System Control

7. Control System Design by the Root-Locus Method

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Control system design

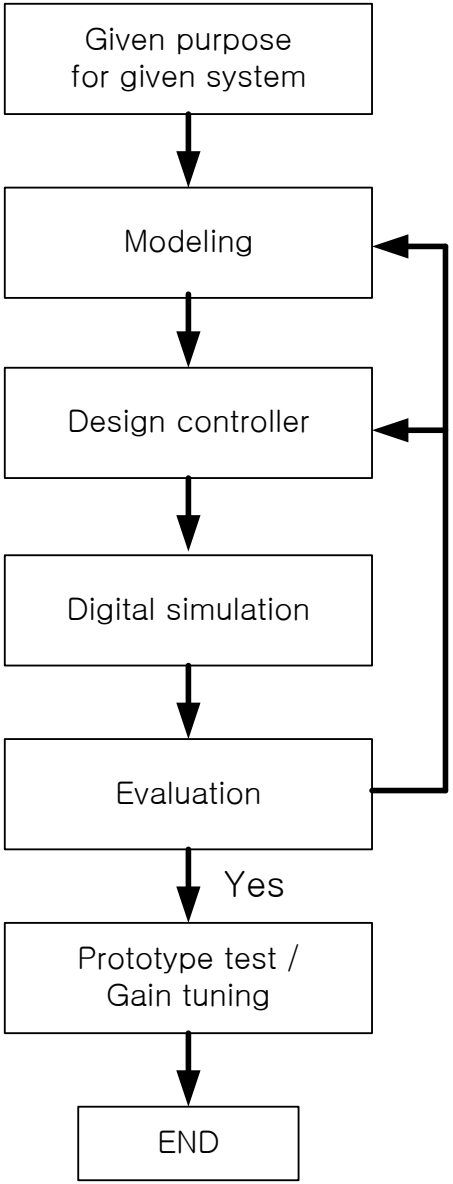
- **Control System Design ; to perform specific tasks**

- **requirements ; performance specifications**
  - **accuracy (steady state error)**
  - **relative stability (ζ , damping ratio)**
  - **speed of response (T , $e^{-\zeta\omega_n t}$, ω_n : natural frequency)**

- **Given purpose → precise performance specifications**
  → an optimal control system

- **Controller ;**
  - **digital controller , in many cases**
  - **microprocessor based controller**
  - **software; control algorithm**
  - **hydraulic**
    pneumatic
    electronic (analog)
    mechanical

Design procedure



1) Build up → test month to years

Suspension

- **spring, damper**
- **tune the parameter**
- **test/evaluation (road tests)**

2) Lab tests (simulator) week to months

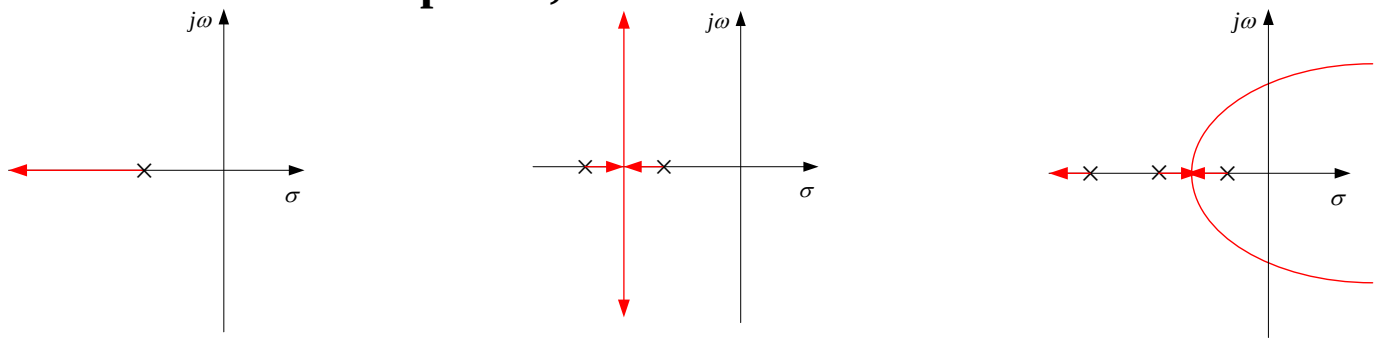
3) Computer simulation (software) day to weeks
/ test 검증 (vehicle test/final tuning)

자동차	48개월 개발기간 ;	80년대 초반
	36개월	80년대 말
	30개월	90년대 초반
	24개월	2000년대

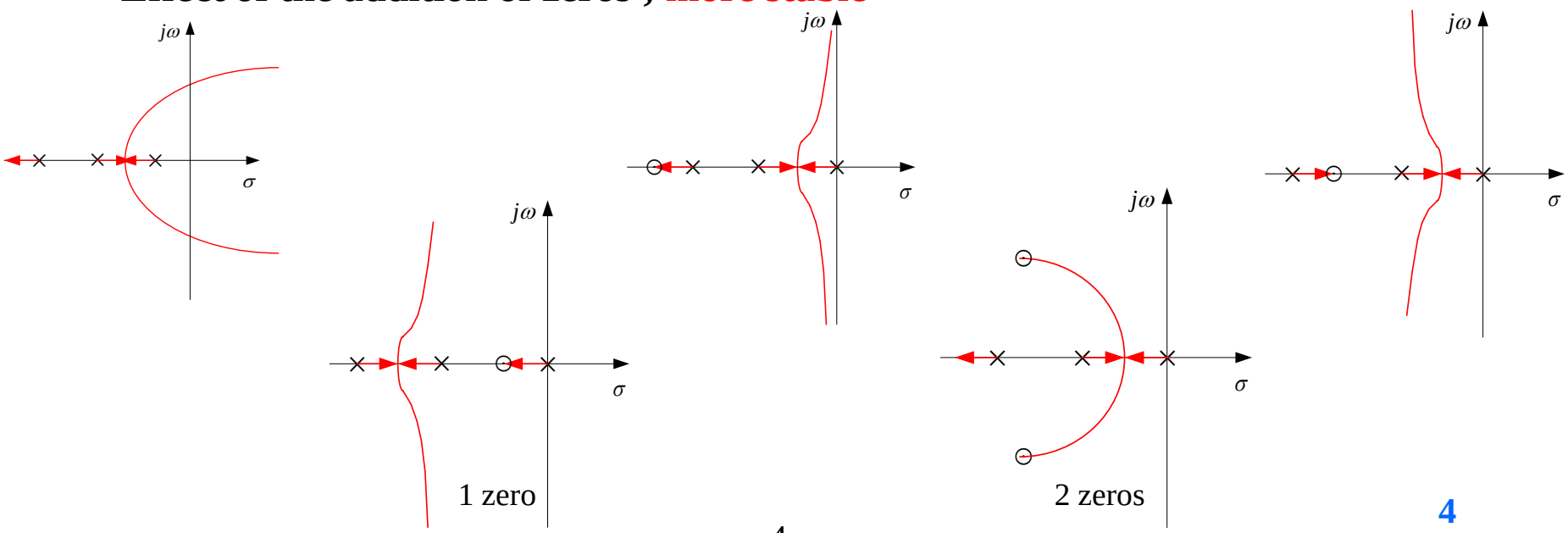
Preliminary Design Considerations

- **Root – Locus approach ;** **- graphical method**
  - gain or parameter variations

