)

- Basis set of linear pathband modulation technique
  - $$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t), \qquad \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$
  - With $f_c T >> 1$ for orthonormality
- Transmitted signal

$$s_i(t) = s_{i1}\sqrt{\frac{2}{T}}\cos(2\pi f_c t) + s_{i2}\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

- The basis set can include a bandpass pulse-shaping filter $g(t)$ to improve the spectral characteristics of the transmitted signal

$$s_i(t) = s_{i1} g(t)\cos(2\pi f_c t) + s_{i2} g(t)\sin(2\pi f_c t)$$

- The pulse shape $g(t)$ must maintain the orthonormal property of basis functions

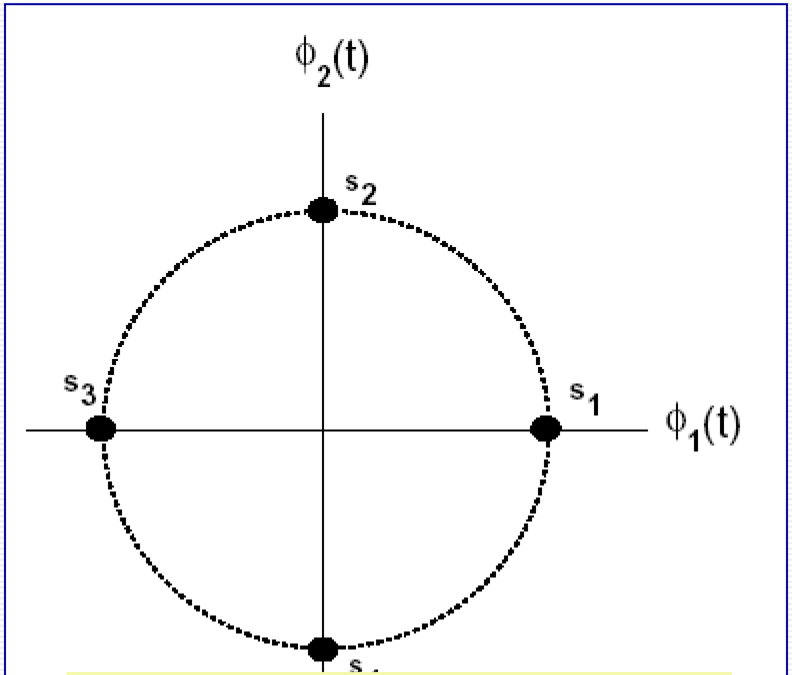
$$\int_0^T g^2(t)\cos^2(2\pi f_c t)dt = 1, \qquad \int_0^T g^2(t)\cos(2\pi f_c t)\sin(2\pi f_c t)dt = 0$$

Geometric Representation of Signals (3)

- Signal space representation
  - Signal constellation point of the signal $s_i(t)$
    - $\mathbf{s}_i = (s_{i1}, \cdots s_{iN}) \in \mathrm{R}^N$
    - the vector of coefficients in the basis representation of $s_i(t)$, that is, $s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t)$
    - One-to-one correspondence between $s_i(t)$ and $\mathbf{s}_i$
  - The distance between two signal constellation points $\boldsymbol{s}_i$ and $\boldsymbol{s}_k$

$$\|s_i - s_k\| = \sqrt{\sum_{j=1}^{N} (s_{ij} - s_{kj})^2}$$

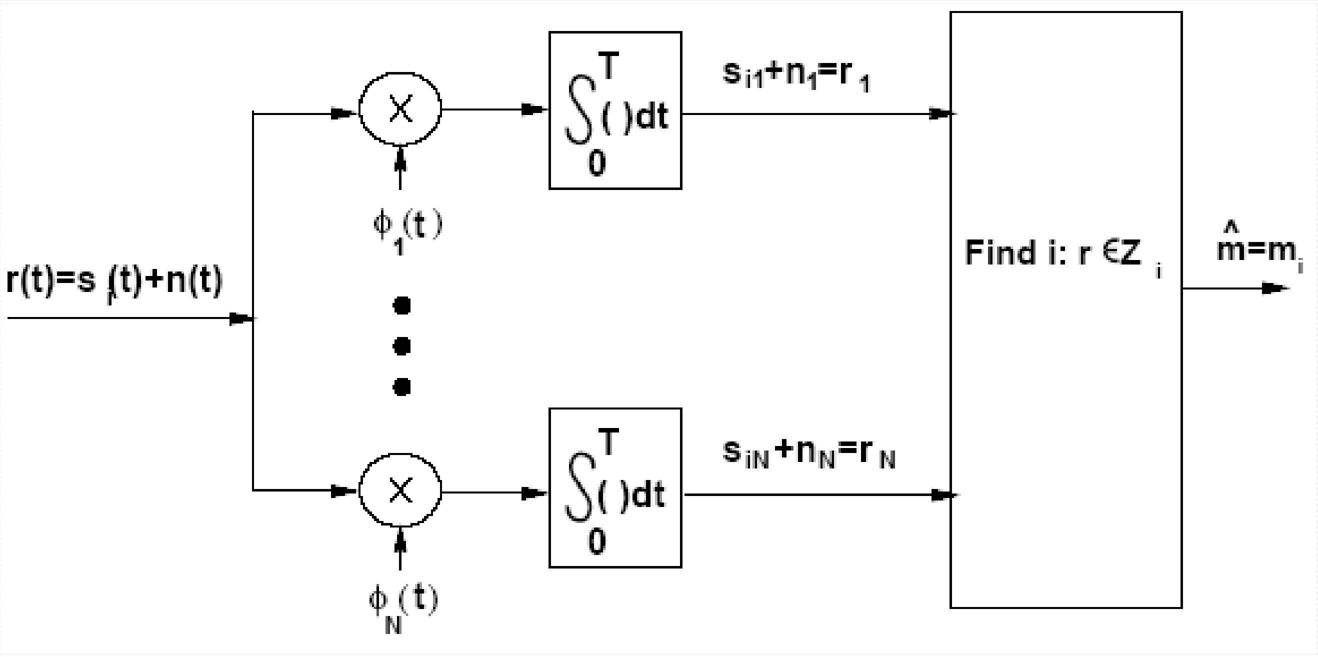
$$= \sqrt{\int_0^T (s_i(t) - s_k(t))^2 \, dt}$$



Signal space of MQAM or MPSK is two-dimensional

Receiver Structure and Sufficient Statistics (1)

- Given the channel output $r(t)=s_i(t)+n(t)$, $0 \le t < T$, the receiver determines which constellation point (or message) is sent over time interval $[0, T\,)$



Receiver Structure and Sufficient Statistics (2)

- In this receiver structure
  - $$\begin{aligned} r(t) &= s_i(t) + n(t) \\ &= \sum_{j=1}^{N} s_{ij}\phi_j(t) + \sum_{j=1}^{N} n_j\phi_j(t) + n_r(t) \\ &= \sum_{j=1}^{N} (s_{ij} + n_j)\phi_j(t) + n_r(t) = \sum_{j=1}^{N} r_j\phi_j(t) + n_r(t) \end{aligned}$$
  - $$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \quad n_j = \int_0^T n(t)\phi_j(t)dt$$
  - $n_r(t)$ is the "remainder" noise which is orthogonal to signal space

Receiver Structure and Sufficient Statistics (3)

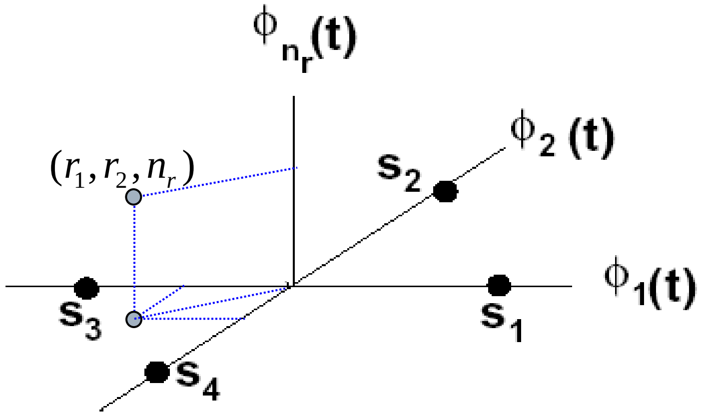


Figure 5.5: Projection of Received Signal onto Received Vector **r**.

$$r = (r_1, r_2)$$

$$[r(t) = r_1\phi_1(t) + r_1\phi_1(t) + n_r\phi_{n_r}(t)]$$

Receiver Structure and Sufficient Statistics (3)

- Goal of receiver design is to minimize the error probability

$$P_e = p(\hat{m} \neq m_i \mid r(t)) = 1 - p(\hat{m} = m_i \mid r(t))$$

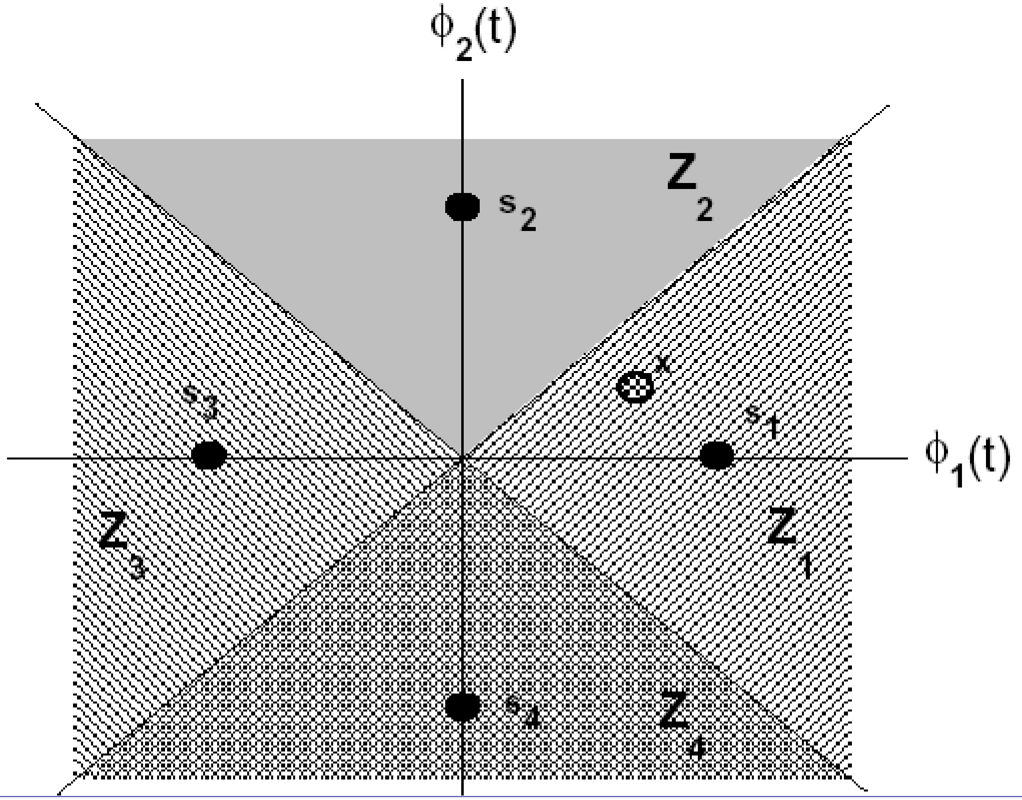
- Maximizing

$$\begin{aligned} p(s_i \text{ sent} \mid r(t)) &= p((s_{i1}, \cdots, s_{iN}) \text{ sent} \mid (r_{i1}, \cdots, r_{iN}), n_r(t)) \\ &= \frac{p((s_{i1}, \cdots, s_{iN}) \text{ sent}, (r_{i1}, \cdots, r_{iN}), n_r(t))}{p((r_{i1}, \cdots, r_{iN}), n_r(t))} \\ &= \frac{p((s_{i1}, \cdots, s_{iN}) \text{ sent}, (r_{i1}, \cdots, r_{iN}))\, p(n_r(t))}{p((r_{i1}, \cdots, r_{iN}))\, p(n_r(t))} \\ &= p((s_{i1}, \cdots, s_{iN}) \text{ sent} \mid (r_{i1}, \cdots, r_{iN})) \end{aligned}$$

