System Control

8. Control Systems Analysis and Design by the Frequency- Response Method

Professor Kyongsu Yi

Vehicle Dynamics and Control Laboratory

Seoul National University

1. Frequency response

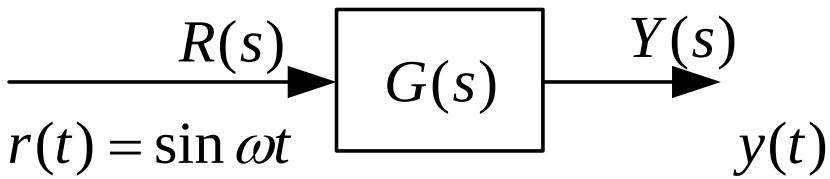
   Bode plots, polar plots

2. Nyquist stability criterion

3. Control systems design by frequency-response approach

   Lead-lag compensation

Frequency Response Analysis

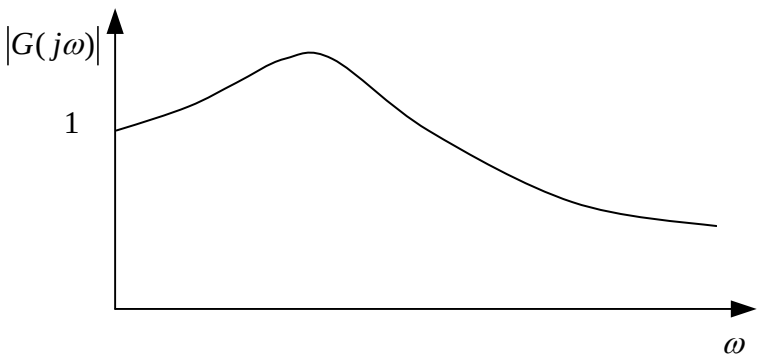


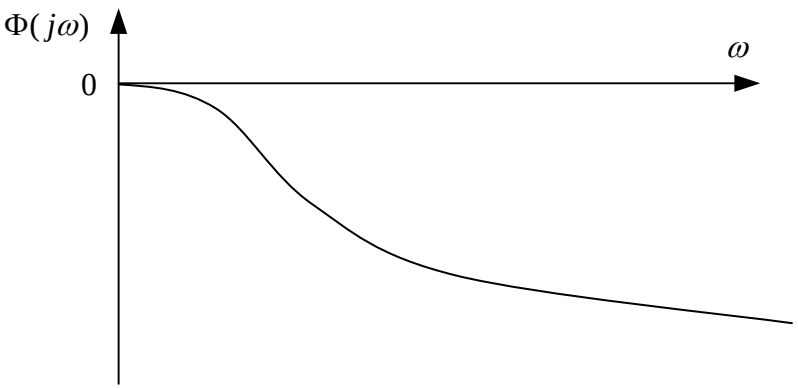
$$Y(s) = G(s)R(s) = G(s)\frac{\omega}{s^2+\omega^2} = \cdots + \cdots + \frac{as+b}{s^2+\omega^2}$$

$$G(s) = \frac{p(s)}{q(s)} \qquad q(s) = 0 \text{ ; poles}$$

- **Steady State**

$$y(t) = A\sin(\omega t + \phi)$$
$$= |G(j\omega)|\sin(\omega t + \phi) \qquad \phi = \angle G(j\omega)$$





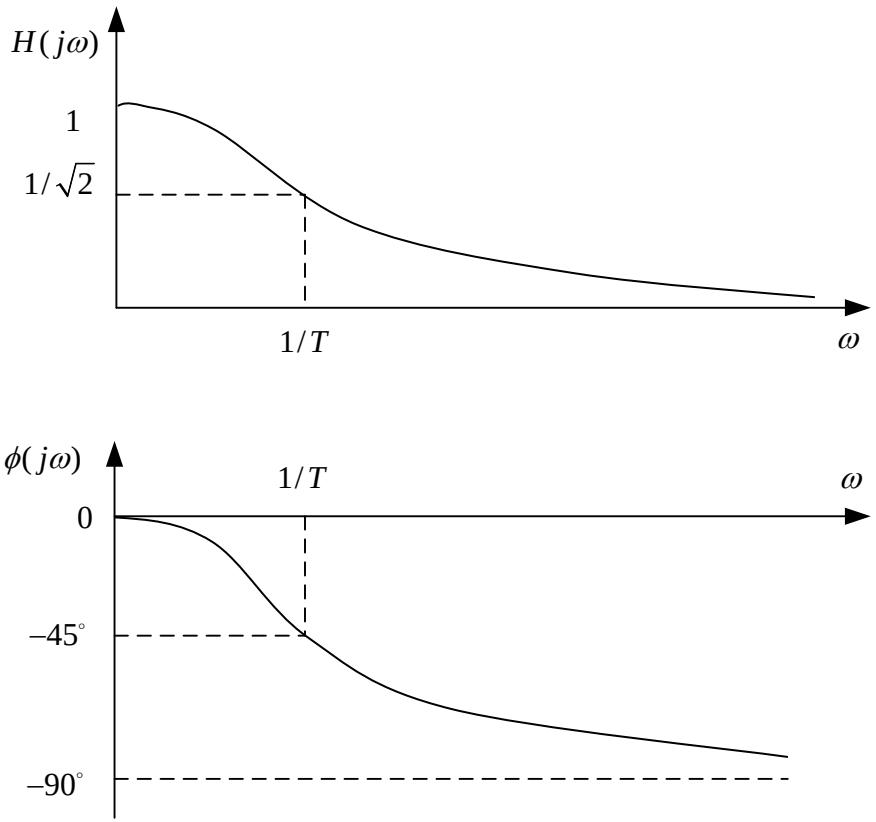
First Order System

$$G(s) = \frac{1}{Ts+1}$$

$$G(j\omega) = \frac{1}{Tj\omega+1}$$

$$H(j\omega) = |G(j\omega)| = \frac{1}{\sqrt{T^2\omega^2+1}}$$

$$\phi(j\omega) = -\angle(Tj\omega+1) = -\tan^{-1} T\omega$$



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Second Order System

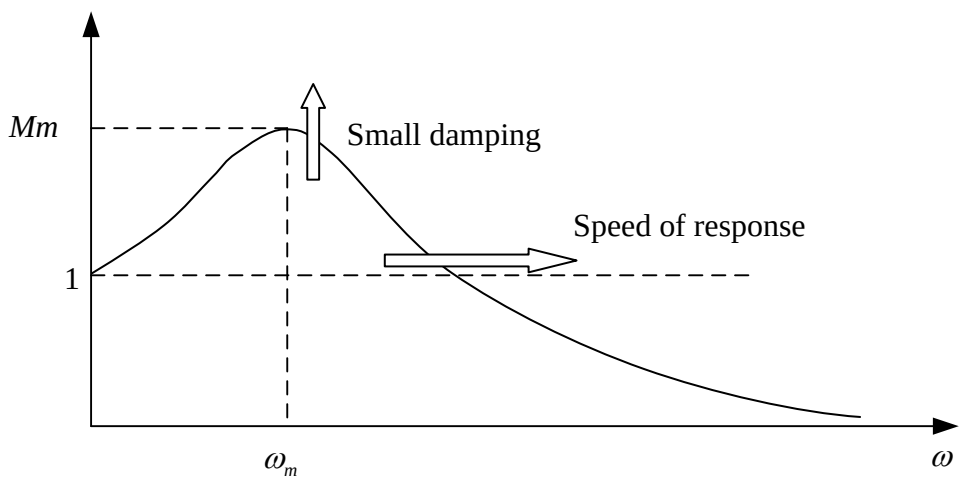
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$H(j\omega) = |G(j\omega)| = \left| \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2\zeta\omega_n \omega j} \right|$$

$$= \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

$$= \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \frac{\omega^2}{\omega_n^2}}}$$

$$\phi(j\omega) = -\tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$

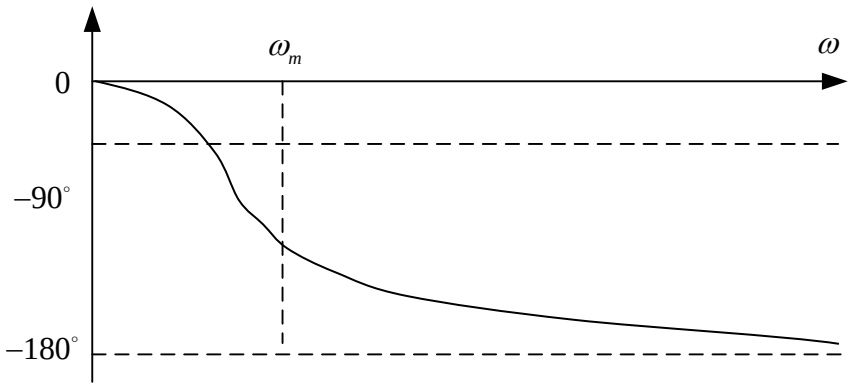


Frequency Response Analysis

$$\frac{dH^2}{d\omega} = 0 = -\frac{\left[2\left(1-\frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right)+8\zeta^2\frac{\omega}{\omega_n^2}\right]}{\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}\right]^2}$$

$$\omega_m = \omega_n\sqrt{1-2\zeta^2} \qquad M_m = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$e^{-\zeta\omega_n t}\sin(\omega_d t + \cdots)$$



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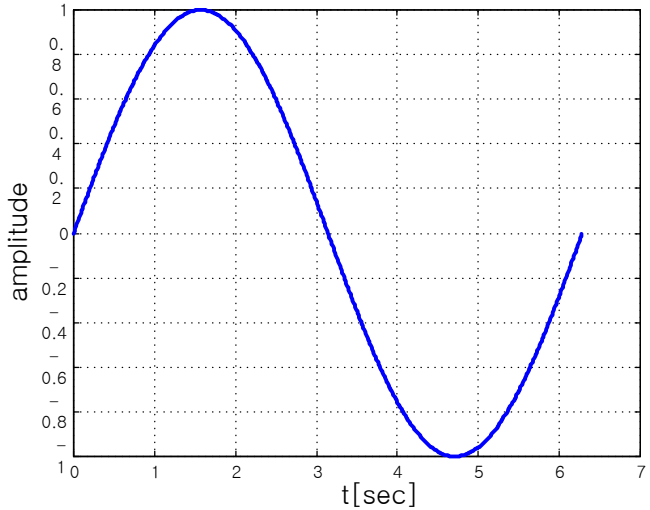
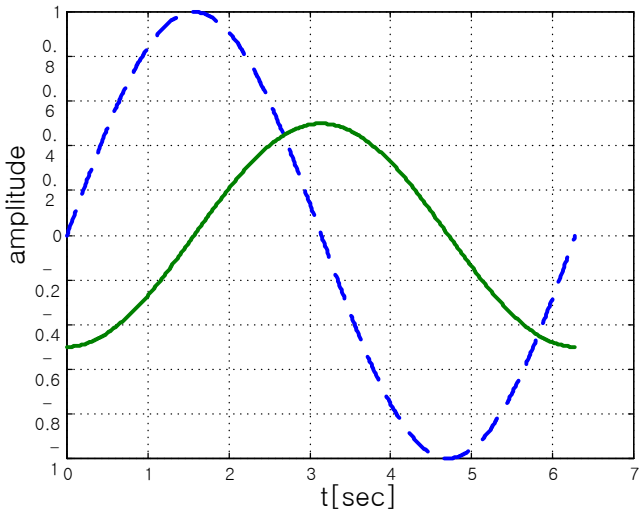
Frequency Response Analysis

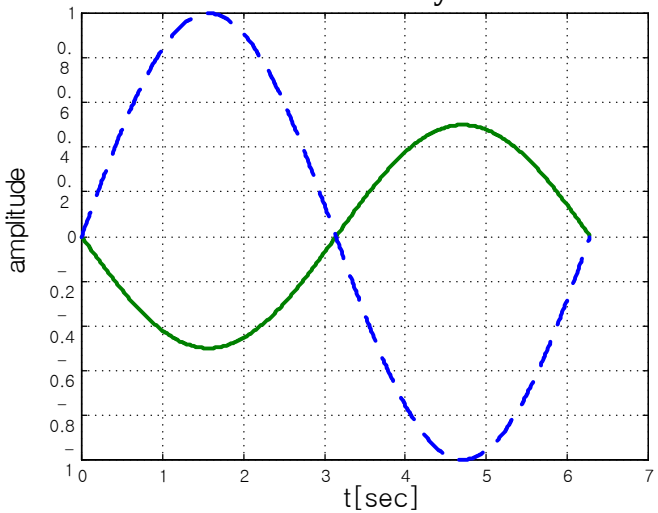
Zero order system



First order system

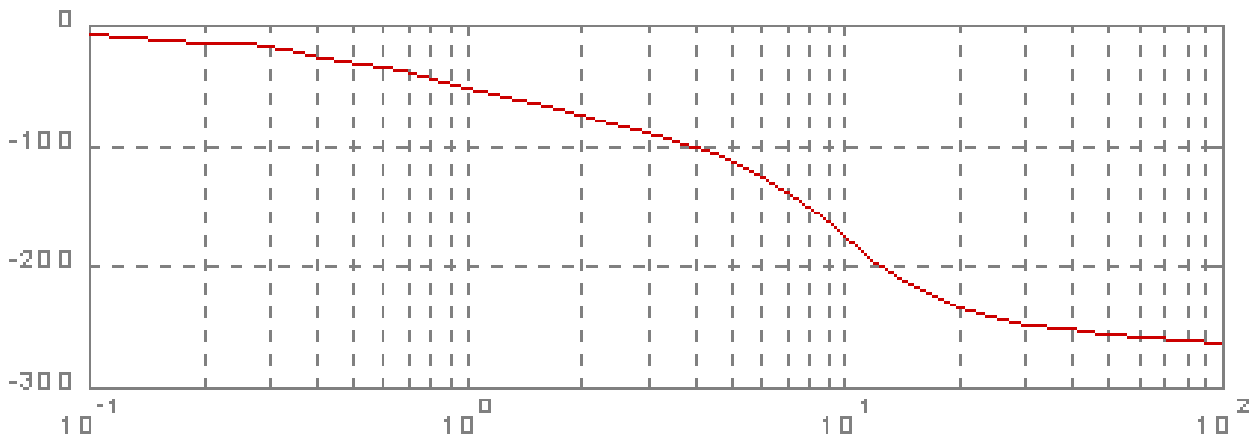


Second order system



Frequency Response Analysis

$$G(s) = \frac{\omega_n^2}{(Ts+1)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$



