# System Control

# 3. Transfer Function and State Equation

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**Seoul National University**

Mathematical models

Simplicity versus accuracy

Linear systems – principle of superposition

Linear time invariant systems and time-varying systems

# Transfer Functions

- **Transfer Functions (Linear Time Invariant Systems)**

-The ratio of the Laplace Transform of the output (response function) to the Laplace Transform of the input (driving function) under the assumption that all initial conditions are zero.

$$\frac{Y(s)}{U(s)} = G(s)$$

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

- **Differential Equation**

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = b_0 x^m + b_1 x^{m-1} + \cdots + b_{m-1} x' + b_m x$$

$$\text{T. F.} \quad G(s) = \frac{\mathrm{Y(s)}}{\mathrm{X(s)}} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \qquad n \geq m$$

# Transfer Functions

$$\frac{Y(s)}{U(s)} = G(s)$$

$$U(s) \longrightarrow \boxed{\;G(s)\;} \longrightarrow Y(s)$$

- **T.F.**

1. A mathematical model

2. A property of a system itself independent of the magnitude and nature of the input

3. T. F. includes the units( input-output relations)

4. however, does not provide any information concerning the physical structure of the system, many different systems can have identical T. F.

- **Convolution integral**

$$\frac{Y(s)}{U(s)} = G(s)$$

$$Y(s) = G(s)U(s)$$

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$
$$= \int_0^t g(t-\tau)u(\tau)d\tau$$

- **Impulse response**

$$\frac{Y(s)}{U(s)} = G(s)$$

$$Y(s) = G(s)$$

$$y(t) = L^{-1}\left[G(s)\right] = g(t)$$

- **Transfer function and block diagram**

$$\frac{Y(s)}{U(s)} = G(s)$$

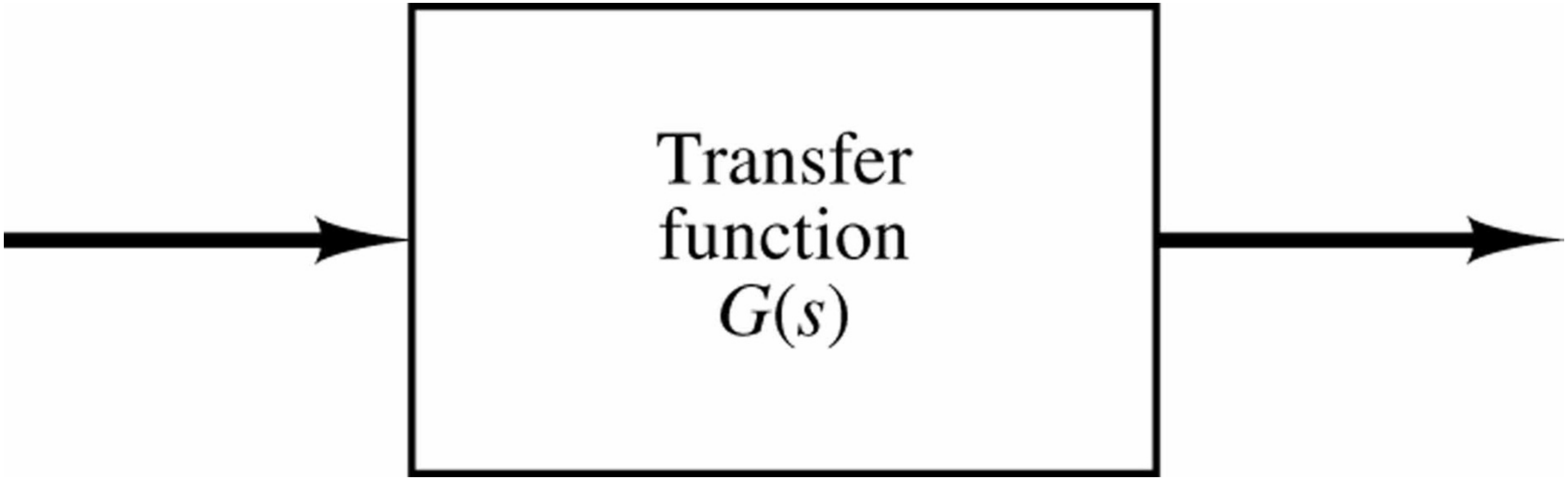

**Figure 2-2** Summing point.

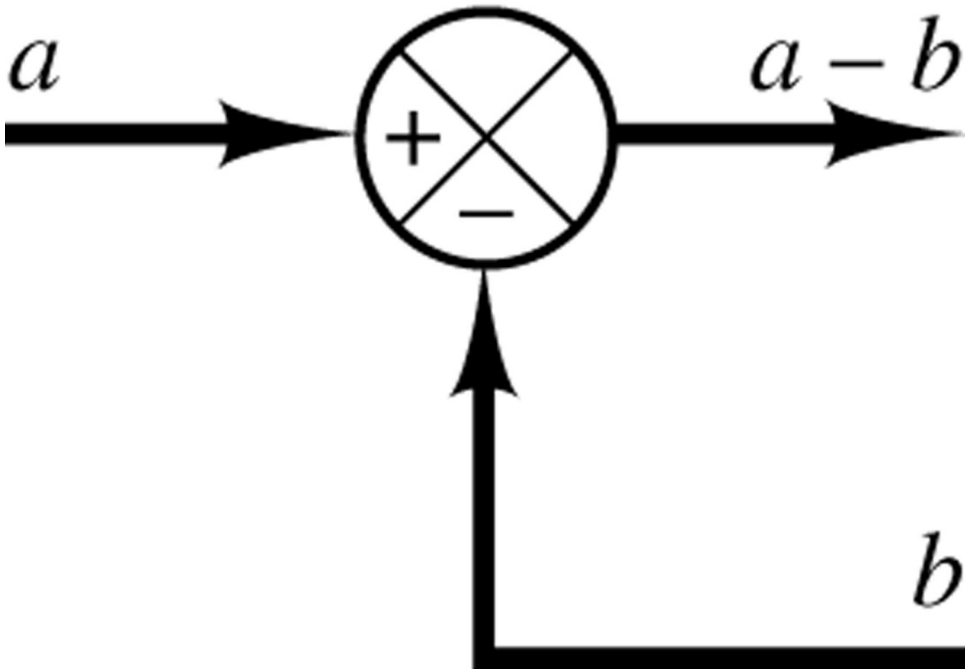




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**Figure 2-3** Block diagram of a closed-loop system.

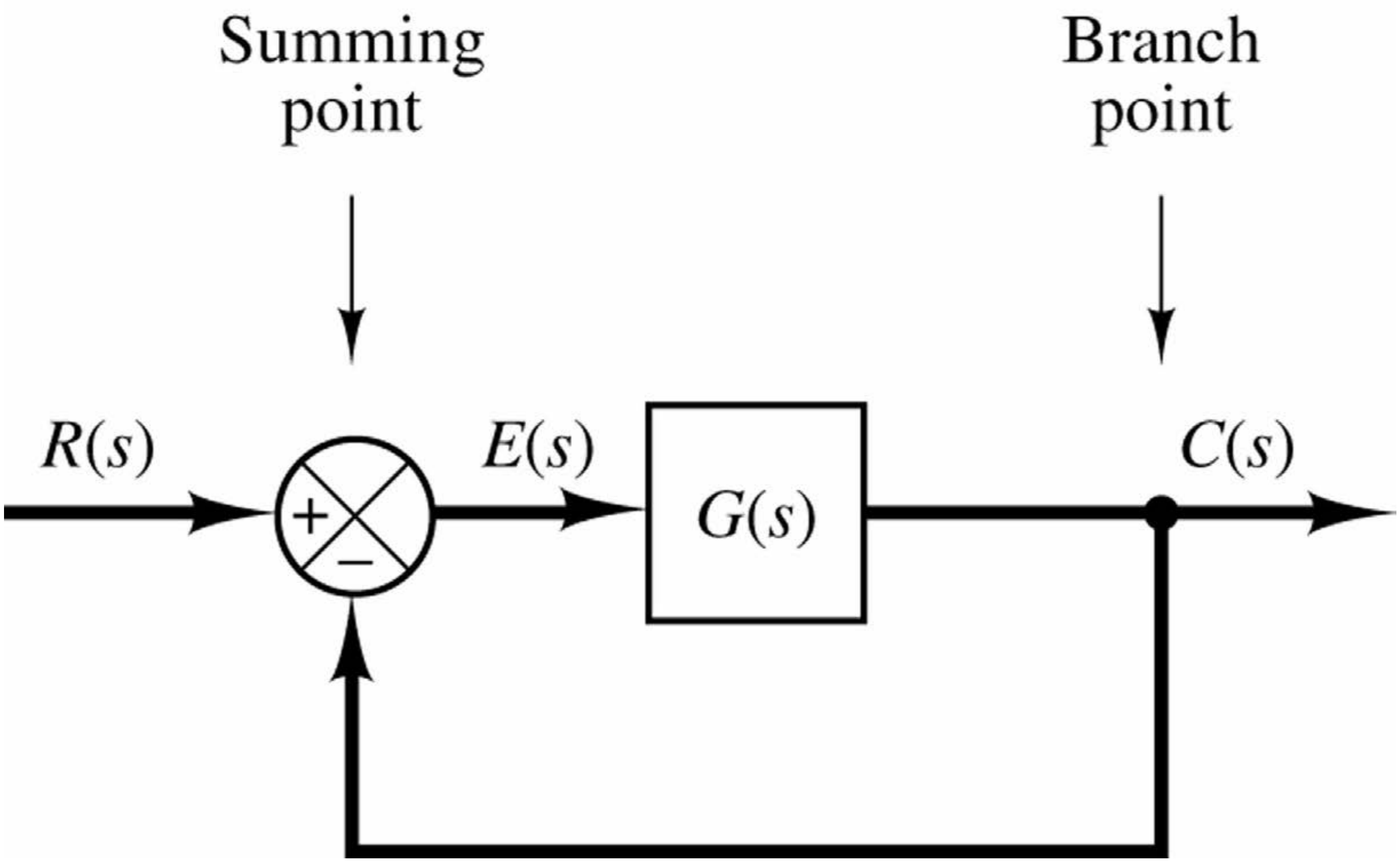




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**Figure 2-4** Closed-loop system.

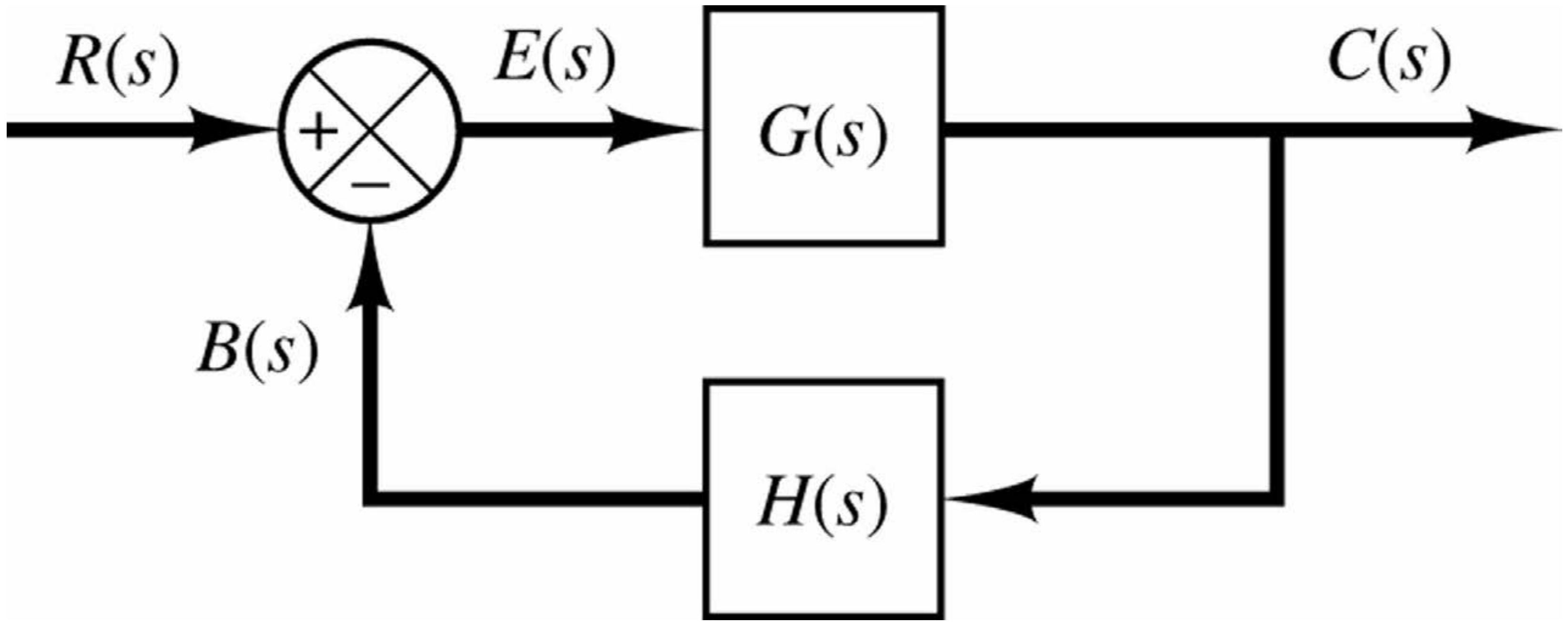




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**Figure 2-5** (a) Cascaded system; (b) parallel system; (c) feedback (closed-loop) system.