- **Effect of the addition of poles ; less stable**



- **Effect of the addition of zeros ; more stable**



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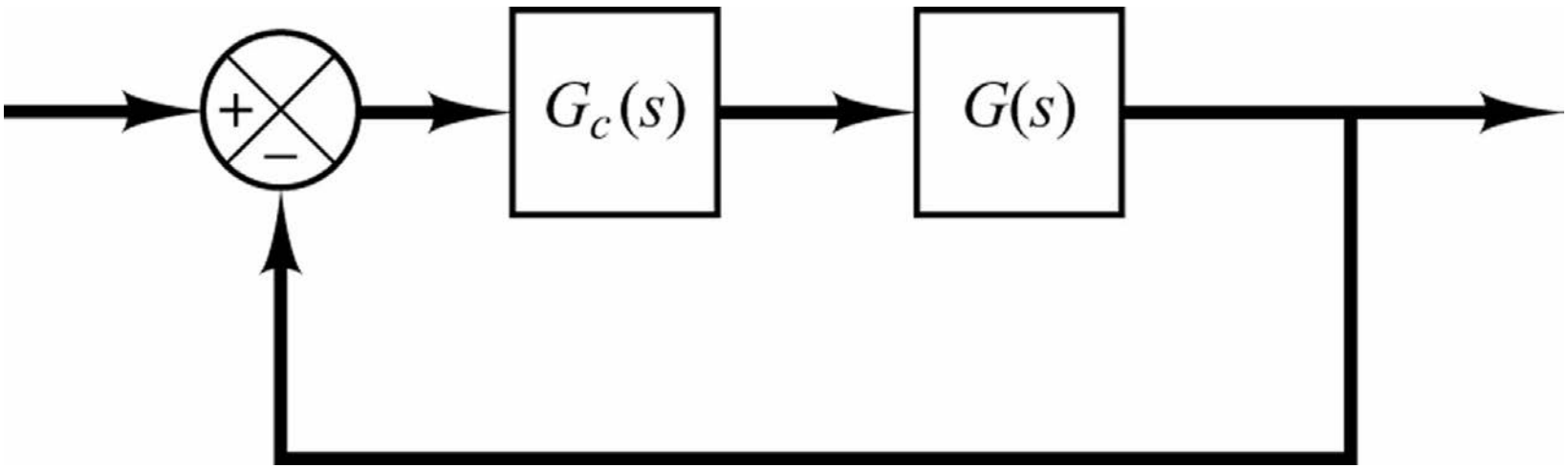
Lead/Lag Compensation



$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

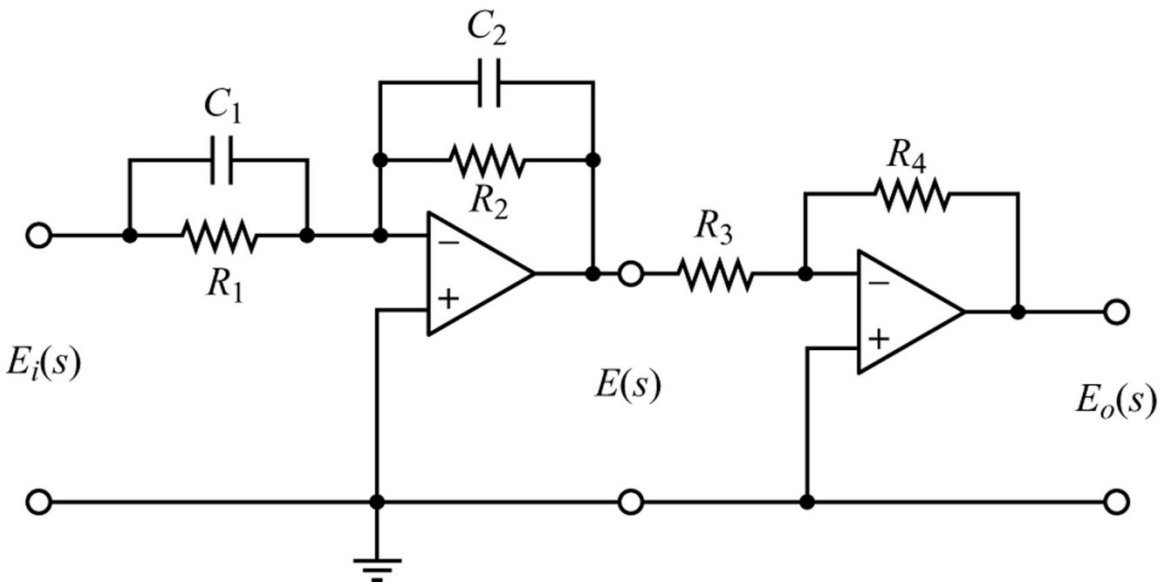
Lead / Lag Compensation

$$\frac{E_o(s)}{E_i(s)} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

$$T = R_1 C_1$$

$$\alpha T = R_2 C_2$$

$$K_c = \frac{R_4 C_1}{R_3 C_2}$$



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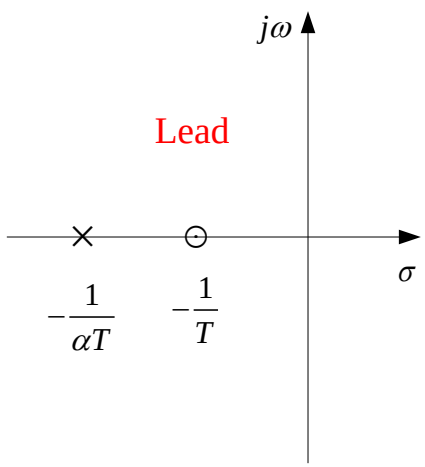
Lead / Lag Compensation

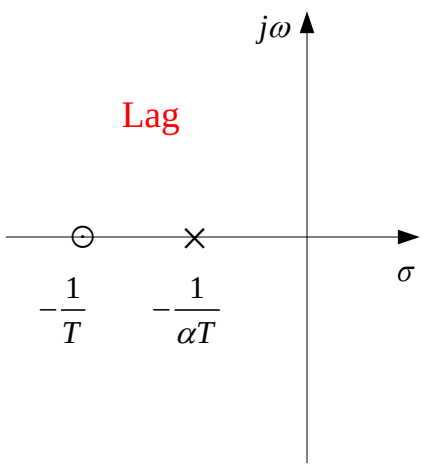
$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

When $s = j\omega$ $\qquad \measuredangle G_c(j\omega) = \measuredangle\left(j\omega + \frac{1}{T}\right) - \measuredangle\left(j\omega + \frac{1}{\alpha T}\right)$

Lead Compensator ← $\quad \frac{1}{T} < \frac{1}{\alpha T} \qquad \measuredangle G_c(j\omega) > 0 \qquad 0 < \alpha < 1$

Lag Compensator ← $\quad \frac{1}{T} > \frac{1}{\alpha T} \qquad \measuredangle G_c(j\omega) < 0 \qquad 1 < \alpha$





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Root Locus / Lead/Lag Compensator

• **Root locus lead/lag compensator : powerful tool**

when ; specification – time domain spec.

➔ • **damping ratio**
• **natural frequency**
• **dominant closed loop poles**
• **rise time etc.**

① **Time domain spec**

② **unstable or**
undesirable transient-response characteristics

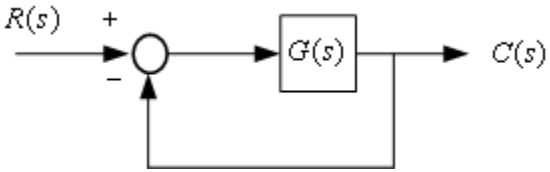
➔ **Lead Compensators (relocate dominant closed loop poles)**

Example

Ex.