Combination of First and Second order system

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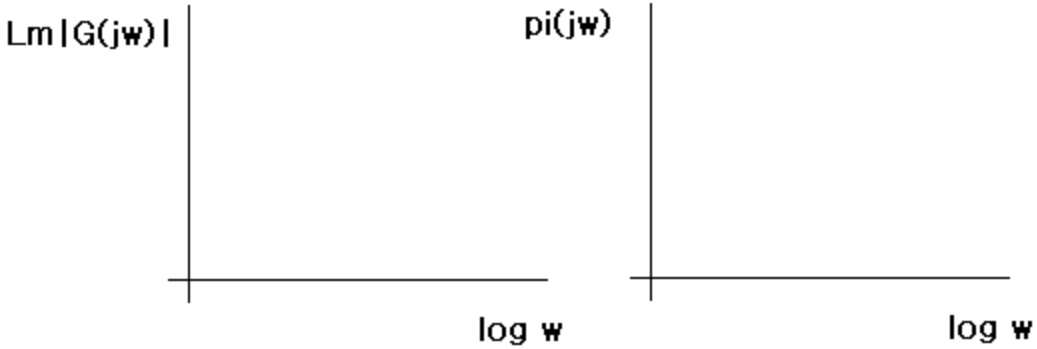
Bode Plots (Logarithmic Plots)

Log magnitude : $LmG(j\omega) = 20\log|G(j\omega)|\ [dB]$



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Bode Plots (Logarithmic Plots)

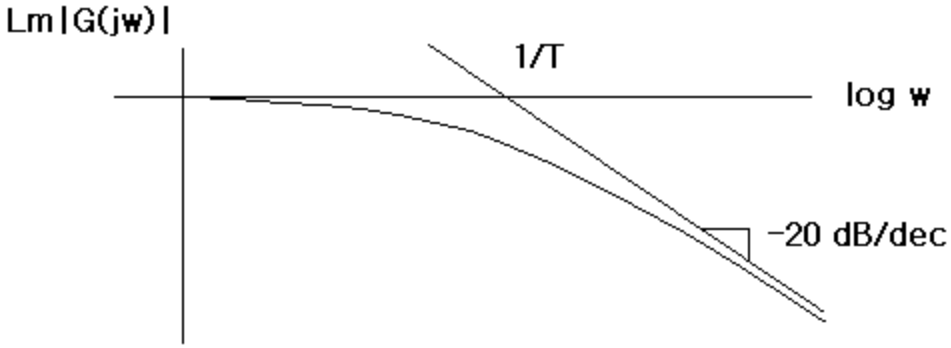
First Order System

$$G(s) = \frac{1}{Ts+1}$$

$$Lm\left|G(j\omega)\right| = 20\log\left|\frac{1}{Tj\omega+1}\right| = -20\log\sqrt{T^2\omega^2+1}$$

$$\text{if } \omega \langle\langle \frac{1}{T} \quad \cong -20\log 1 = 0$$

$$\text{if } \omega \rangle\rangle \frac{1}{T} \quad \cong -20\log T\omega = -20\log T - 20\log\omega$$



$$\omega \quad : 1 \rightarrow 10$$

$$\log\omega \quad : 0 \rightarrow 1$$

$$20\log\omega \quad : 0 \rightarrow 20$$

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Bode Plots (Logarithmic Plots)

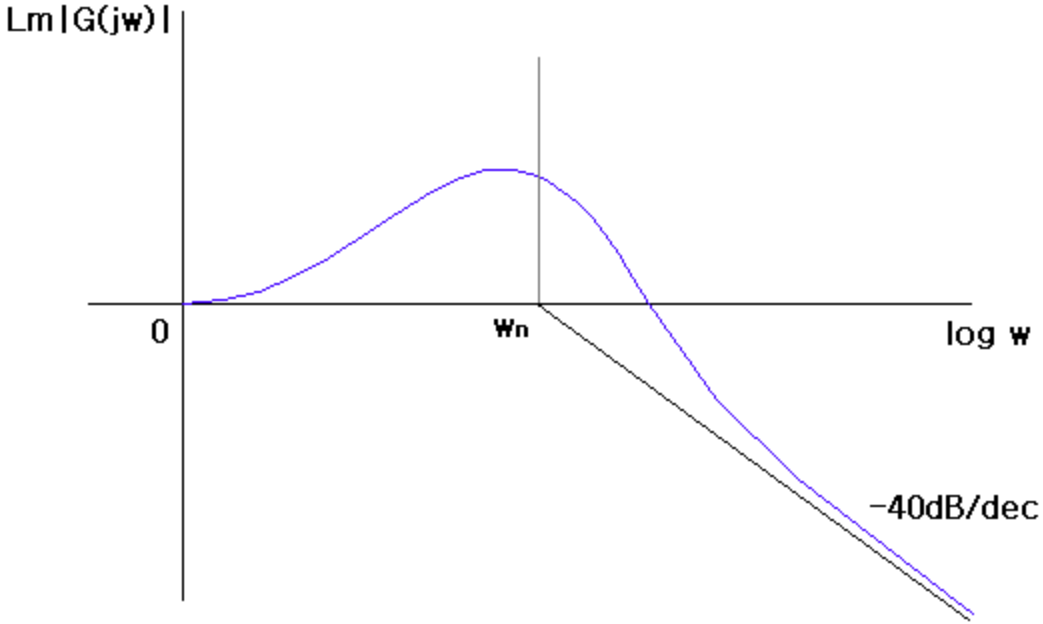
Second Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(j\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n}}$$

$$Lm\left|G(j\omega)\right| = -20\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}$$

$$\begin{cases} \dfrac{\omega}{\omega_n} \ll 1 & \cong -20\log 1 = 0 \\ \dfrac{\omega}{\omega_n} \gg 1 & \cong -20\log\left(\dfrac{\omega}{\omega_n}\right)^2 = -40\log\omega + 40\log\omega_n \end{cases}$$



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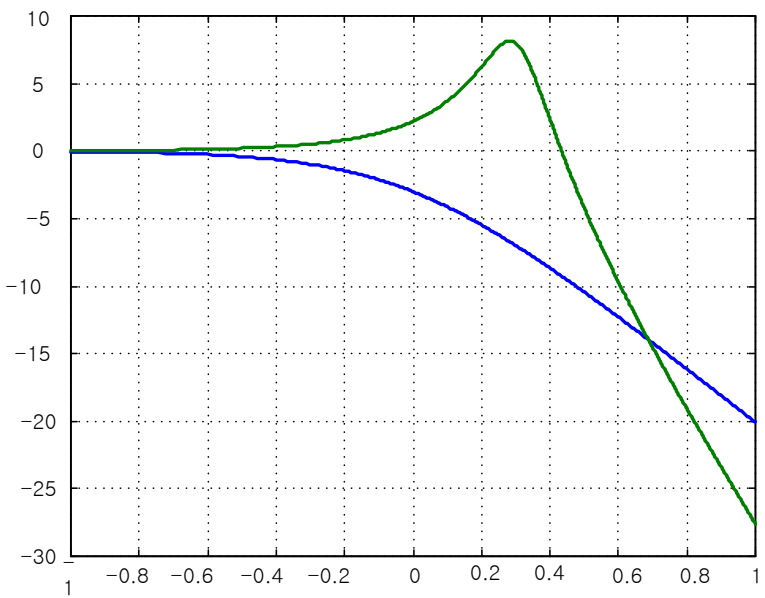
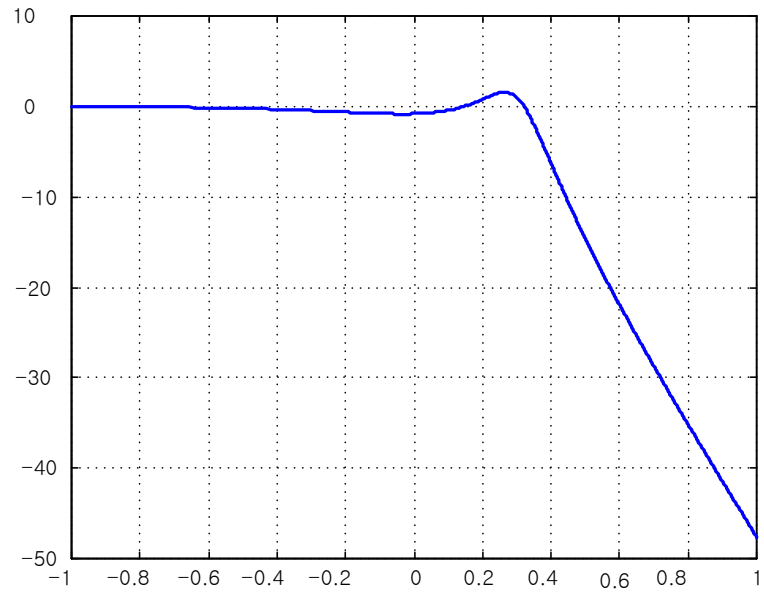
Bode Plots (Logarithmic Plots)

$$G(j\omega) = \frac{1}{Ts+1}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$Lm\left|G(j\omega)\right| = -20\log\left|1+jT\omega\right| - 20\log\left|\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\zeta\frac{\omega}{\omega_n}\right|$$





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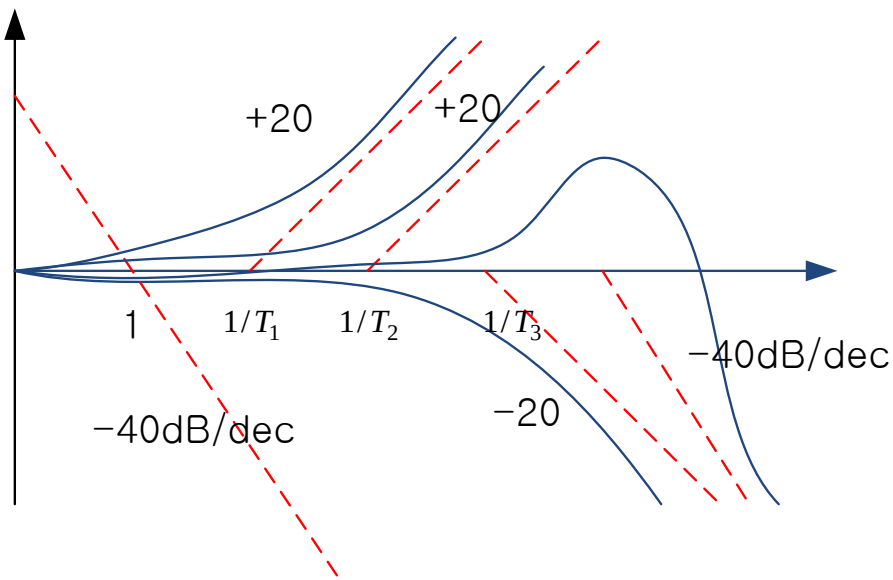
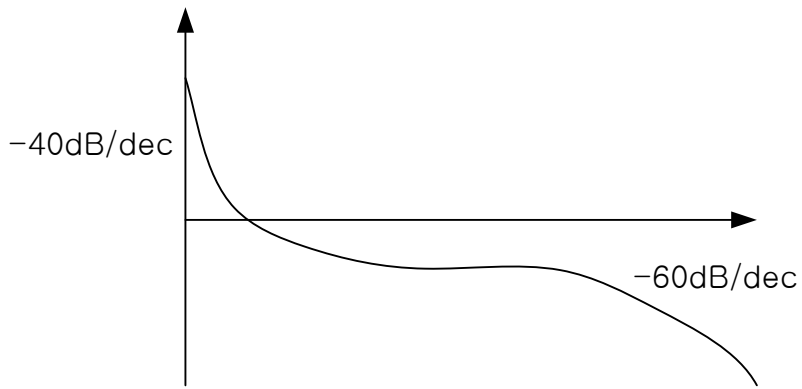
Bode Plots (Logarithmic Plots)

$$G(s)=\frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_3)(1+2as+bs^2)}$$

$$Lm|G(j\omega)|=20\log k+20\log|1+jT_1\omega|+20\log|1+jT_2\omega|$$
$$-20\log(j\omega)^2-20\log|1+jT_3\omega|-20\log|1-b^2\omega^2+j2a\omega|$$





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Plotting Bode Diagrams with MATLAB

```
w=logspace(-1,2)          % 0.1rad/sec~100 rad/sec 50points
w=logspace(0,3,100)       % 1rad/sec~1000 rad/sec 100points

[mag, phase, w]=bode(num, den)
[mag, phase, w]=bode(A,B,C,D)
[mag, phase, w]=bode(A,B,C,D,iu)

[mag, phase, w]=bode(sys)
```

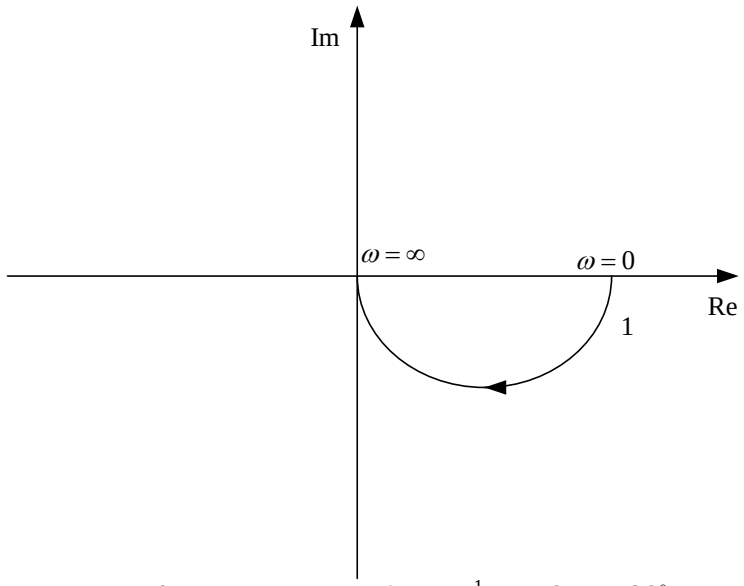
Polar Plot

Polar Plot (Nyquist plots)

- a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity.