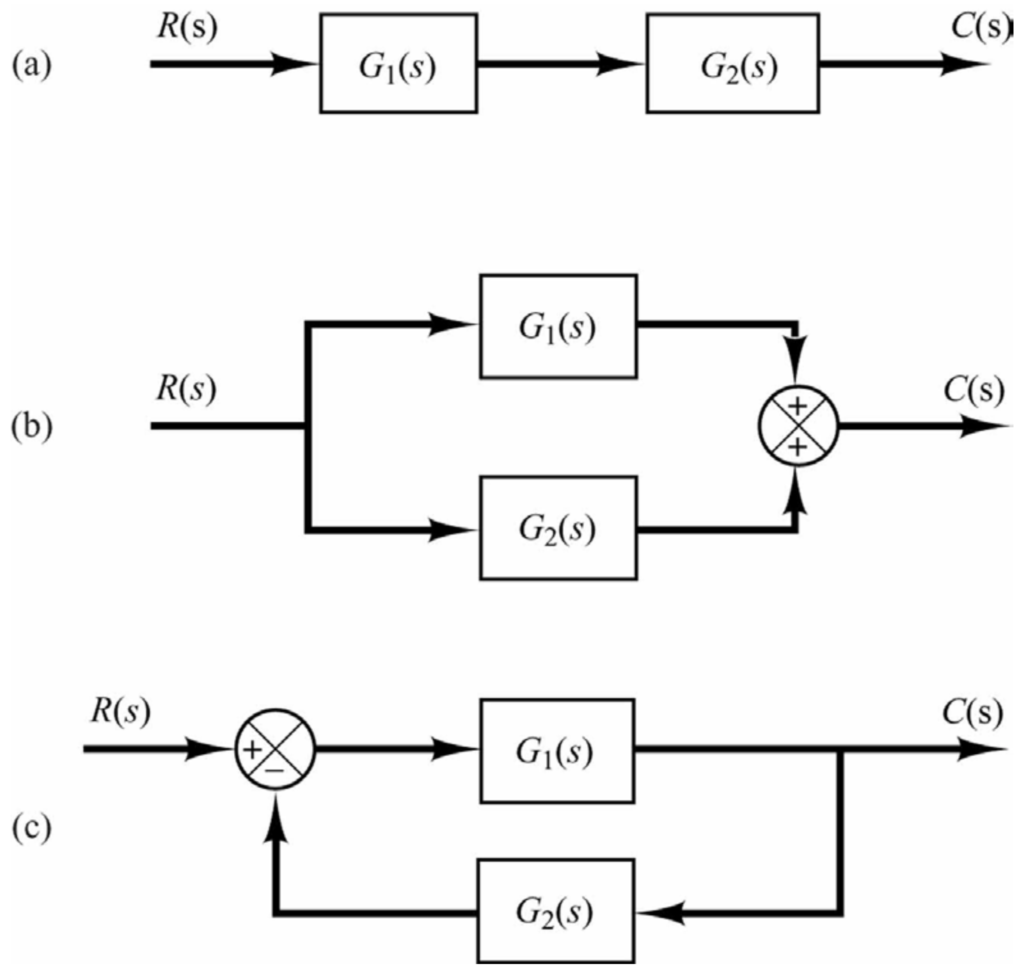




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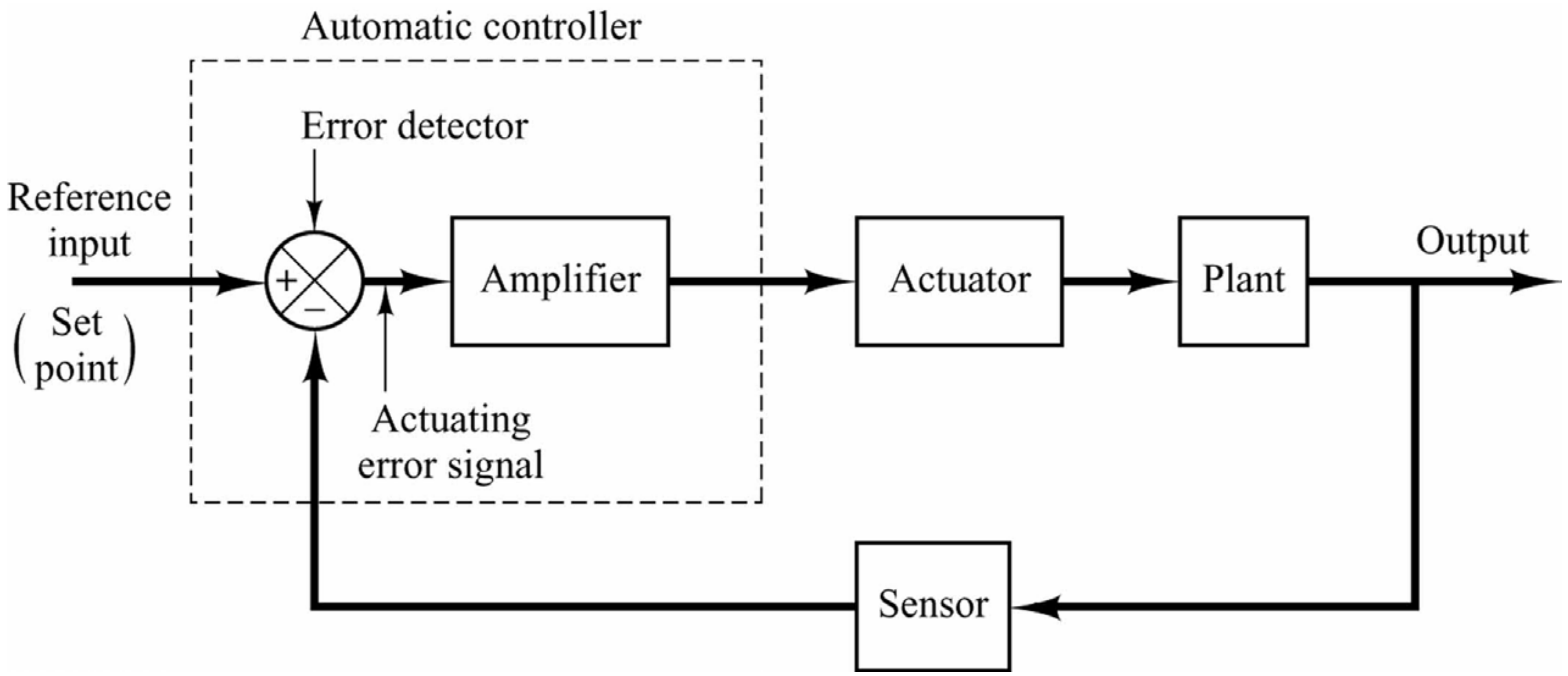
**Figure 2-6** Block diagram of an industrial control system, which consists of an automatic controller, an actuator, a plant, and a sensor (measuring element).



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Controller Block

ON-Off Control



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**Figure 2-8** (a) Liquid-level control system; (b) electromagnetic valve.

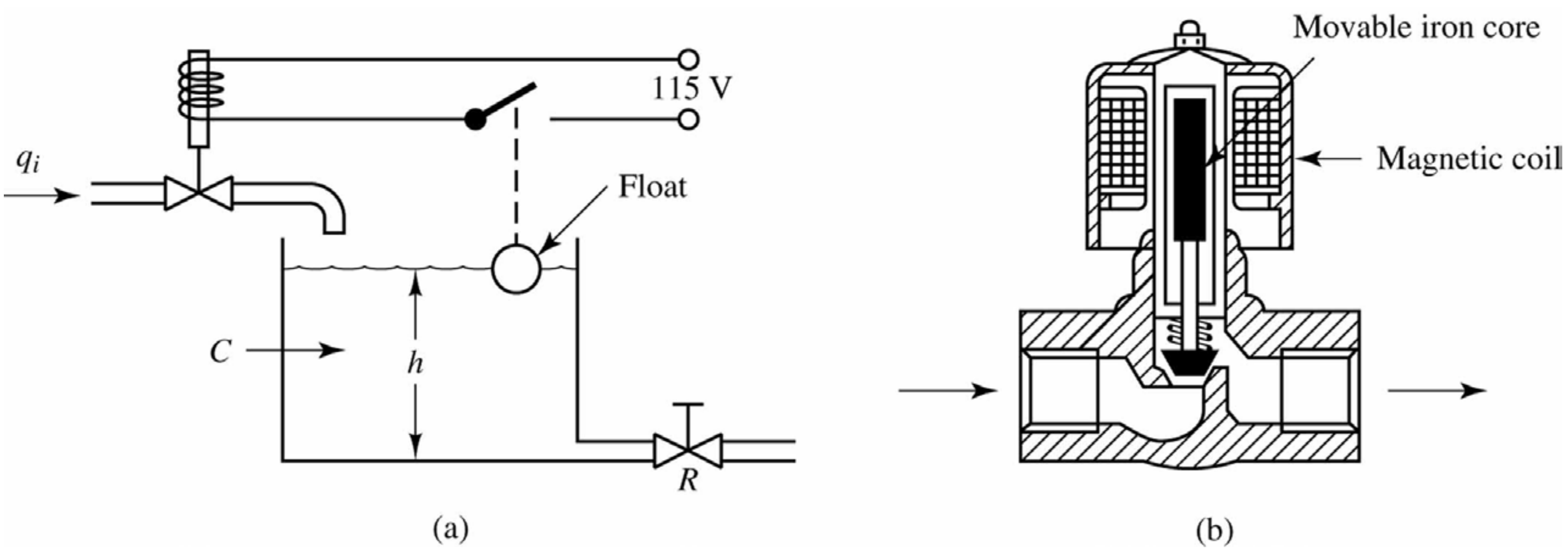




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**Figure 2-7** (a) Block diagram of an on–off controller; (b) block diagram of an on–off controller with differential gap.

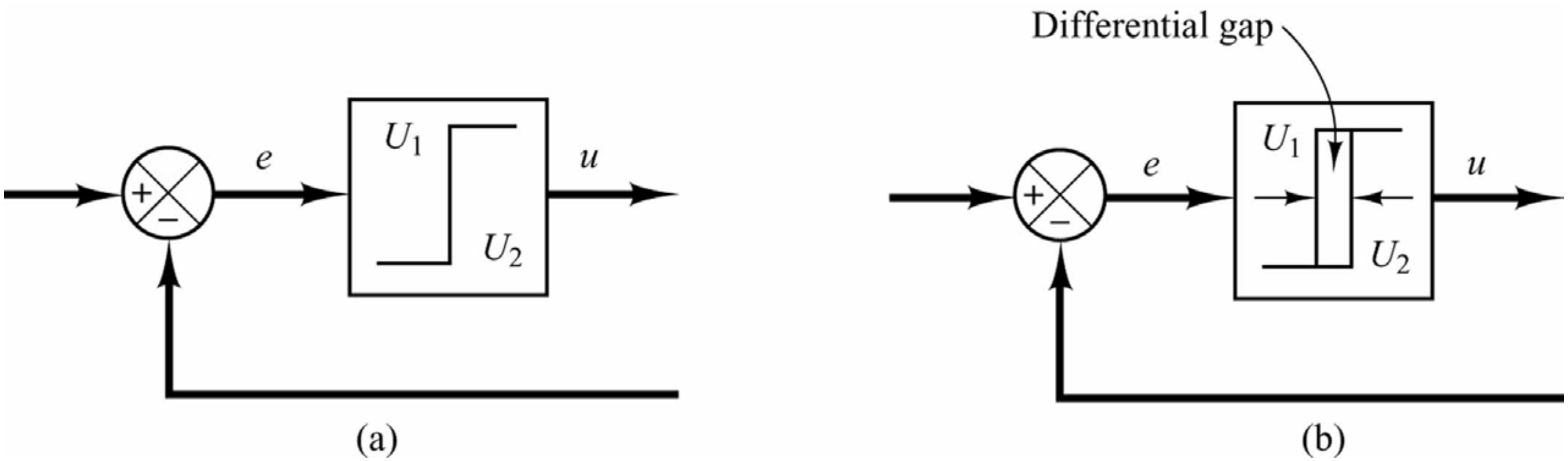




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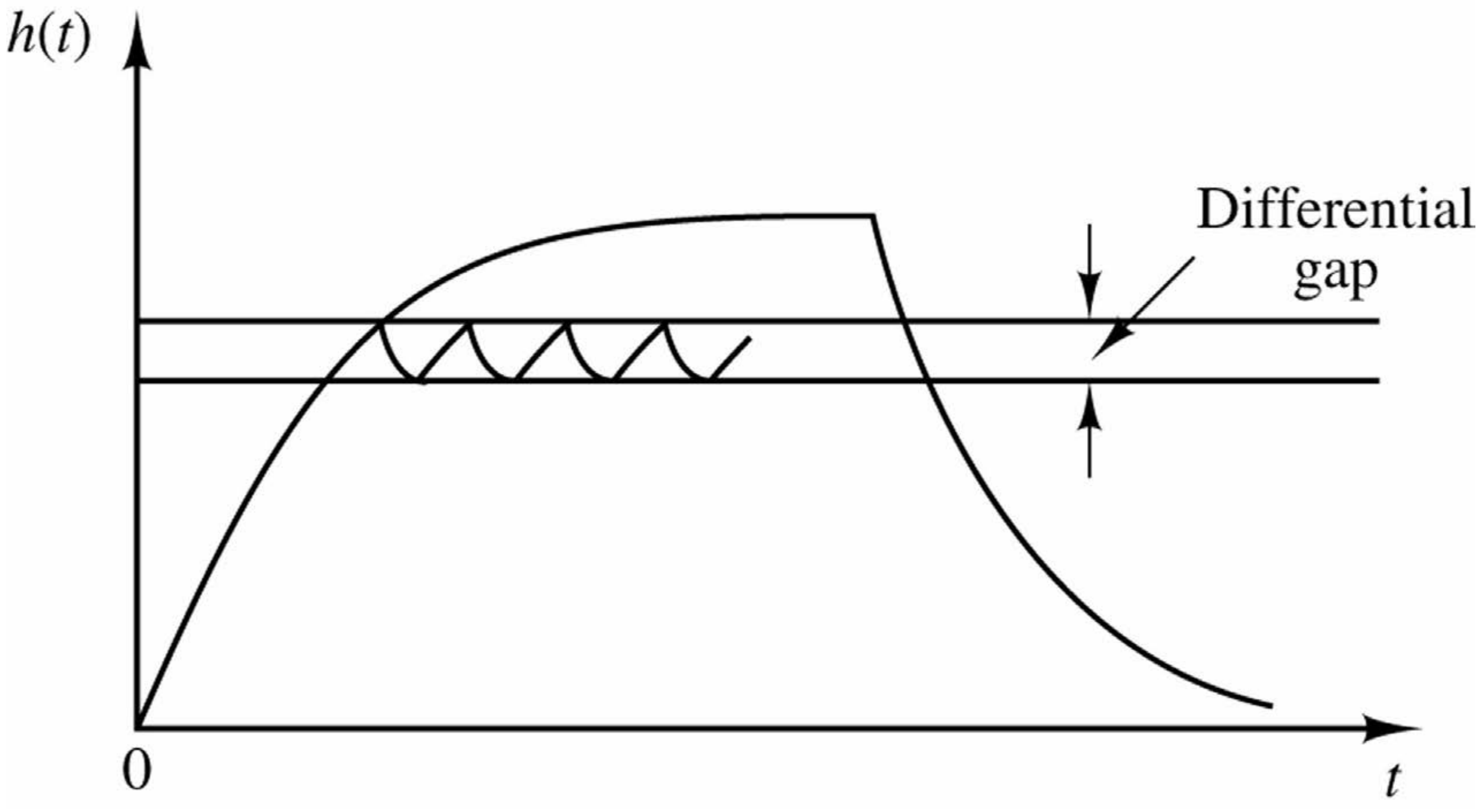
**Figure 2-9** Level $h(t)$-versus-$t$ curve for the system shown in Figure 2–8(a).