- $\boldsymbol{r} = (r_1, \ldots, r_N)$ is a sufficient statistic for r(t) in optimal detection of transmitted message

Decision Region

- Optimal receiver selects $\hat{m} = m_i$ corresponding to constellation s_i that satisfies $p(s_i \text{ sent} \mid r) \geq p(s_j \text{ sent} \mid r)$ for all $j \neq i$
- Design region: $Z_i = \{r : p(s_i \text{ sent} \mid r) > p(s_j \text{ sent} \mid r) \ \forall \ j \neq i\}$



Maximum Likelihood Decision Criteria (1)

- Optimal receiver selects $\hat{m} = m_i$ corresponding to constellation s_i that maximizes

$$p(s_i \mid r) = \frac{p(r \mid s_i)\, p(s_i)}{p(r)}$$

$$\arg\max_{s_i} \frac{p(r \mid s_i)\, p(s_i)}{p(r)} = \arg\max_{s_i} p(r \mid s_i)\, \underline{p(s_i)}$$

$$= \arg\max_{s_i} p(r \mid s_i)$$

Let $p(s_i)=1/M$

- Likelihood function: $L(s_i) = p(r \mid s_i)$
- A maximum likelihood receiver outputs $\hat{m} = m_i$ corresponding to constellation s_i that maximizes $L(s_i)$

Maximum Likelihood Decision Criteria (2)

- Conditional distribution of ***r***
  - Since $n(t)$ is a Gaussian random process, $r(t)=s_i(t)+n(t)$ is also a Gaussian random process and $n(t)$ has a zero mean.
  - $$r_j = s_{ij} + n_j$$

$$\mu_{r_j|s_i} = \mathrm{E}[r_j \mid s_{ij}] = \mathrm{E}[s_{ij} + n_j \mid s_{ij}] = s_{ij}$$

$$\sigma^2{}_{r_j|s_i} = \mathrm{E}[(r_j - \mu_{r_j|s_{ij}})^2] = \mathrm{E}[(s_{ij} + n_j - s_{ij})^2 \mid s_{ij}] = \mathrm{E}[n_j{}^2] = N_0/2$$

$$\mathrm{cov}[r_j r_k \mid s_i] = \mathrm{E}[(r_j - \mu_{r_j})(r_k - \mu_{r_k}) \mid s_i]$$

$$= \mathrm{E}[n_j n_k]$$

$$= \begin{cases} N_0/2 & j = k \\ 0 & j \neq k \end{cases}$$

  - r_j is a Gaussian random variable that is independent of r_k $(j \neq k)$ with mean s_{ij} and variance $N_0/2$.

Maximum Likelihood Decision Criteria (3)

- Likelihood function $L(s_i)$: conditional distribution of ***r***

$$p(r \mid s_i) = \prod_{j=1}^{N} p(r_j \mid s_{ij}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0} \sum_{j=1}^{N} (r_j - s_{ij})^2 \right]$$

- Log likelihood function

$$l(s_i) = -\frac{1}{N_0} \sum_{j=1}^{N} (r_j - s_{ij})^2 = -\frac{1}{N_0} \left\| r - s_i \right\|^2$$

minimizing

Maximum Likelihood Decision Criteria (4)

- A maximum likelihood receiver outputs $\hat{m} = m_i$ corresponding to constellation s_i that satisfies

$$\arg\min_{s_i} \sum_{j=1}^{N} (r_j - s_{ij})^2 = \arg\min_{s_i} \left\| r - s_i \right\|^2$$

- Decision Region

$$Z_i = \{ r : \left\| r - s_i \right\| < \left\| r - s_j \right\| \;\; \forall\, j = 1, \cdots, M,\, j \neq i \}$$

- Constellation point s_i is determined from the decision Z_i that contains ***r***

Union Bound of Error Probability (1)

- A_{ik}: the event that $\|r - s_k\| < \|r - s_i\|$ given that the constellation point s_i was sent
- If the event A_{ik} occurs, the constellation will be decoded in error.
- $$P_e(m_i \text{ sent}) = p\left(\bigcup_{\substack{k=1 \\ k\neq i}}^{M} A_{ik}\right) \leq \sum_{\substack{k=1 \\ k\neq i}}^{M} p(A_{ik})$$

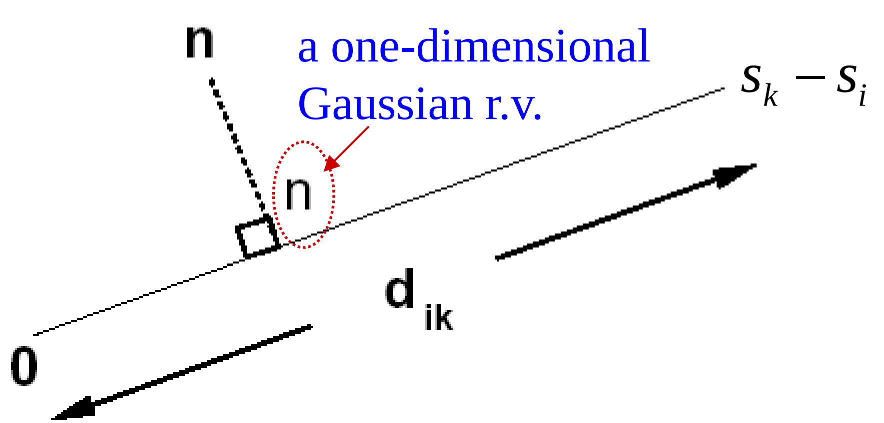


Figure 5.9: Noise Projecion

$$p(A_{ik}) = p(\|r - s_k\| < \|r - s_i\| \mid s_i \text{ sent})$$
$$= p(\|(s_i + n) - s_k\| < \|(s_i + n) - s_i\|)$$
$$= p(\|n - (s_k - s_i)\| < \|n\|)$$

The probability that ***n*** is closer to the vector s_k-s_i than to the origin

Union Bound of Error Probability (2)

- The event A_{ik} occurs if $n > d_{ik}/2$, where $d_{ik} = \|s_i - s_k\|$
- $$p(A_{ik}) = p\left(n > \frac{d_{ik}}{2}\right) = \int_{d_{ik}/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-v^2}{N_0}\right] dv = Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$P_e(m_i \text{ sent}) \le \sum_{\substack{k=1 \\ k \ne i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

- $$P_e = \sum_{i=1}^{M} p(m_i) P_e(m_i \text{ sent}) \le \frac{1}{\mathrm{M}} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \ne i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

Approximation of Error Probability

- The minimum distance of constellation: d_{min}
  - $P_e \le (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$: looser bound
- The number of neighbors at the minimum distance: $M_{d_{\min}}$
  - $$P_e \approx M_{d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$
  - In case of binary modulation (M=2): $P_b = Q\left(d_{\min} / \sqrt{2N_0}\right)$
- Gray code: mistaking a constellation point for one of its nearest neighbors results in a single bit error
  - $$P_b \approx \frac{P_e}{\log_2 M}$$

Amplitude and Phase Modulation

Amplitude and Phase Modulation (1)

- Over the time interval T_s, K ($\log_2 M$) bits are encoded in the amplitude and/or phase of the transmitted signal
  - Signal: $s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$
  - In signal space:

$$s(t) = s_{i1}(t)\phi_1(t) + s_{i2}(t)\phi_2(t)$$

$$\phi_1(t) = g(t)\cos(2\pi f_c t + \phi_0)$$

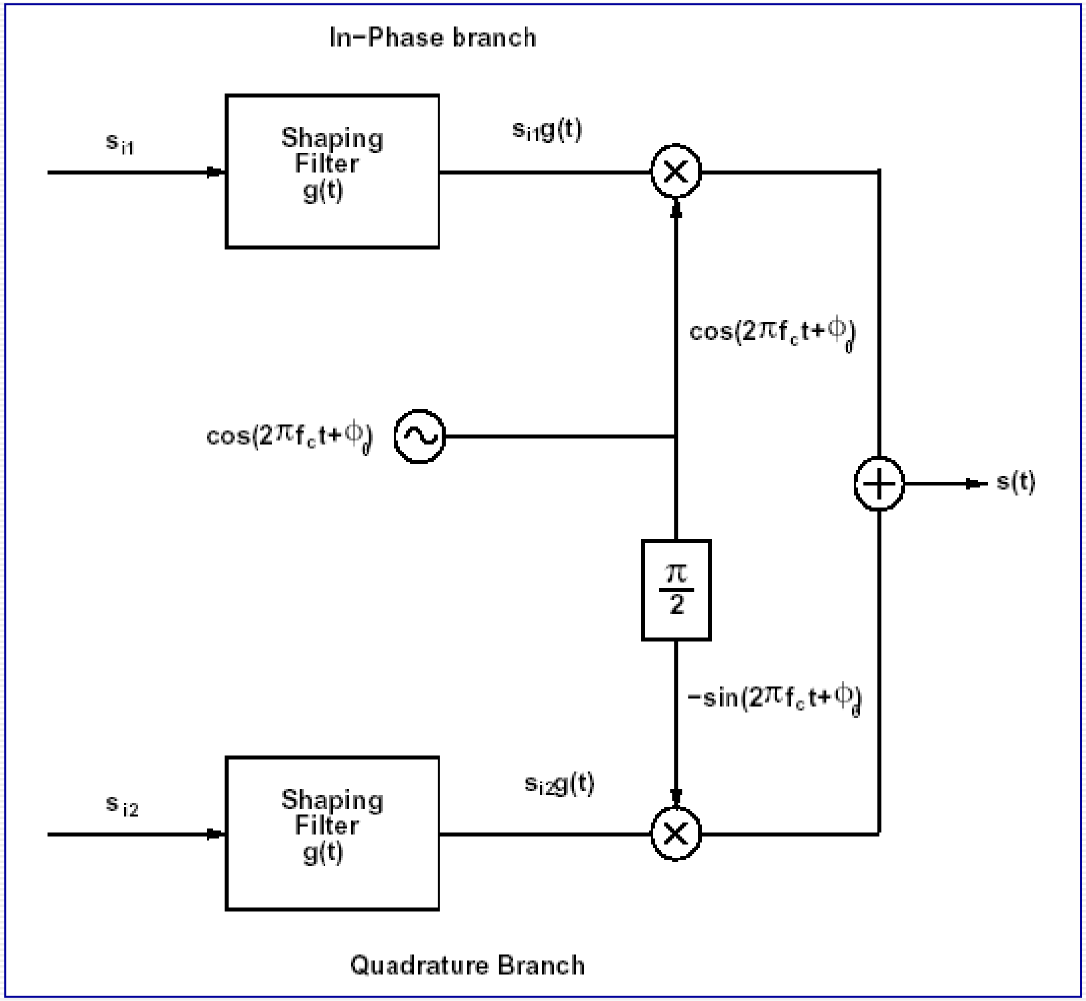
$$\phi_2(t) = -g(t)\sin(2\pi f_c t + \phi_0)$$

- There are three main types of amplitude/phase modulation:
  - Pulse Amplitude Modulation (MPAM) : Uses amplitude only
  - Phase Shift Keying (MPSK) : Uses phase only
  - Quadrature Amplitude Modulation (MQAM)

Amplitude and Phase Modulation (2)

Amplitude/Phase

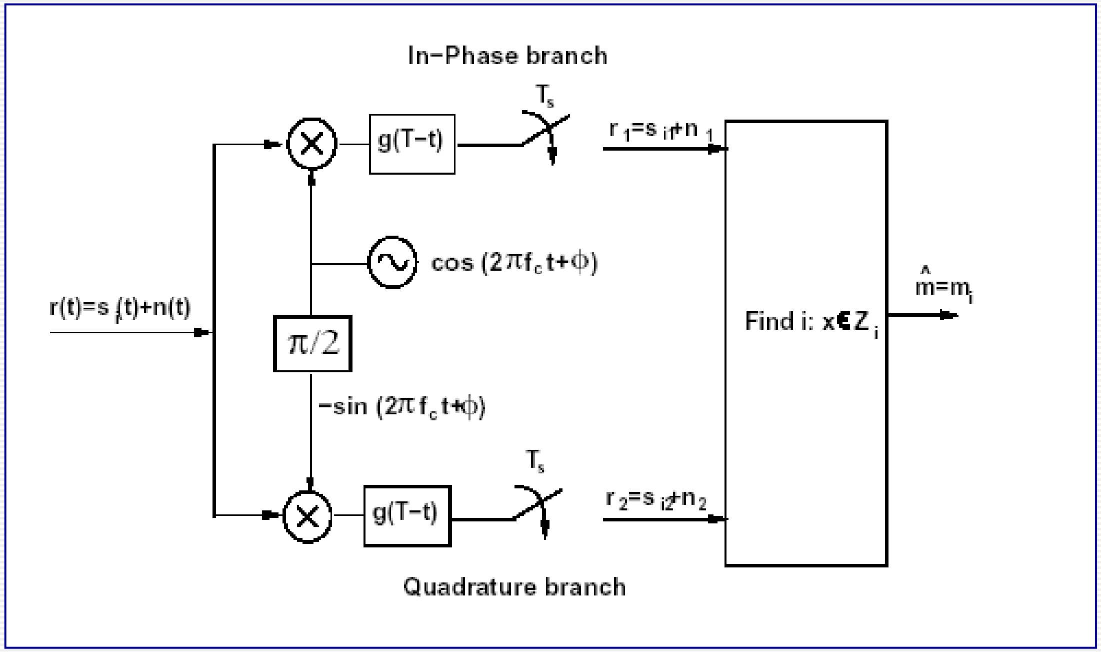
Modulator



Amplitude and Phase Modulation (3)

Amplitude/Phase

Demodulator



- Coherent detection $(\phi = \phi_0)$
  - ✓ if $\phi - \phi_0 = \Delta\phi \neq 0$, $r_1 = s_{i1}\cos(\Delta\phi) + s_{i2}\sin(\Delta\phi) + n_1$ and

    $r_2 = -s_{i1}\sin(\Delta\phi) + s_{i2}\cos(\Delta\phi) + n_2$ => performance degradation
- Synchronization or timing recovery:

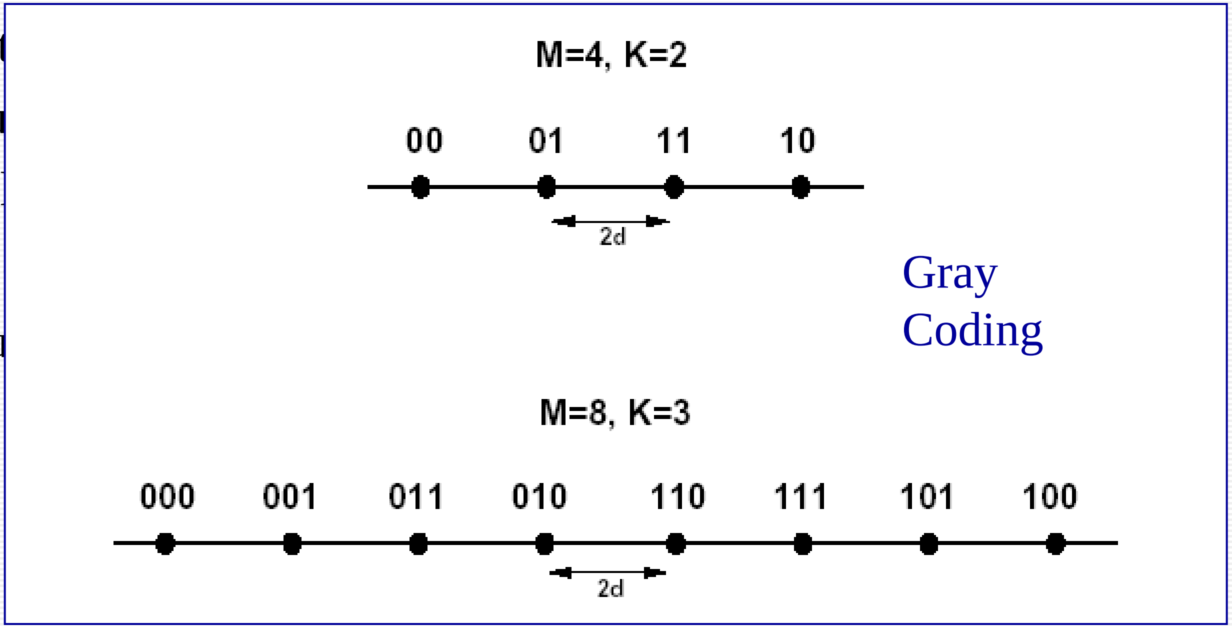
  the sampling function is synchronized to the start of every symbol period.

Amplitude and Phase Modulation (4)

- Pulse Amplitude Modulation
- Phase-Shift Keying
- Quadrature Amplitude Modulation
- Differential Modulation
- Modulator with Quadrature Offset

Pulse Amplitude Modulation (MPAM) (1)

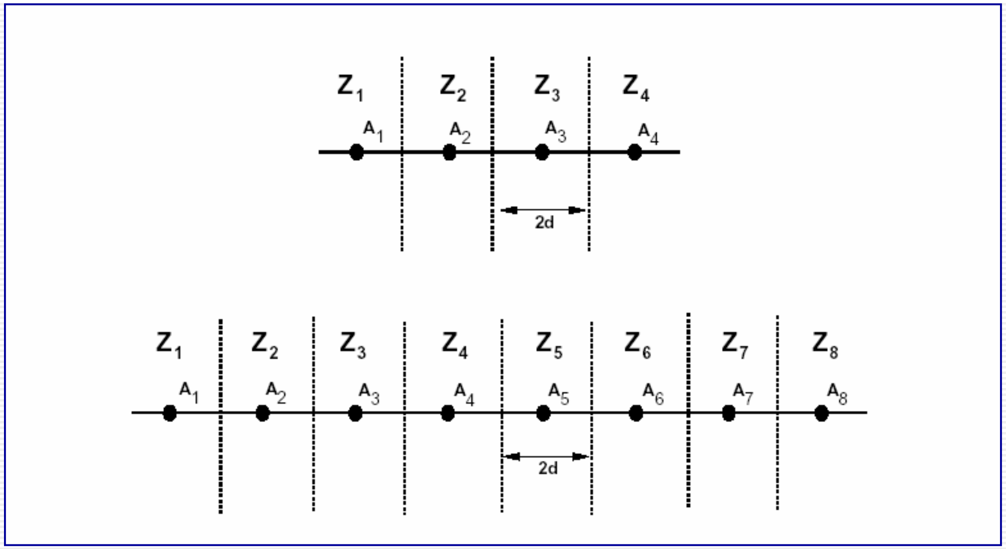
- Encodes all of the information into the signal amplitude (A_i)
- Transmitted signal over time:
  - $s(t) = \mathrm{Re}\left\{ A_i g(t) e^{j2\pi f_c t} \right\} = A_i g(t) \cos(2\pi f_c t), \quad 0 \le t \le T_s \gg 1/f_c$
- Signal constellation: $\{A_i = (2i-1-M)d,\ i = 1, 2, \ldots, M\}$
  - Paramet
  - Minimu
  - Constell
- Amplitude
  - Each pu



Pulse Amplitude Modulation (MPAM) (2)