1) $G(s)=\frac{1}{s}$ $\quad G(j\omega)=\frac{1}{j\omega}=-j\frac{1}{\omega}=\frac{1}{\omega}\left(\measuredangle -90^{\circ}\right)$

2) $G(s)=\frac{1}{1+Ts}$ $\quad G(j\omega)=\frac{1}{1+j\omega T}=\frac{1}{\sqrt{1+\omega^2T^2}}\measuredangle -\tan^{-1}\omega T^{\circ}$



$\omega:0\rightarrow\infty$ $\qquad \measuredangle -\tan^{-1}\omega T:0\rightarrow -90^{\circ}$

$\frac{1}{\sqrt{1+\omega^2T^2}}$ $\quad :1\rightarrow 0$

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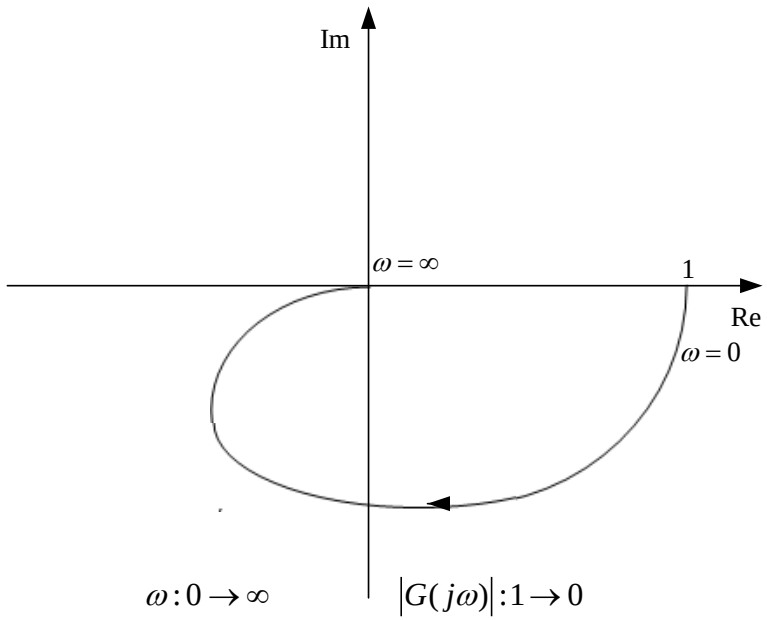
Polar Plot

3) $$G(s)=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}=\frac{1}{\left(\frac{s}{\omega_n}\right)^2+2\zeta\frac{s}{\omega_n}+1}$$

$$G(j\omega)=\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1}=\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+2j\zeta\frac{\omega}{\omega_n}}$$

$$=\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}\measuredangle-\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$



$\omega:0\rightarrow\infty$ $\quad |G(j\omega)|:1\rightarrow 0$

$\measuredangle G(j\omega)\quad :0^\circ\rightarrow-90^\circ\rightarrow-180^\circ$

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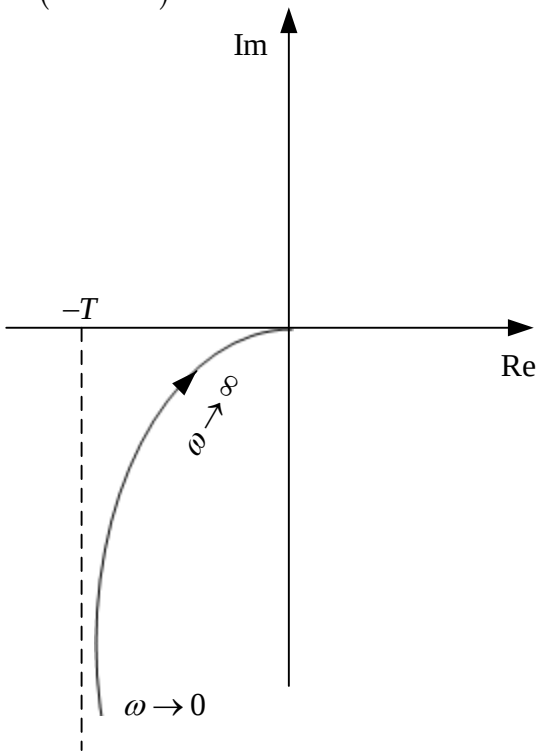
Polar Plot

4) $G(s)=\dfrac{1}{s(Ts+1)}$

$$G(j\omega)=\frac{1}{j\omega(j\omega T+1)}=\frac{1}{-\omega^2T+j\omega}=\frac{-\omega^2T-j\omega}{\omega^4T^2+\omega^2}$$

$$=-\frac{T}{1+\omega^2T^2}-j\frac{1}{\omega\left(1+\omega^2T^2\right)}$$



$$\lim_{\omega\to 0}G(j\omega)=-T-j\infty=\infty\measuredangle-90^\circ$$

$$\lim_{\omega\to\infty}G(j\omega)=-0-j0=0\measuredangle-180^\circ$$

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Polar Plot

5) $G(s)=\dfrac{K(1+j\omega T_a)(1+j\omega T_b)\cdots}{s^{\lambda}(1+j\omega T_1)(1+j\omega T_2)\cdots}$

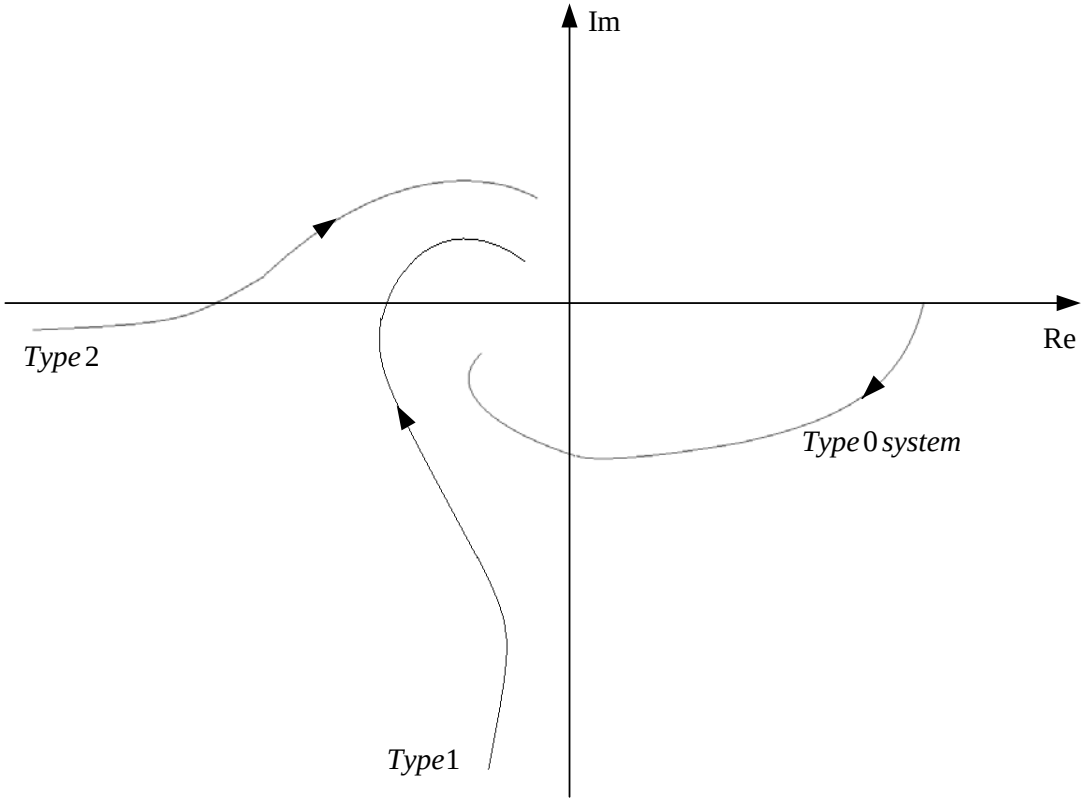
$$=\frac{b_0 s^m + b_1 s^{m-1}+\cdots}{a_0 s^n + a_1 s^{n-1}+\cdots}$$

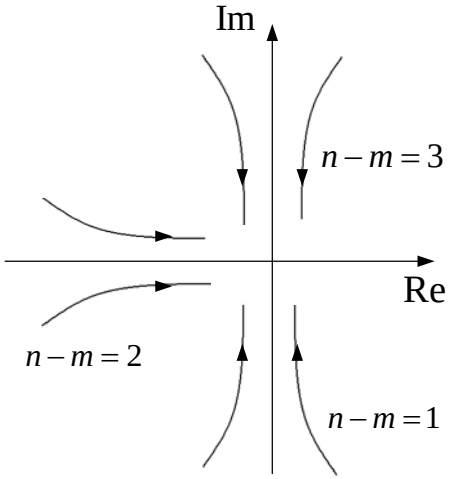
$$=\frac{K(1+T_a s)(1+T_b s)\cdots}{s^{\lambda}(1+T_1 s)(1+T_2 s)\cdots}$$

$\lambda = 0 \quad : \quad Type\ 0$

$\lambda = 1 \quad : \quad Type\ 1$

$\lambda = 2 \quad : \quad Type\ 2$





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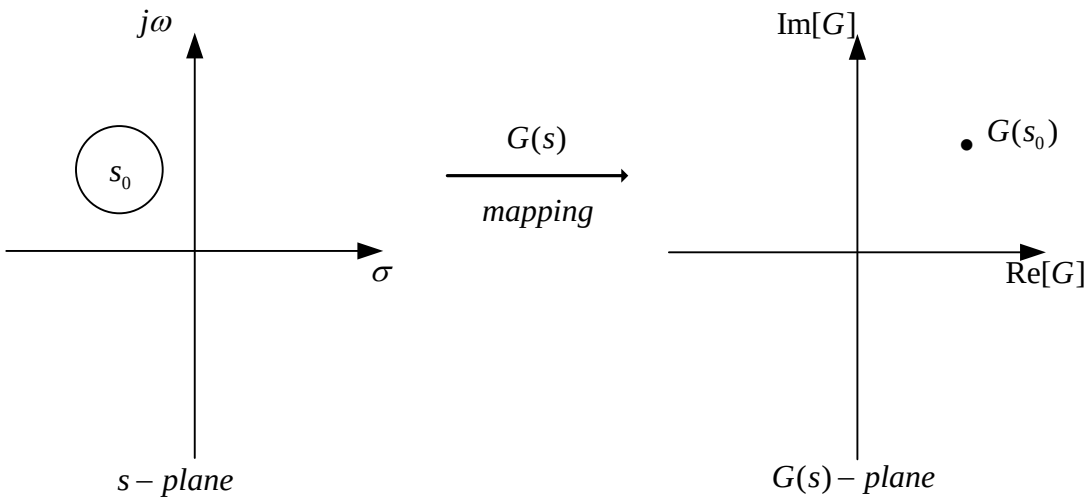
Plotting Nyquist plots with MATLAB

```
w=logspace(-1,2)          % 0.1rad/sec~100 rad/sec 50points
w=logspace(0,3,100)       % 1rad/sec~1000 rad/sec 100points

[re, im, w]=nyquist(num, den)
[re, im, w]=nyquist(A,B,C,D)
[re, im, w]=nyquist(A,B,C,D,iu)

[re, im, w]=nyquist(sys)
```

Nyquist Stability Criterion

Stability : time domain : state eigenvalues
Frequency domain : Nyquist stability criterion

Consider a clockwise contour in the s-plane called C about the point s_0



Cauchy Theorem :

If inside C, there are Z values of S which makes $G(s)=G(s_0)$ and P values which $G(s)=\infty$, then CW encirclement along C will map into Z-P CW encirclement of $G(s_0)$

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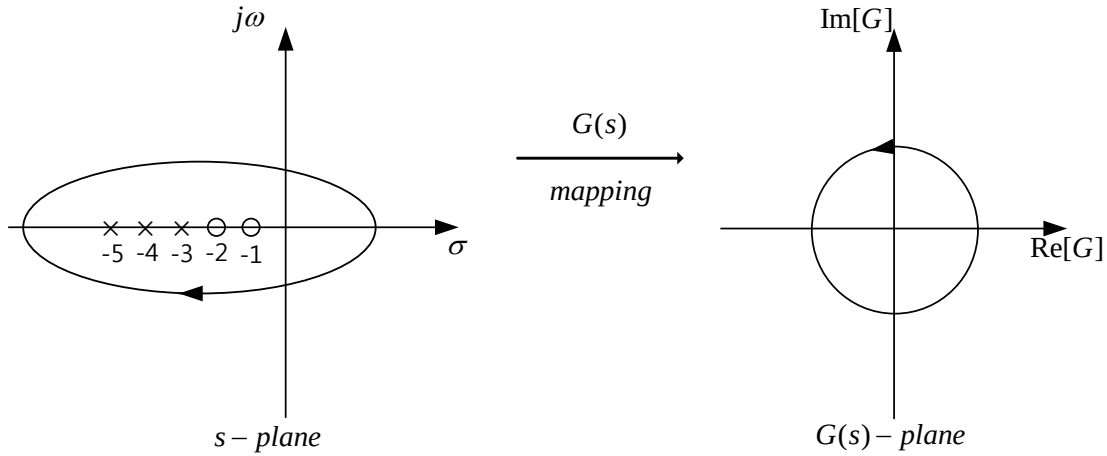
Nyquist Stability Criterion

Ex) $G(s) = \dfrac{(s+1)(s+2)}{(s+3)(s+4)(s+5)}$



$$G(s_0) = 0 \quad : \quad Z = 2$$

$$G(s) = \infty \quad : \quad P = 3$$

$$Z - P = -1 \quad : \quad 1\ CCW\ encirclement\ of\ the\ origin\ in\ G(s)-plane$$

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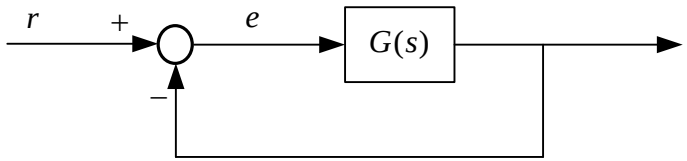
Nyquist Stability Criterion



$$G(s) = \frac{B(s)}{A(s)}$$

characteristic equation : open loop : $\phi_{OL}(s) = A(s) = 0$

: closed loop : $\phi_{CL}(s) = A(s) + B(s) = 1 + G(s) = 0$

$$\phi_{CL}(s) = A(s)\left(1 + \frac{B(s)}{A(s)}\right)$$
$$= \phi_{OL}(1 + G(s))$$

$$\frac{\phi_{CL}(s)}{\phi_{OL}(s)} = 1 + G(s) \qquad 1 + G(s) = 0 : closed\ \ Loop\ \ Poles$$

z : Number of unstable closed loop poles (characteristic roots)
p : Number of unstable open loop poles