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# PID Control



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**Figure 2-10** Block diagram of a proportional-plus-integral-plus-derivative controller.

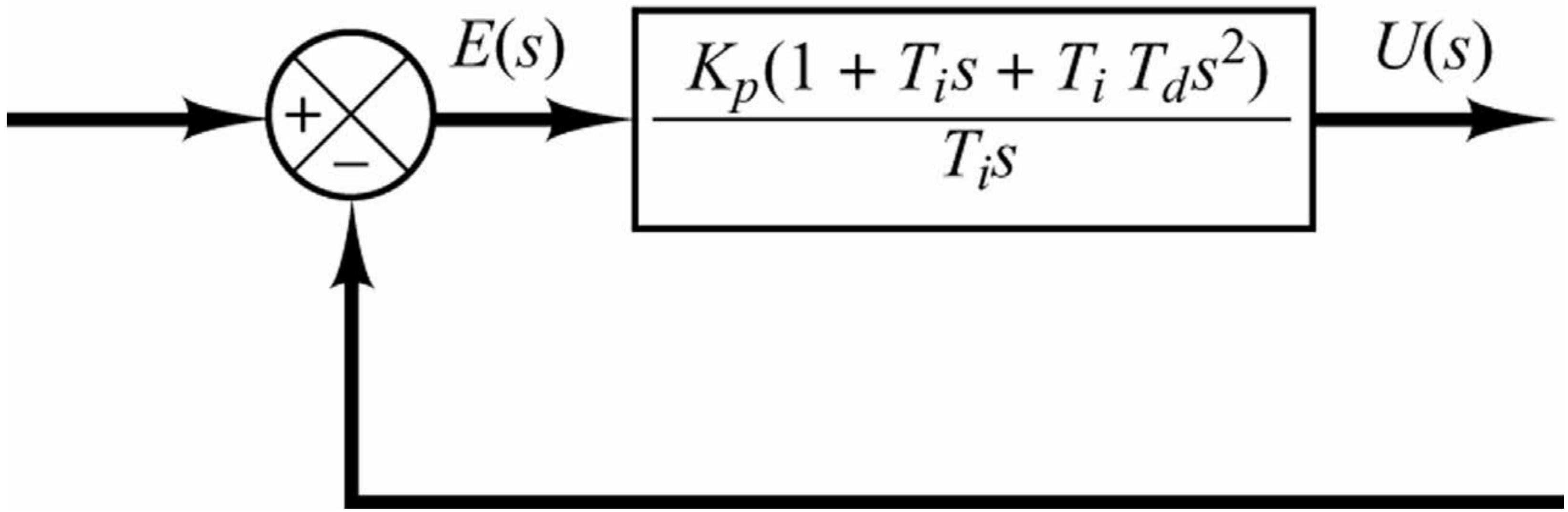




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**Figure 2-11** Closed-loop system subjected to a disturbance.

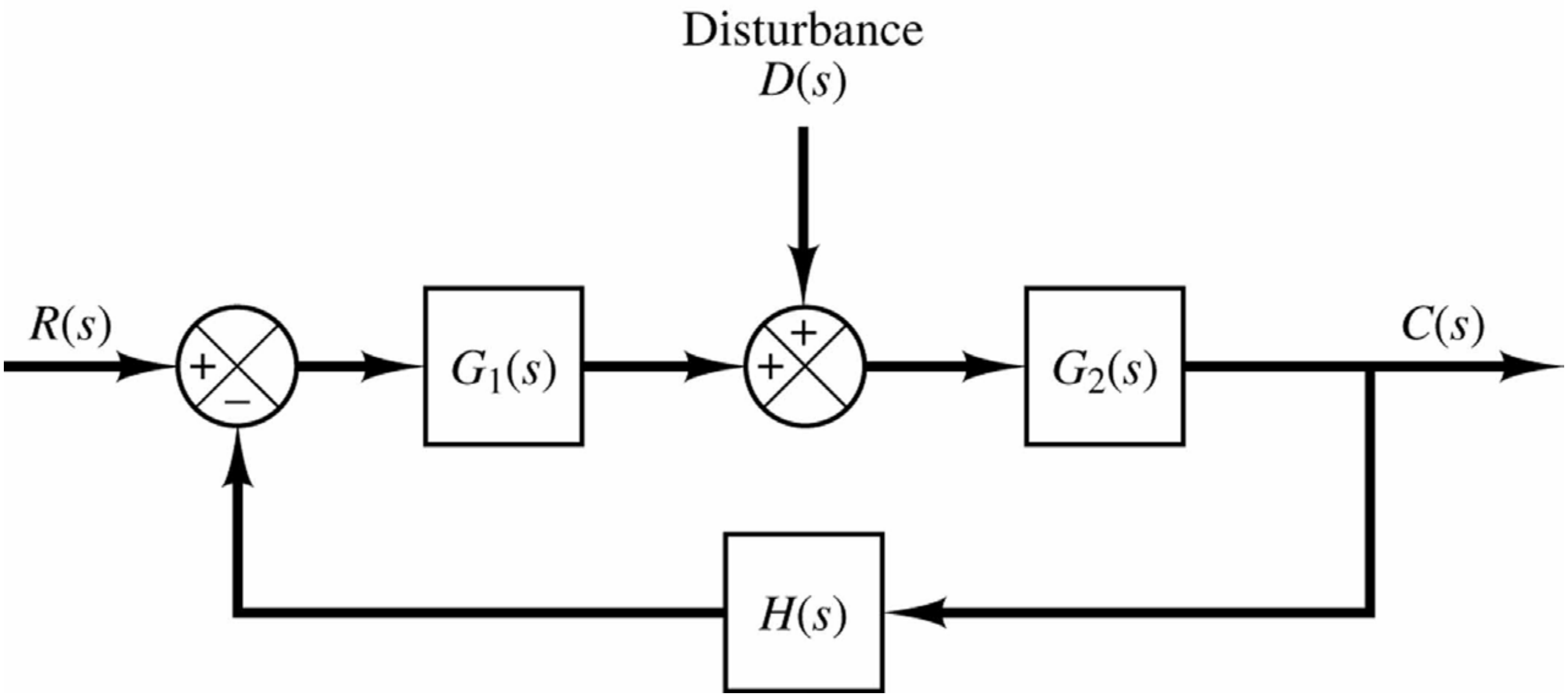




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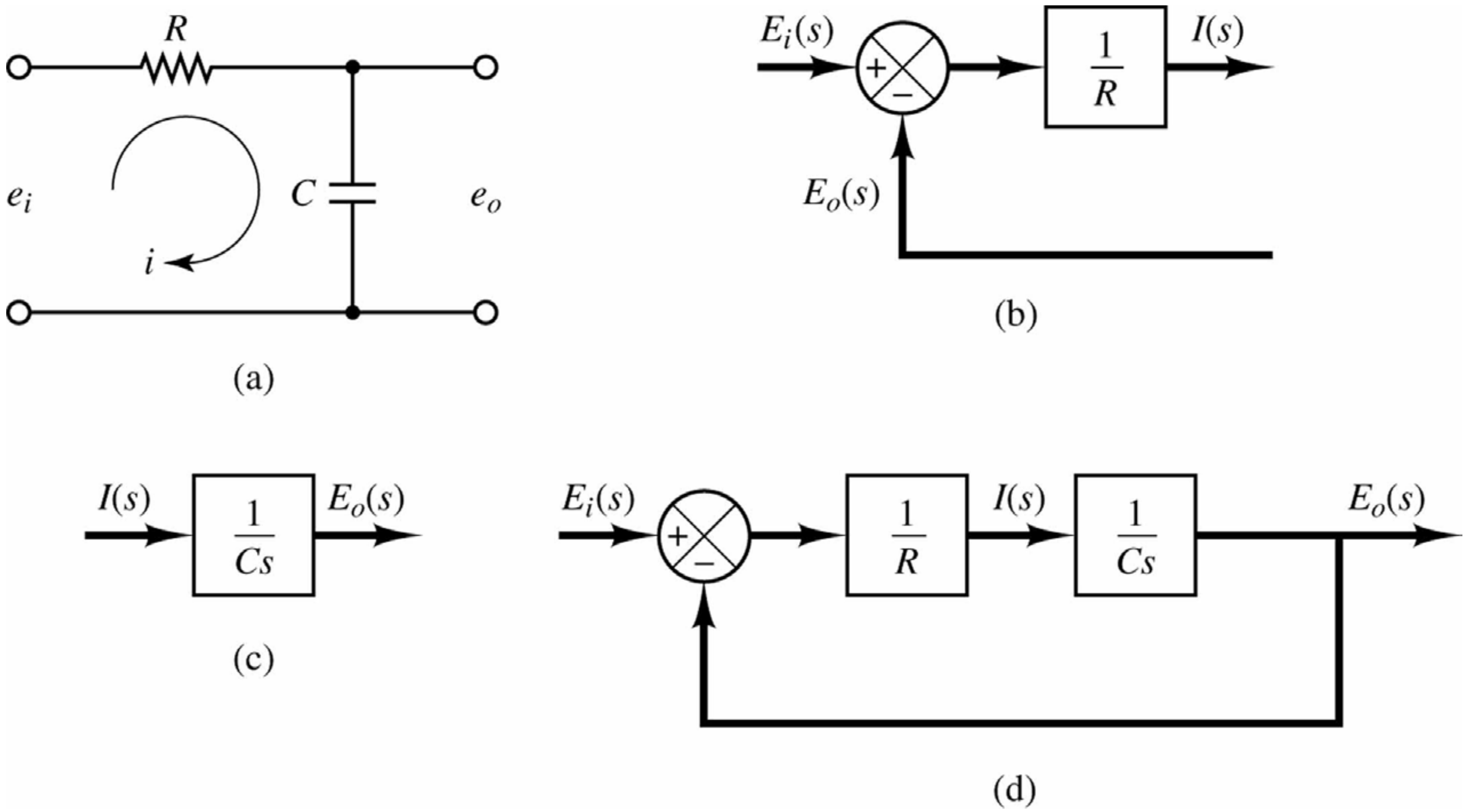
**Figure 2-12** (a) *RC* circuit; (b) block diagram representing Equation (2–6); (c) block diagram representing Equation (2–7); (d) block diagram of the *RC* circuit.



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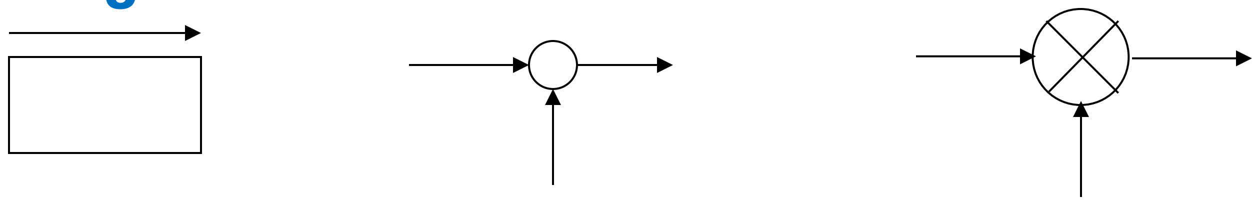
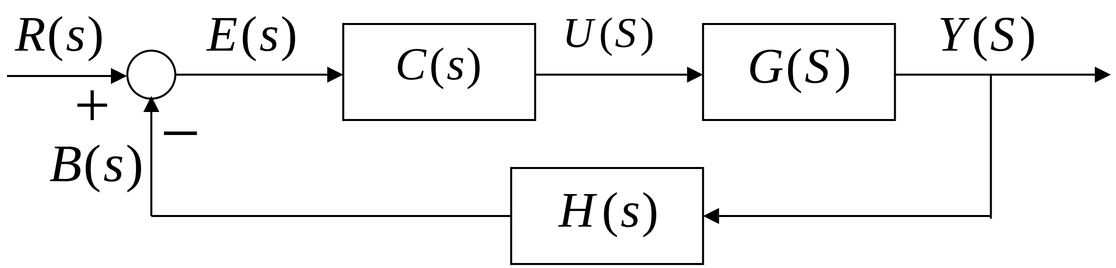
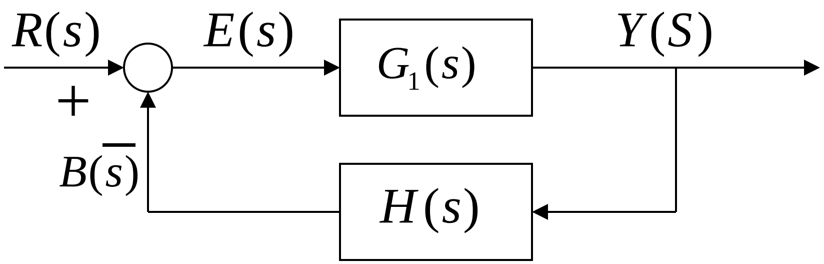
- **Block Diagram**



A block diagram (of a system) : a pictorial representation of the function performed by each component and of the flow of signals.