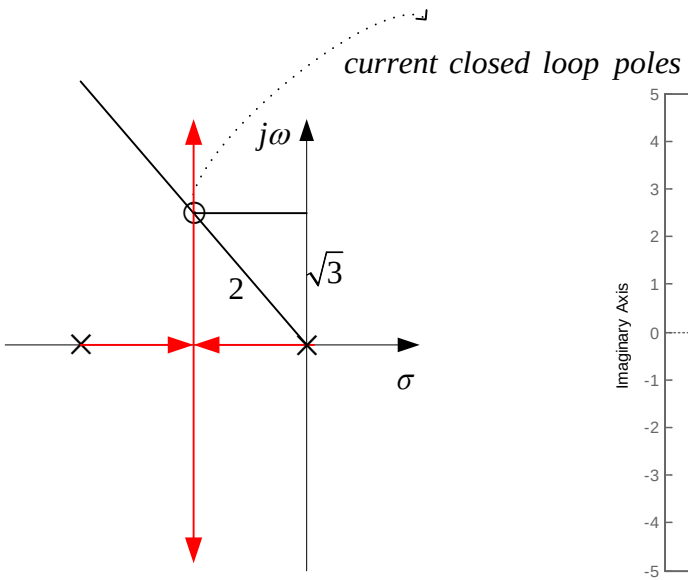
$$G(s)=\frac{4}{s(s+2)}$$

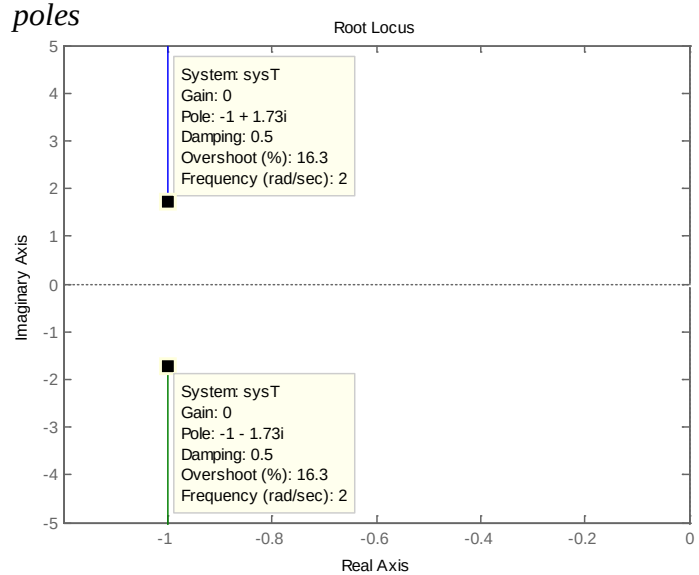
$$\frac{C(s)}{R(s)}=\frac{4}{s^2+2s+4}$$ **, closed loop poles :** $s=-1\pm\sqrt{3}j$



$$\begin{cases}\zeta=0.5\\ \omega_n=2\\ \textbf{The static velocity error constant } K_v=2\end{cases}$$

$$\begin{cases}E(s)=\dfrac{1}{1+G(s)}R(s)\\ e_{ss}(t)=\lim\limits_{s\to 0}s\dfrac{1}{1+G(s)}\dfrac{1}{s^2}\\ \qquad=\lim\limits_{s\to 0}\dfrac{1}{sG(s)}=\dfrac{1}{K_v}=\dfrac{1}{2}=0.5\end{cases}$$





Desired : $\omega_n=4,\zeta=0.5$

$$\Rightarrow s=-2\pm 2\sqrt{3}j$$

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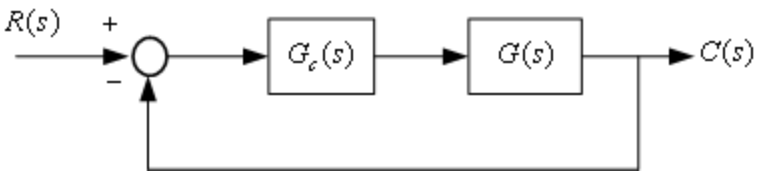
Example

Desired : $\omega_n = 4, \zeta = 0.5$

$$\Rightarrow s = -2 \pm 2\sqrt{3}j$$



1) Angle of deficiency ϕ

$$\measuredangle \frac{4}{s(s+2)}\bigg|_{s=-2\pm 2\sqrt{3}j} = -210^\circ$$

$$\phi = 30^\circ, \ = \measuredangle K \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = 30^\circ, \quad 0 < \alpha < 1$$

➔ **Lead Compensator**

2) Choose $\frac{1}{T}$ $\frac{1}{\alpha T}$ **such that** $\measuredangle \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = 30^\circ$

가능한한 α 가 크게 ➔ K_v 증가 **(Good)**

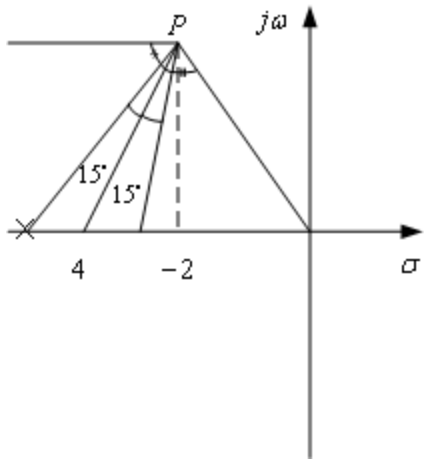
Text 방법 ➔ ploe = -5.4 zero = -2.9 α **=0.536**

$$T = \frac{1}{2.9} = 0.345$$

$$\alpha T = \frac{1}{5.4} = 0.185$$

Example

Pole-zero location for large α



$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+G_c(s)G(s)} \frac{1}{s^2}$$

$$= \lim_{s\to 0} \frac{1}{sG_c(s)G(s)} = \frac{1}{K_v}$$

$$K_v = \lim_{s\to 0} sG_c(s)G(s)$$

$$= \lim_{s\to 0} K \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \frac{4}{s+2}$$

$$= K \cdot \alpha \cdot 2$$

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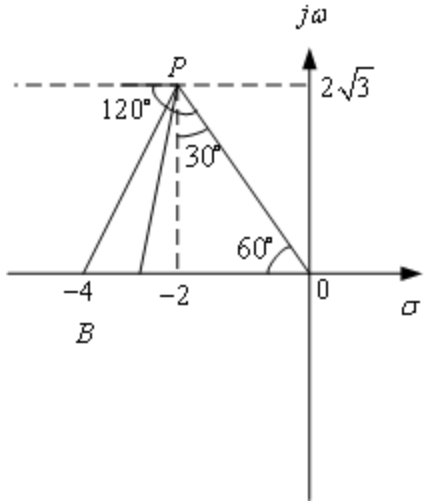
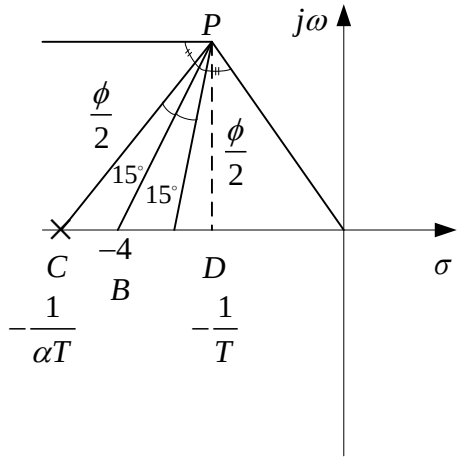
Example

Lead Compensator : pole-zero selection for large α



P **: desired pole**

$$-2+2\sqrt{3}j$$



C **: pole :** $-\dfrac{1}{\alpha T}$

D **: zero :** $-\dfrac{1}{T}$

Lead Compensator

Fig 7-9 Ogata

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Example

3) Magnitude Condition

$$G_c(s)G(s) = K_c \frac{s+2.9}{s+5.4}\frac{4}{s(s+2)}$$

$$1+G_c(s)G(s)\big|_{s=-2+2\sqrt{3}j} = 0$$

$$\left|G_c(s)G(s)\right| = 1 = K_c \left|\frac{s+2.9}{s+5.4}\right|\left|\frac{4}{s(s+2)}\right|_{s=-2+2\sqrt{3}j}$$

