

System Control

5. Basic Control Actions and Response of Control Systems

(1) PID Control

(2) Routh's stability tests

(3) Systems Type

Open loop and closed loop : model sensitivity

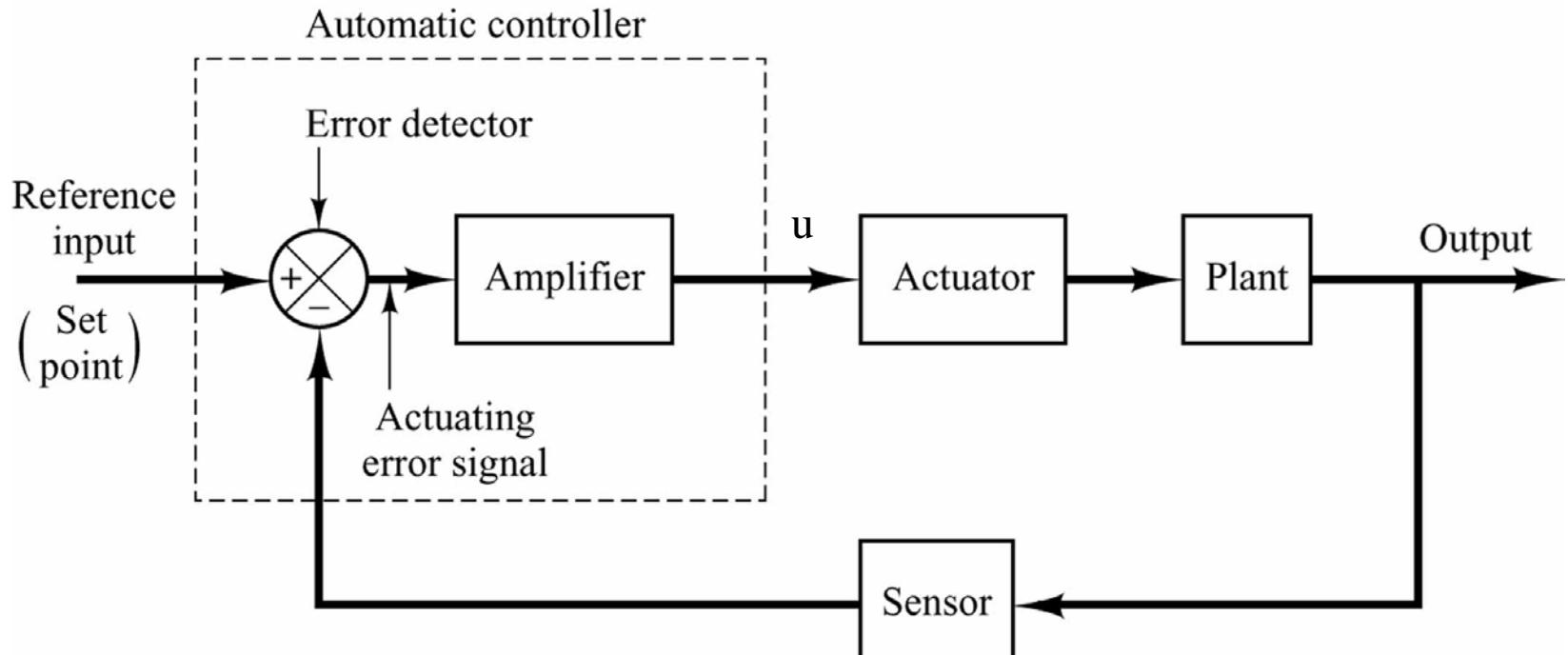
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Seoul National University

- Routh stability criterion
- Effects of Integral and derivative control
- Steady state error in unity feedback control systems
- Open loop and closed loop systems

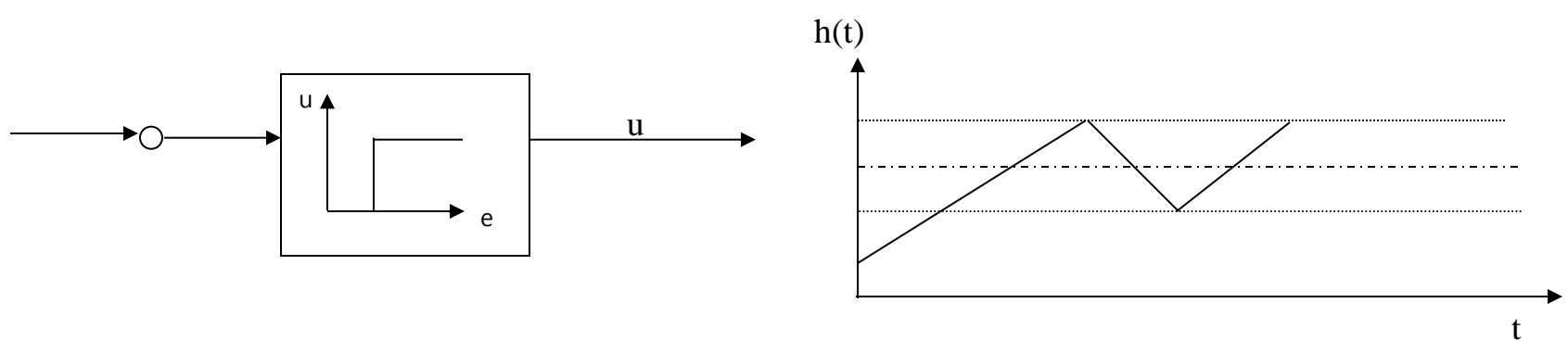
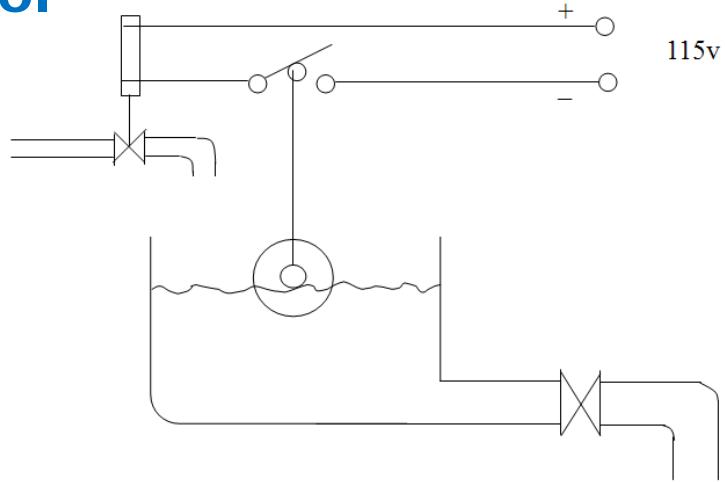
Control Systems