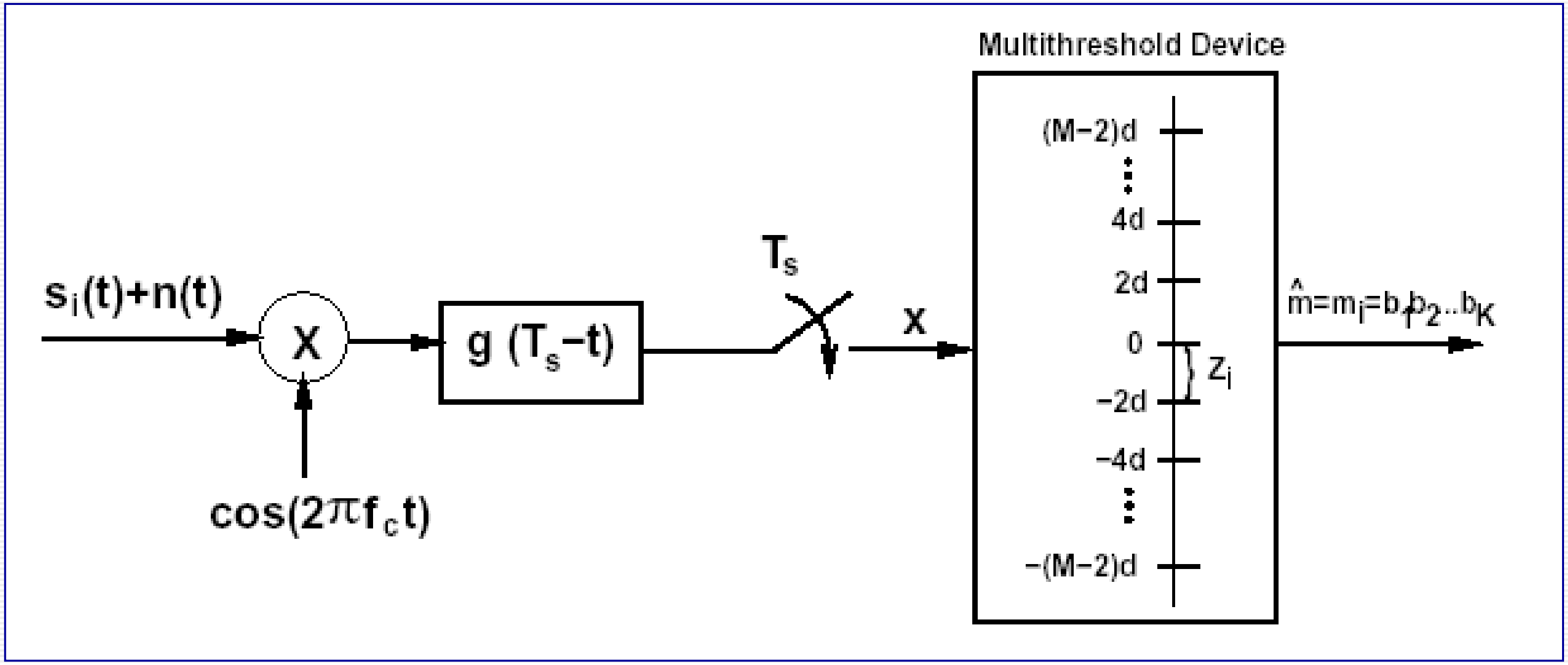
- Over each symbol period, the MPAM signal associated with the *i*th constellation has energy $E_{s_i} = \int_0^{T_s} s_i^2(t)dt = \int_0^{T_s} A_i^2 g^2(t)\cos^2(2\pi f_c t)dt = A_i^2$
  - Average energy: $\overline{E}_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2$
- Decision region associated with signal amplitude $A_i = (2i-1-M)d$

$$Z_i = \begin{cases} (-\infty, A_i + d) & i = 1, \\ [A_i - d, A_i + d) & 2 \le i \le M-1, \\ [A_i - d, \infty) & i = M \end{cases}$$



Pulse Amplitude Modulation (MPAM) (3)

Coherent MPAM demodulator



Phase Shift Keying (MPSK) (1)

- Encodes information in the phase of the transmitted signal
- Transmitted signal over one symbol time:

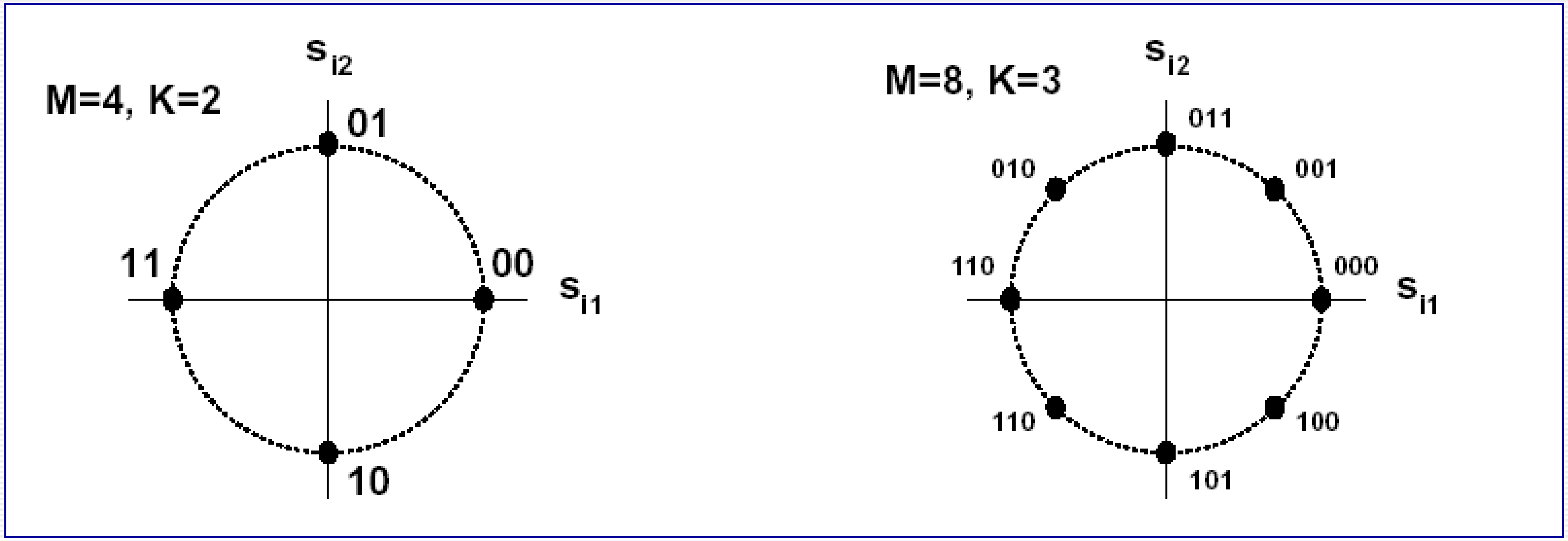
$$s_i(t) = \mathrm{Re}\left\{ Ag(t) e^{j2\pi(i-1)/M} e^{j2\pi f_c t} \right\}$$
$$= Ag(t)\cos\left[2\pi f_c t + \frac{2\pi(i-1)}{M}\right]$$
$$= Ag(t)\cos\left[\frac{2\pi(i-1)}{M}\right]\cos 2\pi f_c t - Ag(t)\sin\left[\frac{2\pi(i-1)}{M}\right]\sin 2\pi f_c t$$

- Constellation points: (s_{i1}, s_{i2})

  $s_{i1} = A\cos[2\pi(i-1)/M], \quad s_{i2} = A\sin[2\pi(i-1)/M] \ \text{ for } i = 1,\ldots,M$
- Minimum distance: $d_{\min} = 2A\sin(\pi/M)$, where A is typically function of signal energy
- All possible signals have equal energy A^2
- Constellation mapping usually uses Grady encoding

Phase Shift Keying (MPSK) (2)

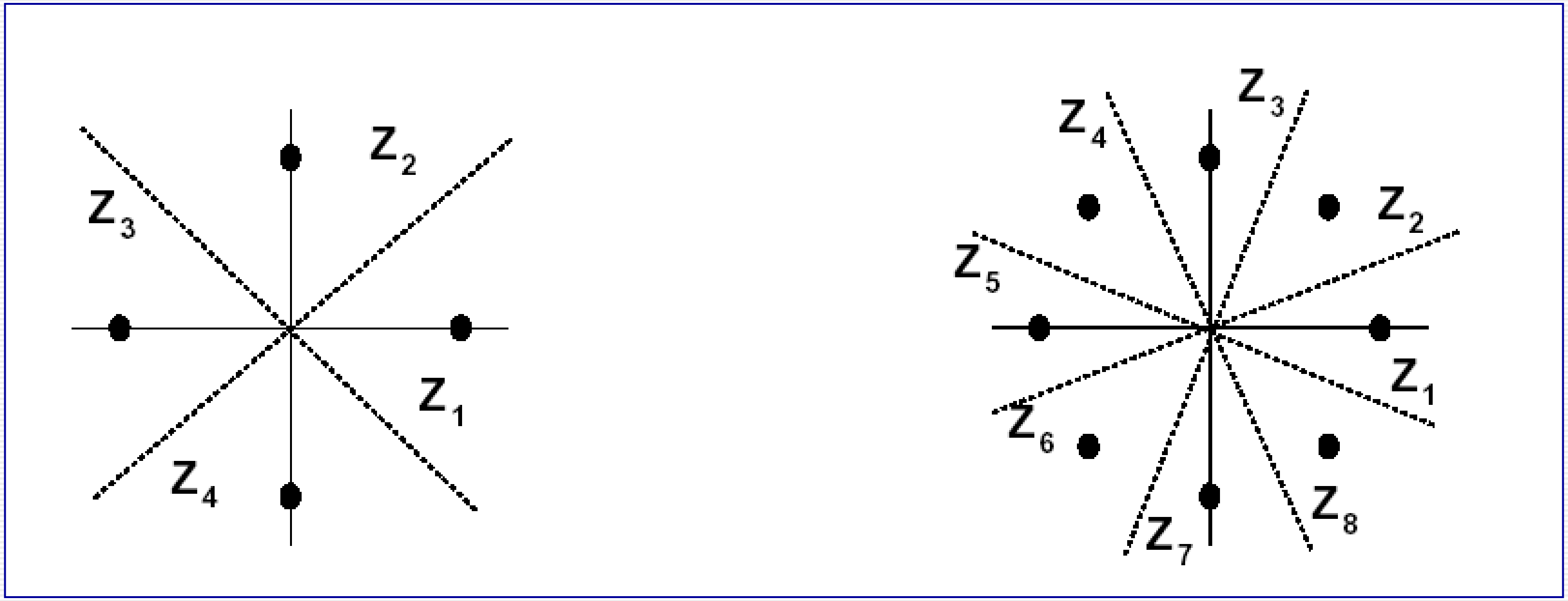
Gray Coding for MPSK



Phase Shift Keying (MPSK) (3)