➔ $K_c = 4.68$

➔ **Lead Compensator**

$$G_c(s) = 4.68\frac{s+2.9}{s+5.4} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = 2.51\frac{0.345s+1}{0.185s+1}$$

<u>Analog controller</u>

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3}\frac{R_1C_1s+1}{R_2C_2s+1}$$

Arbitrarily chosen $\begin{cases} C_1 = C_2 = 10\mu F \\ R_3 = 10k\Omega \end{cases}$

<u>Digital controller</u>

$$\frac{u}{e} = G_c(s) = K_1 + \frac{a}{0.185s+1}$$

$$u = u_1 + u_2, \begin{cases} u_1 = K_1 e \\ 0.185\dot{u}_2 + u_2 = ae \\ \sim> u_2 \end{cases}$$

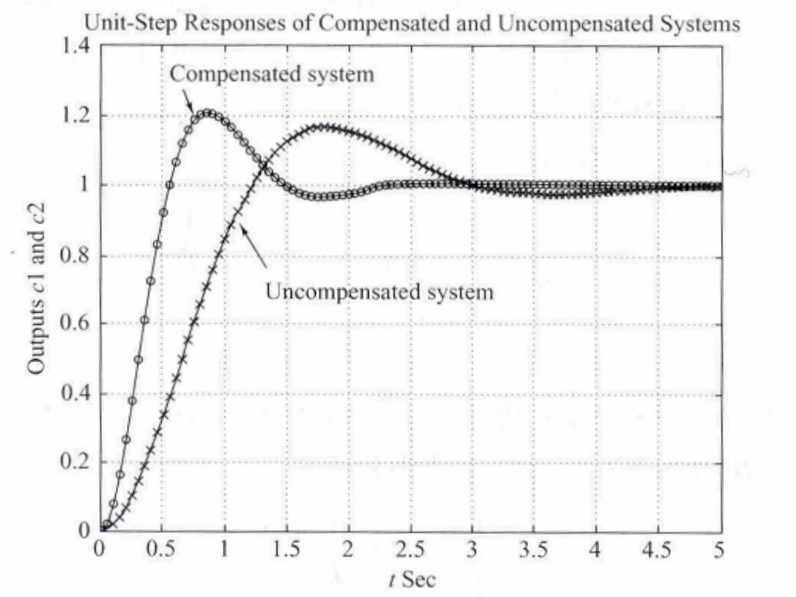

그림 7-11 보상된 시스템과 보상되지 않은 시스템의 단위계단응답

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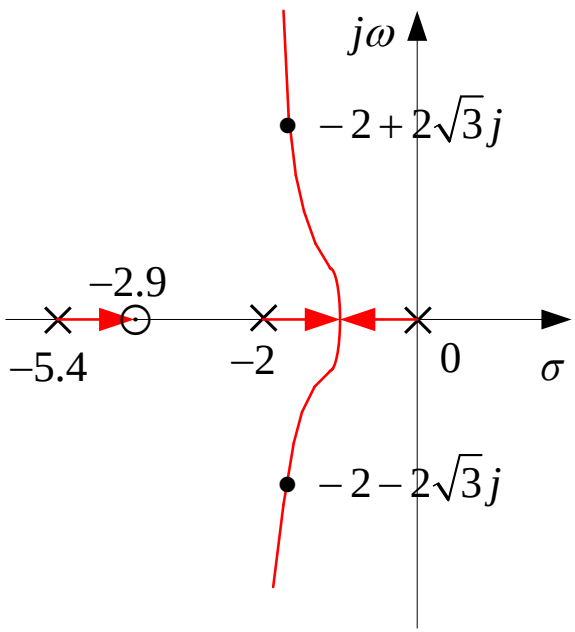
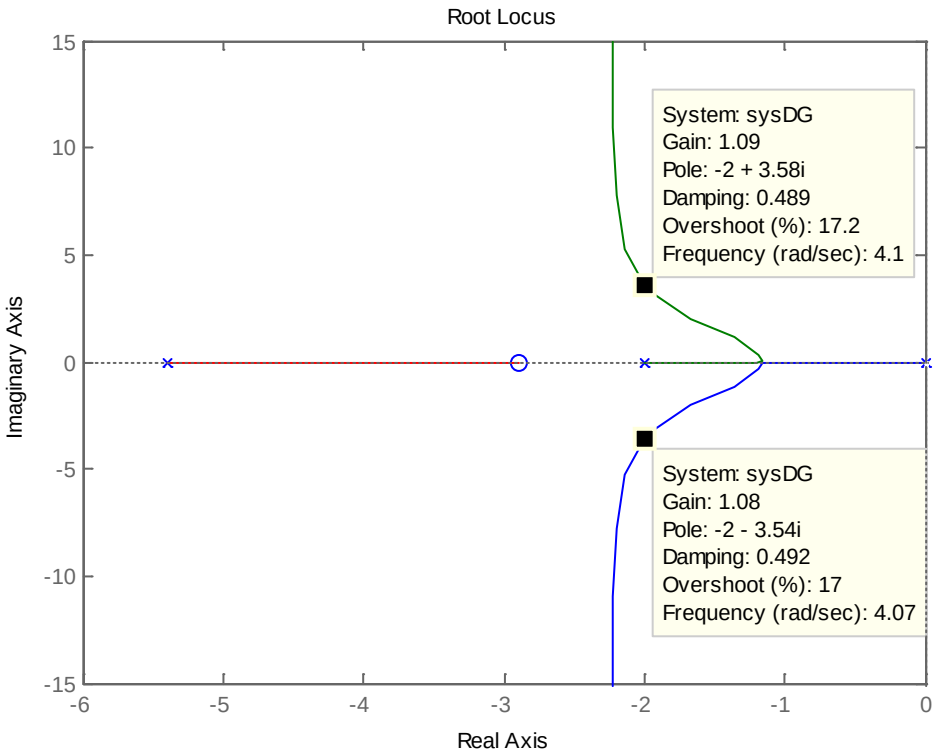
Example

Root locus of the compensated system

$$K_c \frac{s+2.9}{s+5.4} \frac{4}{s(s+2)}$$





➔ **Comparison of step responses**
- **original system**
- **compensated system**

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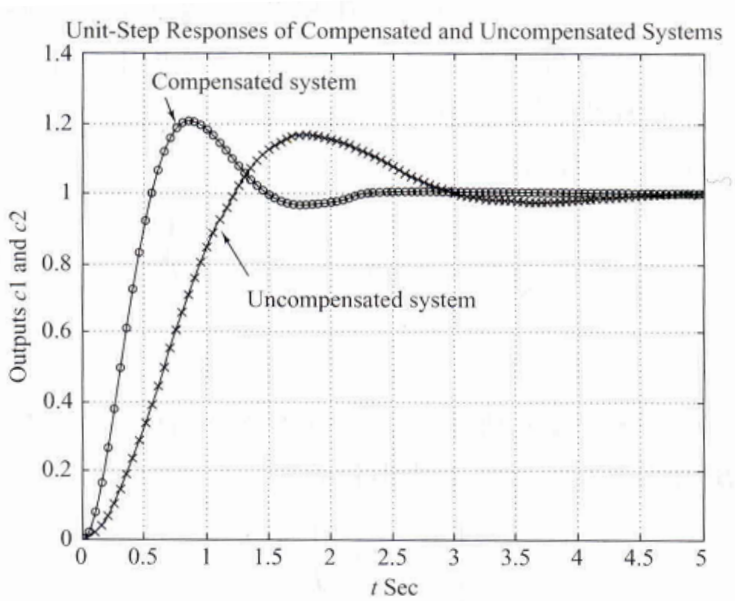
Example