$$u=f(r,c)$$

Classifications of controllers

1. ON-OFF control

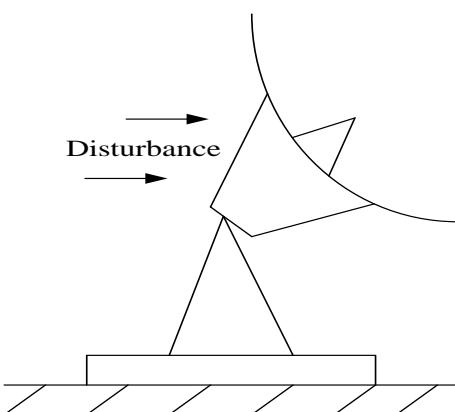
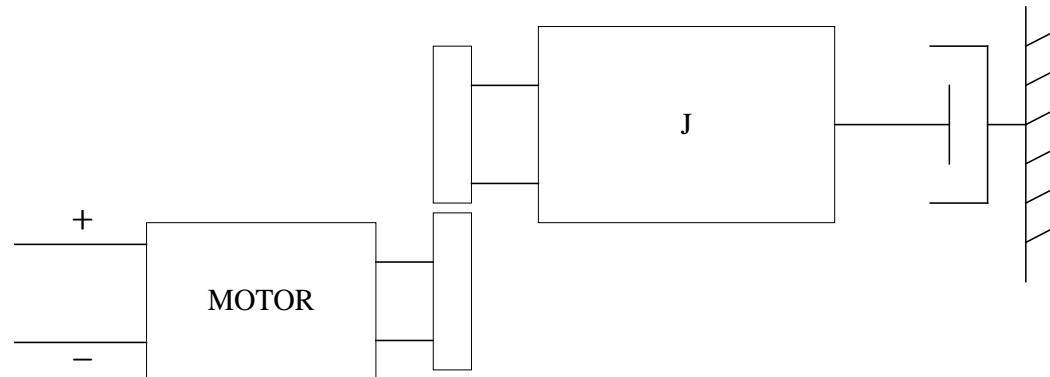


Classifications of controllers

2. Proportional Control

$$U = k_p e$$

k_p : The proportional Gain



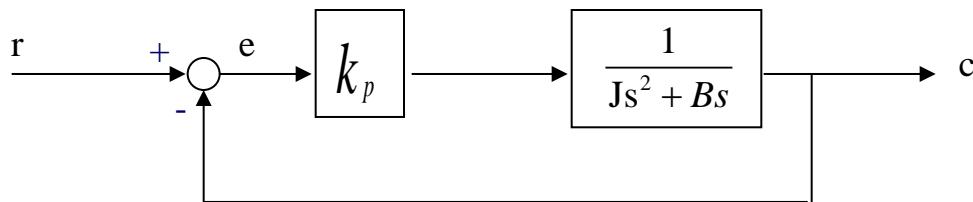
Antenna Position
Control System

Classifications of controllers

2. Proportional Control

$$U = k_p e$$

k_p : The proportional Gain



$$\frac{c(s)}{r(s)} = \frac{\frac{K_p}{Js^2 + Bs}}{1 + \frac{K_p}{Js^2 + Bs}} = \frac{K_p}{Js^2 + Bs + K_p}$$

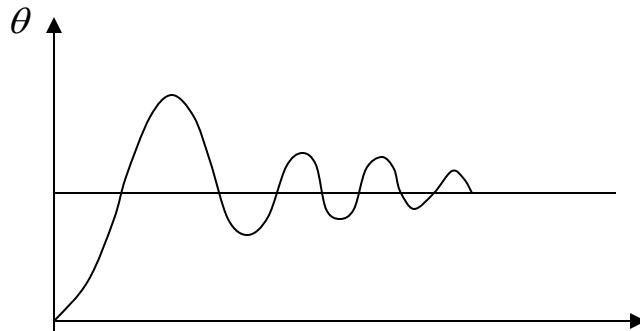
Classifications of controllers

2. Proportional Control

- Step input response $r(s) = \frac{1}{s}$

- steady state response $c(t) = \lim_{s \rightarrow 0} sC(s)$

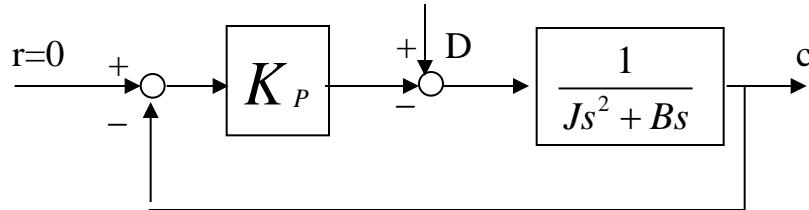
$$= \lim_{s \rightarrow 0} s \frac{K_p}{Js^2 + Bs + K_p} \frac{1}{s} = 1$$



Classifications of controllers

2. Proportional Control

- Response to torque disturbance



$$C(s) = \frac{K_p}{J s^2 + B s + K_p} r(s) + \frac{1}{J s^2 + B s + K_p} D(s)$$

Assume that $D(s) = \frac{1}{s}$ steady state response

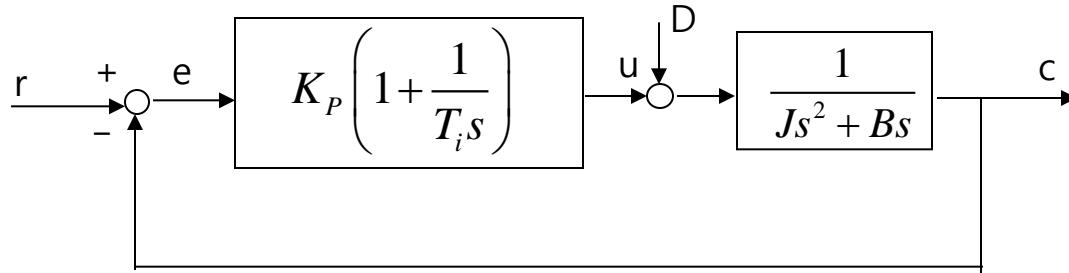
$$c(t) = \lim_{s \rightarrow \infty} s \cdot \frac{1}{J s^2 + B s + K_p} \cdot \frac{1}{s} = \frac{1}{K_p} : \text{steady-state error}$$