**or**




$$G_1(S) = C(s) \cdot G(s)$$

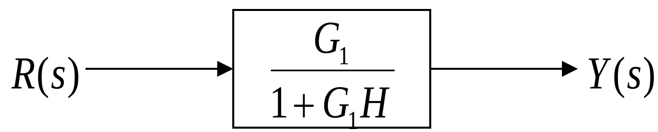
# Transfer Functions

- **Open loop T.F.** $= \dfrac{B(s)}{E(s)} = \dfrac{H(s)\cdot Y(S)}{E(s)} = \dfrac{H(s)G_1(s)E(s)}{E(S)}$

  $= H(s)\cdot G_1(s)$

- **Feed forward T.F.** $= \dfrac{Y(s)}{E(s)} = G_1(s)$

- **Closed loop T.F.** $= \dfrac{Y(s)}{R(s)}$

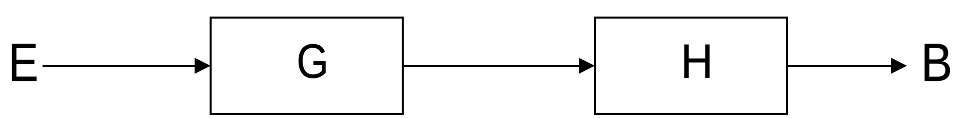



$$Y(s) = G_1(s)E(s) = G_1(s)[R(s) - B(s)]$$

$$= G_1(s)[R(s) - H(s)\cdot Y(s)]$$

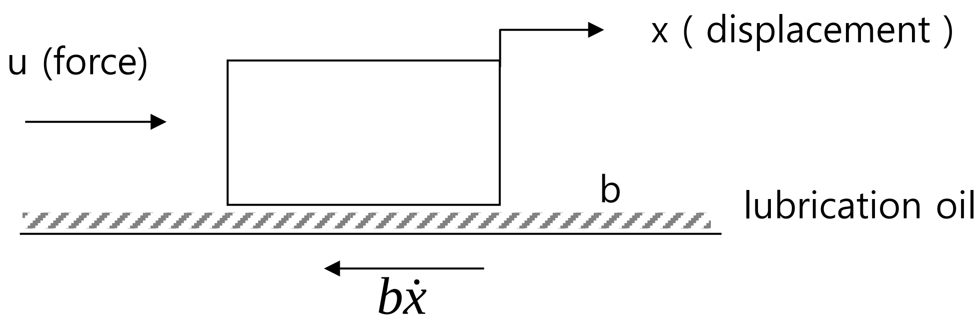
$$[1 + G_1(s)H(s)]Y(s) = G_1(s)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)H(s)}$$




$$\frac{G_{\text{feed forward}}}{1 + G_{\text{open}}(s)}$$

## Ex)



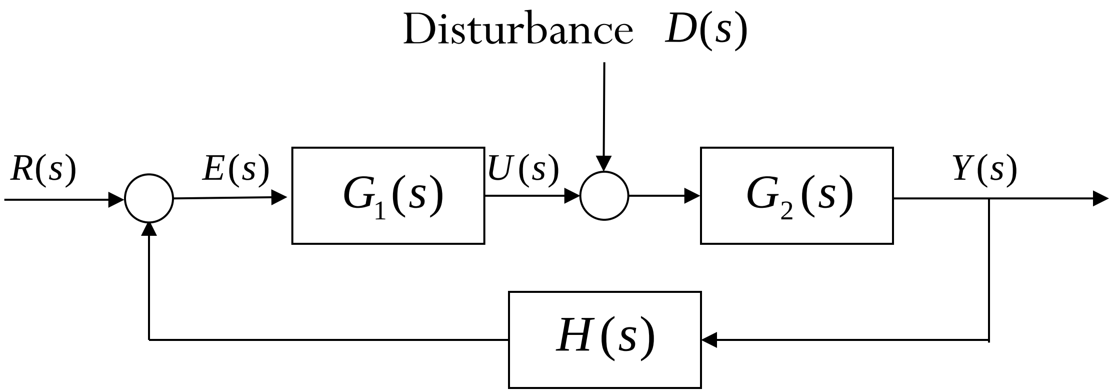

$$m\ddot{x} = u - b\dot{x} \qquad\qquad m\ddot{x} + b\dot{x} = u$$

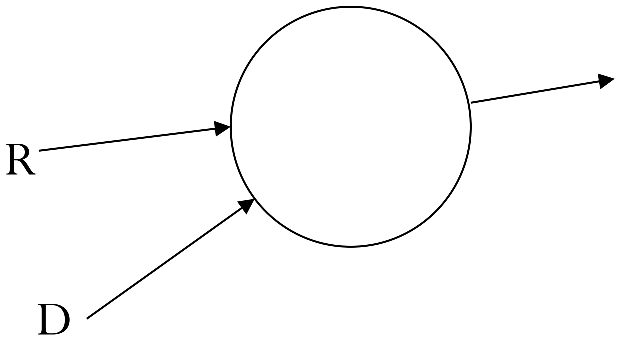
$$x(0) = 0$$
$$\dot{x}(0) = 0$$

$$ms^2 X(s) + bsX(s) = U(s)$$
$$(ms^2 + bs)X(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{s(ms+b)}$$ : Transfer Function

# Closed-loop system subjected to a disturbance




$Y = Y_R + Y_D$ : linear system

$$
\begin{aligned}
Y_D(s) &= G_2(s)[D(s) + U(s) \\
&= G_2(s)D(s) + G_2(s)G_1(s)E(s) \\
&= G_2(s)D(s) + G_2(s)G_1(s)[-H(s)Y_D(s)]
\end{aligned}
$$

# Closed-loop system subjected to a disturbance

$$\text{R=0 :} \quad \frac{Y_D(s)}{D(s)} = \frac{G_2}{1+G_1G_2H} = G_D(s)$$

$$\text{D=0 :} \quad \frac{Y_R(s)}{R(s)} = \frac{G_1G_2}{1+G_1G_2H} = G_R(s)$$

$$Y(s) = \frac{G_1G_2}{1+G_1G_2H}R + \frac{G_2}{1+G_1G_2H}D$$

$$= \frac{G_2}{1+G_1G_2H}[G_1R + D]$$

- $G_1G_2H >> 1$

$$G_D(s) \cong \frac{G_2}{G_1G_2H} = \frac{1}{G_1H}$$

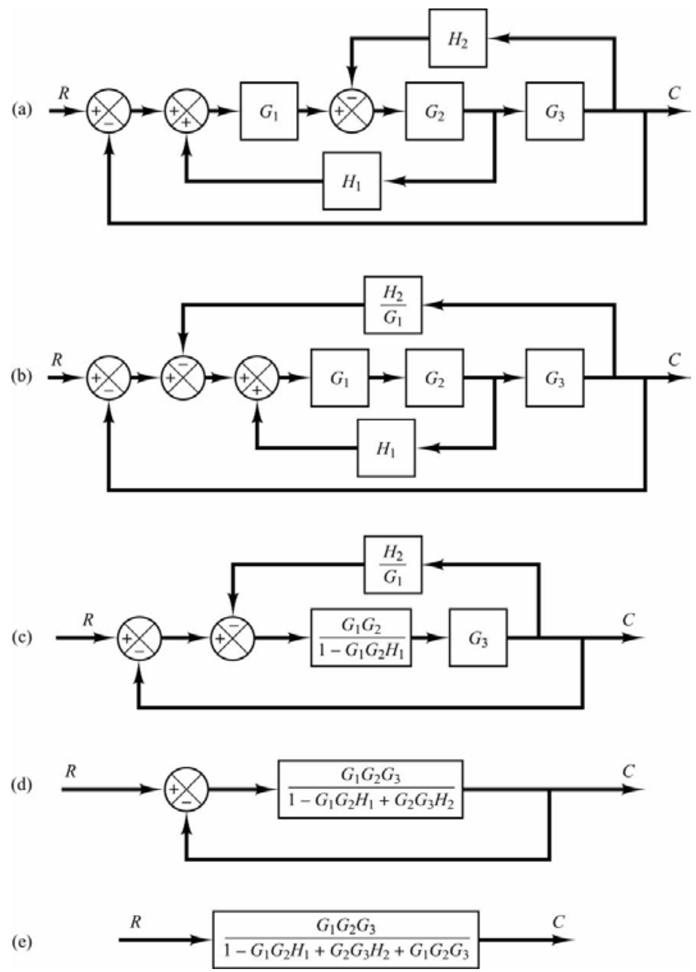
$$G_1H >> 1 \qquad G_D = \varepsilon << 1$$

The effect of the disturbance is reduced → Advantage of the closed-loop system

$$G_R(s) \approx \frac{G_1G_2}{G_1G_2H} = \frac{1}{H}$$

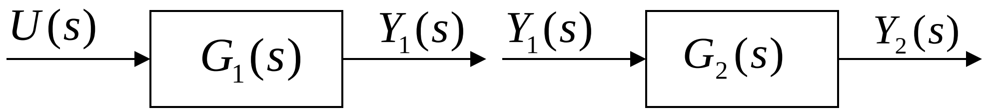
# Block diagram reduction

**Figure 2-13** (a) Multiple-loop system; (b)–(e) successive reductions of the block diagram shown in (a).



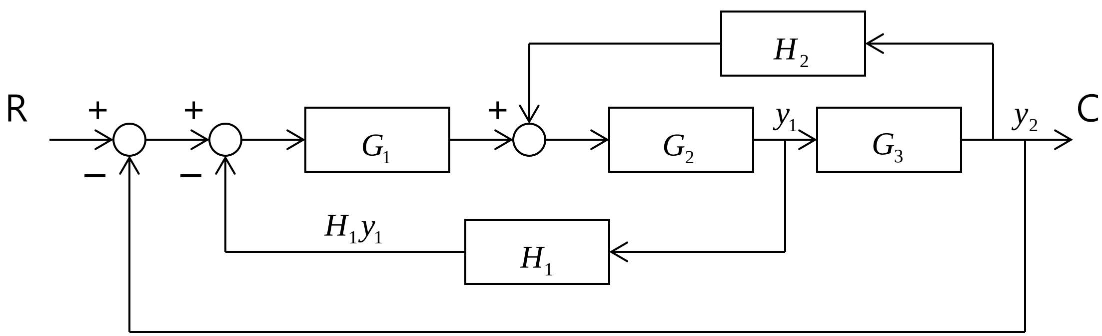
# Block Diagram Reduction

ex1)




$$Y_2(s) = G_2(s)Y_1(s) = G_2(s)G_1(s)U(s)$$

ex2)

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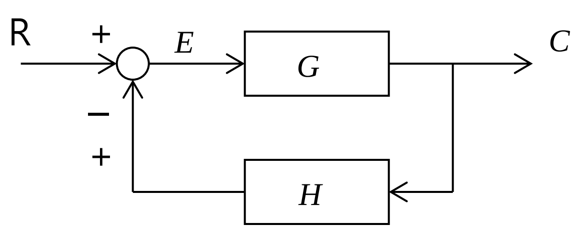
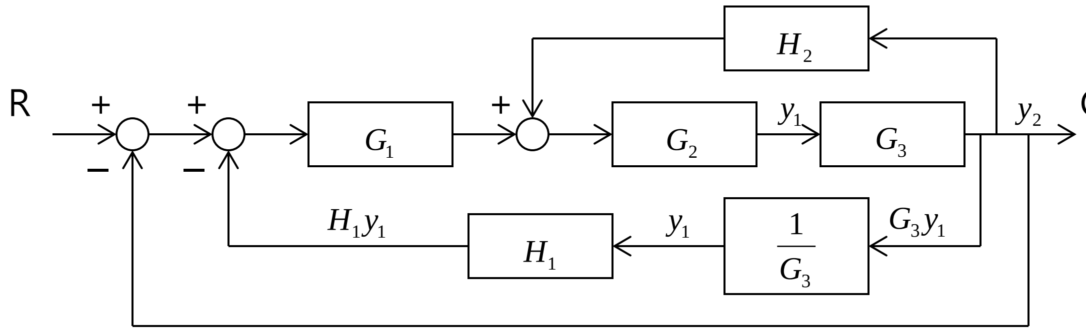
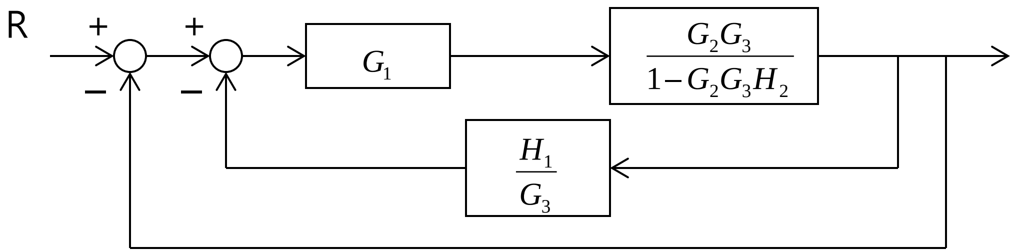
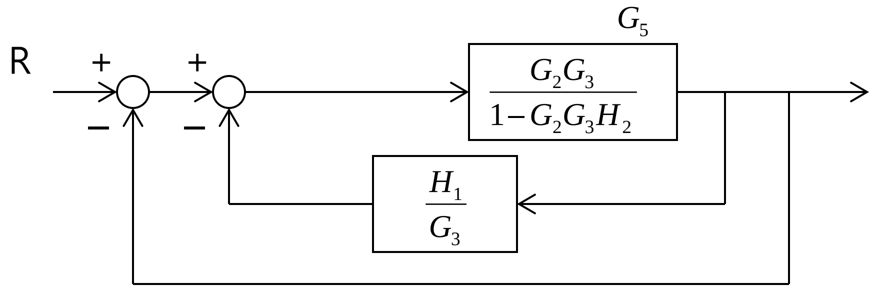



$$\frac{C}{R} = \frac{G}{1 \pm GH}$$

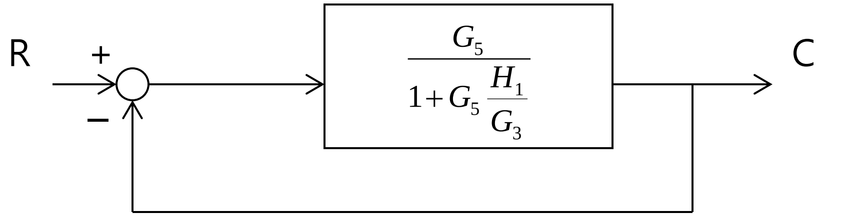
# Block Diagram Reduction



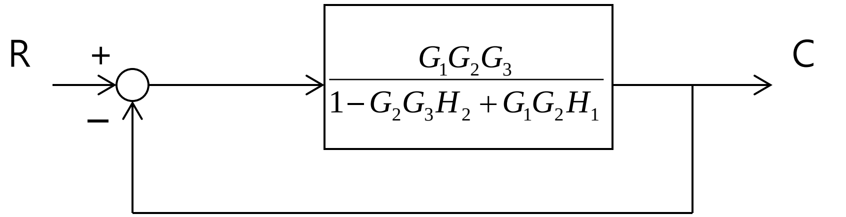



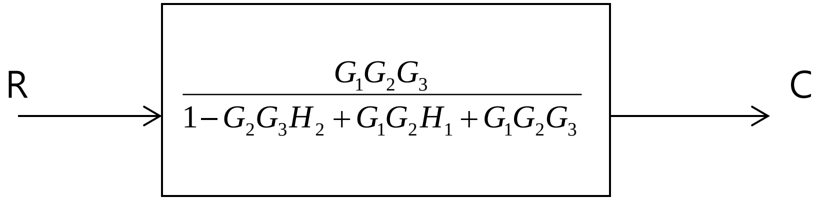
# Block Diagram Reduction




$$\frac{\dfrac{G_1G_2G_3}{1-G_2G_3H_2}}{1+\dfrac{G_1G_2G_3}{1-G_2G_3H_2}\dfrac{H_1}{G_3}}=\frac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1}$$




$$\frac{\dfrac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1}}{1+\dfrac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1}}=\frac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1+G_1G_2G_3}$$