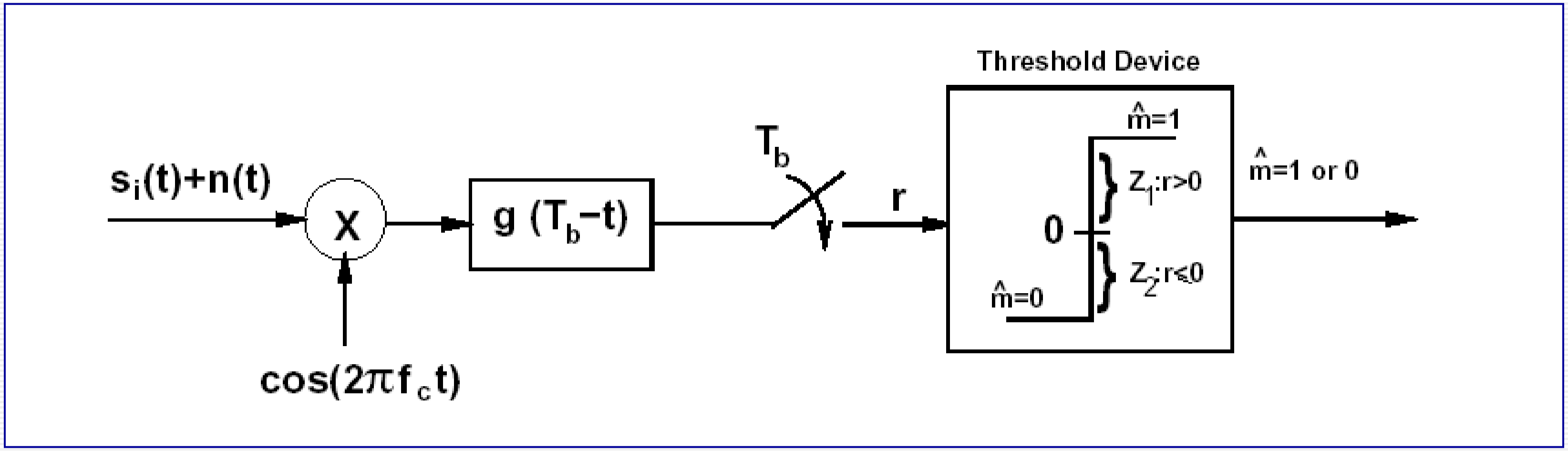
- Decision Regions for MPSK

$$Z_i = \left\{ re^{j\theta} : 2\pi\left(i - 3/2\right)/M < \theta < 2\pi\left(i - 1/2\right)/M \right\}$$



Phase Shift Keying (MPSK) (4)

Coherent Demodulator for BPSK



Quadrature Amplitude Modulation (MQAM) (1)

- Information bits are encoded in both amplitude and phase of the transmitted signal
- MPSK and MPAM have only one degree-of-freedom, but MQAM has two degree-of-freedom. Thus, MQAM is more spectral-efficient
- Transmitted signal:

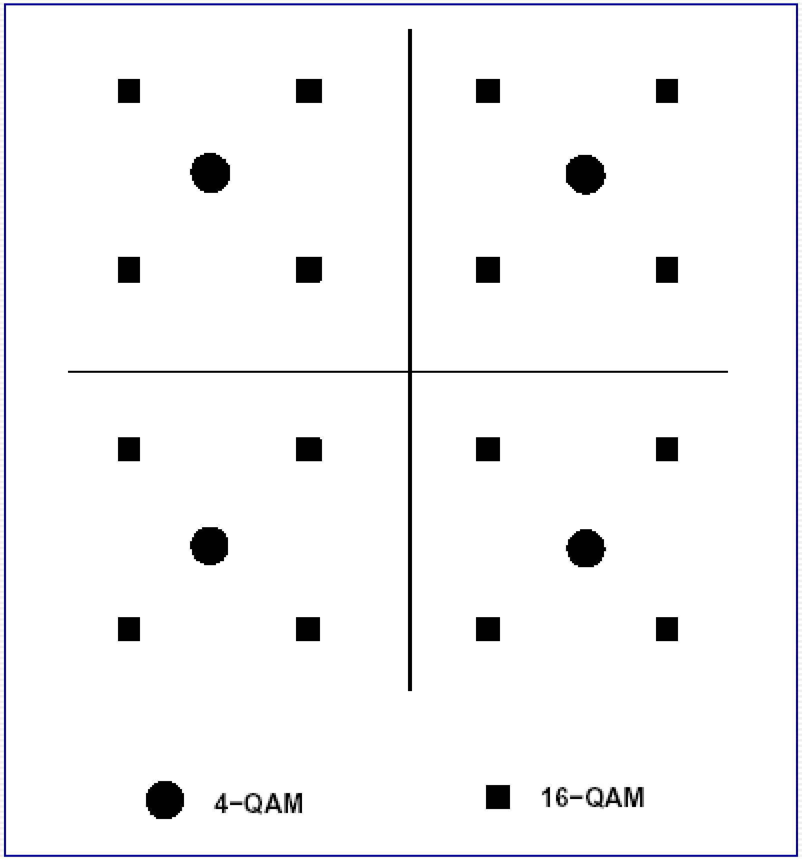
$$S_i(t) = \mathrm{Re}\left\{A_i e^{j\theta_i} g(t) e^{j2\pi f_c t}\right\}$$
$$= A_i \cos(\theta_i) g(t) \cos\left(2\pi f_c t\right) - A_i \sin(\theta_i) g(t) \sin\left(2\pi f_c t\right), \quad 0 \le t \le T_s$$

- Signal energy in $s_i(t)$: $E_{si} = \int_0^{T_s} s_i^2(t)dt = A_i^2$
- Distance between constellation points:

$$d_{ij} = \left\| s_i - s_j \right\| = \sqrt{\left(s_{i1} - s_{j1}\right)^2 + \left(s_{i2} - s_{j2}\right)^2}$$

Quadrature Amplitude Modulation (MQAM) (2)

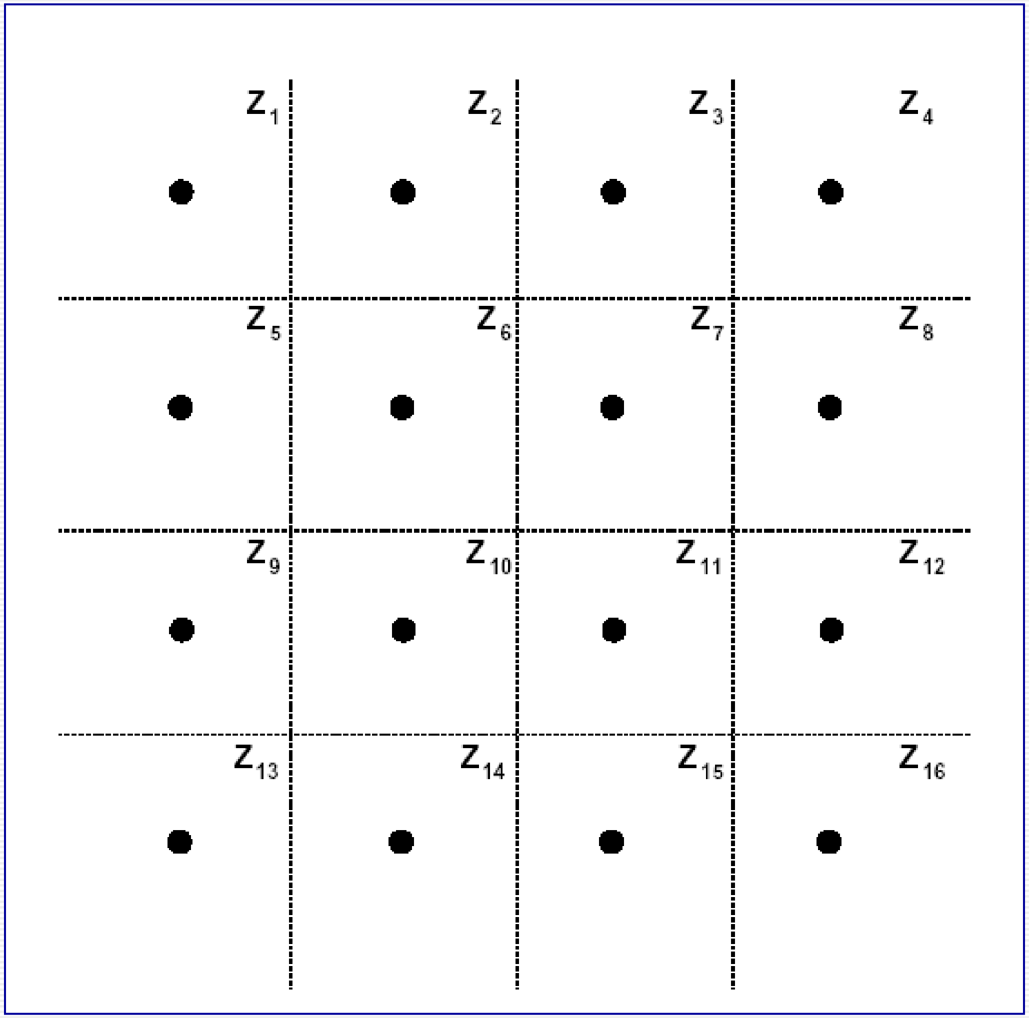
- For square signal constellation,
  - Values on $(2i-1-L)d$ for $i = 1, \ldots, L$
  - $d_{min} = 2d$
- Good constellation mapping can be hard to find for QAM signal



4-QAM and 16-QAM constellations

Quadrature Amplitude Modulation (MQAM) (3)

Decision Regions for 16-QAM



Differential Modulation (1)

- Information in MPSK and MQAM signals is carried in the signal phase
  - MPSK and MQAM require coherent detection
  - Phase recovery mechanism required in receiver
  - Coherent demodulation
    - makes receiver complex
    - is hard in rapidly fading channel
    - is more susceptible to phase drift of the carrier
- The principle of differential modulation is to use the previous symbol as a phase reference for current symbol for avoiding the need for a coherent phase modulation
  - Differential BPSK (DPSK)
    - 0-bit: no change in phase, 1-bit: a phase change of π
  - Differential QPSK (DQPSK)
    - 00: no change in phase, 01: a phase change of $\pi/2$
    - 10: a phase change of $-\pi/2$, 11: a phase change of π

Differential Modulation (2)