- Large K_p
- small steady-state error
 - large motor power is needed
 - oscillations
 - large w_n $\left(w_n = \sqrt{\frac{K_p}{J}} \right)$
 - small damping ratio $\zeta = \frac{B}{2\sqrt{JK_p}}$

Classifications of controllers

3. Proportional-Integral Control (PI control)

- Response to torque disturbance



$$\frac{U(s)}{E(s)} = K_P \left(1 + \frac{1}{T_i s} \right)$$

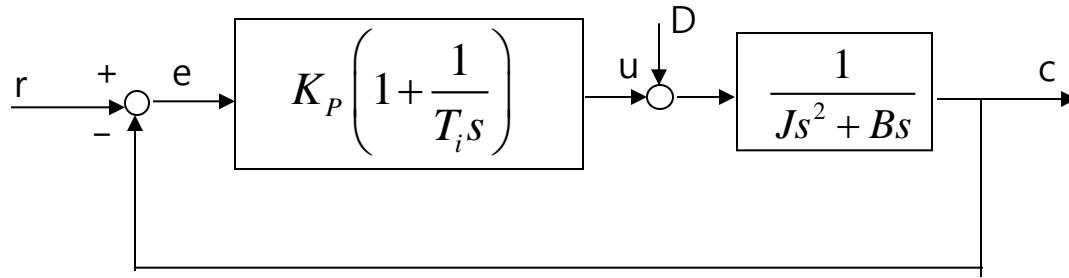
$$u(t) = K_p e(t) + \frac{K_p}{T_i} \int_0^t e(t) dt$$

$$\frac{C(s)}{R(s)} = \frac{K_p s + \frac{1}{T_i} K_p}{J s^3 + B s^2 + K_p s + \frac{K_p}{T_i}}$$

No steady-state error for reference input

Classifications of controllers

3. Proportional-Integral Control (PI control)



$$\frac{C(s)}{D(s)} = \frac{\frac{1}{Js^2 + Bs}}{1 + K_P \left(1 + \frac{1}{T_i s}\right) \frac{1}{Js^2 + Bs}}$$

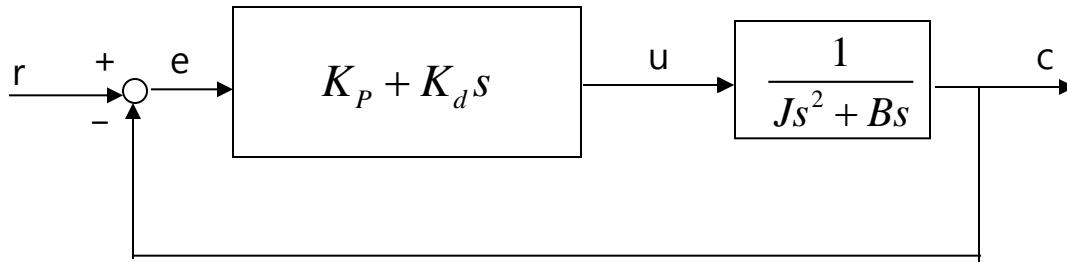
$$= \frac{s}{Js^3 + Bs^2 + K_P s + \frac{K_P}{T_i}}$$

$$D(s) = \frac{1}{s}$$
$$C(t) = \lim_{s \rightarrow 0} s \frac{s}{Js^3 + Bs^2 + K_P s + \frac{K_P}{T_i} s} \frac{1}{s} = 0$$

No steady-state error for step disturbance

Classifications of controllers

4. Proportional-Derivative Control (PD control)



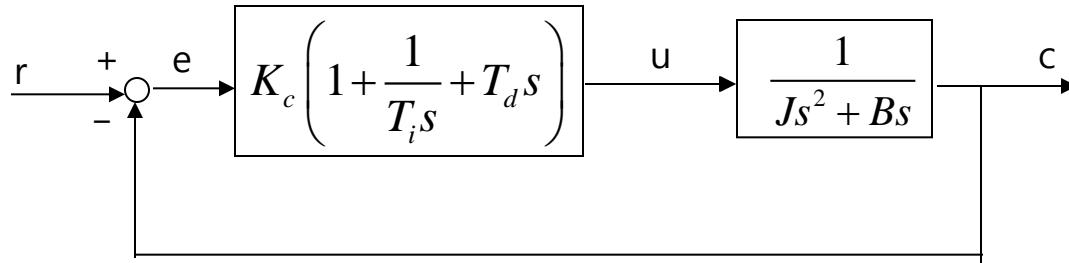
$$u(t) = K_p e(t) + K_d \frac{d}{dt} e(t)$$

$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{J s^2 + (B + K_d) s + K_p}$$

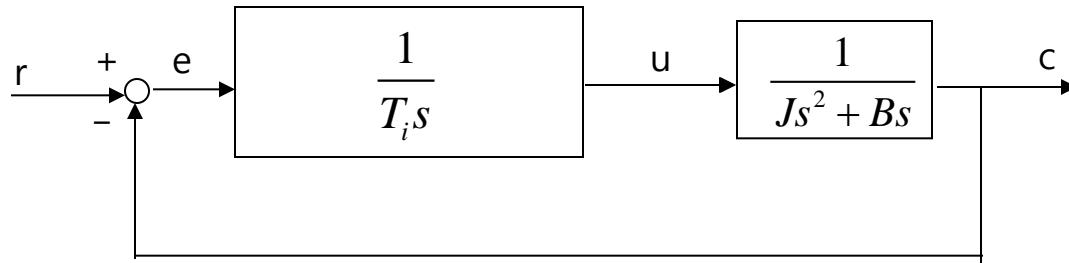
$$\zeta = \frac{B + K_d}{2\sqrt{J K_p}} : \text{increased effective damping}$$