# Mason's Gain Formula

The overall gain

$$P = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

$P_k$ = path gain of k-th forward path

$\Delta$ = determinant

$$= 1 - \sum_a L_a - \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + \cdots$$

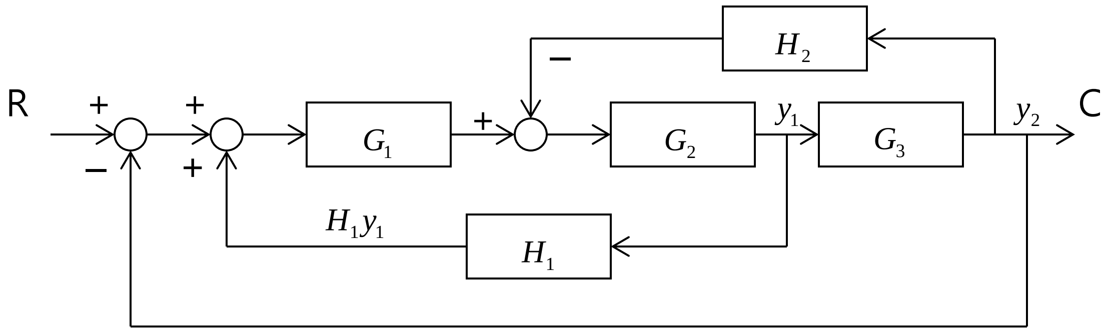
$\sum_a L_a$ = sum of all individual loop gains

$\sum_{b,c} L_b L_c$ = sum of gain products of all possible combination of two non touching loops

$\sum_{d,e,f} L_d L_e L_f$ = sum of gain products of all possible combination of three non touching loops

$\Delta_k$ = cofactors of the k-th forward path determinant of the graph with the loops touching the k-th forward path removed, that is, the cofactor $\Delta_k$ is obtained from $\Delta$ by removing the loops that touch path $P_k$

# Mason's Gain Formula




① **One Forward path :** $P_1 = G_1 G_2 G_3$

② **Three Individual Loops :** $L_1 = G_1 G_2 H_1$

$$L_2 = -G_2 G_3 H_2$$

$$L_3 = -G_1 G_2 G_3$$

③ **No Non-touching Loops :** $\Delta = 1 - (L_1 + L_2 + L_3)$

$$= 1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3$$

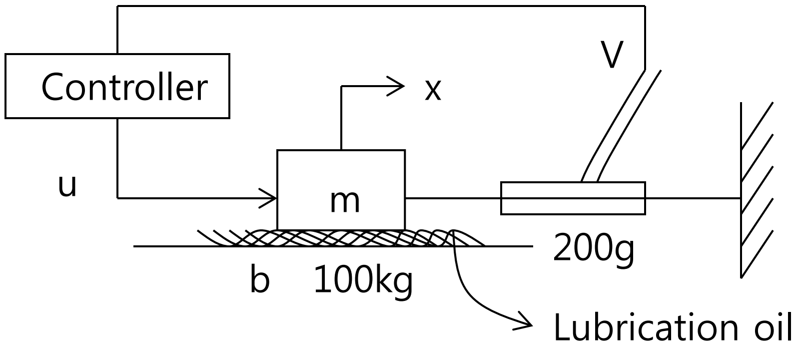
④ $\Delta_1$ **:** $P_1$ **touches all loops**

$$\Delta_1 = 1$$

⑤
$$\frac{C(s)}{R(s)} = \frac{1}{\Delta}(P_1 \Delta_1)$$

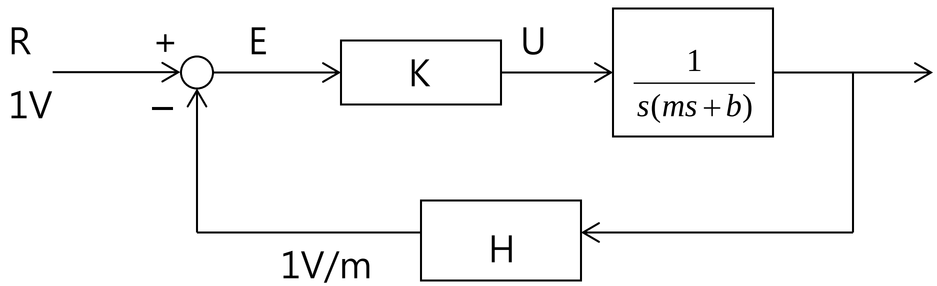
$$= \frac{G_1 G_2 G_3}{1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

# State Equation




$$m\ddot{x} = u - b\dot{x}$$

$$(ms^2 + bs)X(s) = U(s)$$



# State Equation

- $m\ddot{x} + b\dot{x} = u$

$$x_1 = x \qquad \dot{x}_1 = \dot{x} = x_2$$

$$x_2 = \dot{x} \qquad \dot{x}_2 = \ddot{x} = -\frac{b}{m}\dot{x} + \frac{u}{m} = -\frac{b}{m}x_2 + \frac{1}{m}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

First order matrix differential Eq.
➔ State Equation

# State Equation

## • State $x$

The smallest set of variables such that knowledge of these variables at $t = t_0$ , together with the knowledge of the input for $t \geq t_0$ , completely determines the behavior of the system at any time $t \geq t_0$

## • State Variables

The variables making up the smallest set of variables that determine the state of the dynamic system

ex) $x_1$ : displacement $x_2$ : velocity

## • State Vector

$x_1$ , $x_2$ , $x_3$ ... state variables

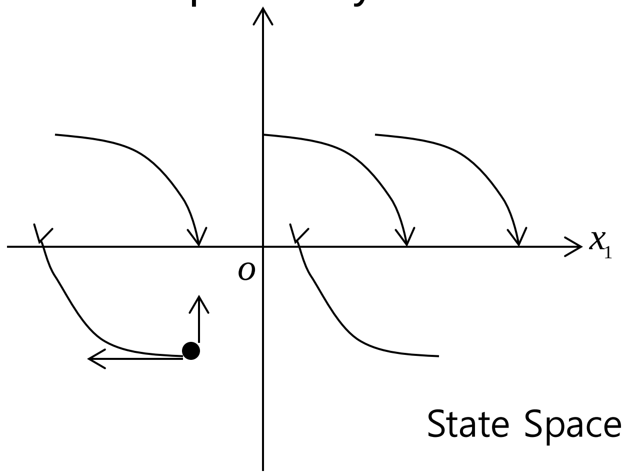
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

A vector that determines uniquely the system state x(t) for any time once the state at $t = t_0$ is given and the input u(t) for $t \geq t_0$ is specified

# State Equation

## • State Space

The n-dimensional space, whose coordinate axes consist of the $x_1$ axis, $x_2$ axis, ..., $x_n$ axis is called a state space. Any state can be represented by a point in the state space




$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\ \dot{x}_2 = f_2(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\ \vdots \\ \dot{x}_n = f_n(x_n, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \end{cases}$$

$$\begin{cases} y_1 = g_1(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\ \vdots \\ y_m = g_m(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \end{cases}$$

$$\dot{x} = f(x, u, t)$$
$$y = g(x, u, t)$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$$

# Linear Systems

- **Linear Systems**

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- **Linearization (of nonlinear system)**

$$\dot{x} = f(x,u,t) = f(x,u)$$

$$x = x_0, u = u_0, \dot{x} = \dot{x}_0 = f_0 = 0$$

$$x = x_0 + \Delta x, u = u_0 + \Delta u$$

$$\dot{x} = f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0 u_0} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0 u_0} \Delta u + \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0 u_0} \Delta x^2 + \left.\frac{\partial^2 f}{\partial u^2}\right|_{x_0 u_0} \Delta u^2 + \cdots$$

$$\cong f_0 + K_1 \Delta x + K_2 \Delta u$$

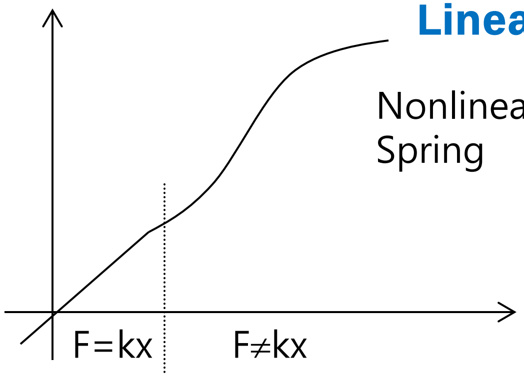
$$\dot{x} - \dot{x}_0 = K_1 \Delta x + K_2 \Delta u \qquad \Delta x, \Delta u : small$$

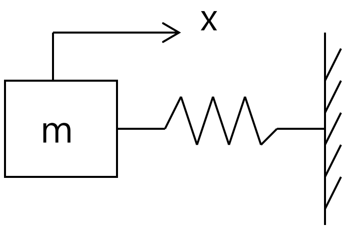
$$\Delta x = K_1 \Delta x + K_2 \Delta u \qquad Approximation$$

# Linear Systems

- **Nonlinear Systems** $\longrightarrow$ **Linear systems**

**Linearization**




- **State : mathematical concept, not physical meaning**

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\hat{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x + \dot{x} \\ \dot{x} \end{bmatrix} = Tx$$

$$x = T^{-1}\hat{x}$$

$$T^{-1}\dot{\hat{x}} = AT^{-1}\hat{x} + BU$$

$$\begin{cases} \dot{\hat{x}} = TAT^{-1}\hat{x} + TBU \\ y = CT^{-1}\hat{x} \end{cases}$$

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}U \\ y = \hat{C}\hat{x} \end{cases}$$

# Linear Systems

- **Model :**

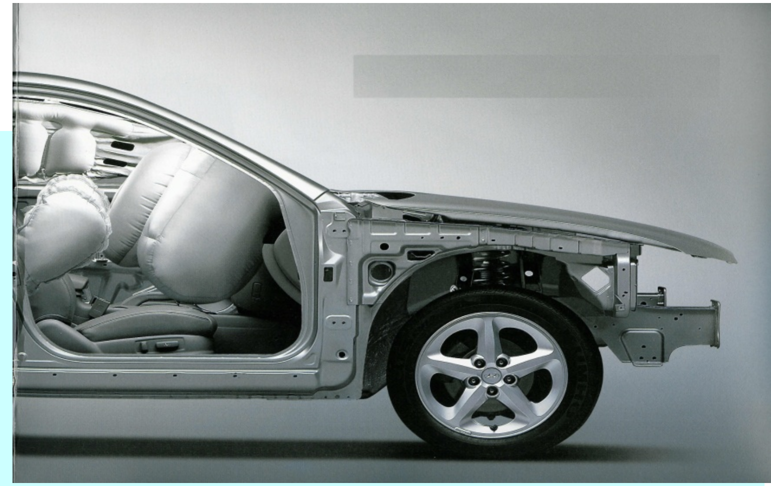
  Differential eq.
  Transfer Functions
  State eq.