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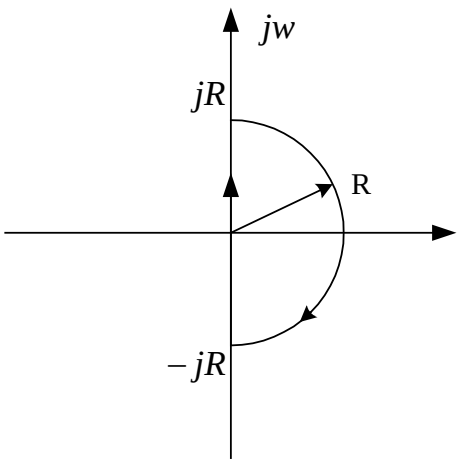
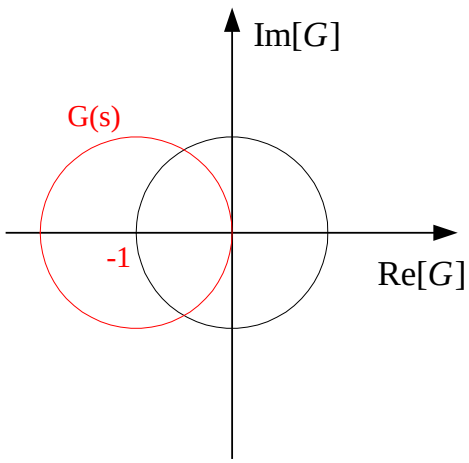
Nyquist Stability Criterion





$1+G(s)-Plane$

If $R \to \infty \to C_N$

$N(0,\ 1+G(s),\ C_N) = Z - P$; **0을 중심으로 몇 번 돌았는가?**

Stability :

$$N(0,\ 1+G(s),\ C_N) = -P \quad (z=0)$$

$$N(-1,\ G(s),\ C_N) = -P \quad (z=0)$$

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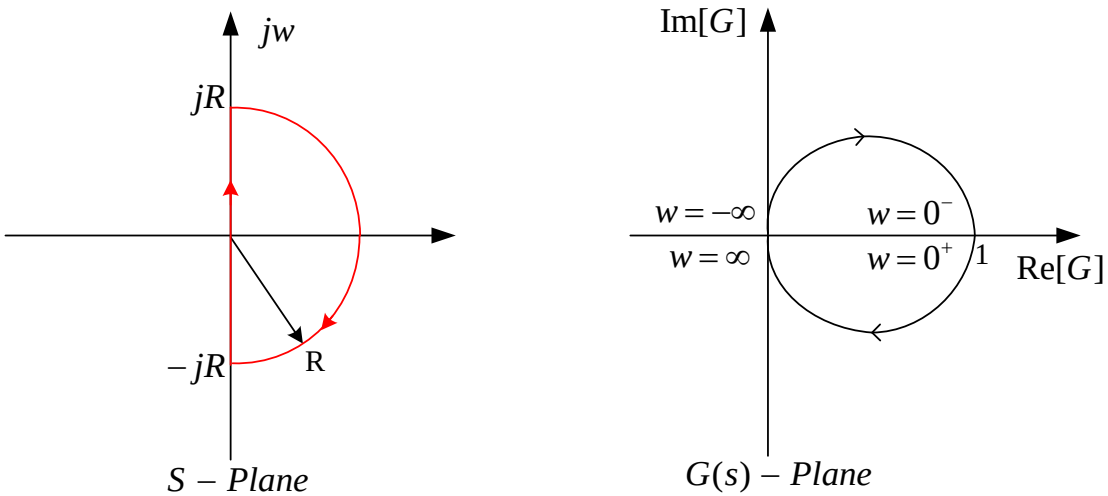
Nyquist Stability Criterion

Ex1)

$$G(s) = \frac{1}{Ts+1}$$



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Nyquist Stability Criterion

Open loop poles : $s = -\frac{1}{T}$

<u>Ex1)</u>

1) $G(j\omega) = \frac{1}{j\omega T + 1}$: $\omega \approx 0,\ |G| = 1,\ \angle G = 0^\circ$

$\omega \approx \infty,\ |G| = 0,\ \angle G = -90^\circ$

2) $s = \mathrm{Re}^{j\theta}$: $\theta : 90^\circ \to 0 \to -90^\circ$

$R \to \infty$

3) $s = \mathrm{Re}^{j\theta}$: $\theta : -90^\circ \to 0^\circ \to 90^\circ$

$r \to 0$

$$G(s) = \frac{1}{re^{j\theta}T + 1}$$

$$= 1 \qquad \phi : 180^\circ \to 0^\circ \to -180^\circ$$

$N(-1, G, C_N) = 0 = Z - P(P = 0,\ no\ open\ loop\ poles\ in\ RHP)$

$\Rightarrow$ 0 CW $encirclement\ of\ -1$

$\Rightarrow$ $Z = 0,\ no\ closed\ loop\ poles\ in\ RHP$

$\Rightarrow$ $stable$

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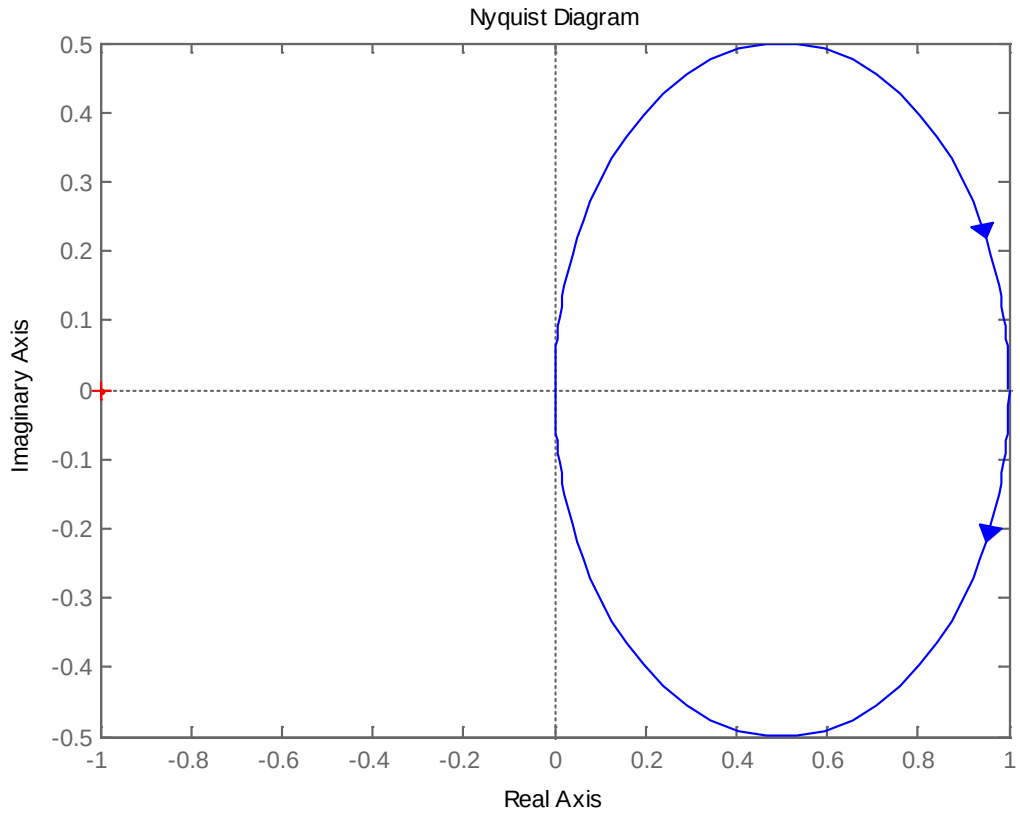
Nyquist Stability Criterion

Ex1)

$$G(s) = \frac{1}{Ts+1}$$



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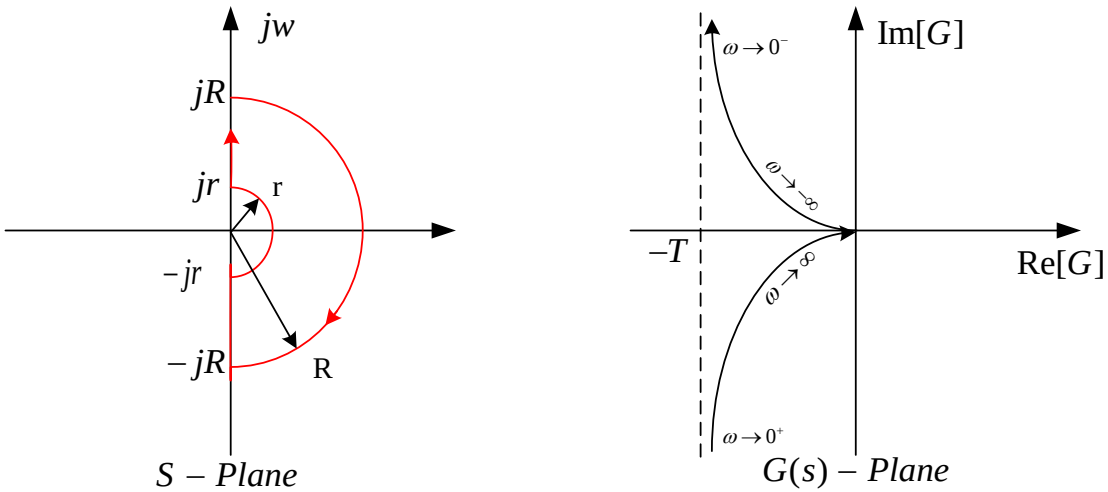
Nyquist Stability Criterion

Ex2)

$$G(s) = \frac{1}{s(Ts+1)}$$



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Nyquist Stability Criterion

Open loop poles : $s = 0, -\frac{1}{T}$

Ex2)

1) $G(j\omega) = \frac{K}{j\omega(j\omega T + 1)}$: $\omega \approx 0, |G| = \infty, \angle G = \pm 90^\circ$

$\omega \approx \infty, |G| = 0, \angle G = \pm 180^\circ$

2) $\omega : -\infty \to -r$

$|G| : 0 \to \infty$

$\angle G : -\angle(-jR) - \angle(-jRT + 1)$

3) $s = \mathrm{Re}^{j\theta}$: $\theta : 90^\circ \to 0 \to -90^\circ$

$R \to \infty$

4) $s = \mathrm{Re}^{j\theta}$: $\theta : -90^\circ \to 0^\circ \to 90^\circ$

$r \to 0$

$$G(s) = \frac{K}{re^{j\theta}(re^{j\theta}T + 1)}$$

$$= \frac{K}{r} e^{-j\theta} \quad : -\theta \Rightarrow \phi$$

$$= \infty \times e^{j\phi} \quad : \phi : 180^\circ \to 0^\circ \to -180^\circ$$

$N(-1, G, C_N) = 0 = Z - P(P = 0,\ no\ open\ loop\ poles\ in\ RHP)$

$\Rightarrow$ 0 CW $encirclement$ of -1

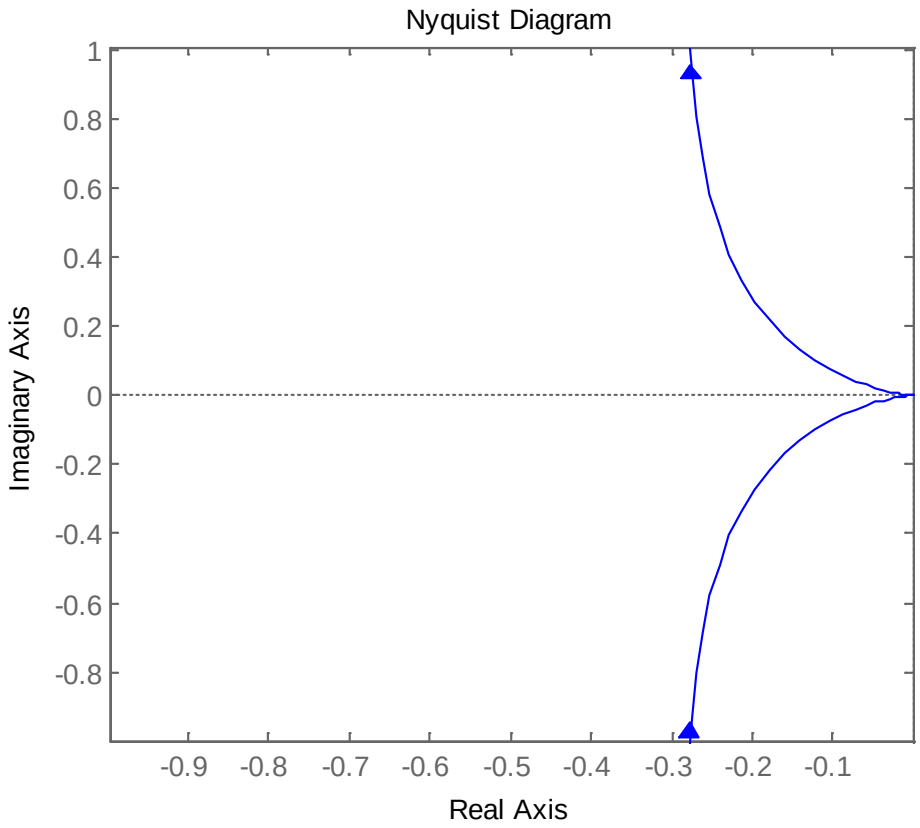
$\Rightarrow$ $Z = 0$, no $closed$ $loop$ $poles$ in RHP

$\Rightarrow$ $stable$

Nyquist Stability Criterion

Ex2)

$$G(s) = \frac{1}{s(Ts+1)} \qquad T = 0.3$$



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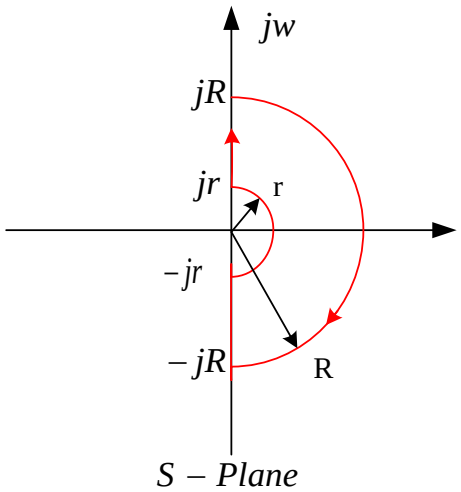
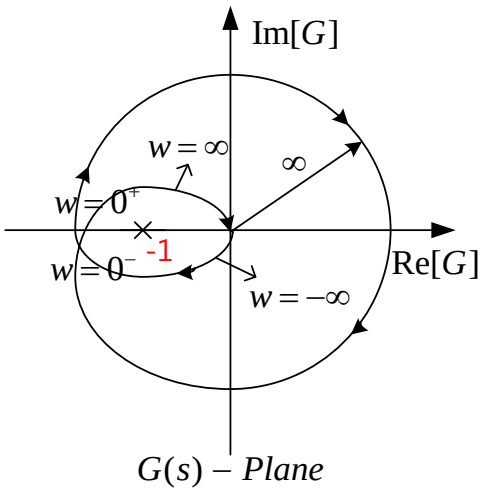
Nyquist Stability Criterion