Classifications of controllers

5. Proportional-Integral-Derivative Control (PID control)

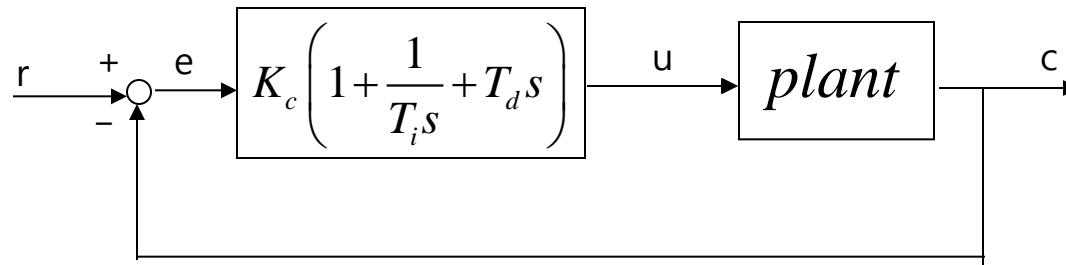


6. I control

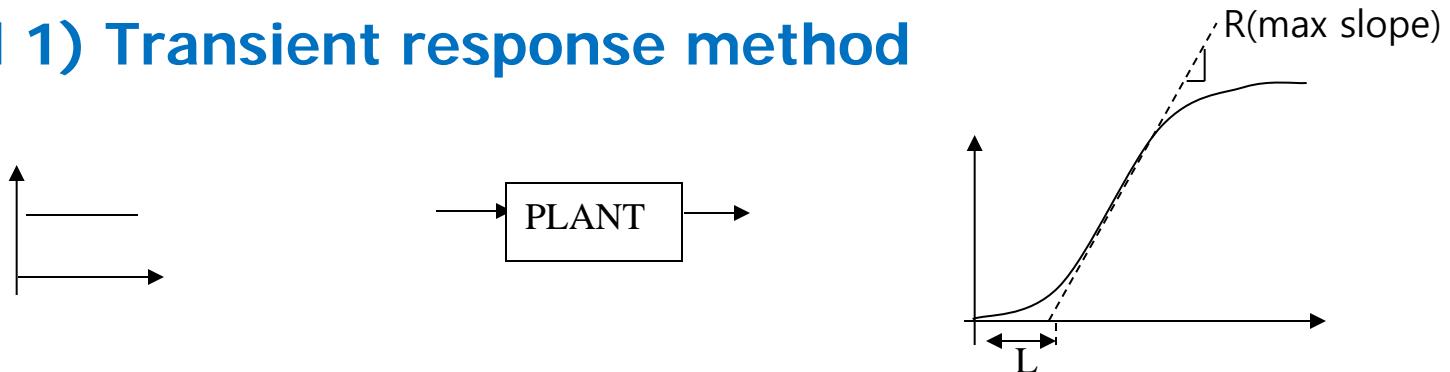


$$\frac{C(s)}{R(s)} = \frac{\frac{1}{T_i s} \frac{1}{J s^2 + B s}}{1 + \frac{1}{T_i s} \frac{1}{J s^2 + B s}} = \frac{\frac{1}{T_i}}{J s^3 + B s^2 + \frac{1}{T_i}}$$

Ziegler-Nicholas Tuning Rules for PID controllers



Method 1) Transient response method

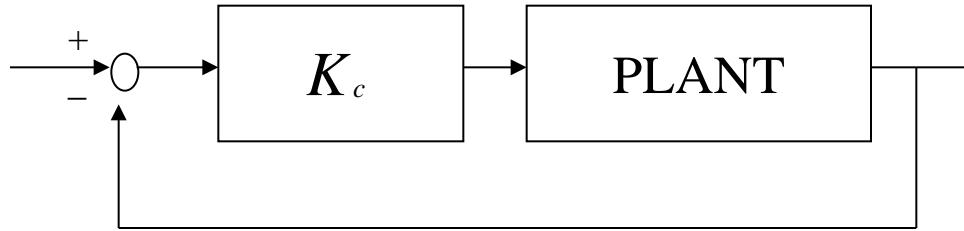


Step input $\left\{ \begin{array}{l} K_c = \frac{1}{RL} \quad \text{for } P\text{-control} \\ K_c = \frac{0.9}{RL}, \quad T_i = 3.3L \quad \text{for } PI\text{-control} \\ K_c = \frac{1.2}{RL}, \quad T_i = 2L, \quad T_d = 0.5L \quad \text{for } PID\text{-control} \end{array} \right.$

- This method works good if the unit step response is \int (sigmod) shaped

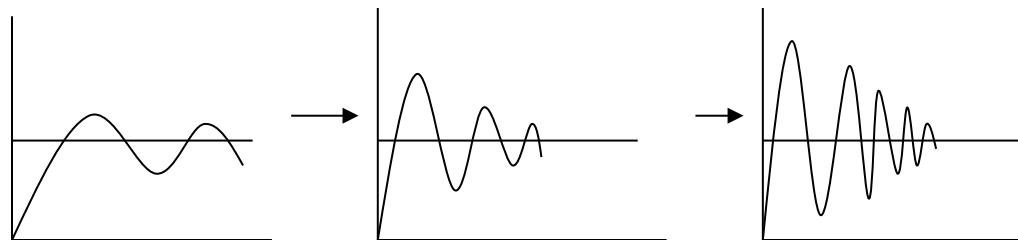
Ziegler-Nicholas Tuning Rules for PID controllers