Root locus of the compensated system $K_c \dfrac{s+2.9}{s+5.4} \dfrac{4}{s(s+2)}$



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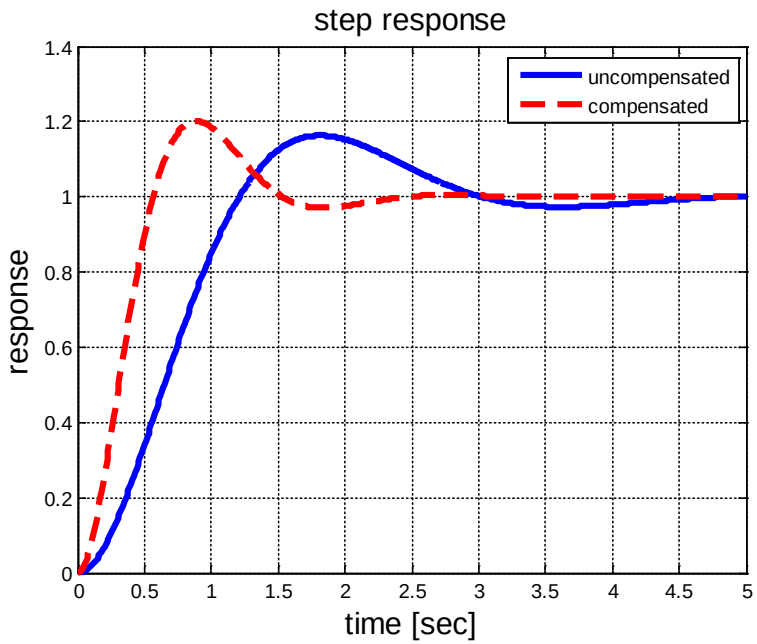

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Example

4) Static velocity error constant ;

➔ 가능한한 큰 α 가 K_v 증가에 기여 , K_v 가 **lead Compensator**로 충분히 증가하지 못한 경우 **Lag compensator** 추가 ➔ **Lead-Lag Compensator**

$$e_{ss}(t) = \lim_{s\to 0} s\frac{1}{1+G_c(s)G(s)}R(s) = \lim_{s\to 0} s\frac{1}{1+G_c(s)G(s)}\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{sG_c(s)G(s)} = \frac{1}{\cancel{s}\cdot 2.51\left(\frac{0.345s+1}{0.185s+1}\right)\frac{4}{\cancel{s}(s+2)}}$$

$$= \frac{1}{2.52\times 2} = 0.1984$$

$$K_v = 5.04 \;\; (old \;\; K_v = 2)$$

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The Procedure for design a Lead compensator

1) Performance Spec.

→ **desired locations for the dominant closed-loop poles**

2) Draw the root locus

(1) Check whether or not the gain adjustment alone → satisfactory results

(2) If not calculate the angle deficiency ϕ

3) $$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = K_c \alpha \frac{Ts + 1}{\alpha Ts + 1}$$

dc gain ; $K_c \alpha$

α, T **; compensate angle deficiency**

4) Tune α, T **to locate the closed – loop poles at the desired locations**

$-\alpha$ **; as large as possible**

5) Tune K_c **from the magnitude condition**

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Lag Compensation

$$G_c(s) = \hat{K}_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} \quad , \beta > 1$$

① **satisfactory transient response**
② **unsatisfactory steady state characteristics**
i) closed loop poles : no change
$\beta > 1$
ii) open loop gain increase
➔ **Lag compensator**

Procedure

1) Draw the root-locus plot for the uncompensated system
→ **locate the dominant closed loop poles**

2)
$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}$$

calculate $\measuredangle G_c(s)$

3) Evaluate the particular static error constant specified ;
The static velocity error constant

4) Determine pole and zero of the lag compensator s.t.
① **the static error constant is sufficiently large**
② **without altering the original root loci.**

$$\left| \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} \right| \simeq 1$$

Angle $\measuredangle G_c(s) \le 50$
pole, zero close together and near the origin of the s-plane

5) Draw a new root-locus plot

6) Adjust $\hat{K}_c$ **from the magnitude condition**

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Lag Compensation

The static velocity gain ;

Uncompensated system $K_v = \lim_{s \to 0} sG(s)$

Compensated system $\hat{K}_v = \lim_{s \to 0} sG_c(s)G(s) = \lim_{s \to 0} G_c(s)K_v = \underbrace{\hat{K}_c \cdot \beta}\cdot K_v$

$\beta = factor$

$\hat{K}_c \approx 1$

Increased by a factor of $\hat{K}_c \cdot \beta$

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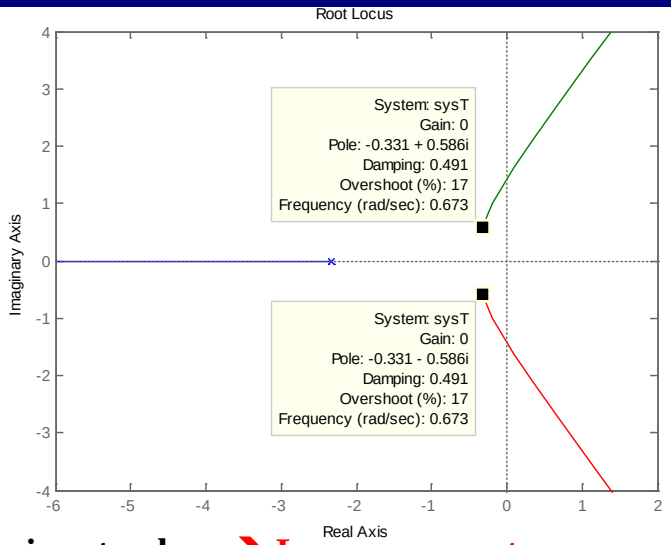
Example

Ex.

$$G(s)=\frac{1.06}{s(s+1)(s+2)}$$

$$\frac{C(s)}{R(s)}=\frac{1.06}{s(s+1)(s+2)+1.06}$$



The dominant closed-loop poles ; $s=-0.3307\pm 0.5864j$

$\zeta=0.491$

$\omega_n=0.673 rad/\sec$

$K_v=0.53\sec^{-1}$

Increase $K_v\to 5\sec^{-1}$ **without change of the location of dominant poles** ➔ **Lag compensator**

1)

$$G_c(s)=\hat{K}_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad \hat{K}_v=\hat{K}_c\beta K_v$$

2) Let $\beta=10,\ \frac{1}{T}=0.05,\ \frac{1}{\beta T}=0.005$

$$G_c(s)=\hat{K}_c\frac{s+0.05}{s+0.005}$$

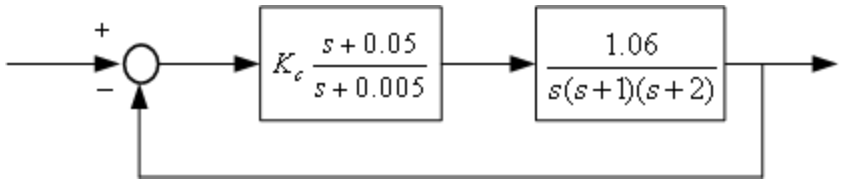
Angle contribution near a dominant closed loop poles