Ex3)

$$G(s) = \frac{1}{s^2(Ts+1)}$$





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Nyquist Stability Criterion

Open loop poles : $s = 0,\ 0,\ -\dfrac{1}{T}$

Ex3)

1) $G(j\omega) = \dfrac{K}{(j\omega)^2(j\omega T + 1)}$: $\omega \approx 0, |G| = \infty, \angle G = -180^\circ$

$\omega \approx \infty, |G| = 0, \angle G = -270^\circ$

2) $\omega : -\infty \to -r$

$|G| : 0 \to \infty$

$\angle G : -\angle(-jR) - \angle(-jR) - \angle(-jRT + 1)$

3) $s = \mathrm{Re}^{j\theta}$: $\theta : 90^\circ \to 0 \to -90^\circ$

$R \to \infty$

4) $s = \mathrm{Re}^{j\theta}$: $\theta : -90^\circ \to 0^\circ \to 90^\circ$

$r \to 0$

$$G(s) = \frac{K}{r^2 e^{2j\theta}(re^{j\theta}T + 1)}$$

$$= \frac{K}{r^2} e^{-2j\theta} \quad : -2\theta \Rightarrow \phi$$

$$= \infty \times e^{j\phi} \quad : \phi : 180^\circ \to 0^\circ \to -180^\circ$$

$N(-1, G, C_N) = 2 = Z - P(P = 0,\ no\ open\ loop\ poles\ in\ RHP)$

$\Rightarrow$ 2 CW $encirclement$ of -1

$\Rightarrow$ $Z = 2$, 2 $closed$ $loop$ $poles$ in RHP

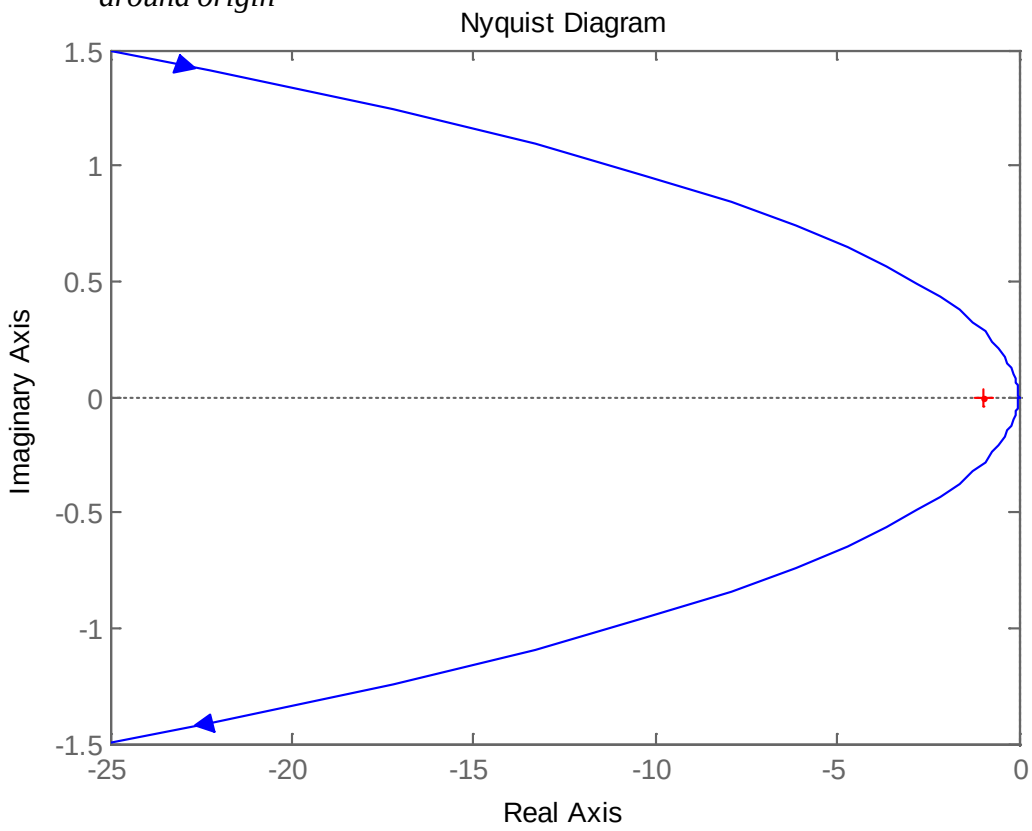
$\Rightarrow$ $unstable$

Nyquist Stability Criterion

Ex3)

$$G(s) = \frac{1}{s^2(Ts+1)} \qquad T = 0.3$$

around origin



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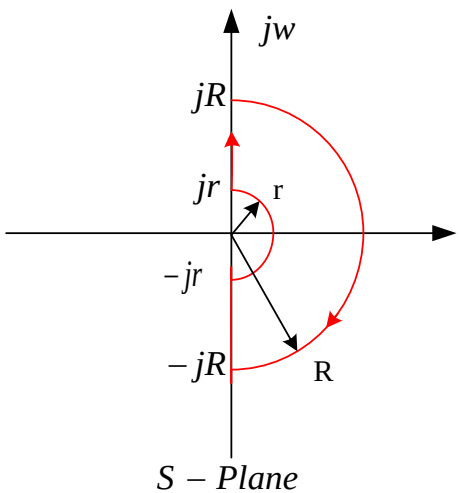
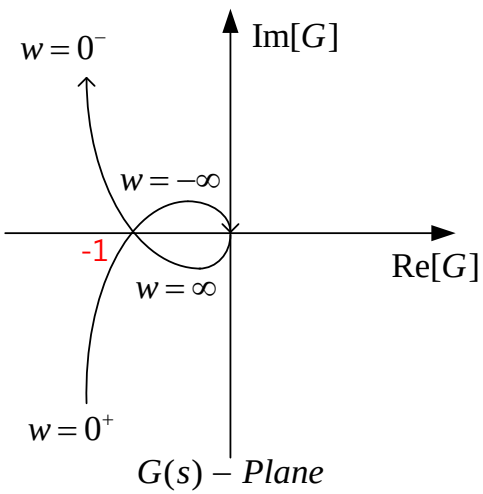
Nyquist Stability Criterion

Ex4)

$$G(s) = \frac{1}{s(s^2 + 2\zeta\omega_n s + \omega_n{}^2)} \qquad 0 < \zeta < 1$$





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Nyquist Stability Criterion

Open loop poles : $s = 0, -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

1) $$G(j\omega) = \frac{K}{(j\omega)\left\{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^{\ 2}\right\}}$$

$$\omega \approx 0, |G| = \infty, \angle G = \mp 90^\circ$$
$$\omega \approx \infty, |G| = 0, \angle G = \mp 270^\circ$$

2) $\omega : -\infty \to -r$

$|G| : 0 \to \infty$

$\angle G : -\angle(-jR) - \angle(-j(2\zeta\omega_n R) + \omega_n^{\ 2} - \omega^2)$

3) $s = \mathrm{Re}^{j\theta} \quad : \quad \theta : 90^\circ \to 0 \to -90^\circ$

$R \to \infty$

4) $s = \mathrm{Re}^{j\theta} \quad : \quad \theta : -90^\circ \to 0^\circ \to 90^\circ$

$r \to 0$

$$G(s) = \frac{1}{re^{j\theta}(2\zeta\omega_n(re^{j\theta}) + \omega_n^{\ 2} - \omega^2)}$$

$$= \frac{K}{r(\omega_n^{\ 2} - \omega^2)} e^{-j\theta} \quad : -\theta \Rightarrow \phi$$

$$= \infty \times e^{j\phi} \quad : \phi : 180^\circ \to 0^\circ \to -180^\circ$$

$N(-1, G, C_N) = 0 = Z - P(P = 0,\ no\ open\ loop\ poles\ in\ RHP)$

$\Rightarrow$ 0 CW $encirclement$ of -1

$\Rightarrow$ $Z = 0$, no $closed$ $loop$ $poles$ in RHP

$\Rightarrow$ $stable$

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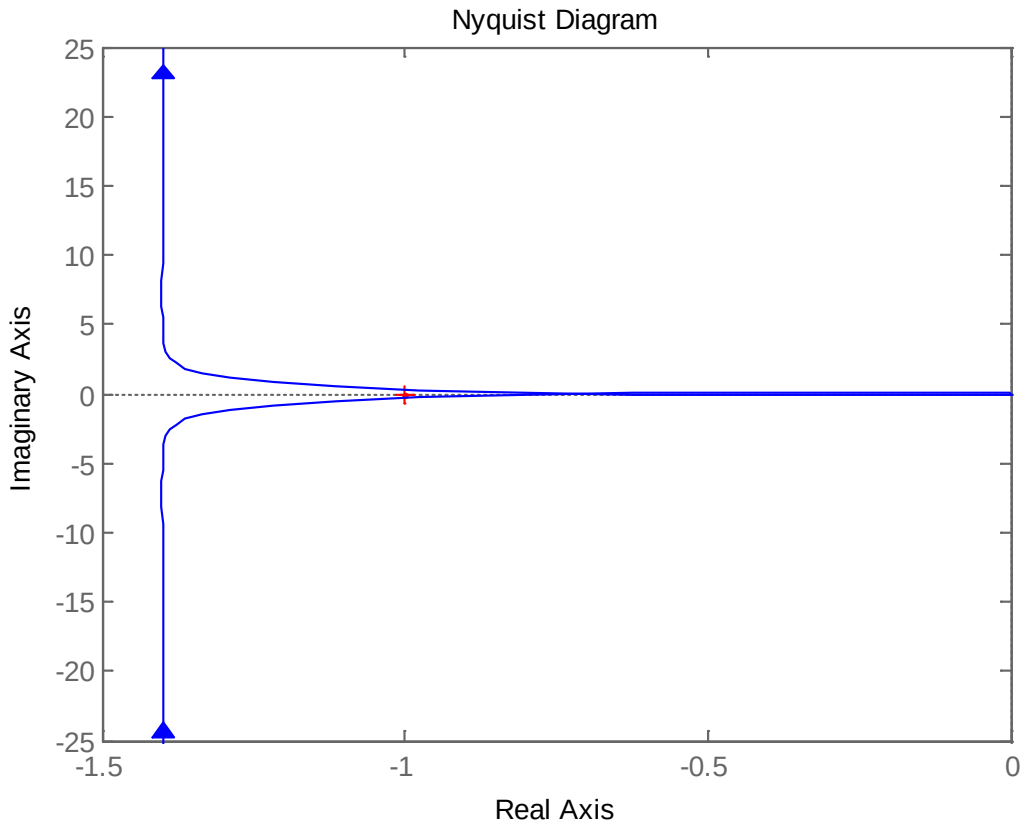
Nyquist Stability Criterion

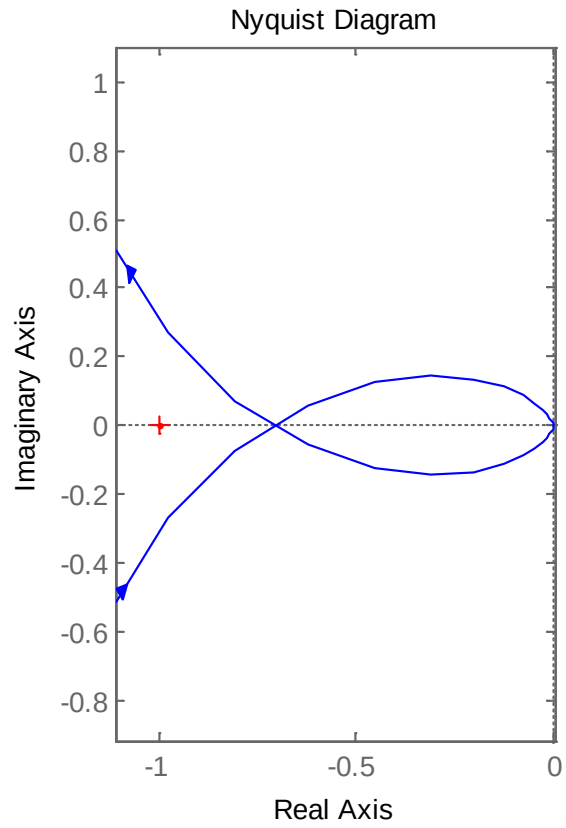
Ex4)

$$G(s) = \frac{1}{s(s^2 + 2\zeta\omega_n s + \omega_n{}^2)}$$

$\zeta = 0.7$

$\omega_n = 1$





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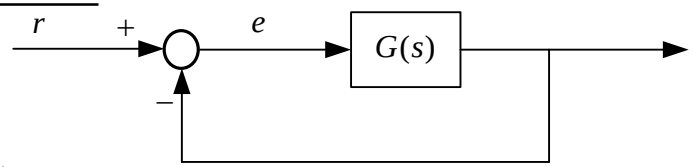
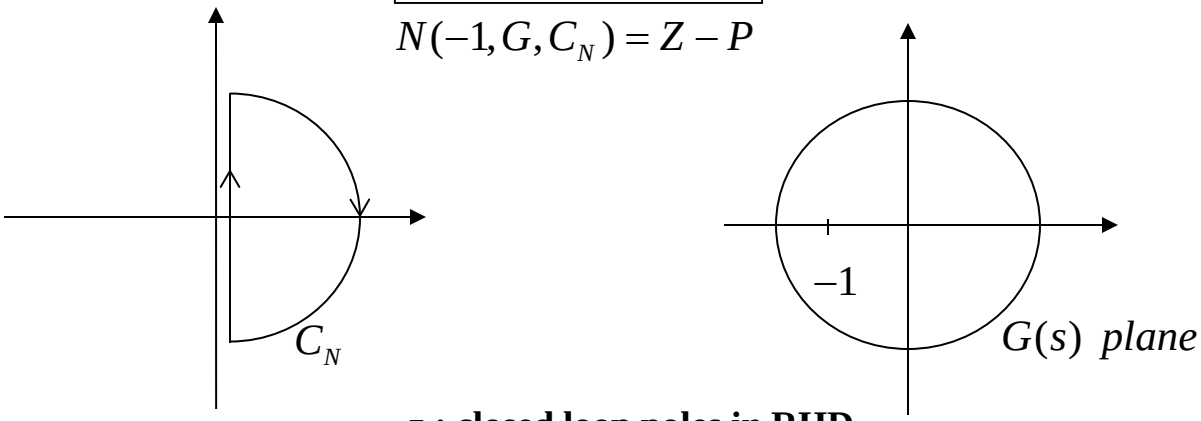
Nyquist Stability Criterion