Method 2) Ultimate sensitivity method



$K_c \gg 1, \omega_n \rightarrow \text{increase}$

$\zeta \rightarrow \text{decrease}$

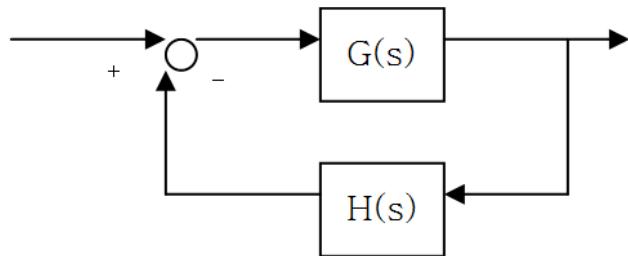


- Increase K_c until you hit the stability limit

$$\begin{cases} K_c = 0.5K_U & \text{for } P\text{-control} \\ K_c = 0.45K_U, \quad T_i = 0.83P_U & \text{for } PI\text{-control} \\ K_c = 0.6K_U, \quad T_i = 0.5P_U \quad T_d = 0.125P_U & \text{for } PID\text{-control} \end{cases}$$

Routh's stability criterion

Stability of Feedback Systems



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{P(s)}{Q(s)}$$

Poles : s_i

$\text{Re}(s_i) < 0$; stable

Characteristic equation $Q(s) = 0$

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0$$

Routh's Stability Criterion

Characteristic equation

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0$$

s^n	b_n	b_{n-2}	b_{n-4}	\dots
s^{n-1}	b_{n-1}	b_{n-3}	b_{n-5}	\dots
s^{n-2}	c_1	c_2		
s^{n-3}	d_1	d_2		
.	.	.		
s_0				

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}}, \quad c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}$$

$$d_1 = \frac{c_1 b_{n-3} - b_{n-1} c_2}{c_1}, \quad d_2 = \frac{c_1 b_{n-5} - b_{n-1} c_3}{c_1}$$

- Routh's Criterion

Number of the characteristic roots with positive real parts

=Number of sign changes of the first column have the same sign

Routh's Stability Criterion

Example 1)

$$= \frac{C}{R} = \frac{P(s)}{s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240}$$

s^5	1	10	152
s^4	1	72	240
s^3	-62	-88	
s^2	70.6	240	
s^1	122.6		
s^0	240		

Two sign change

- $Q(s)=0$ has two roots with positive real parts
- unstable

Routh's Stability Criterion

Example 2)

s^4	1	2	5
s^3	1	2	
s^2	0		
s^1	?		

→ prevents completion of the array

method 1) $s = \frac{1}{x}$

$$Q(s) = \frac{5x^4 + 2x^3 + 2x^2 + x + 1}{x^4}$$

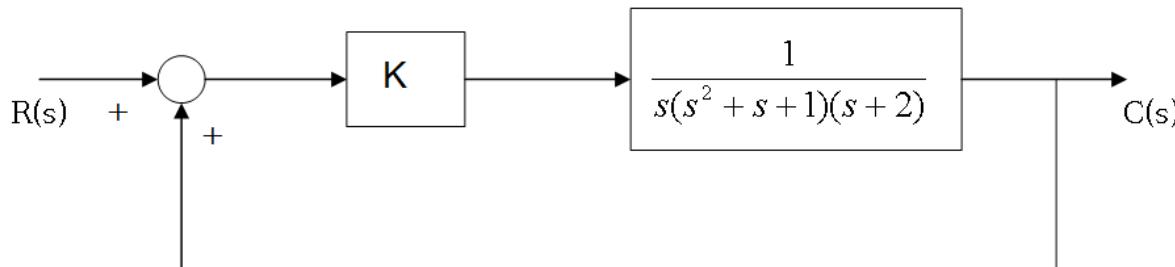
method 2)

$$\text{set } Q(s)(s+1) = 0$$

x^4	5	2	1
x^3	2	1	
x^2	-0.5	1	
x^1	-1	2	
x^0	5	two sign change → unstable	

Routh's Stability Criterion

Example 3)



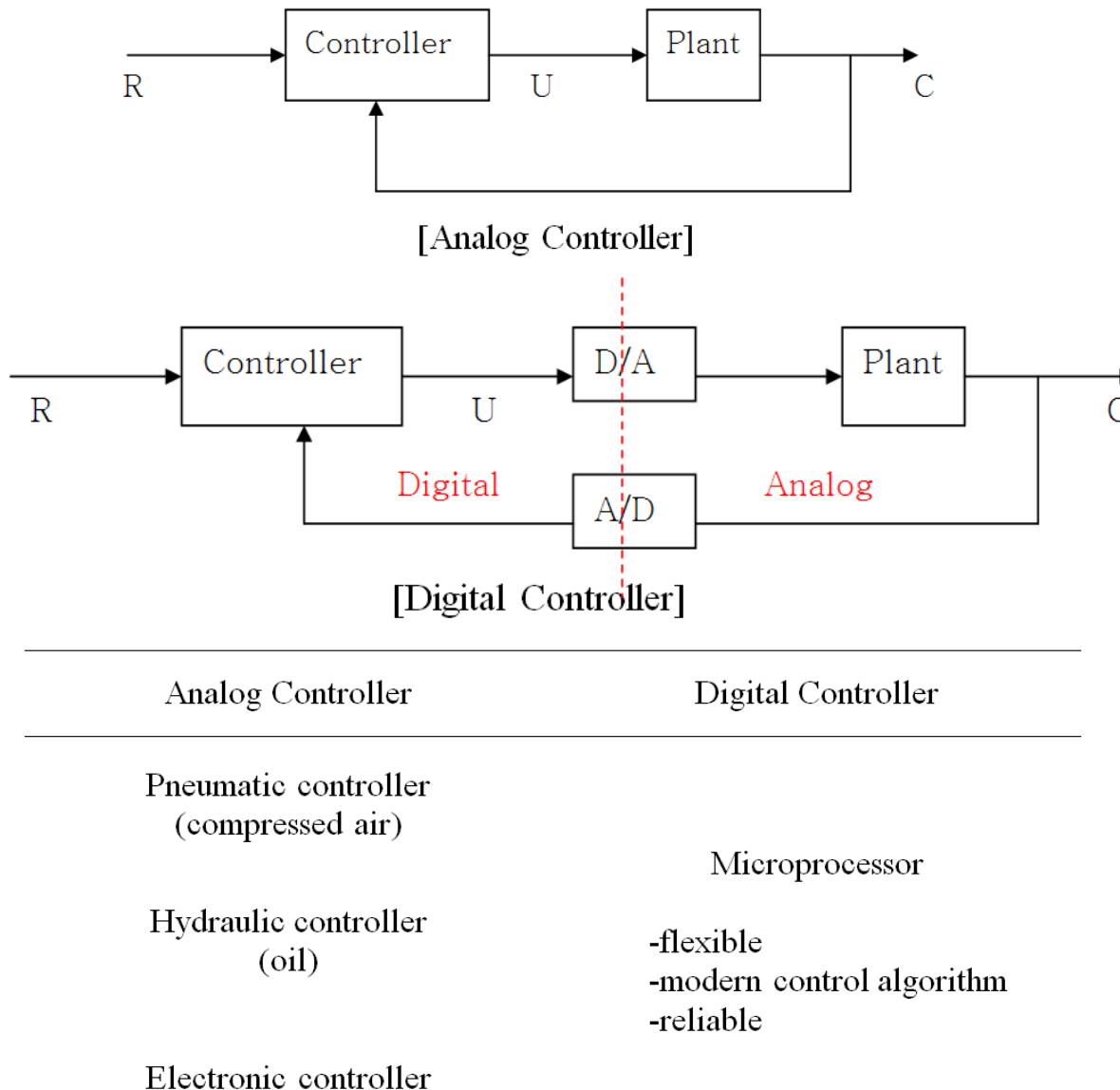
$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s^2 + s + 1)(s + 2)}}{1 + \frac{K}{s(s^2 + s + 1)(s + 2)}} = \frac{K}{s(s^2 + s + 1)(s + 2) + K} = \frac{K}{s^4 + 3s^3 + 3s^2 + 2s + K}$$