- Phase Comparator
  - Transmitted signal: $s(k) = Ae^{j(\theta(k)+\phi_0)}$
  - Received signal at time *k*: $r(k) = r_1(k) + jr_2(k) = Ae^{j(\theta(k)+\phi_0-\phi)} + n(k)$
  - Received signal at time (*k*-1):

$$r(k-1) = r_1(k-1) + jr_2(k-1) = Ae^{j(\theta(k-1)+\phi_0-\phi)} + n(k-1)$$

  - Phase difference

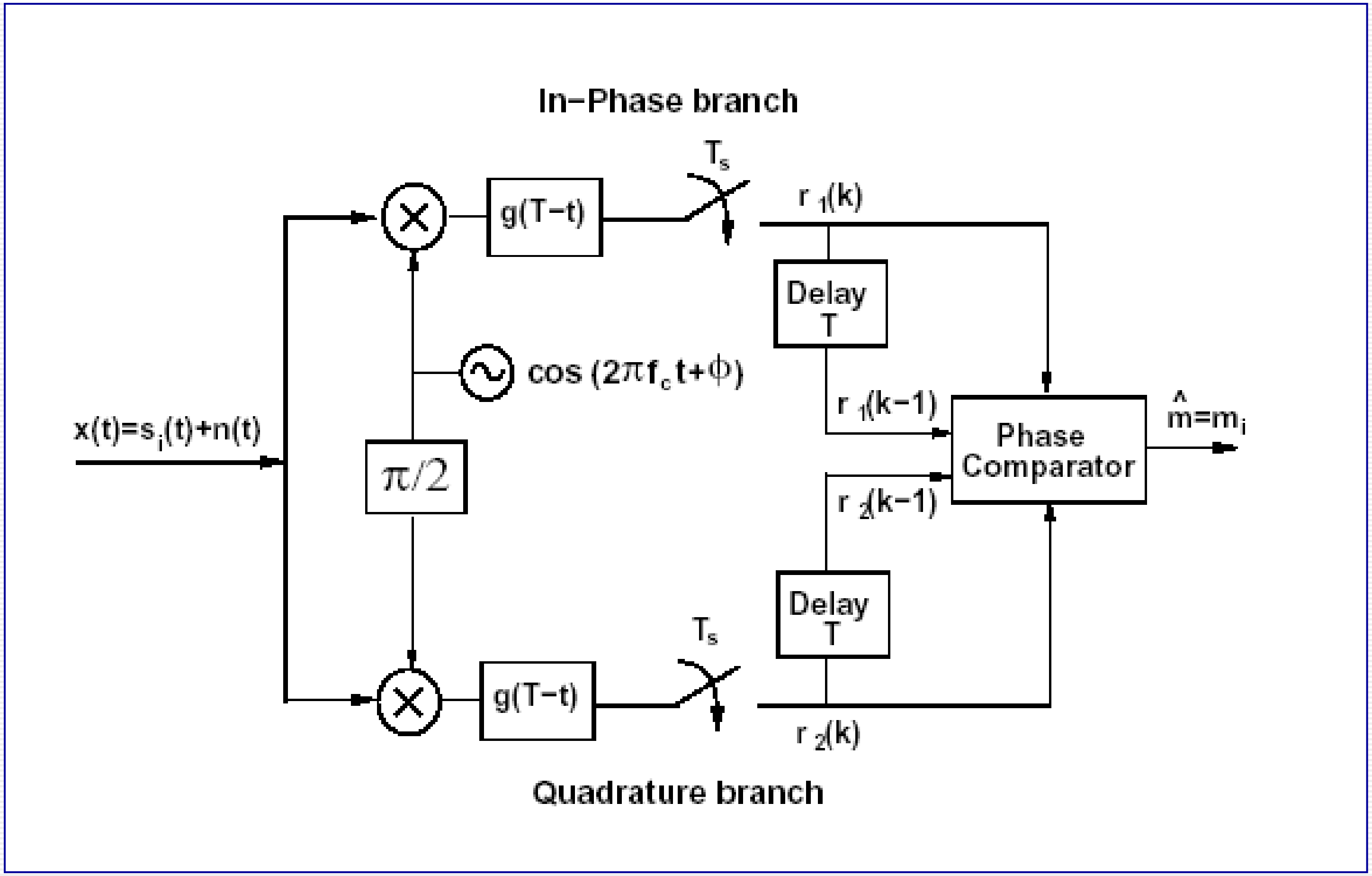
$$r(k)r^*(k-1) = A^2e^{j(\theta(k)-\theta(k-1))} + Ae^{j(\theta(k)+\phi_0-\phi)}n^*(k-1)$$
$$+ Ae^{-j(\theta(k-1)+\phi_0-\phi)}n(k) + n(k)n^*(k-1)$$

Phase difference in the absence of noise ($n(k) = n(k\text{-}1) = 0$)

- modulation with memory
- less sensitive to random drift in the carrier phase
- With non-zero Doppler frequency, previous symbol is not good for phase reference

Differential Modulation (3)

DPSK demodulator

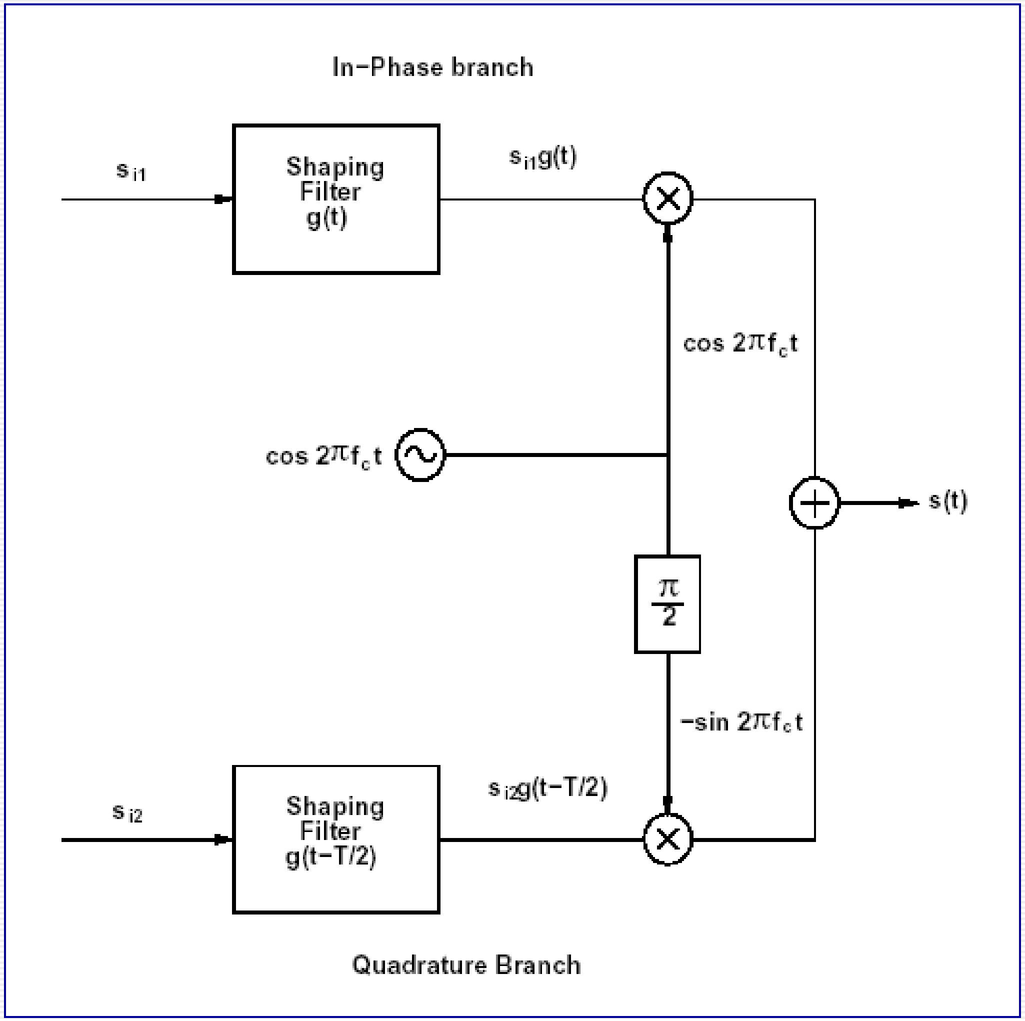


Modulation with Quadrature Offset (1)

- Linearly modulated signal may cause transition to symbol which makes phase change up to π, and signal amplitude to cross zero point
  - Abrupt phase transition and amplitude variations can be distorted by non-linear filters and amplifiers
- To avoid the above problem
  - Offsetting the quadrature branch pulse g(t) half a symbol period
  - Phase can change maximum $\pi/2$

Modulation with Quadrature Offset (2)

Modulator with quadrature offset



Pulse Shaping (1)

- Bandwidth of the baseband and passband modulated signal is a function of the bandwidth of the pulse shape $g(t)$.
- The effective received pulse: $p(t) = g(t) * c(t) * g^*(-t)$
  - $c(t)$: the channel impulse response
  - $g^*(-t)$: the matched filter
  - In AWGN channel ($c(t)=\delta(t)$), $p(t) = g(t) * g^*(-t)$
- To avoid ISI between the received pulses, $p(t)$ must satisfy Nyquist criterion, which requires the pulse to equal zero at the ideal sampling point associated with past or future symbols.
- Pulse shapes that satisfy the Nyquist criterion
  - Rectangular pulse
  - Cosine pulse
  - Raised Cosine pulse

Pulse Shaping (2)

- Raised Cosine Pulse
  - These pulses are designed in the frequency domain

$$P(f) = \begin{cases} T_s & 0 \le |f| \le \dfrac{1-\beta}{2T_s} \\ \dfrac{T_s}{2}\left[1 - \sin\dfrac{\pi T_s}{\beta}\left(f - \dfrac{1}{2T_s}\right)\right] & \dfrac{1-\beta}{2T_s} \le |f| \le \dfrac{1+\beta}{2T_s} \end{cases}$$