Nyquist Stability Criterion



$$N(-1, G, C_N) = Z - P$$



z : closed loop poles in RHD
p : open loop poles in RHD

$$1 + G(s) = 0$$

$$G(s) = \frac{B(s)}{A(s)}$$

$$1 + \frac{B(s)}{A(s)} = 0 \quad \rightarrow \quad \frac{A(s) + B(s)}{A(s)} = 0$$

$$1 + G(s) = \frac{\phi_{CL}(s)}{\phi_{OL}(s)} = 0$$

$$N(0, 1 + G, C_N) = Z - P$$

$$N(-1, G, C_N) = Z - P$$

$$Stability\ Condition : Z = 0$$

$$N(-1, G, C_N) = -P$$

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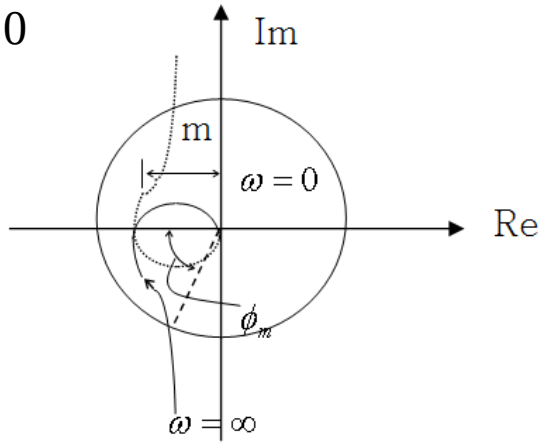
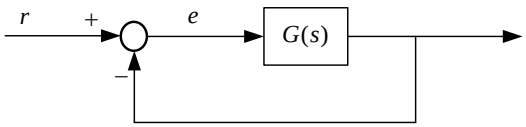
Nyquist Stability Criterion

Minimum phase System : no pole and zero in RHD

$$N(-1, G, C_N) = 0$$



$G(s)$: *open loop Transfer function*



$$G(j\omega) = me^{(-\pi)}$$

$$K_g G(j\omega) = K_g me^{(-\pi)}$$

$K_g = \dfrac{1}{m}$ **: stability 가 유지되는 gain margin**

G_m *or* K_g **: gain margin**
reciprocal of $|G(j\omega)|$ **when phase angle is -180˚**

ϕ_m **: phase margin**

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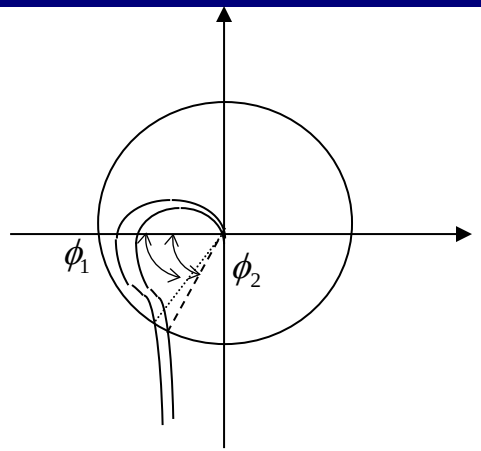
Nyquist Stability Criterion

Ex)



$\phi_1 < \phi_2$

Relative stability

Stability for minimum phase systems

$G_m > 1,\ \phi_m > 0$

The bigger G_m **, the more stable**

Rule of thumb : $G_m = 3 \sim 10$

$\phi_m = 30 \sim 60^\circ$

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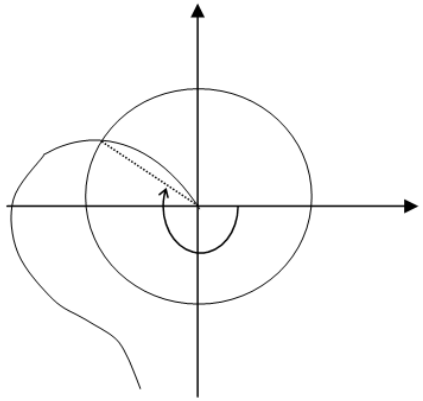
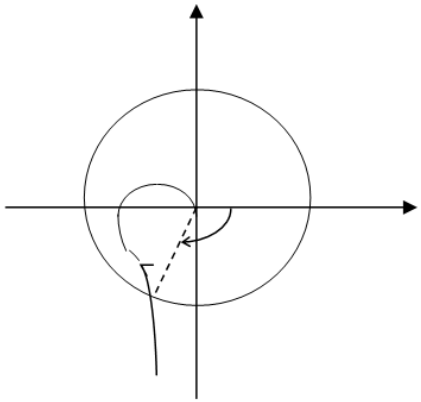
Nyquist Stability Criterion

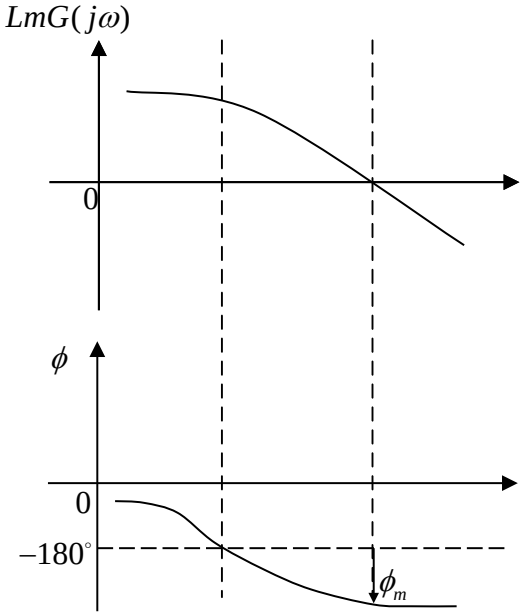
Unstable Systems

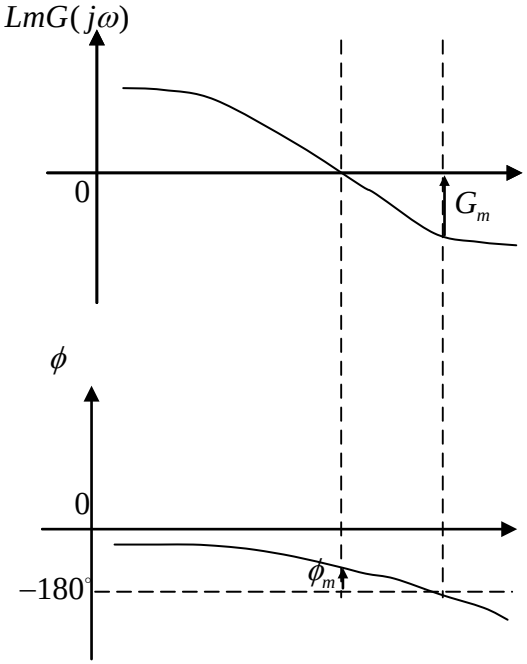


Stable Systems



Bode plot





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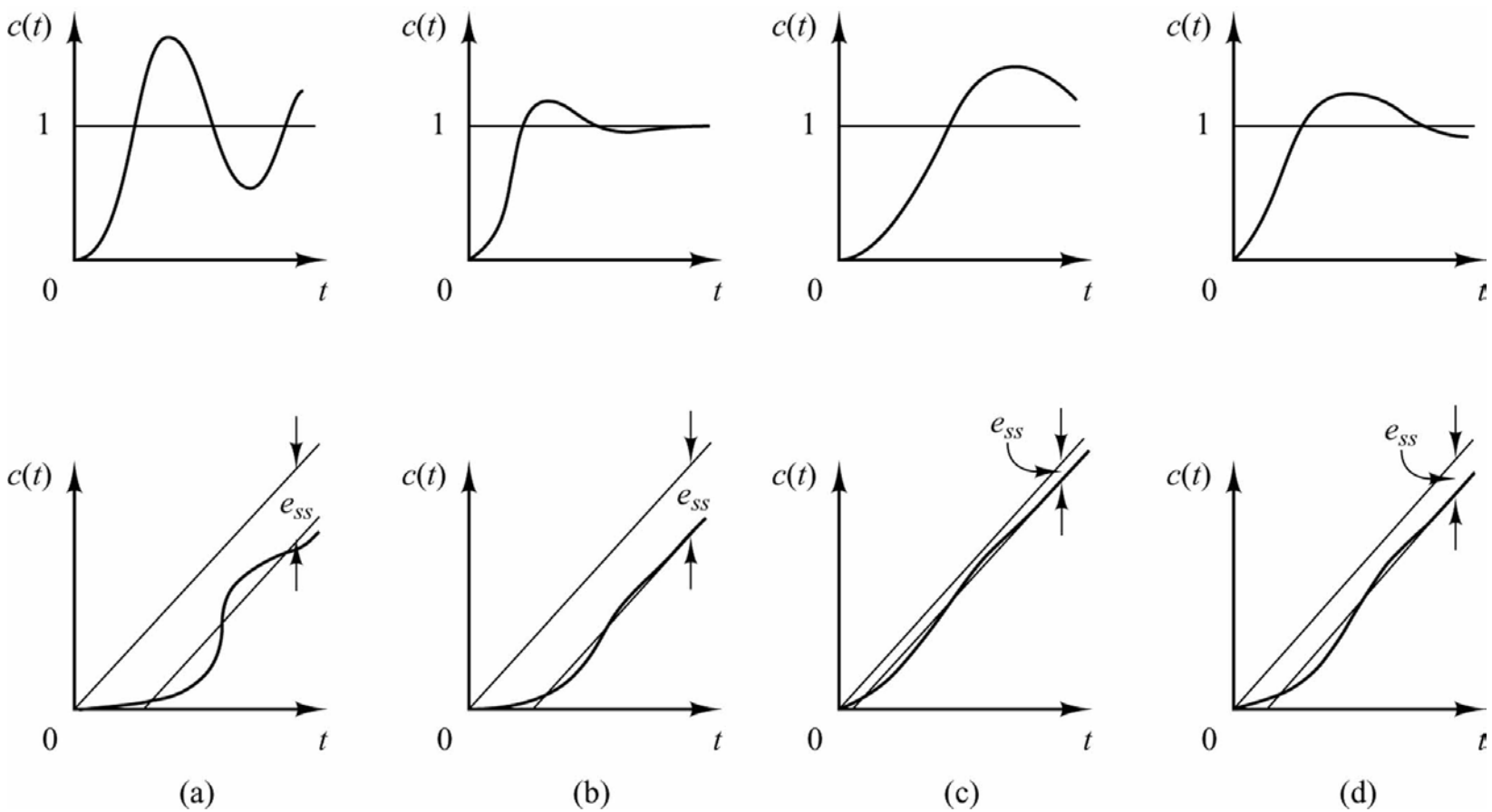
Control Systems Design by Frequency Response

Lead-Lag Compensation

Design using Openloop transfer function

Unit-step response curves and unit-ramp response curves.



(a) Uncompensated system; (b) lead compensated system; (c) lag compensated system; (d) lag–lead compensated system

Control System Design by Frequency Response

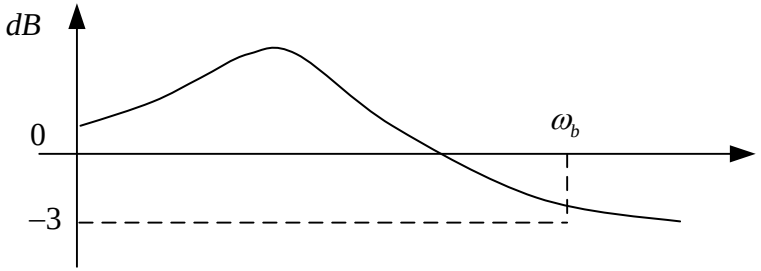
Frequency Response

- the phase margin
  gain margin
  resonant peak magnitude

} A rough estimate of the system damping

the gain crossover frequency
resonant frequency
bandwidth

} Speed of response



the static error constant
→ the steady state accuracy

$$20\log M = -3$$

$$\log M = -\frac{3}{20} = -0.15$$

$$M = 10^{-0.15} = 0.7$$

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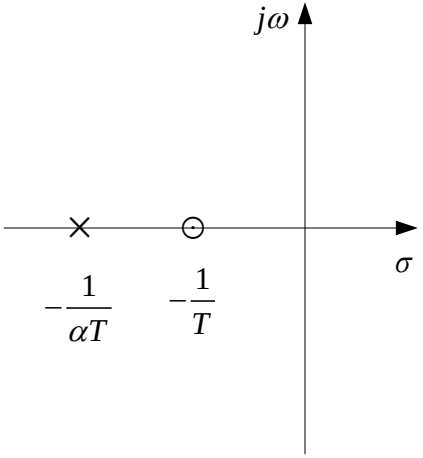
Lead Compensation (Transient response 개선)

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = K_c \alpha \frac{Ts + 1}{\alpha Ts + 1} \quad (0 < \alpha < 1)$$

α **: the attenuation factor of the lead compensator**



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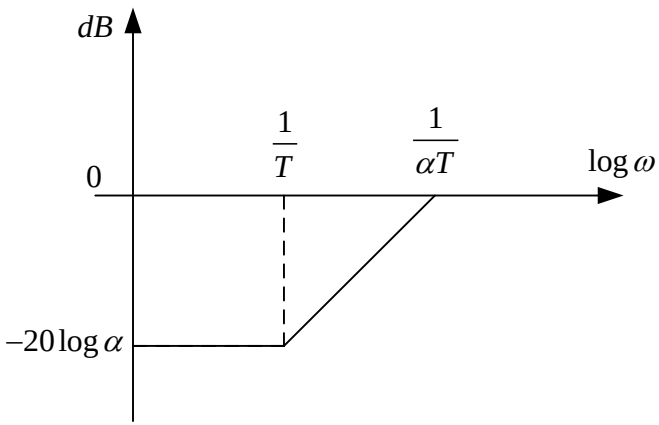
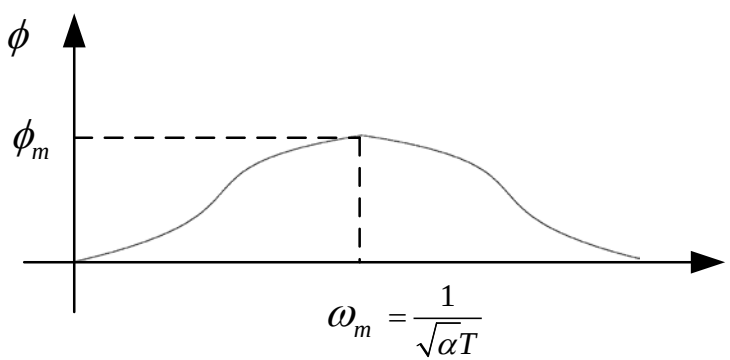
Lead Compensation