s^4	1	3	K
s^3	3	2	
s^2	$\frac{7}{3}$		K
s^1	$\left(2 - \frac{9}{7}K\right)$		
s^0	K		

$$2 - \frac{9}{7}K > 0 \implies K < \frac{14}{9} \quad K > 0$$

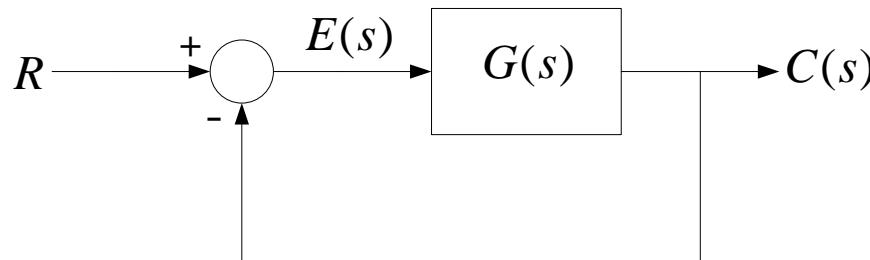
$$\therefore 0 < K < \frac{14}{9}$$

Analog/Digital Controller



Steady-state error and system types

Steady state Error in Unity Feedback control systems



$$G(s) = \frac{K(T_a s + 1)(T_b s + 1) \cdots (T_n s + 1)}{S^N (T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)}$$

N=0 ; Type 0, N=1 ; Type 1, N=2 ; Type 2

Steady state Error

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

Steady state Error in Unity Feedback control systems

=
1) $R(s) = \frac{1}{s}$ 일 경우 (unit step input),

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1+G(s)} \frac{1}{s} = \frac{1}{1+G(0)}$$

Static position error constant k_p

$$k_p = \lim_{s \rightarrow 0} G(s) = G(0) = \begin{cases} k & ; Type\ 0\ system \\ \infty & ; Type\ 1\ system \\ \infty & ; Type\ 2\ system \end{cases}$$

$$e_{ss} = \frac{1}{1+k_p} = \begin{cases} \frac{1}{1+k_p} & ; Type\ 0\ system \\ 0 & ; Type\ 1,\ 2,.. system \end{cases}$$

Steady state Error in Unity Feedback control systems

2) $R(s) = \frac{1}{s^2}$ 일 경우 (unit-ramp input),

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1+G(s)} \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{1}{s(G(s)+1)} = \lim_{s \rightarrow 0} \frac{1}{sG(s)}$$

Static velocity error constant k_v

$$k_v = \lim_{s \rightarrow 0} SG(s) = \begin{cases} 0 & ; Type\ 0\ system \\ k & ; Type\ 1\ system \\ \infty & ; Type\ 2,3..\ system \end{cases}$$

$$e_{ss} = \frac{1}{k_v} = \begin{cases} \infty & ; Type\ 0\ system \\ \frac{1}{k} & ; Type\ 1\ system \\ 0 & ; Type\ 2,3..\ system \end{cases}$$

Steady state Error in Unity Feedback control systems

3) $R(s) = \frac{1}{s^3}$ 일 경우 (unit-acceleration input),

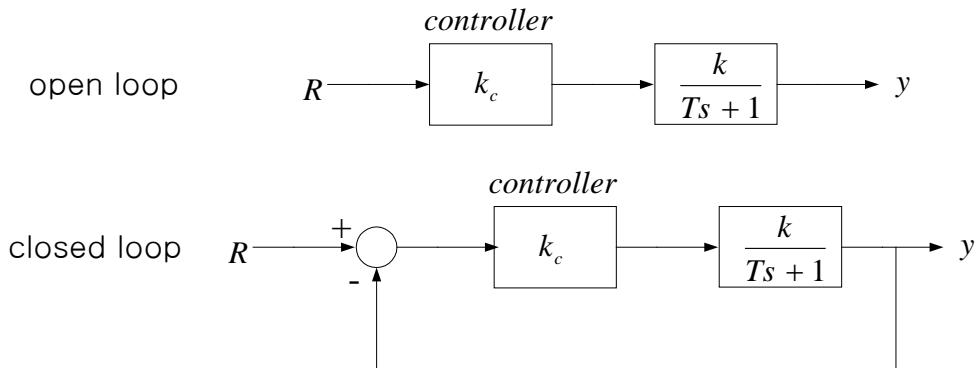
$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + G(s)} \frac{1}{s^3} = \lim_{s \rightarrow 0} \frac{1}{s^2 G(s)}$$