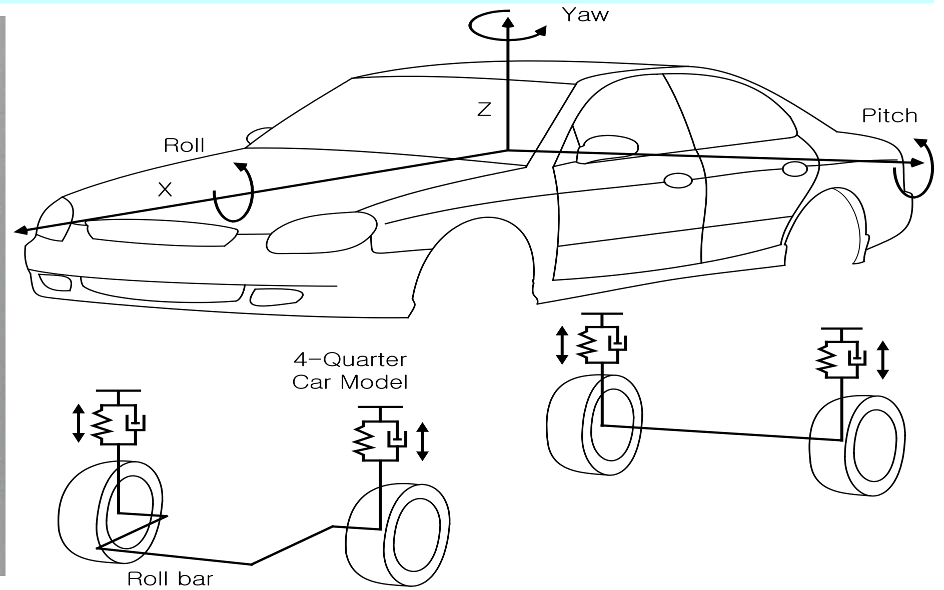
- **State Eq.** $\xrightarrow{\text{Unique}}$ **T.F.**; **State Eq.** $\xleftarrow[\text{Not unique}]{}$ **T.F.**

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$sX(s) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1} BU(s)$$
$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= \left[C(sI - A)^{-1} B + D\right] U(s) \\ &= G(s)U(S) \end{aligned}$$

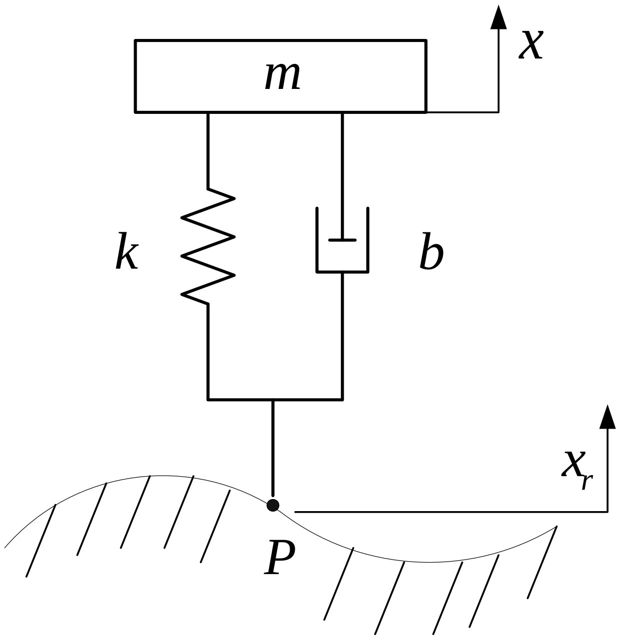
# Vehicle Suspension





# Vehicle Suspension

## Ex 1) Simplified Quarter Car Model




$$m\ddot{x} = -b(\dot{x} - \dot{x}_r) - k(x - x_r)$$

$$m\ddot{x} + b(\dot{x} - \dot{x}_r) + k(x - x_r) = 0$$

$$m\ddot{x} + b\dot{x} + kx = b\dot{x}_r + kx_r$$

Laplace Transform

$$(ms^2 + bs + k)X(s) = (bs + k)X_r(s)$$

The transfer function

$$\frac{X(s)}{X_r(s)} = \frac{bs + k}{ms^2 + bs + k}$$

State Eq : $\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{b}{m}\dot{x}_r + \frac{k}{m}x_r$

let $x_1 = x \qquad u = \dot{x}_r$

$x_2 = \dot{x}$

$x_3 = x_r$

then $\dot{x}_1 = \dot{x} = x_2$

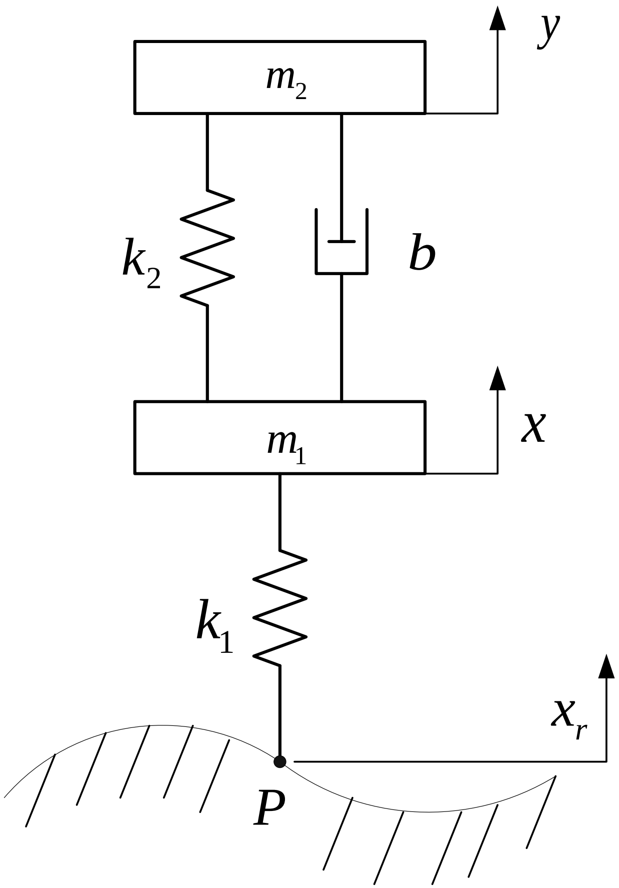
$$\dot{x}_2 = \ddot{x} = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{k}{m}x_3 + \frac{b}{m}u$$

$$\dot{x}_3 = \dot{x}_r = u$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{b}{m} \\ 1 \end{bmatrix}u$$

# Vehicle Suspension

## Ex 2) Another Quarter Car Model ( 2 DOF ¼ Car model)




Applying the Newton's second law to the system, we obtain

$$m_1\ddot{x} = -k_2\left(x-y\right) - b\left(\dot{x}-\dot{y}\right) - k_1\left(x-x_r\right)$$

$$m_2\ddot{y} = k_2\left(x-y\right) + b\left(\dot{x}-\dot{y}\right)$$

Hence we have

$$m_1\ddot{x} + b\dot{x} + \left(k_1+k_2\right)x = b\dot{y} + k_2 y + k_1 x_r$$

$$m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$$

Taking Laplace Transform

$$\left(m_1 s^2 + bs + k_1 + k_2\right)X\left(s\right) = \left(bs+k_2\right)Y\left(s\right) + k_1 X_r\left(s\right)$$

$$\left(m_2 s^2 + bs + k_2\right)Y\left(s\right) = \left(bs+k_2\right)X\left(s\right)$$

Eliminating X(s) from the last two equations, we have

$$\frac{Y\left(s\right)}{X_r\left(s\right)} = \frac{k_1\left(bs+k_2\right)}{m_1 m_2 s^4 + \left(m_1+m_2\right)bs^3 + \left[k_1 m_2 + \left(m_1+m_2\right)k_2\right]s^2 + k_1 bs + k_1 k_2}$$

# Vehicle Suspension

State Equation :

let

$$x_1 = y - x$$
$$x_2 = \dot{y}$$
$$x_3 = x - x_r$$
$$x_4 = \dot{x}$$

$$\dot{x}_1 = \dot{y} - \dot{x} = x_2 - x_4$$
$$x_2 = \ddot{y} = -\frac{k_2}{m_2} x_1 - \frac{b}{m_2}(x_2 - x_4)$$
$$\dot{x}_3 = \dot{x} - \dot{x}_r = x_4 - \dot{x}_r$$
$$\dot{x}_4 = \ddot{x} = \frac{k_2}{m_1} x_1 + \frac{b}{m_1}(x_2 - x_4) - \frac{k_1}{m_1} x_3$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_2}{m_2} & -\frac{b}{m_2} & 0 & -\frac{b}{m_2} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & \frac{b}{m_1} & -\frac{k_1}{m_1} & -\frac{b}{m_1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} u \qquad u = \dot{x}_r$$

Block Diagram and Signal flow graph

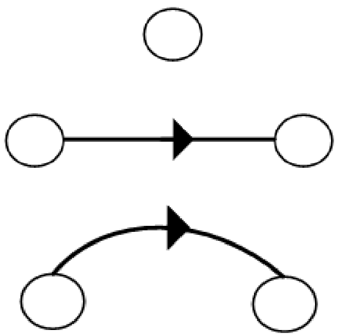
Transfer function and state equation

State space representation of Linear Differential Equations

# Signal Flow

$$\dot{x}(t) = Ax + Bu$$

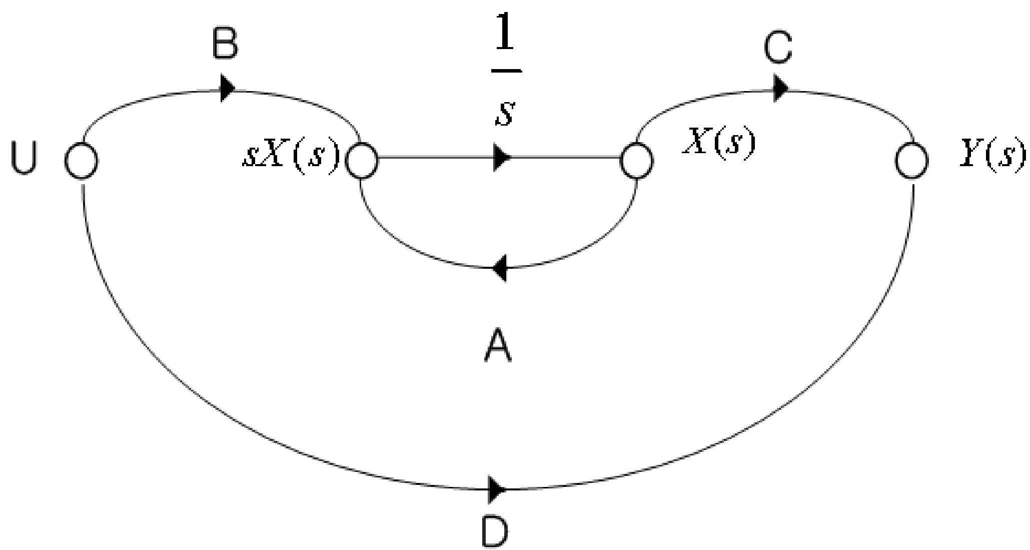
$$y = Cx + Du$$



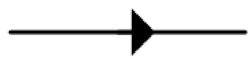
: signal

: relation dynamic $\left(\frac{1}{s}, z^{-1}\right)$

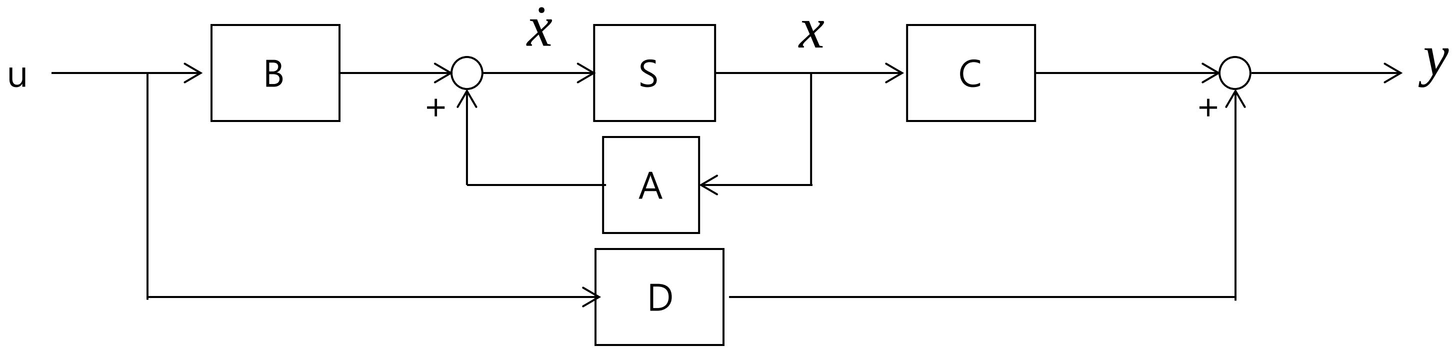
: static relation

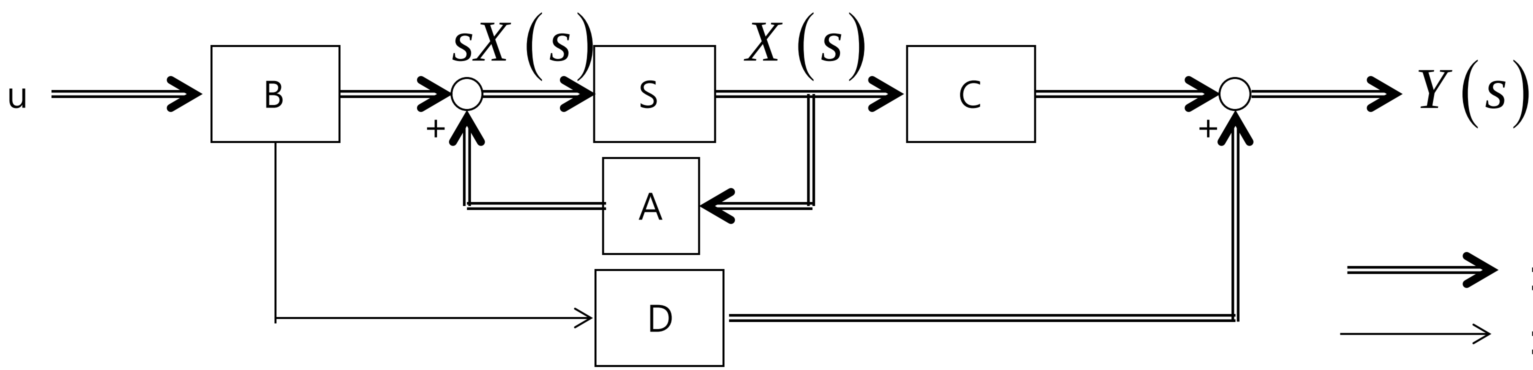


# Block Diagram


: Signals on Line


: Transfer function inside the box (block)