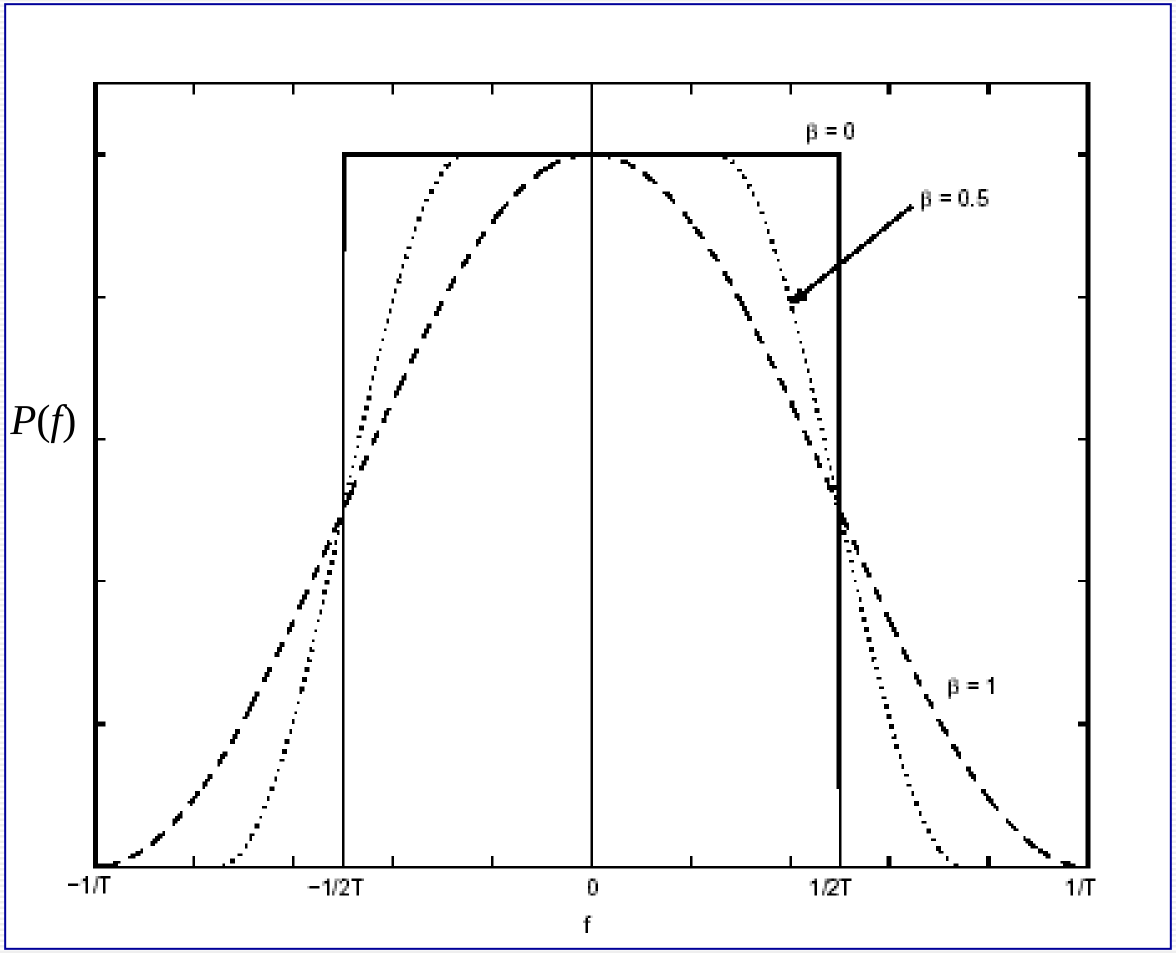
where β is a rolloff factor

  - The pulse $p(t)$ in the time domain:

$$p(t) = \frac{\sin \pi t / T_s}{\pi t / T_s} \times \frac{\cos \beta\pi t / T_s}{1 - 4\beta^2 t^2 / T_s^2}$$

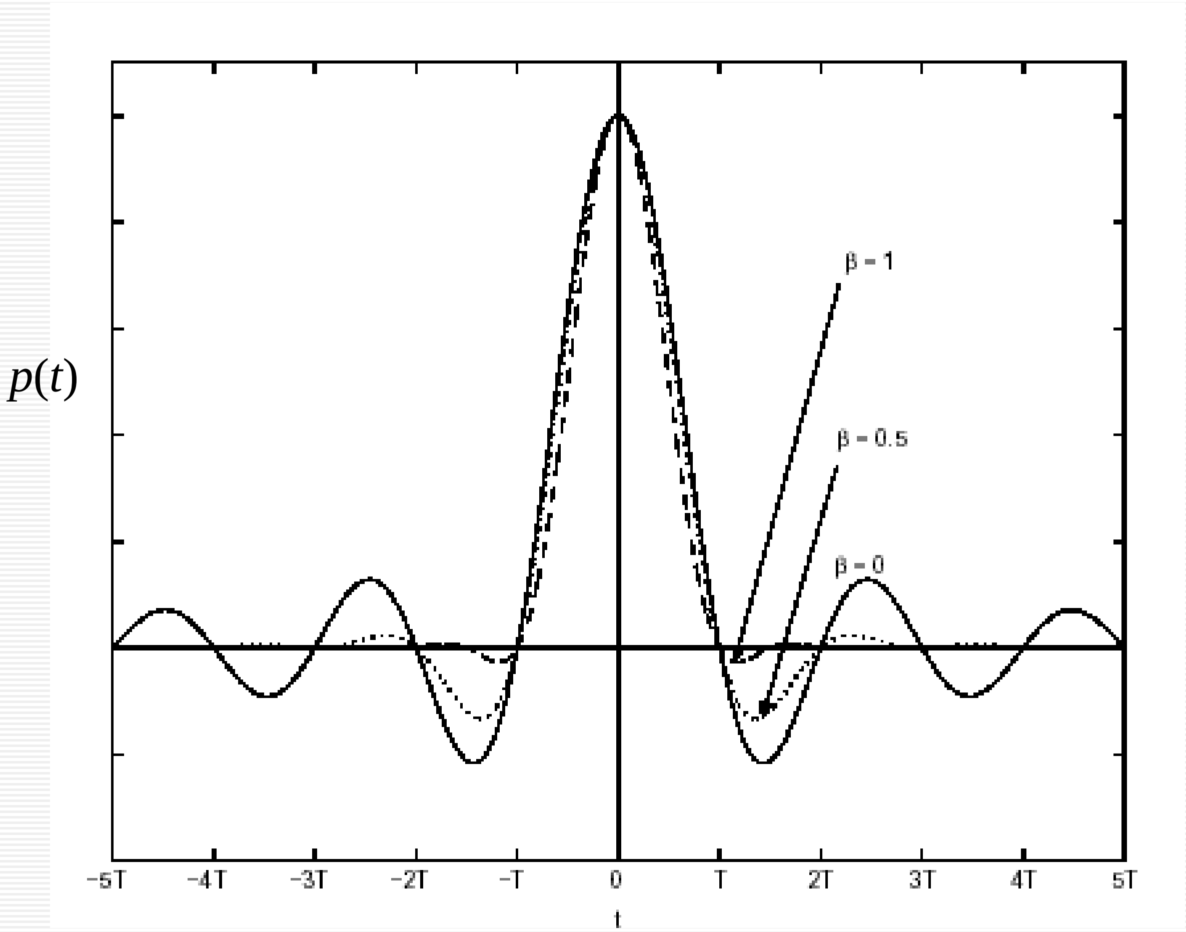
Pulse Shaping (3)

Raised Cosine Pulse in frequency domain



Pulse Shaping (4)

Raised Cosine Pulse
in time domain



Error Probability of Digital Modulation over AWGN Channel

- BPSK and QPSK
- MPSK
- MPAM and MQAM

Signal-to-Noise Power Ratio (SNR)

- In an AWGN channel
  - Modulated (transmitted) signal: $s(t) = \mathrm{Re}\{u(t)e^{j2\pi f_c t}\}$
  - Received signal: $r(t) = s(t) + n(t)$
  - $n(t)$: a white Gaussian random process with mean zero and power spectral density $N_0/2$
- SNR
  - Ratio of the received signal power P_r to the power of the noise within the bandwidth of the transmitted signal
  - $SNR = \frac{P_r}{N_0 B} = \frac{E_s}{N_0 B T_s} = \frac{E_b}{N_0 B T_b}$
  - In system with interference
    - $SINR = \frac{P_r}{N_0 B + P_I}$

Bit/Symbol Errors

- For pulse shaping with $T_s = 1/B$ (e.g., raised cosine pulse with $\beta = 1$), SNR=E_s/N_0
- For general pulse, $T_s = k/B$ and $\text{SNR} = E_s / N_0 \times 1/k$
- Define
  - SNR per symbol: $\gamma_s = E_s / N_0$
  - SNR per bit: $\gamma_b = E_b / N_0$
- We are interested in bit error probability P_b as a function of γ_b
- Approach
  - First, compute the symbol error probability P_s as a function of γ_s
  - Then, obtain bit error probability as a function of SNR per bit using assumptions.
    - The symbol energy is divided equally among all bits,
    - Gray encoding is used
  - These assumptions for M-ary signaling lead to the approximations

$$\gamma_b \approx \frac{\gamma_s}{\log_2 M} \quad \text{and} \quad P_b \approx \frac{P_s}{\log_2 M}$$

Error Probability for BPSK

- For binary modulation (M=2),
  - $$P_b = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$
  - $$d_{\min} = \left\| s_1 - s_0 \right\| = 2A,$$

$$E_b = \int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} A^2 g^2(t)\cos^2(2\pi f_c t)dt = A^2$$

- $$P_b = P_s = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2\gamma_b}\right)$$

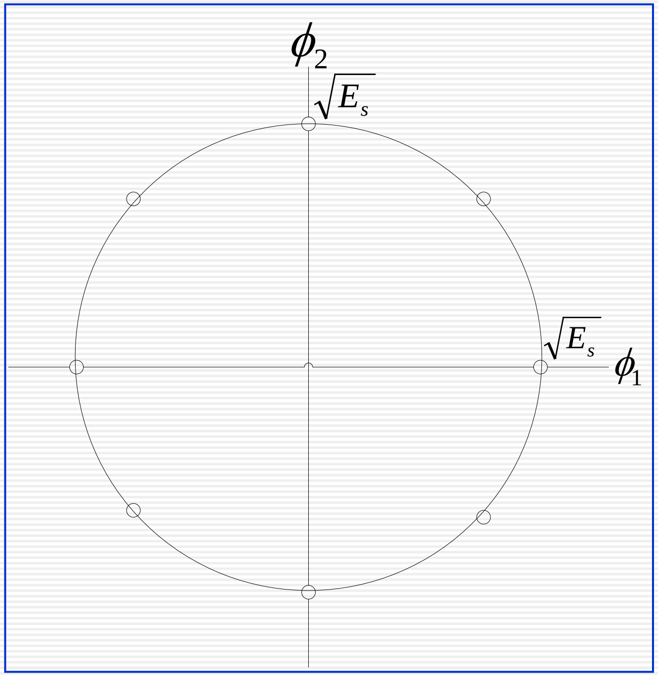
Error probability for QPSK

- The QPSK system is equivalent to the system consisting of BPSK modulation on both the in-phase and quadrature components of the signal.
- The bit error probability on each component is the same as for BPSK
  - $P_b = Q(\sqrt{2\gamma_b})$
- The symbol error probability is $P_s = 1 - \left[1 - Q\left(\sqrt{2\gamma_b}\right)\right]^2$
- Since the transmitted symbol energy E_s is split between each branch, the signal energy per branch is $(E_s/2)$
- The symbol error probability is the probability that either branch has a error
  - $P_s = 1 - \left[1 - Q\left(\sqrt{\gamma_s}\right)\right]^2$
- For the same E_b/N_0 and therefore the same average probability of bit error, QPSK system transmits data at twice the bit rate of a BPSK system for the same channel bandwidth.
  - $P_s = 1 - \left[1 - Q\left(\sqrt{\gamma_s}\right)\right]^2 \approx 2Q\left(\sqrt{\gamma_s}\right) = 2Q\left(\sqrt{2\gamma_b}\right)$