Static acceleration error constant k_a

$$k_a = \lim_{s \rightarrow 0} s^2 G(s) = \begin{cases} 0 & ; Type\ 0\ system \\ 0 & ; Type\ 1\ system \\ k & ; Type\ 2\ system \\ \infty & ; Type\ 3\ system \end{cases}$$

$$e_{ss} = \frac{1}{k_a} = \begin{cases} \infty & ; Type\ 0,1\ system \\ \frac{1}{k} & ; Type\ 2\ system \\ 0 & ; Type\ 3,4,\dots\ system \end{cases}$$

Open/Closed loop control systems



Open loop Transfer function 에서

$$\frac{y}{R} = k_c \frac{k}{Ts + 1} \rightarrow k_c = \frac{1}{k} \text{ 이 되어 } \text{error}=0 \text{ 이 된다.}$$

Closed loop Transfer function 에서

$$\frac{y}{R} = \frac{k_p \frac{k}{Ts + 1}}{1 + k_p \frac{k}{Ts + 1}} = \frac{k_p k}{Ts + 1 + k_p k}$$

$$\text{Steady/ state error for } R(s) = \frac{1}{s}$$

$$\text{Open loop Transfer function 에서 } e_{ss} = 0$$

$$\text{Closed loop Transfer function 에서 } e_{ss} = 1 - \frac{k_p k}{1 + k_p k} = \frac{1}{1 + k_p k}$$

Open/Closed loop control systems

Model error sensitivity

$$k = 10, \Delta k = 1 \quad \text{model error}$$

Open loop Systems

$$\frac{y}{R} = \frac{1}{k} \frac{k + \Delta k}{Ts + 1}, \quad y(t) = \frac{k + \Delta k}{k} = 1.1 \quad (10\% \text{ steady state error})$$

Closed loop Systems

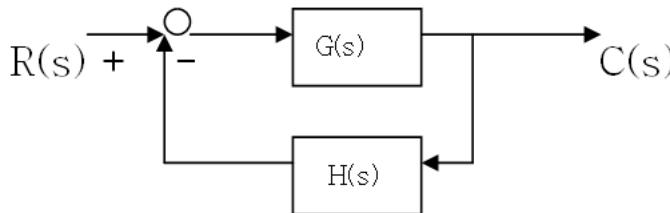
$$e_{ss} = \frac{1}{1 + k_p k} \quad \text{let} \quad k_p = \frac{100}{k} \Rightarrow e_{ss} = \frac{1}{1 + \frac{100}{k} k} = \frac{1}{101} = 0.0099$$

$$k + \Delta k,$$

$$e_{ss} = \frac{1}{1 + \frac{100}{k} (k + \Delta k)} = \frac{1}{1 + 10(11)} = \frac{1}{111} = 0.00901$$

End of section 5

Higher Order Systems



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

- unit step response $R(s) = \frac{1}{s}$

$$C(s) = \frac{b_0 s^m + \cdots + b_m}{a_0 s^n + \cdots + a_n} \cdot \frac{1}{s}$$

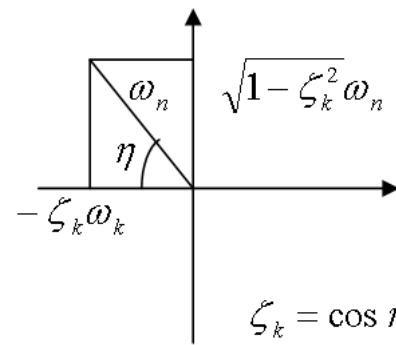
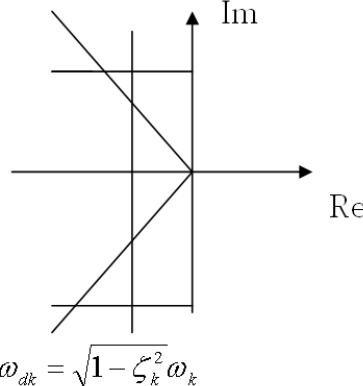
characteristic equation

$$\begin{aligned} a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n &= 0 \\ s = p_i \quad i &= 1, \dots, q \\ s = -\zeta_k \omega_k \pm \sqrt{1 - \zeta_k^2} \omega_k j \quad k &= 1, \dots, r \end{aligned}$$

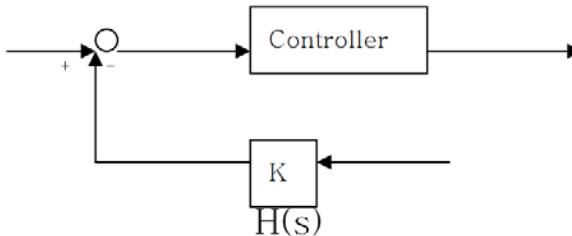
zero ; $s = Z_i \quad i = 1, \dots, m$

$$C(s) = \frac{k \prod_{i=1}^m (s - z_i)}{s \prod_{j=1}^q (s - p_j) \prod_{k=1}^r (s^2 + 2\zeta_k \omega_k s + \omega_k^2)} = \frac{a}{s} + \sum_{j=1}^q \frac{a_j}{s - p_j} + \sum_{r=1}^r \frac{b_r (s + \zeta_r \omega_r) + C_r \omega_r \sqrt{1 - \zeta_r^2}}{s^2 + 2\zeta_r \omega_r s + \omega_r^2}$$

$$C(t) = a + \sum_{j=1}^q a_j e^{p_j t} + \sum_{k=1}^r b_k e^{-\zeta_k \omega_k t} \cos \omega_k \sqrt{1 - \zeta_k^2} t + \sum_{k=1}^r C_k e^{-\zeta_k \omega_k t} \sin \omega_k \sqrt{1 - \zeta_k^2} t$$