Bode plot of a lead compensator

$$\alpha \frac{Ts+1}{\alpha Ts+1} \qquad (0<\alpha<1)$$





$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$$\omega_m = \frac{1}{\sqrt{\alpha}T}$$

the geometric mean of the two corner freq.

$$\log\omega_m = \frac{1}{2}\left(\log\frac{1}{T} + \log\frac{1}{\alpha T}\right)$$

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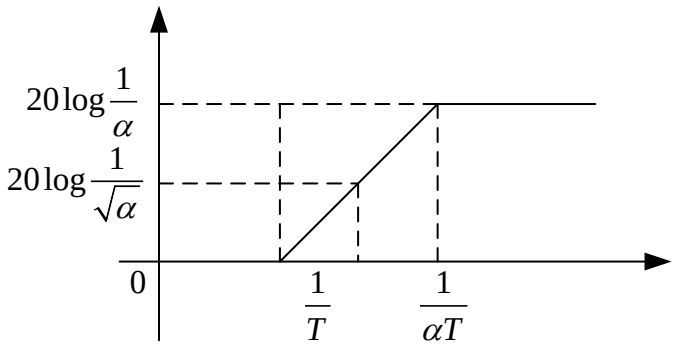
Lead Compensation

$$\frac{Ts+1}{\alpha Ts+1}$$



At ω_m,

$$\frac{Ts+1}{\alpha Ts+1} = 20\log\frac{1}{\sqrt{\alpha}}$$

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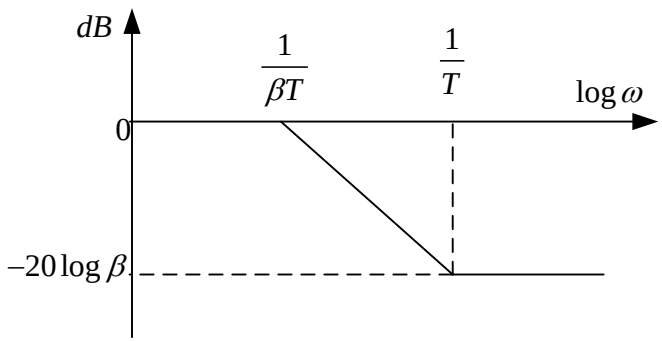
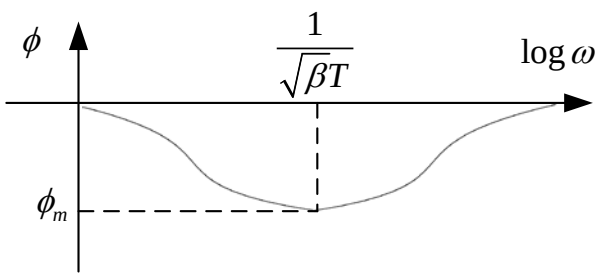
Lag Compensation

Lag Compensation

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \frac{Ts + 1}{\beta Ts + 1} \quad (\beta > 1) \qquad (K_c \approx 1) \quad (K_c \beta = \textit{Static velocity error constant})$$





$$\phi_m = \sin^{-1} \frac{1-\beta}{1+\beta}$$

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Lead Compensation – Frequency Response Approach

Performance Requirements

i) K_v **: static velocity error constant**

ii) $G_m,\ \phi_m$

1) $$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \quad (0<\alpha<1)$$

Define $K = K_c\alpha$

Then $G_c(s) = K\dfrac{Ts+1}{\alpha Ts+1}$

Open loop T.F. of the compensated system

$$G_c(s)G(s) = K\frac{Ts+1}{\alpha Ts+1}G(s) = \frac{Ts+1}{\alpha Ts+1}\underbrace{KG(s)}_{G_1(s)} = \frac{Ts+1}{\alpha Ts+1}G_1(s)$$

Determine gain K **to satisfy the requirement on the given static error constant**

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Lead Compensation – Frequency Response Approach

2) **Using the gain K , draw a bode diagram of $G_1(j\omega)$**

Evaluate the phase margin

3) **Determine the necessary phase phase-lead angle to be added to the system**

add an additional 5° to 12° to the phase lead angle required.

4) **Determine α by** $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha}$

5) **Determine the new gain crossover frequency**

$$\omega_m = \frac{1}{\sqrt{\alpha}T}, \quad \left|G(j\omega)\right| = -20\log\frac{1}{\sqrt{\alpha}}$$

➔ find T

6) **Determine the corner frequency of the lead compensator**

zero of the compensator : $\omega = \dfrac{1}{T}$

pole of the compensator : $\omega = \dfrac{1}{\alpha T}$

7) **Find K_c using the value of K and α**

8) **Check the gain margin to be sure it is satisfactory**
If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

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Lead Compensation – Frequency Response Approach

Ex

$$G(s) = \frac{4}{s(s+2)}$$

Design spec. $K_v = 20\sec^{-1}$

$\phi_m = 50^\circ$

$G_m = 10dB \quad at\ least$

The static velocity error coefficient of the original system

$$K_{v,0} = \lim_{s\to 0} sG(s) = \lim_{s\to 0}\frac{4}{s+2} = 2$$

1) Consider lead compensator

$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1}$$

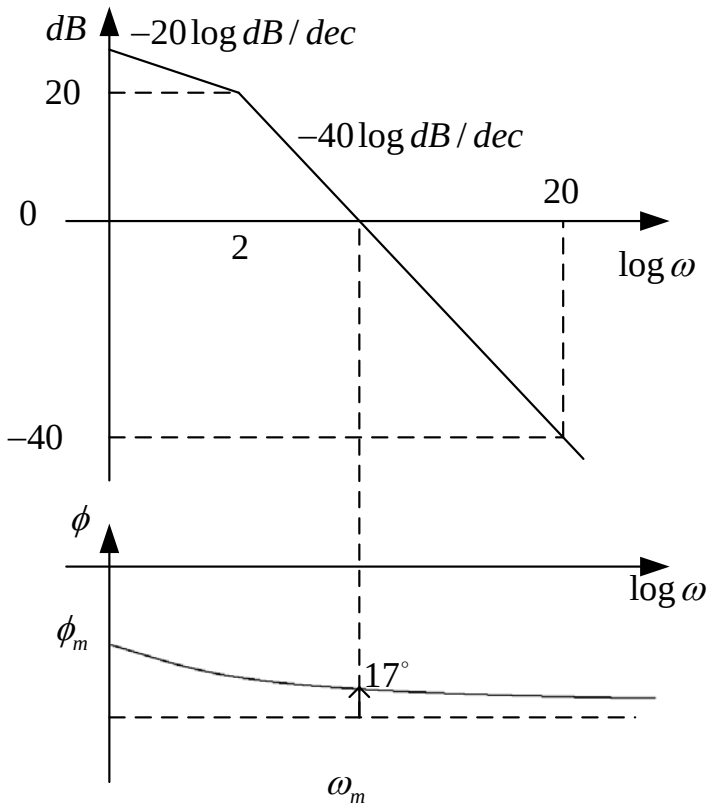
Let $G_1(s) = K_c\alpha G(s) = K\cdot G(s)$

$$K_v = \lim_{s\to 0} sK\cdot G(s) = 2K = 20 \quad \Rightarrow \quad K = 10$$

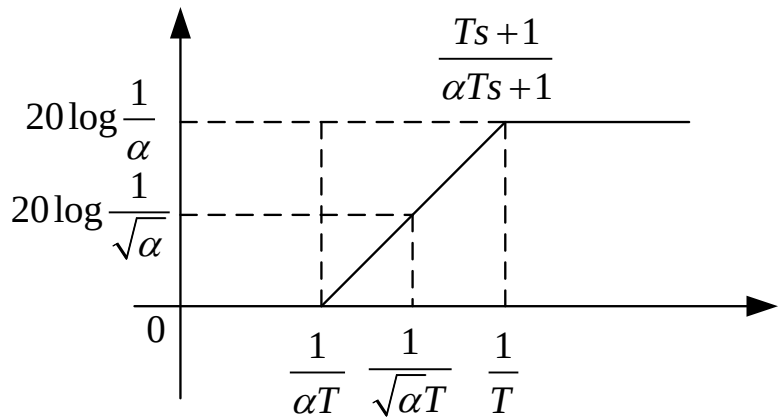
2) With $K = 10$ **plot bode diagram of** $G_1(j\omega)$

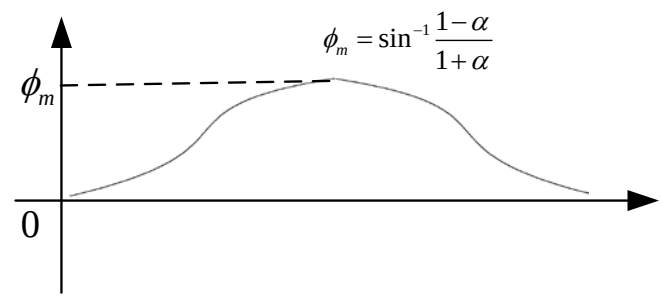
$$G_1(j\omega) = \frac{40}{j\omega(j\omega+2)} = \frac{20}{j\omega(0.5j\omega+1)}$$

Lead Compensation – Frequency Response Approach



Phase lead compensator





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Lead Compensation – Frequency Response Approach

3) Necessary phase-lead angle : 33° **to have** $\phi_m = 50^\circ$

Add 5° ; $\phi_m = 38^\circ$

4) $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha} = \sin 38^\circ$ $\qquad \alpha = 0.24$

5) Since lead compensator

$$\left|\frac{Ts+1}{\alpha Ts+1}\right|_{\substack{s=j\omega \\ \omega=\frac{1}{\sqrt{\alpha}T}}} = \left|\frac{jT\dfrac{1}{\sqrt{\alpha}T}+1}{\alpha T\dfrac{1}{\sqrt{\alpha}T}j+1}\right| = \frac{1}{\sqrt{\alpha}}$$

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.24}} = \frac{1}{0.49} = 6.2dB$$

Find $\left|G_1(j\omega)\right| = -6.2dB \;\Rightarrow\; \omega = 9\,rad/\sec = \omega_c$ $\qquad G_c(s)G(s)\big|_{\omega_c} = 1$

$$\frac{1}{\sqrt{\alpha}T} = \omega,\; T = \frac{1}{\sqrt{\alpha}\omega} = \frac{1}{4.41}$$

Lead Compensation – Frequency Response Approach

6) $$\frac{1}{T} = \sqrt{\alpha}\,\omega = \sqrt{0.24}\cdot 9 = 4.41$$

$$\frac{1}{\alpha T} = \frac{1}{\alpha}\frac{1}{T} = \frac{1}{0.24}\cdot 4.41 = 18.4$$

7) $$G_c(s) = K_c\frac{s+4.41}{s+18.4} = K_c\alpha\frac{0.227s+1}{0.054s+1}$$

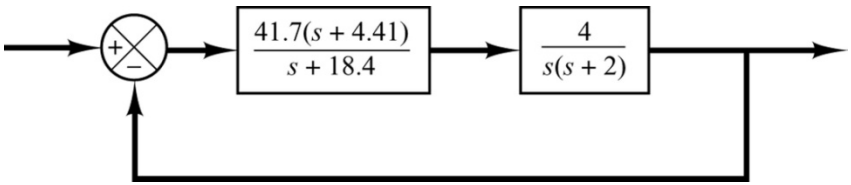
$$K_c = \frac{K}{\alpha} = \frac{10}{0.24} = 41.7$$

8) Check the gain margin

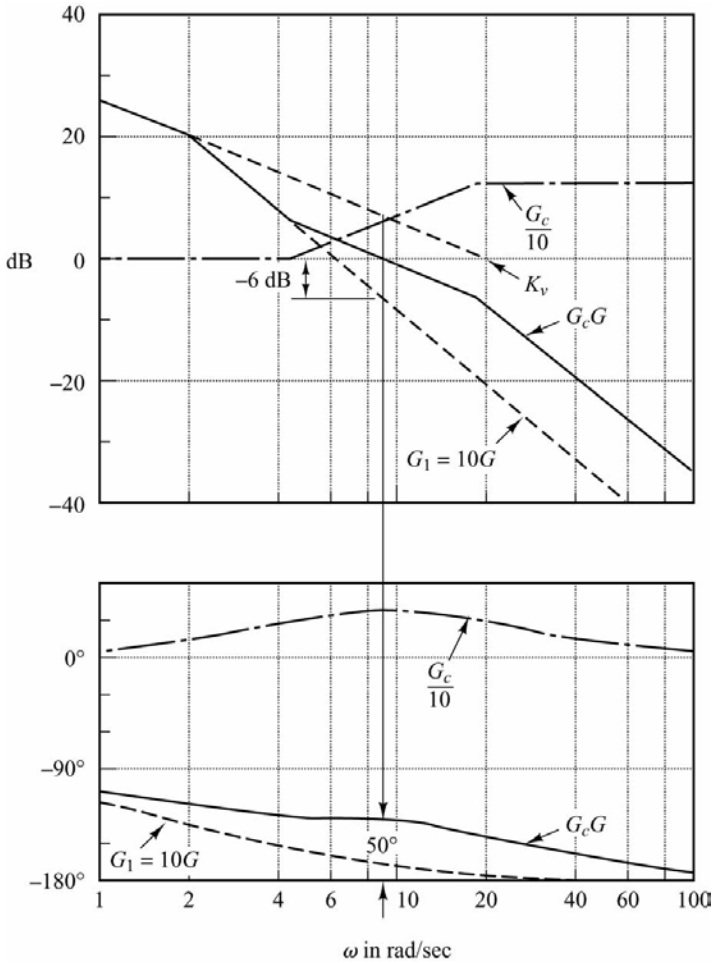
$$G_c(s)G(s) = 41.7\frac{s+4.41}{s+18.4}\frac{4}{s(s+2)}$$