$$P_s \approx 2Q\left(\sqrt{\gamma_s}\right)$$

Error Probability for MPSK

- Signal-space diagram for 8PSK



- When using the union bound of error probability and the nearest neighbor approximation,

$$P_s \approx M_{d_{\min}} Q\left(d_{\min} / \sqrt{2N_0}\right)$$

- For MPSK,

$$d_{\min} = 2\sqrt{E_s} \sin(\pi/M),$$
$$M_{d_{\min}} = 2$$

-

$$P_s = 2Q\left(\sqrt{2\gamma_s} \sin(\pi/M)\right)$$

Error Probability for MPAM

- The constellation for MPAM is $A_i = (2i - 1 - M)d, \quad i = 1,2,\ldots,M$
- Since each of the M-2 inner constellation points has two nearest neighbor at distance 2d, $P_s(s_i) = p(|\mathrm{n}| > d), \quad i = 2,\ldots,M-1$
- For outer constellation points, there is only one nearest neighbor.
- The average energy per symbol for MPAM is $\overline{E_s} = \frac{1}{M}\sum_{i=1}^{M} A_i^2 = \frac{1}{3}(M^2 - 1)d^2$
- The symbol error probability P_s in terms of the average energy as

$$P_s = \frac{1}{M}\sum_{i=1}^{M} P_s(s_i) = \frac{M-2}{M} \times 2 \times Q\left(2d/\sqrt{2N_0}\right) + \frac{2}{M} \times Q\left(2d/\sqrt{2N_0}\right)$$

$$P_s = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\overline{\gamma_s}}{M^2 - 1}}\right)$$

Error Probability for MQAM

- MQAM system can be viewed as two MPAM systems with signal constellations of size $L=\sqrt{M}$ transmitted over the in-phase and quadrature signal components, each with half the energy of the original MQAM system.
- The constellation points in the in-phase and quadrature branches take values

$$A_i = (2i-1-L)d, \quad i = 1,2,\ldots,L$$

- The symbol error probability for each branch is

$$P_{s,\text{branch}} = \frac{2(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$$

- The probability of symbol error the for MQAM system is $P_s = 1-(1-P_{s,\text{branch}})^2$
- If we take a conservative approach and set the number of nearest neighbors to be four,

$$P_s \approx 4\times Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$$

Summary in Error Probability for Coherent Modulation (1)

- Many of the exact or approximation values for P_s derived for coherent modulation are in the following form:

$$P_s(\gamma_s) \approx \alpha_M Q\left(\sqrt{\beta_M \gamma_s}\right)$$

  - α_M : the number of nearest neighbors at the minimum distance d_{min}
  - β_M : a constant that relates the minimum distance to average symbol energy
- Performance specifications are generally most concerned with the bit error rate as a function of the bit energy.

$$P_b(\gamma_b) \approx \hat{\alpha}_M Q\left(\sqrt{\hat{\beta}_M \gamma_b}\right)$$

  - With Gray coding and high SNR, $\hat{\alpha}_M = \alpha_M / \log_2 M$ and $\hat{\beta}_M = \beta_M \times \log_2 M$

Summary in Error Probability for Coherent Modulation (2)

Modulation	$P_s(\gamma_s)$	$P_b(\gamma_b)$
BPSK		$P_b = Q\left(\sqrt{2\gamma_b}\right)$
QPSK	$P_s \approx 2Q\left(\sqrt{\gamma_s}\right)$	$P_b \approx Q\left(\sqrt{2\gamma_b}\right)$
MPAM	$P_s = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\bar{\gamma}_s}{M^2-1}}\right)$	$P_b \approx \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\bar{\gamma}_b \log_2 M}{M^2-1}}\right)$
MPSK	$P_s \approx 2Q\left(\sqrt{2\gamma_s}\sin\left(\frac{\pi}{M}\right)\right)$	$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M}\sin\left(\frac{\pi}{M}\right)\right)$
MQAM	$P_s \approx 4Q\left(\sqrt{\frac{3\bar{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\bar{\gamma}_b \log_2 M}{M-1}}\right)$

Flat Fading Channel

- Outage Probability
- Average Probability of Error

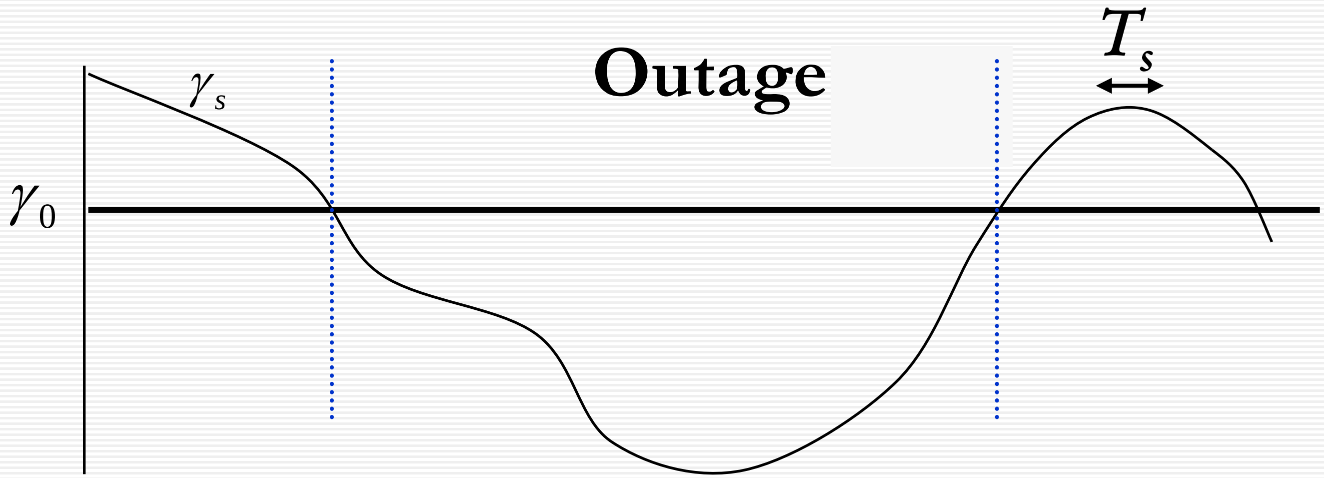
Performance Criteria (1)

- In a fading environment, the received signal power varies randomly over distance or time due to shadowing and/or multipath fading.
  - In fading γ_s is a random variable with distribution $f_{\gamma_s}(\gamma)$, and therefore $P_s(\gamma_s)$ is also random.
- Performance criteria
  - The outage probability, P_{out} , defined as the probability that γ_s falls below a given value corresponding to the maximum allowable P_s
  - The average error probability, $\overline{P_s}$, averaged over the distribution of γ_s

Performance Criteria (2)

- When the fading coherence time is on the order of a symbol time $(T_s \approx T_c)$
  - The signal fading level is roughly constant over a symbol period
  - The error correction coding techniques can recover from a few bit errors
  - An average error probability is a reasonably good figure
- When the signal fading is changing slowly $(T_s << T_c)$
  - A deep fade affects simultaneously many symbols
  - Large error bursts that cannot be corrected for with coding of reasonable complexity
  - Outage probability
  - When the channel is modeled as a combination of fast and slow fading (e.g., log-normal shadowing with fast Rayleigh fading), outage and average error probability is often combined
- When $T_s >> T_c$, the fading will be averaged out by the matched filter in the demodulator
  - For very fast fading, performance is the same as in AWGN

Outage Probability (1)



- Probability that γ_s is below a target γ_0, which is the minimum SNR required for acceptable performance.
  - $P_{out} = p(\gamma_s < \gamma_0) = \int_0^{\gamma_0} p_{\gamma_s}(\gamma)d\gamma$

Outage Probability (2)

- In Rayleigh fading with mean zero and variance σ^2 (dB)
  - The received signal power is exponentially distributed with average $2\sigma^2$
  - The received SNR γ_s also has an exponential distribution with average $\bar{\gamma}_s$
    - $\bar{\gamma}_s = \frac{\bar{E}_s}{N_0} = \frac{2\sigma^2 T_s}{N_0}$
    - The probability density function of γ_s : $p_{\gamma_s}(\gamma) = \frac{1}{\overline{\gamma_s}} e^{-\gamma/\overline{\gamma_s}}$
  - Outage probability

$$P_{out} = \int_0^{\gamma_0} \frac{1}{\overline{\gamma_s}} e^{-\gamma/\overline{\gamma_s}} d\gamma_s = 1 - e^{-\gamma_0/\overline{\gamma_s}}$$

  - Average SNR
    - $$\overline{\gamma_s} = \frac{-\gamma_0}{\ln(1 - P_{out})}$$

Average Error Probability

- The averaged probability of error is computed by integrating the error probability in AWGN over the fading distributions.

$$\overline{P_s} = \int_0^\infty P_s(\gamma) p_{\gamma_s}(\gamma) d\gamma$$

  - An error probability in AWGN with SNR γ : $P_s(\gamma) \approx \alpha_M Q\left(\sqrt{\beta_M \gamma_s}\right)$

- In Rayleigh fading,

$$\overline{P_s} = \int_0^\infty \alpha_M Q\left(\sqrt{\beta_M \gamma}\right) \cdot \frac{1}{\overline{\gamma_s}} e^{-\gamma/\overline{\gamma_s}} d\gamma = \frac{\alpha_M}{2}\left(1 - \sqrt{\frac{0.5\beta_M \overline{\gamma_s}}{1 + 0.5\beta_M \overline{\gamma_s}}}\right) \approx \frac{\alpha_M}{2\beta_M \overline{\gamma_s}}$$

  - BPSK: $P_b(\gamma_b) = Q(\sqrt{2\gamma_b})$

$$\overline{P_b} = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma_b}}{1 + \overline{\gamma_b}}}\right)$$

Average P_b for MQAM in Rayleigh Fading and AWGN