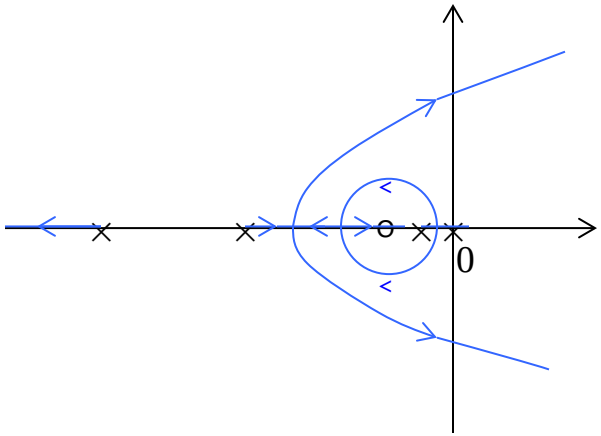
$$\measuredangle \hat{K}_c\frac{s+0.05}{s+0.005}\bigg|_{s=-0.3307\pm j0.5864}\simeq 4^\circ$$

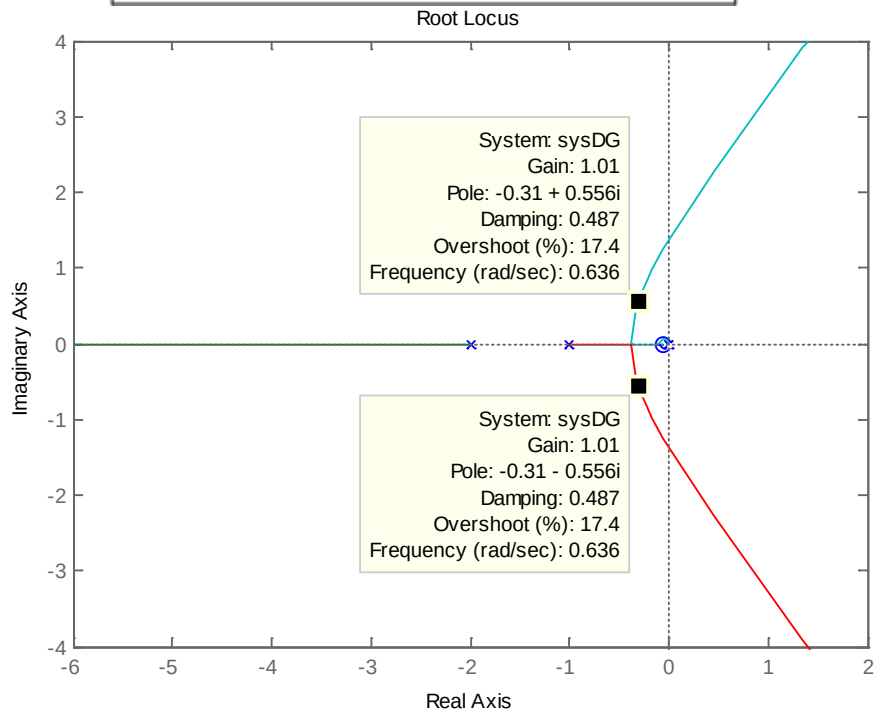
➔ **No significant change**

Example

3) $G_c(s)G(s) = \hat{K}_c \dfrac{s+0.05}{s+0.005} \cdot \dfrac{1.06}{s(s+1)(s+2)}$







- **dominant closed loop poles same damping ratio**

➔ $s = -0.31 \pm j0.55$

4) Magnitude condition

$$\left| \hat{K}_c \frac{s+0.05}{s+0.005} \cdot \frac{1.06}{s(s+1)(s+2)} \right|_{s=-0.31 \pm j0.55} = 1$$

$$\Rightarrow \hat{K}_c = 0.9656$$

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Example

Verification ;

$$\frac{1}{\hat{K}_v} = \lim_{s\to 0} s \frac{1}{1+G_c(s)G(s)} \frac{1}{s^2}$$

$$\hat{K}_v = \lim_{s\to 0} sG_c(s)G(s)$$

$$= 0.9656 \frac{s+0.05}{s+0.005} \frac{1.06}{(s+1)(s+2)}$$

$$= 5.12\,\text{sec}^{-1} \simeq \beta K_{v,old}$$

5) $s_3 = -2.326 \quad s_4 = -0.0549$

A long tail of small amplitude

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Example

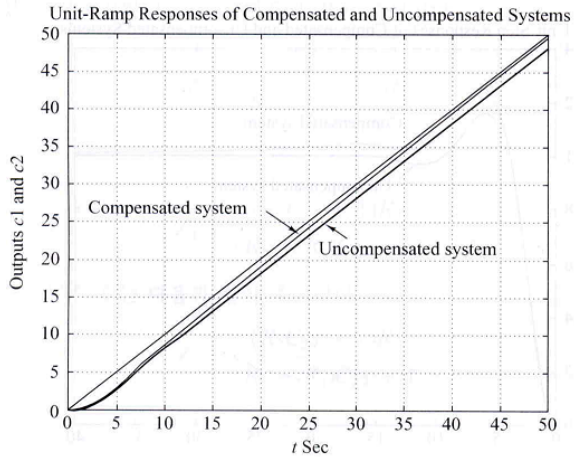
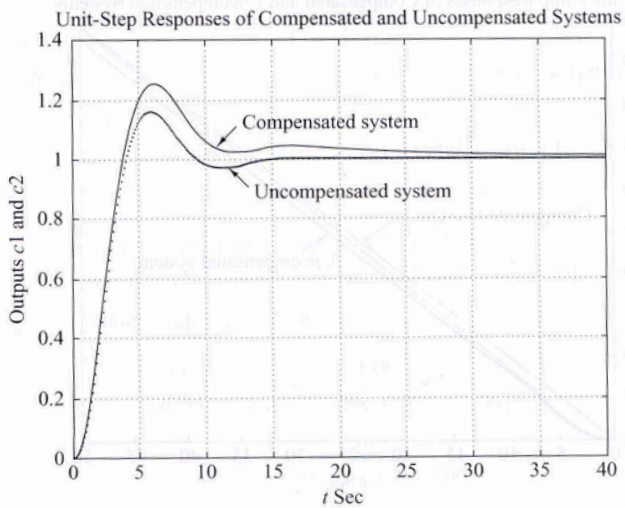


그림 7-16 보상된 시스템과 보상되지 않은 시스템의 단위램프응답 [보상기는 식 (7-3)으로 주어짐]



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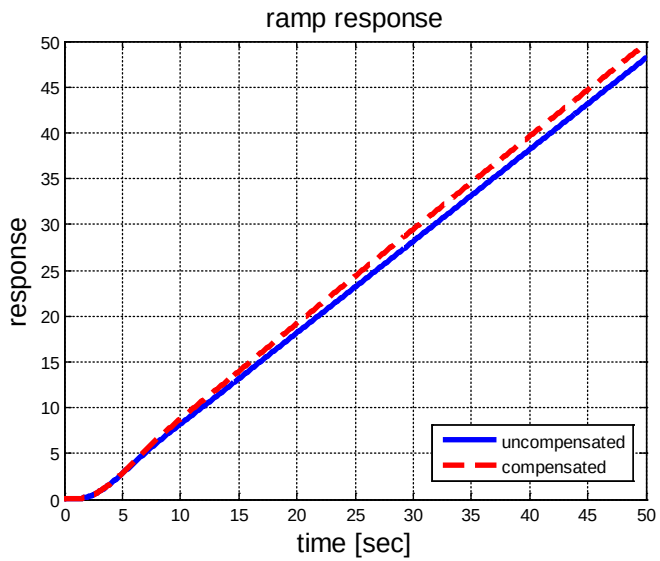