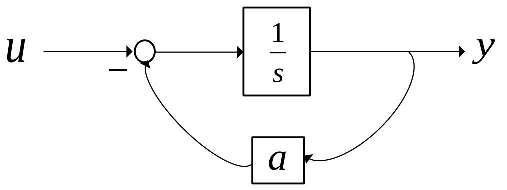

: vector
: scalar

# Block Diagram

ex1. First order

$$\dot{y} + ay = u$$

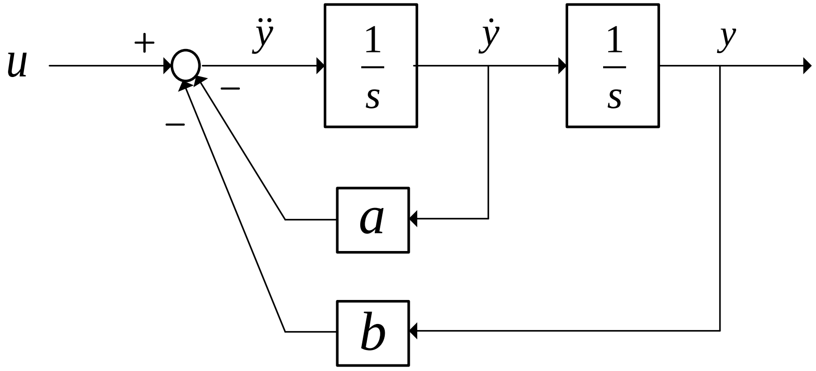
$$\frac{Y}{U} = \frac{1}{s+a}$$




ex2. Second order

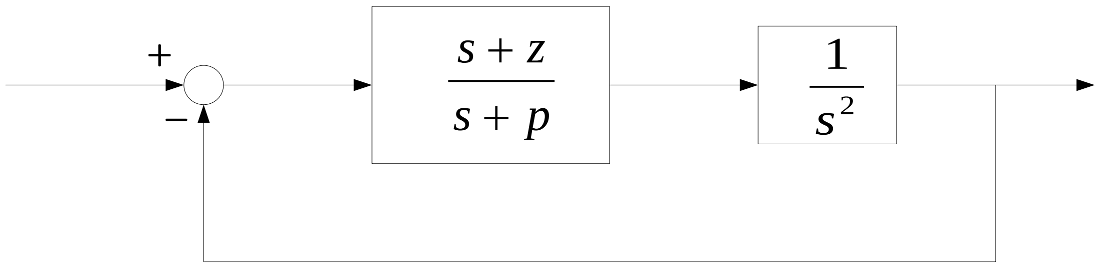
$$\ddot{y} + a\dot{y} + by = u$$

$$\frac{Y}{U} = \frac{1}{s^2 + as + b} = \frac{1}{s(s+a)+b}$$



# Block Diagram

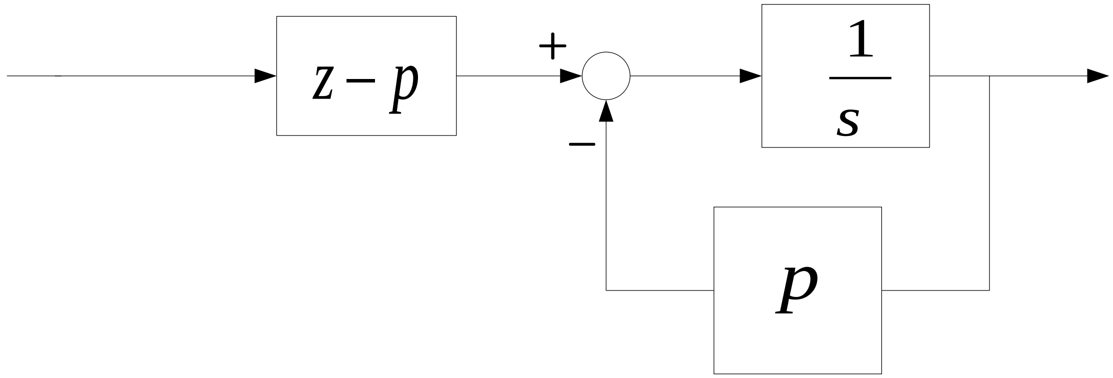
ex3.



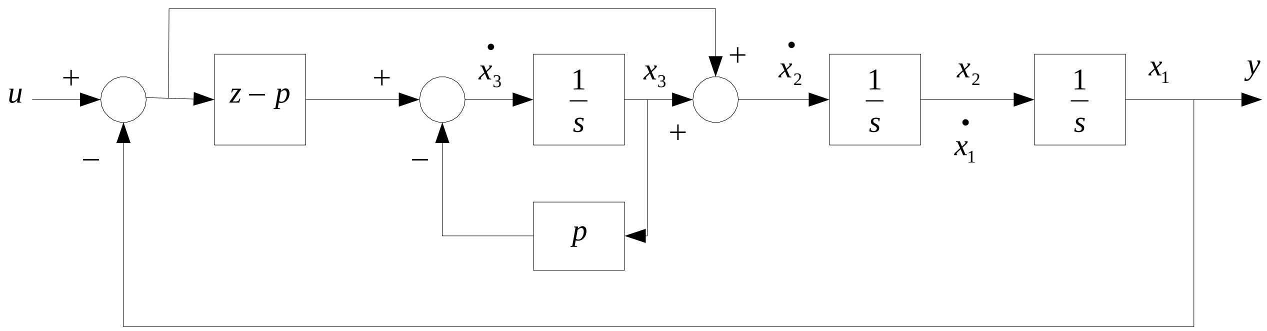

$$\frac{s+z}{s+p} = \frac{s+p+z-p}{s+p} = 1 + \frac{z-p}{s+p}$$

let $\quad \frac{y}{u} = \frac{z-p}{s+p}$

then $\quad sy + py = (z-p)u$



# Block Diagram




Thus, $\dot{x}_1 = x_2$

$$\dot{x}_2 = x_3 + (u - x_1)$$

$$\dot{x}_3 = (z - p)\cdot(u - x_1) - p x_3$$

The state representation is as follows

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ p - z & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z - p \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_1$$

## State-Space Representation of *nth-Order Systems of Linear Differential Equations*

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = x_3 \\ &\vdots \\ \dot{x}_{n-1} &= y^{(n-1)} = x_n \\ \dot{x}_n &= -a_n x_1 - \cdots - a_1 x_n + u \end{aligned}$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \bullet \\ \bullet \\ \bullet \\ x_n \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

# Canonical Forms

- **Canonical Forms (T.F. → State Eq.)**

$$\begin{cases} \text{Controllable Canonical Form} & \leftarrow \text{Direct Programming Method} \\ \text{Observable Canonical Form} & \leftarrow \text{Nested Programming Method} \\ \text{Diagonal (Jordan) Canonical Form} & \leftarrow \text{Partial Fraction Expansion} \end{cases}$$

- **Controllable Canonical Form**

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

Ex) n=3

$$G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$= b_0 + \frac{(b_1 - b_0 a_1) s^2 + (b_2 - b_0 a_2) s + (b_3 - b_0 a_3)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\begin{cases} b_1' = b_1 - b_0 a_1 \\ b_2' = b_2 - b_0 a_2 \\ b_3' = b_3 - b_0 a_3 \end{cases}$$

# Canonical Forms

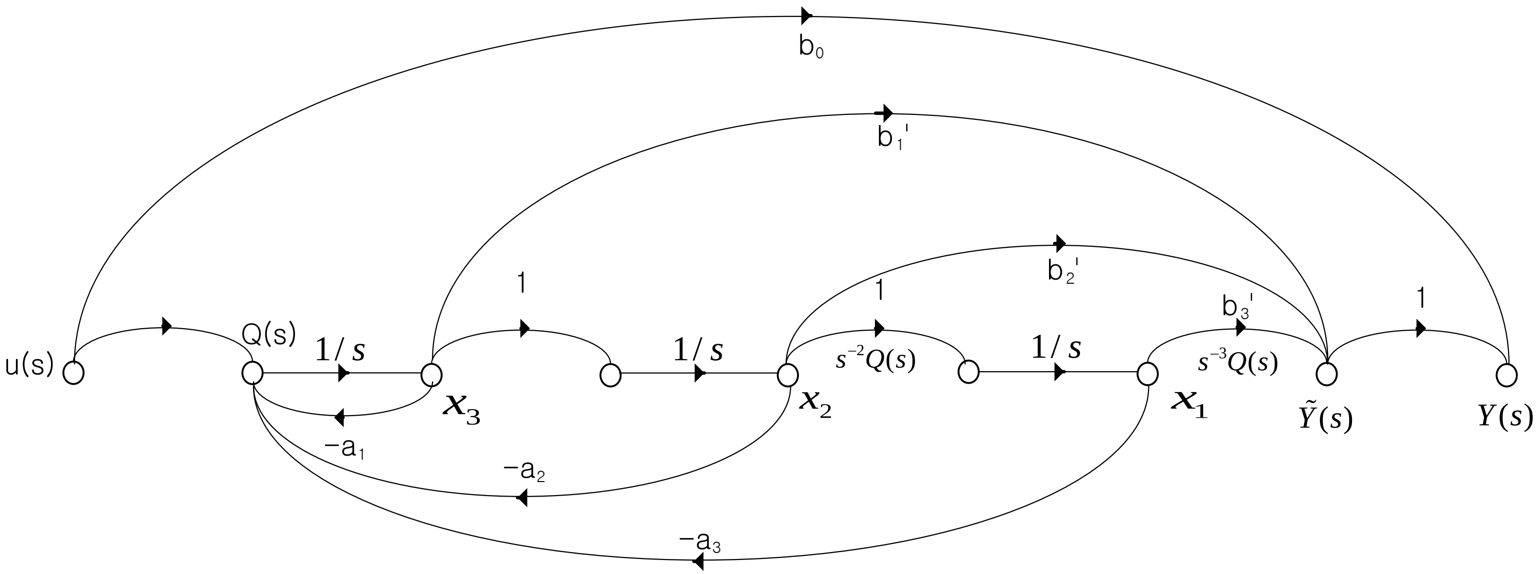
$$\tilde{Y}(s)=\left(\frac{b_1's^2+b_2's+b_3'}{s^3+a_1s^2+a_2s+a_3}\right)u(s)\times\frac{s^{-3}}{s^{-3}} \qquad Y(s)=b_0u(s)+\tilde{Y}(s)$$

$$Y(s)=\left[C\left(sI-A\right)^{-1}B+D\right]u(s)$$

$$\Rightarrow \frac{\tilde{Y}(s)}{b_1's^{-1}+b_2's^{-2}+b_3's^{-3}}=\frac{u(s)}{1+a_1s^{-1}+a_2s^{-2}+a_3s^{-3}}=Q(s)$$

$$Q(s)=u(s)-a_1s^{-1}Q(s)-a_2s^{-2}Q(s)-a_3s^{-3}Q(s)$$

$$\tilde{Y}(s)=\left(b_1's^{-1}+b_2's^{-2}+b_3's^{-3}\right)Q(s)$$



# Canonical Forms

- **Controllable Canonical Form**

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u$$

$$y = b_3' x_1 + b_2' x_2 + b_1' x_3 + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_3' & b_2' & b_1' \end{bmatrix} x + \begin{bmatrix} b_0 \end{bmatrix} u$$

# Canonical Forms

- **Observable Canonical Form (Nested Programming)**

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u$$

$$y = b_3' x_1 + b_2' x_2 + b_1' x_3 + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_0 \end{bmatrix} u$$

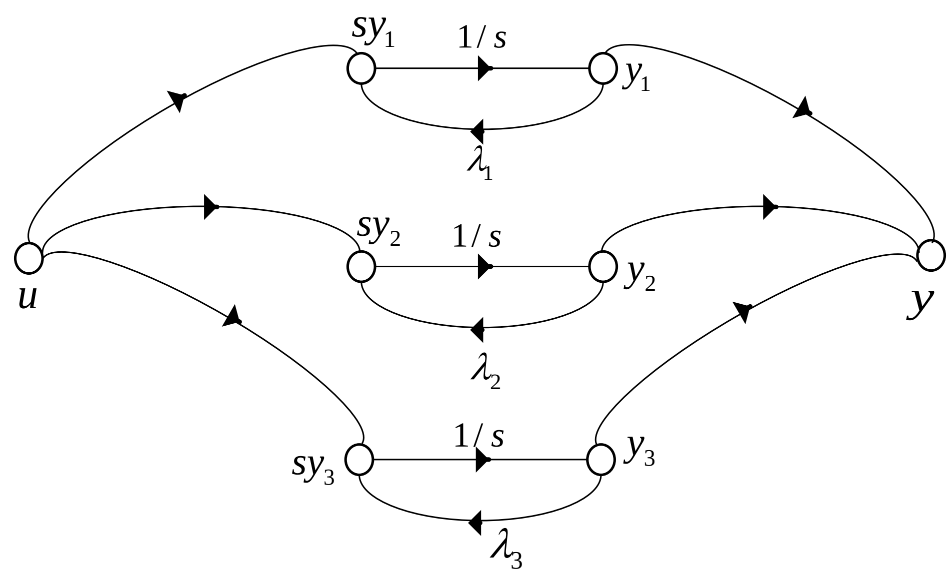
(Note : x in the controllable canonical form $\neq$ x in the observable canonical form)

- **Diagonal (or Jordan) Canonical Form** **(Partial Fraction Expansion)**

Case 1. Distinct Roots $(\lambda_1 \neq \lambda_2 \neq \lambda_3)$

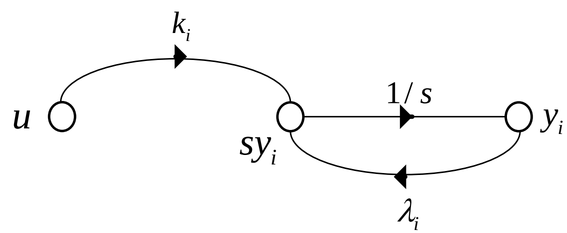
$$G(s) = \frac{B(s)}{A(s)} = \frac{K_1}{s-\lambda_1} + \frac{K_2}{s-\lambda_2} + \frac{K_3}{s-\lambda_3}$$

$$Y(s) = \sum_{i=1}^{3} \frac{K_i}{s-\lambda_i} u(s) = y_1 + y_2 + y_3$$




$$y_i = \frac{K_i}{s-\lambda_i} u$$

$$sy_i = \lambda_i y_i + K_i u$$




let $x_1 = y_1, x_2 = y_2, x_3 = y_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

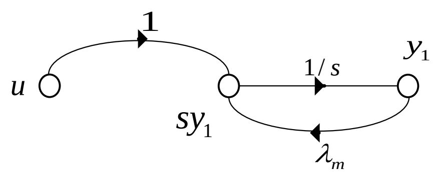
- **Diagonal (or Jordan) Canonical Form** (Partial Fraction Expansion)