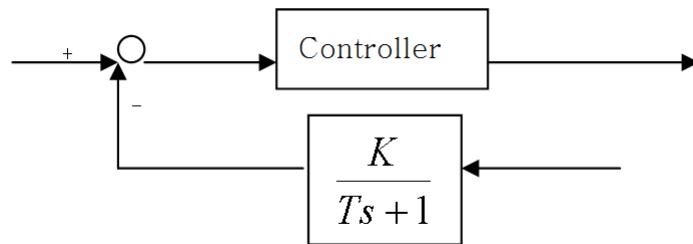


Effect of sensors on system performance

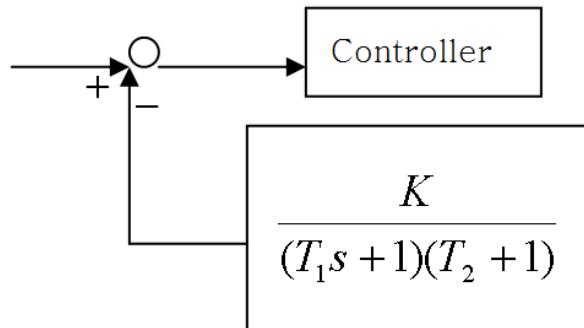


- **Fast sensor dynamics** ; $H(s)=\text{constant}$

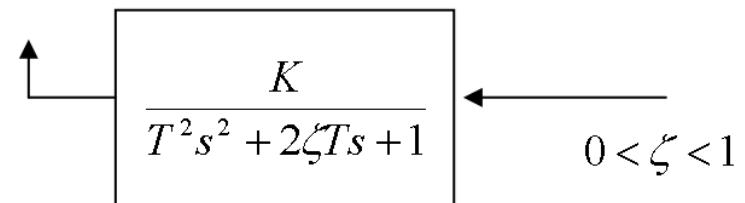
- First order sensor



- Overdamped second order

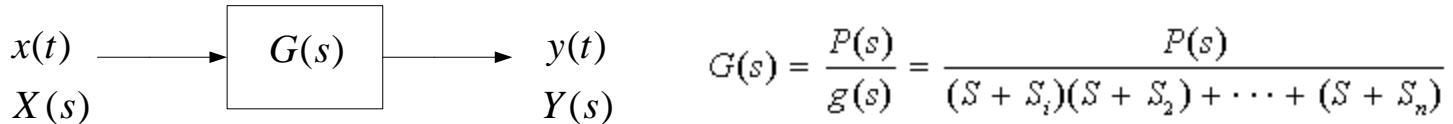


- underdamped second order



Phase Lead and Phase Lag in Sinusoidal Response

Linearsystem



위 system에 $x(t) = X \sin \omega t$ 를 넣어주면 $y(t)$ 는 ?

$$X(s) = \frac{X\omega}{s^2 + \omega^2}$$

$$\begin{aligned} \Rightarrow Y(s) &= G(s)X(s) = G(s) \frac{X\omega}{s^2 + \omega^2} \\ &= \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \dots + \frac{b_n}{s + s_n} + \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} \end{aligned}$$

1) Transient response → Stable system인 경우 = 0

2) Stable system인 경우 Steady state response 는

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$a = G(s) \frac{\omega}{s^2 + \omega^2} (-s + j\omega)_{s=-j\omega} = -\frac{XG(-j\omega)}{2j}$$

$$\bar{a} = G(s) \frac{X\omega}{s^2 + \omega^2} (s - j\omega)_{s=j\omega} = \frac{XG(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)|e^{j\phi}$$

$$G(-j\omega) = |G(j\omega)|e^{-j\phi}, \phi = phase$$

$$\therefore y(t) = X|G(j\omega)| \frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j} = X|G(j\omega)| \sin(\omega t + \phi)$$

$$= Y \sin(\omega t + \phi), Y = X|G(j\omega)|, \phi = \angle G(j\omega)$$

Phase Lead and Phase Lag in Sinusoidal Response

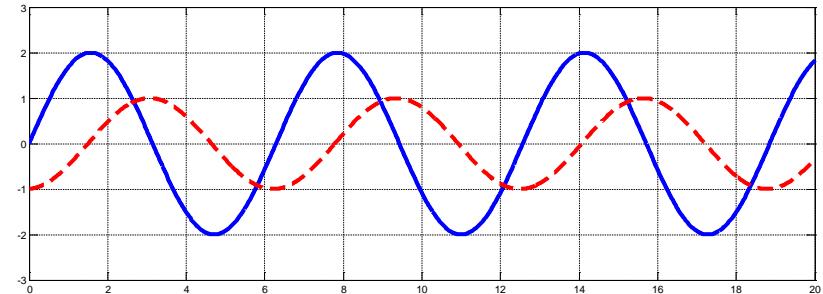
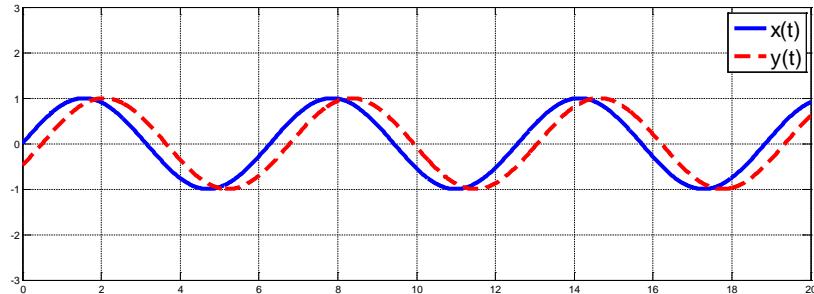
Example) First order system

$$G(s) = \frac{K}{Ts+1}$$

$$G(j\omega) = \frac{K}{Tj\omega+1}, |G(j\omega)| = \frac{K}{\sqrt{1+T^2\omega^2}}$$

$$G(j\omega) = \frac{K}{Tj\omega+1}, |G(j\omega)| = \frac{K}{\sqrt{1+T^2\omega^2}}, \angle G(j\omega) = -\tan^{-1} T\omega$$

$$\therefore y_{ss}(t) = X \frac{K}{\sqrt{1+T^2\omega^2}} \sin(\omega t + \phi), \quad \phi = -\tan^{-1} T\omega$$



$\phi < 0$: phase lag, $\phi > 0$: phase lead

End of section 5

5-6 routh stability criterion

5-7 Effects of Integral and derivative control

5-8 steady state error in unity feedback
control systems

Next Root locus analysis