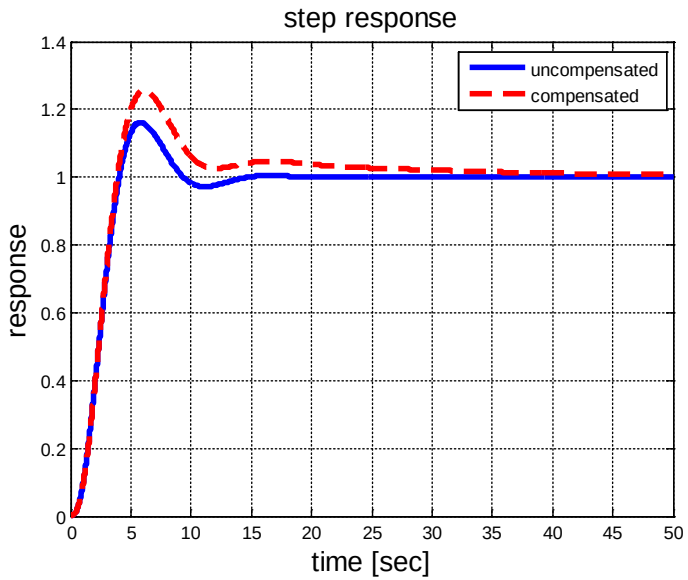

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Lag-Lead Compensation

Lead compensator → speeds up, increase the stability

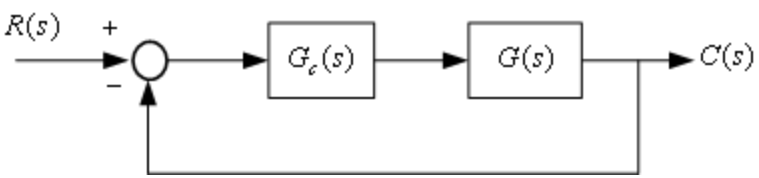
Lag compensator → improves the steady state accuracy but reduce the speed

→Lag-Lead compensators

Design procedure

1) **Design lead compensator to relocate the dominant poles**
2) **Design lag compensator to improve the s.s. gain**

$$G_c(s) = K_c \underbrace{\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right)}_{\substack{Lead \\ \gamma>1}} \underbrace{\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)}_{\substack{Lag \\ \beta>1}}$$



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Lag-Lead Compensation

<u>Spec.</u> **1) dominant poles**
2) K_v **(steady state velocity error constant)**

Step 1. Lead Compensator

1) choose desired poles s_1

2) angle condition

$$\measuredangle K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \right) \left(G(s_1) \right) \Bigg|_{s=s_1} = 180^\circ \left(2k+1 \right)$$

→ **determine the angle deficiency** ϕ

$$\measuredangle \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \Bigg|_{s=s_1} = \phi = 180^\circ \left(2k+1 \right) - \measuredangle G(s_1)$$

(infinitely many $\gamma,\ T$ **) ; choose one**

3) magnitude condition

$$\left| K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} G(s_1) \right| = 1$$

→ K_c

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Lag-Lead Compensation

Step 2. Lag Compensator, K_v spec

1) $$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sK_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)G(s) = \lim_{s\to 0} sK_c \frac{\beta}{\gamma}G(s)$$

$\rightarrow \beta$

2) choose T_2 such that

i) $$\left|\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right| \doteq 1$$

ii) $$\underbrace{-5^\circ < \measuredangle \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} < 0^\circ}$$

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Example

Ex

$$G(s) = \frac{4}{s(s+0.5)}$$

The closed-loop poles ;

$$s = -0.25 \pm 1.9843j$$
$$\zeta = 0.125$$
$$\omega_n = 2 rad/\sec$$
$$K_v = 8\sec^{-1}$$

Desired spec :

$$\begin{cases} \zeta = 0.5 \\ \omega_n = 5 \\ K_v = 80 \end{cases}$$

➔ **Lead – Lag compensator**

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right), \ \gamma > 1, \ \beta > 1$$

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Example

Step 1. Lead Compensator

1) choose desired poles $s_1 = -2.50 \pm j4.33$

2) angle condition

$$\measuredangle \frac{4}{s(s+0.5)}\bigg|_{s=s_1} = -235^\circ$$

→ the angle deficiency = 55° $\measuredangle G_c(s) + \measuredangle G(s) = -180^\circ(2k+1)$

choose T_1 **, such that pole-zero cancelation happens**

let $s + \frac{1}{T_1} = s + 0.5 \quad \Rightarrow \quad T_1 = 2$

→ angle condition

$$\gamma = 10.04$$

$$\measuredangle(s_1 + 0.5) + \measuredangle(s_1 + \gamma 0.5) = 55^\circ$$

3) magnitude condition

$$\left| K_c \frac{s+0.5}{s+5.021} \frac{4}{s(s+0.5)} \right|_{s=s_1} = 1$$

$$\Rightarrow K_c = 6.26$$

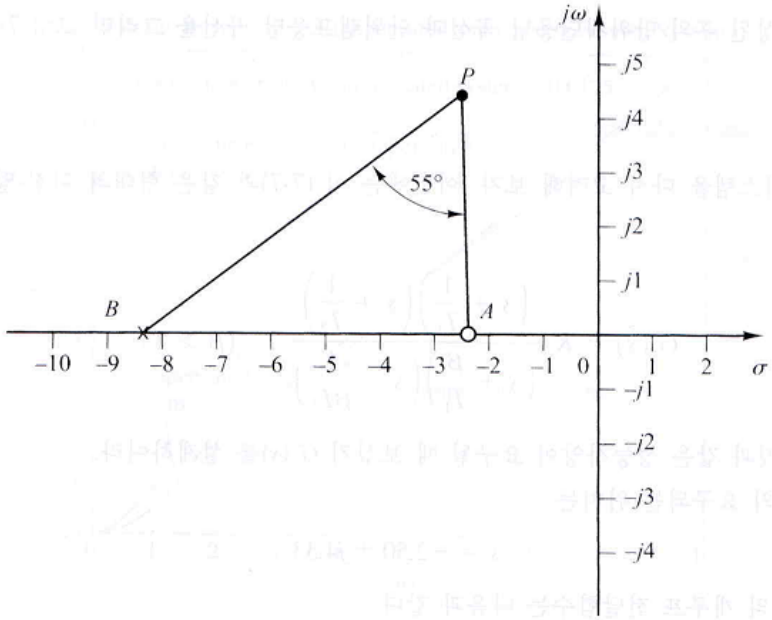

그림 7-23 요구되는 극점-영점 위치의 결정

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Example

Step 2. Lag Compensator

1) $$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sK_c \frac{\beta}{\gamma} G(s)$$

$$= \lim_{s\to 0} s\left(6.26\right)\frac{\beta}{10.04}\frac{4}{s\left(s+0.5\right)} = 4.988\beta = 80$$

$$\Rightarrow \beta = 16.04$$

2) choose T_2 such that

i) $$\left|\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right|_{s=s_1} \doteq 1$$

ii) $$-5^\circ < \measuredangle \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} < 0$$

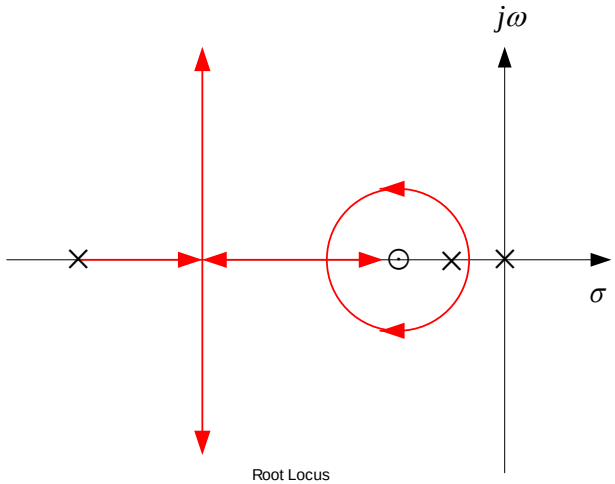
$$\Rightarrow T_2 \geq 5 \quad \Rightarrow let\ T_2 = 5$$