Case 2. Multiple Roots

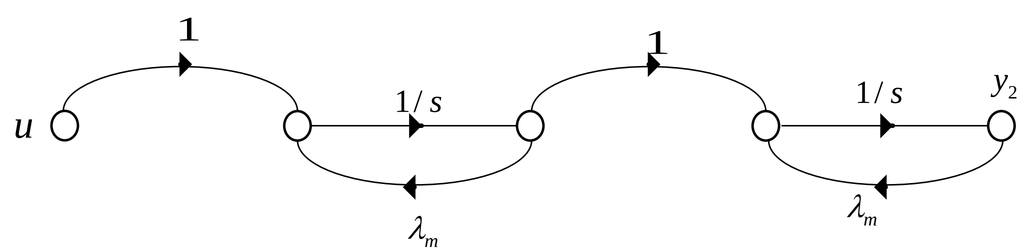
$$G(s)=\frac{B(s)}{(s-\lambda_m)^3}=\frac{K_1}{s-\lambda_m}+\frac{K_2}{(s-\lambda_m)^2}+\frac{K_3}{(s-\lambda_m)^3}\qquad \lambda_1=\lambda_2=\lambda_3=\lambda_m$$

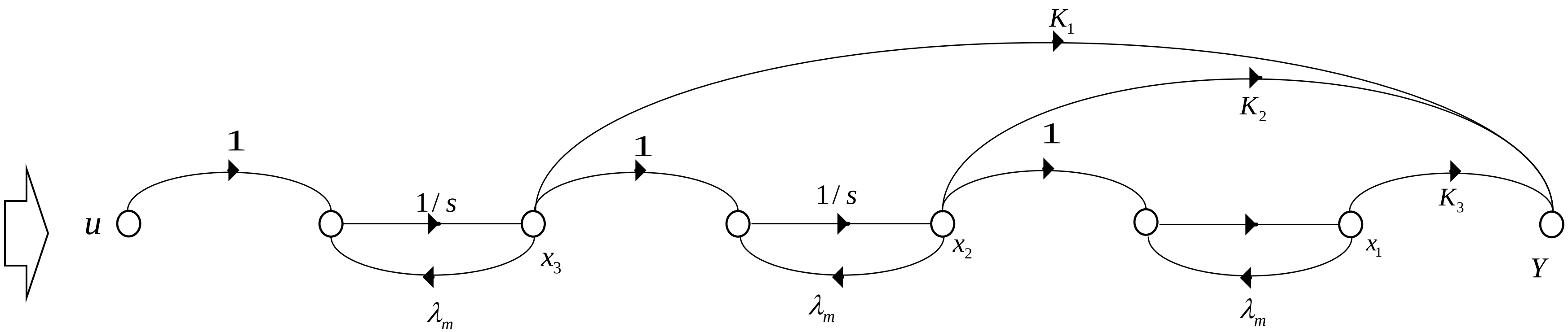
$$y_1=\frac{1}{s-\lambda_m}u$$

$$sy_1=\lambda_m y_1+u$$




$$y_2=\frac{1}{(s-\lambda_m)^2}u$$



- **Diagonal (or Jordan) Canonical Form** **(Partial Fraction Expansion)**

Case 2. Multiple Roots

$$\dot{x}_1 = \lambda_m x_1 + x_2$$
$$\dot{x}_2 = \lambda_m x_2 + x_3$$
$$\dot{x}_3 = \lambda_m x_3 + u$$
$$y = K_3 x_1 + K_2 x_2 + K_1 x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} K_3 & K_2 & K_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Canonical Forms

## • Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

$$G(s)=\frac{1}{s^2-2\sigma s+\sigma^2+\omega^2}=\frac{1}{\left[s-(\sigma+j\omega)\right]\left[s-(\sigma-j\omega)\right]}$$

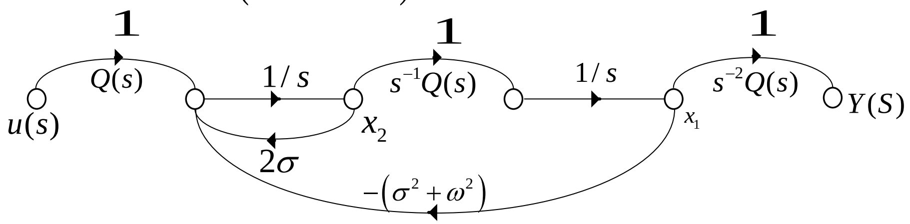
$$\frac{Y}{u}=G(s)=\frac{1}{s^2-2\sigma s+\sigma^2+\omega^2}$$

$$=\frac{s^{-2}}{1-2\sigma s^{-1}+\left(\sigma^2+\omega^2\right)s^{-2}}$$

$$\frac{Y}{s^{-2}}=\frac{u}{1-2\sigma s^{-1}+\left(\sigma^2+\omega^2\right)s^{-2}}=Q(s)$$

$$\begin{cases} Y=s^{-2}Q(s) \\ Q(s)\left(1-2\sigma s^{-1}+\left(\sigma^2+\omega^2\right)s^{-2}\right)=u \end{cases}$$

$$Q(s)=u+2\sigma s^{-1}Q(s)-\left(\sigma^2+\omega^2\right)s^{-2}Q(s)$$



# Canonical Forms

- **Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)**

Case 3. Complex Roots

$$AP = P\Lambda$$

Note : Complex Roots, Complex State x

$$\dot{x} = \Lambda x + bu$$

$$y = Cx$$

→ Complex case의 diagonalization 방법 이용

$$\Lambda K = KJ$$

$$\Lambda = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} \qquad K = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix} \qquad K^{-1} = \frac{2}{j}\begin{bmatrix} \frac{j}{2} & \frac{j}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

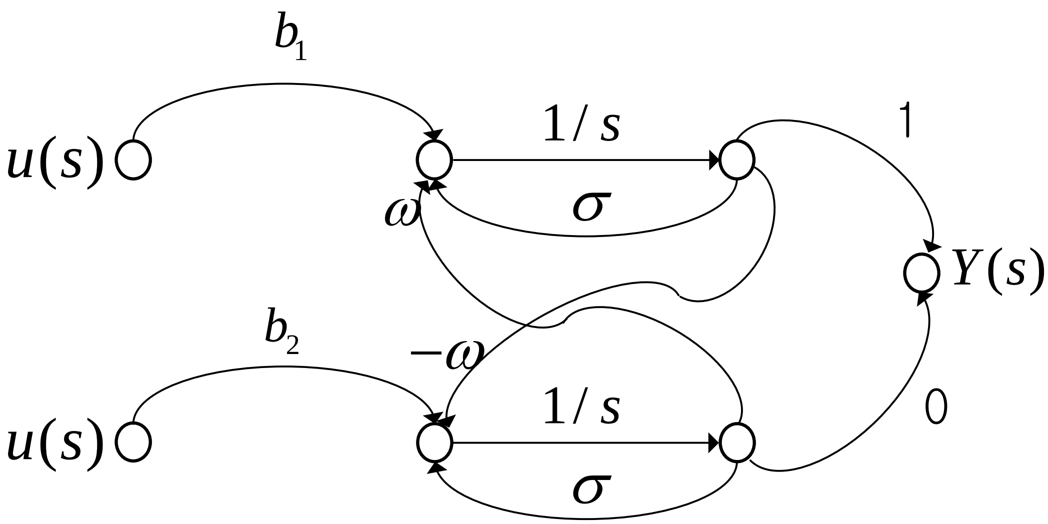
$$\begin{aligned} J &= K^{-1}\Lambda K \\ &= K^{-1}P^{-1}APK \end{aligned} \qquad J = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

# Canonical Forms

- **Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)**

Case 3. Complex Roots

Ex)
$$\dot{z} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$



# Canonical Forms

- **Diagonal (or Jordan) Canonical Form** **(Partial Fraction Expansion)**

Case 3. Complex Roots

Step 1 $\dot{x} = Ax + Bu$

Step 2 let $x = P\xi$

$$\dot{\xi} = \underbrace{P^{-1}AP}_{\Lambda}\xi + P^{-1}Bu$$ : diagonal

Step 3 let $\xi = Kz$

$$\dot{z} = \underbrace{K^{-1}\Lambda K}_{J}z + K^{-1}P^{-1}Bu$$

$$= \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$\begin{cases} x = P\xi = PKz \\ \dot{z} = \underbrace{K^{-1}\Lambda K}_{J}z + K^{-1}P^{-1}bu \\ y = CPKz \end{cases}$$

## Transformation of mathematical models with MATLAB

## Sec. 2-6 pp.49-52

**TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB**

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in s}}{\text{denominator polynomial in s}} = \frac{\text{num}}{\text{den}}$$

the MATLAB command

[A,B,C,D] = tf2ss(num,den)

There are many (infinitely many) state-space representations for the same system.The MATLAB command gives one possible such state-space representation.

## Transformation from Transfer Function to State Space Representation

$$\frac{Y(s)}{U(s)} = \frac{s}{(s+10)(s^2+4s+16)} = \frac{s}{s^3+14s^2+56s+160}$$

possible state-space representations (among infinitely many alternatives)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + [0]u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + [0]u$$

**MATLAB Program 2–2**

```
num = [1 0];
den = [1 14 56 160];
[A,B,C,D] = tf2ss(num,den)

A =
    -14  -56  -160
      1    0     0
      0    1     0
B =
    1
    0
    0
C =
      0 1 0
D =
      0
```

**Transformation from State Space Representation to Transfer Function.**

[num,den] = ss2tf(A,B,C,D,iu)

systems with more than one input

[num,den] = ss2tf(A,B,C,D)

**MATLAB Program 2–3**

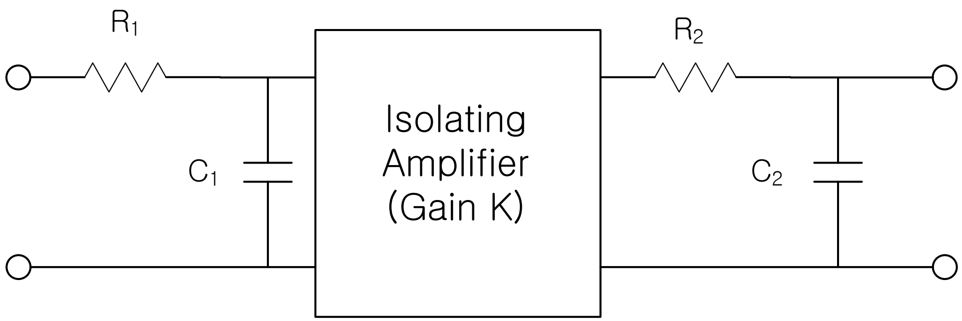
```
A = [0 1 0; 0 0 1; -5 -25 -5];
B = [0; 25; -120];
C = [1 0 0];
D = [0];
[num,den] = ss2tf(A,B,C,D)
num =
        0   0.0000   25.0000   5.0000
den=
        1.0000   5.0000   25.0000   5.0000
% ***** The same result can be obtained by
entering the following command: *****
[num,den] = ss2tf(A,B,C,D,1)
num =
        0 0.0000 25.0000 5.0000
den =
        1.0000 5.0000 25.0000 5.0000
```

End of section 3
(Ch. 2 of Ogata)

# Loading Effect

- ## No Loading Effect

Block can be connected in series only if the output of one block is not affected by the next following block.



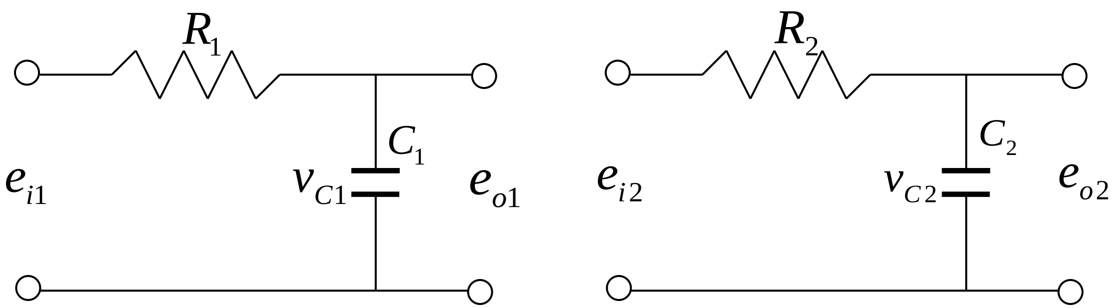

$$V(s) = I(s)Z(s)$$

$Z(s)$ : complex impedance

If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element.

# No Loading Effect

ex1)




$$e_{i1} = R_1 i_1 + e_{o1}$$

$$\frac{dv_{C1}}{dt} = \frac{1}{C_1} i_1, \qquad v_{C1} = e_{o1}$$

$$\Rightarrow \frac{de_{o1}}{dt} = \frac{1}{C_1}\left[\frac{1}{R_1}(e_{i1} - e_{o1})\right] = -\frac{1}{C_1 R_1} e_{o1} + \frac{1}{C_1 R_1} e_{i1}$$

$$\frac{E_{o1}(s)}{E_{i1}(s)} = \frac{\frac{1}{C_1 R_1}}{S + \frac{1}{C_1 R_1}} \qquad a_1$$

$$\frac{E_{o2}(s)}{E_{i2}(s)} = \frac{\frac{1}{C_2 R_2}}{S + \frac{1}{C_2 R_2}} \qquad a_2$$

# No Loading Effect

ex1)

$$\frac{E_{o1}(s)}{E_{i1}(s)} = \frac{\frac{1}{C_1R_1}}{S + \frac{1}{C_1R_1}} \rightarrow a_1 \qquad \frac{E_{o2}(s)}{E_{i2}(s)} = \frac{\frac{1}{C_2R_2}}{S + \frac{1}{C_2R_2}} \rightarrow a_2$$

$$E_{i1} \rightarrow \boxed{\frac{a_1}{s+a_1}} \rightarrow E_{o1} \qquad E_{i2} \rightarrow \boxed{\frac{a_2}{s+a_2}} \rightarrow E_{o2}$$

$$E_{i1} \rightarrow \boxed{\frac{a_1}{s+a_1} \cdot \frac{a_2}{s+a_2}} \rightarrow E_{o2}$$

**No !!**
**Incorrect**

P.90.Ogata

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + \left(R_1C_1 + R_2C_2 + R_1C_2\right)s + 1}$$