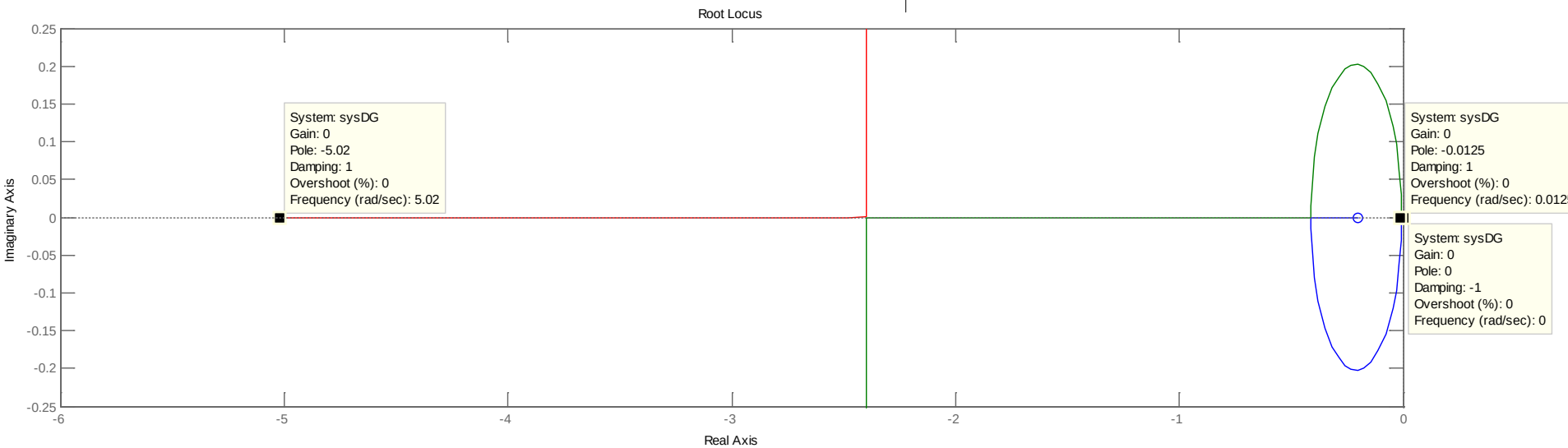
$$G_c(s) = 6.26\left(\frac{s+\frac{1}{2}}{s+\frac{10.04}{2}}\right)\left(\frac{s+\frac{1}{5}}{s+\frac{1}{16.04\times 5}}\right)$$

$$G_c(s)G(s) = \frac{25.04(s+0.2)}{s(s+5.02)(s+0.01247)}$$

Example

Root locus of the compensated system





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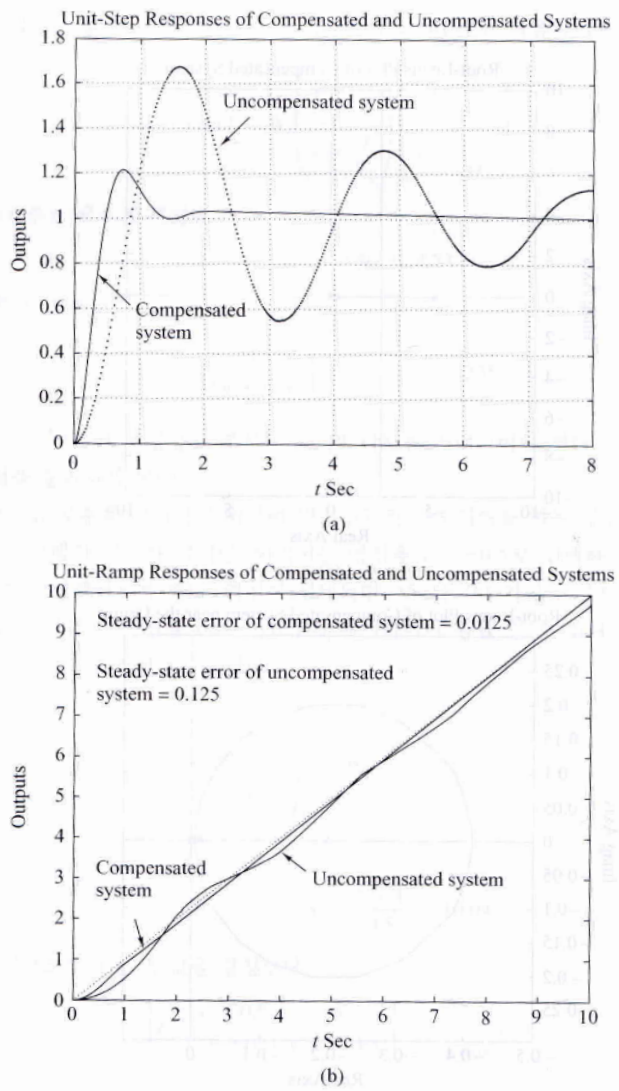
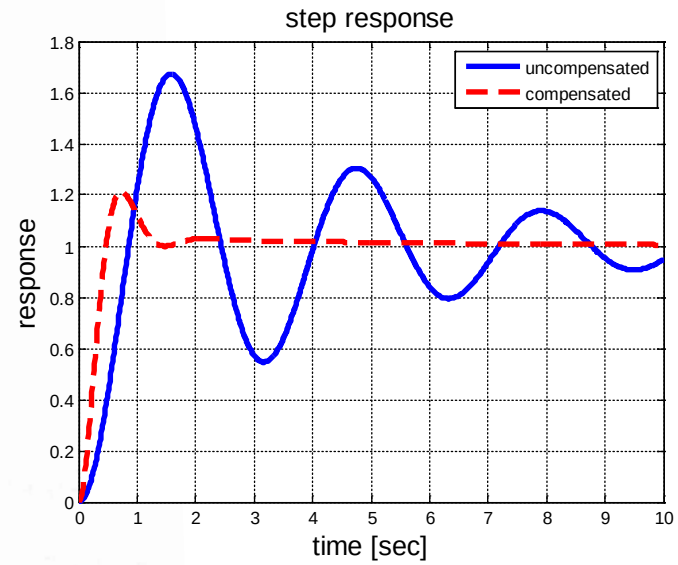
Example





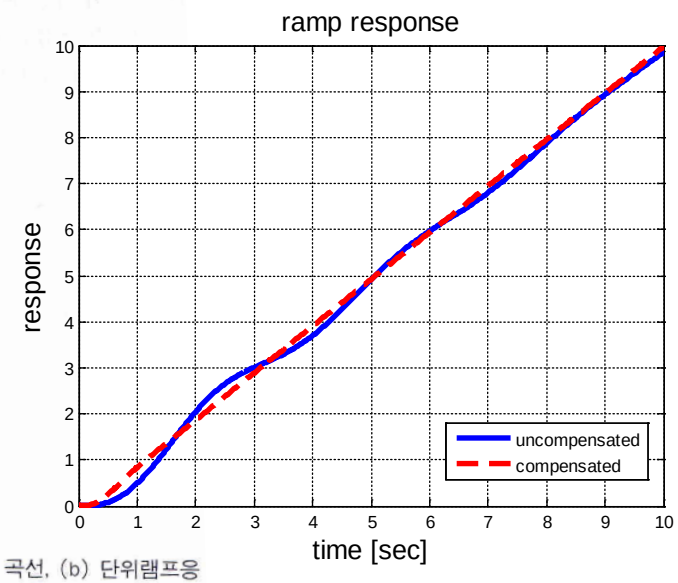


그림 7-22 보상된 시스템과 보상되지 않은 시스템의 과도응답 곡선: (a) 단위계단응답 곡선, (b) 단위램프응답 곡선.

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End of lecture note 7