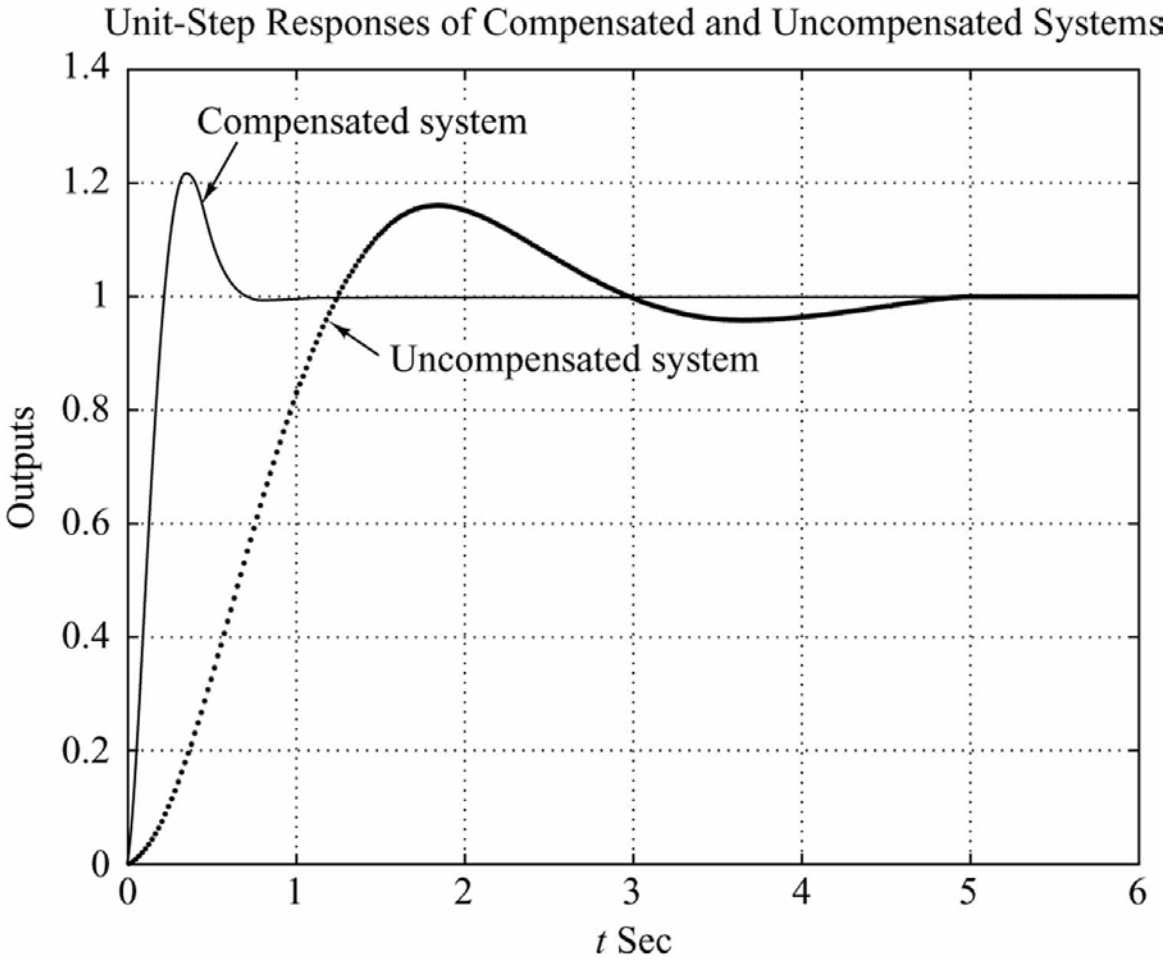
$$= \frac{G_c(s)}{K}K\cdot G(s) = \frac{G_c(s)}{K}G_1(s)$$

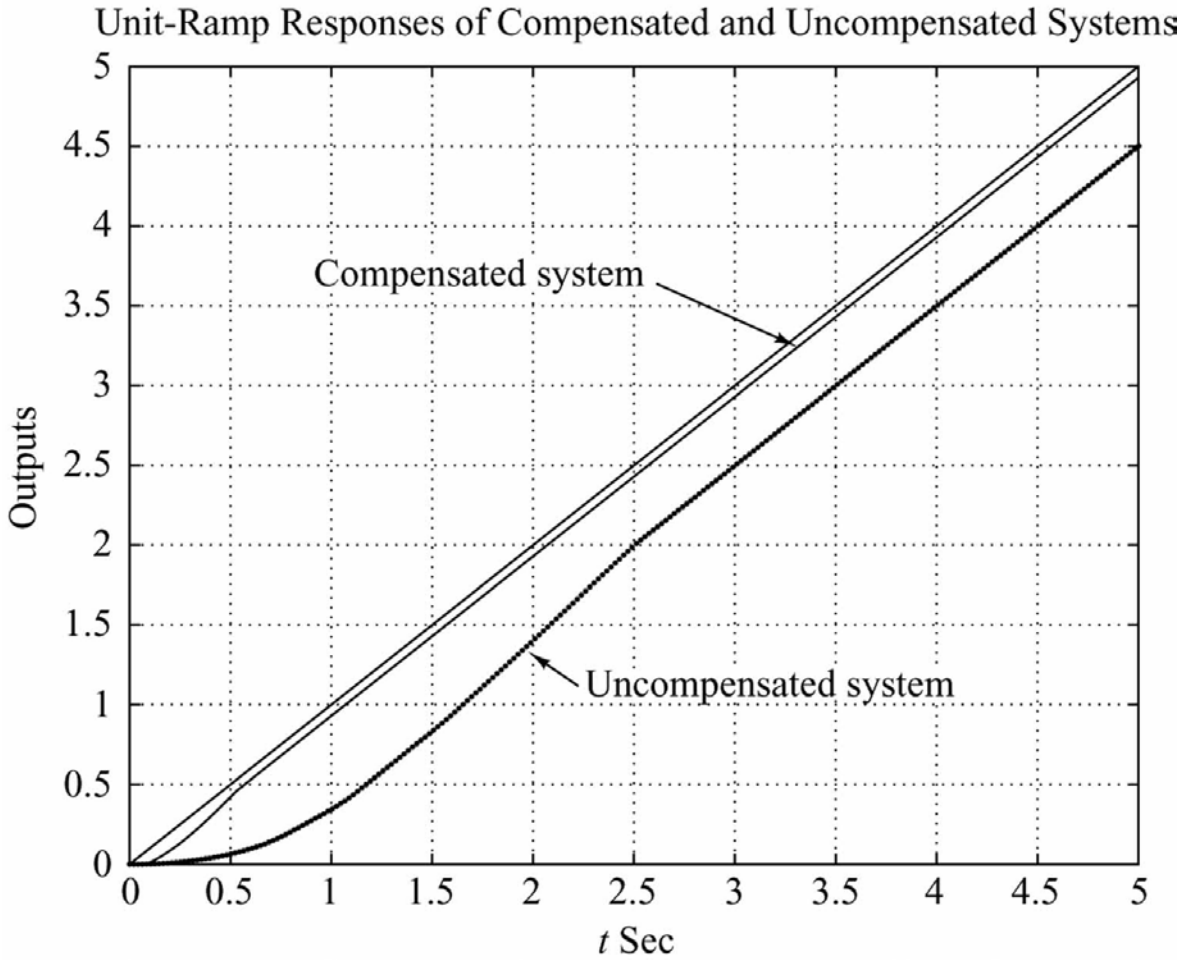
Lead Compensation



Bode diagram for the compensated system




Unit-Step Responses of Compensated and Uncompensated Systems
Compensated system
Uncompensated system
Outputs
t Sec
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6


Unit-Ramp Responses of Compensated and Uncompensated Systems
Compensated system
Uncompensated system
Outputs
t Sec
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Lead Compensation – Frequency Response Approach

Bode plot $G_c(s)G(s) = \frac{G_c(s)}{K} G_1(s), \qquad K = 10$

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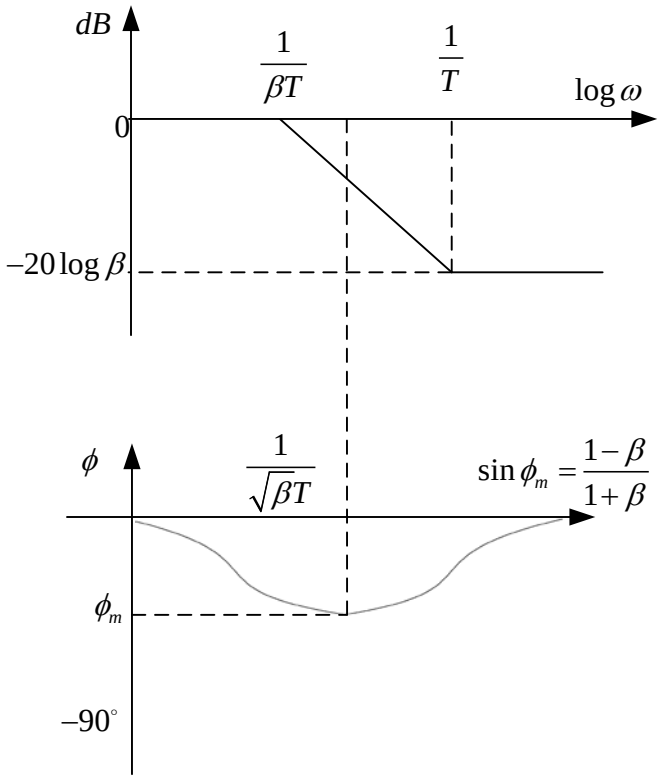
Lag Compensation – Frequency Response Approach

Lag Compensation

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \frac{Ts + 1}{\beta Ts + 1} \qquad (\beta > 1)$$



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Lag Compensation – Frequency Response Approach

Lag Compensation Technique

: to provide "attenuation in the high freq. range" to give a system sufficient phase margin

1) $$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \frac{Ts+1}{\beta Ts+1} \quad (\beta > 1)$$

Define $K = K_i \beta$

$$G_c(s)G(s) = K \frac{Ts+1}{\beta Ts+1} G(s) = \frac{Ts+1}{\beta Ts+1} G_1(s)$$

$$G_1(s) = K \cdot G(s)$$

K

Find K **to satisfy the requirement on the given static velocity error constant.**

2) **The gain adjusted system** $G_1(j\omega) = K \cdot G(j\omega)$

Find the frequency where

$$\phi = -180^\circ + \left(\phi_m + 5 \sim 12^\circ\right)$$

ω_c

Choose this freq. as the new gain crossover freq.

3) **Choose** $\frac{1}{T}$ **1 octave to 1 decade below the new gain crossover freq.**

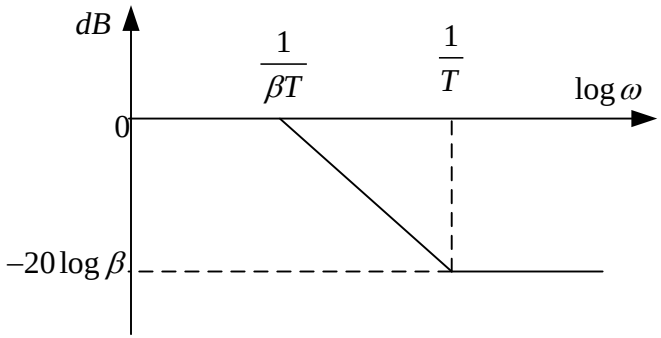
(To minimize phase lag effect)

T

Lag Compensation – Frequency Response Approach

4) **Determine the attenuation necessary to bring the magnitude curve down to 0 dB at the new gain crossover freq.**
Determine the value of β

β



5) $$K_c = \frac{K}{\beta}$$

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Lag Compensation – Frequency Response Approach

Ex

$$G(s)=\frac{1}{s(s+1)(0.5s+1)}$$

Design spec. $K_v = 5\sec^{-1}$

$\phi_m = 40^\circ$

$G_m = 10dB \quad at\ least$

1) $G_c(s)=K_c\beta\dfrac{Ts+1}{\beta Ts+1}\ (\beta>1)$

$$G_1(s)=K\cdot G(s)=\frac{K}{s(s+1)(0.5s+1)}$$

$$K_v=\lim_{s\to 0} sG_c(s)G(s)=K=5$$

$$\therefore K=5=K_c\beta$$

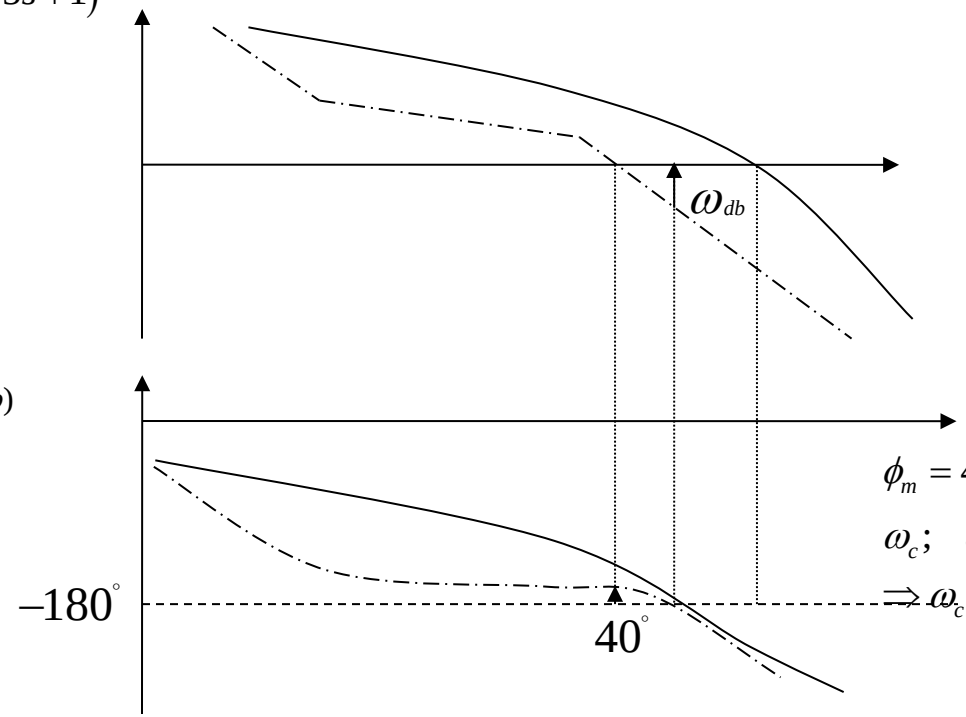
2) $G_1(s)=\dfrac{5}{s(s+1)(0.5s+1)}$

Plot bode diagram of $G_1(j\omega)$

Fig 9-16

$60^\circ \sim 70^\circ$ *angle plus*

if not $\Rightarrow$ *Lag compensator*



$\phi_m = 40^\circ$

$\omega_c;\quad \phi = 40^\circ + 12^\circ = 52^\circ$

$\Rightarrow \omega_c \approx 0.5\ rad/\sec$

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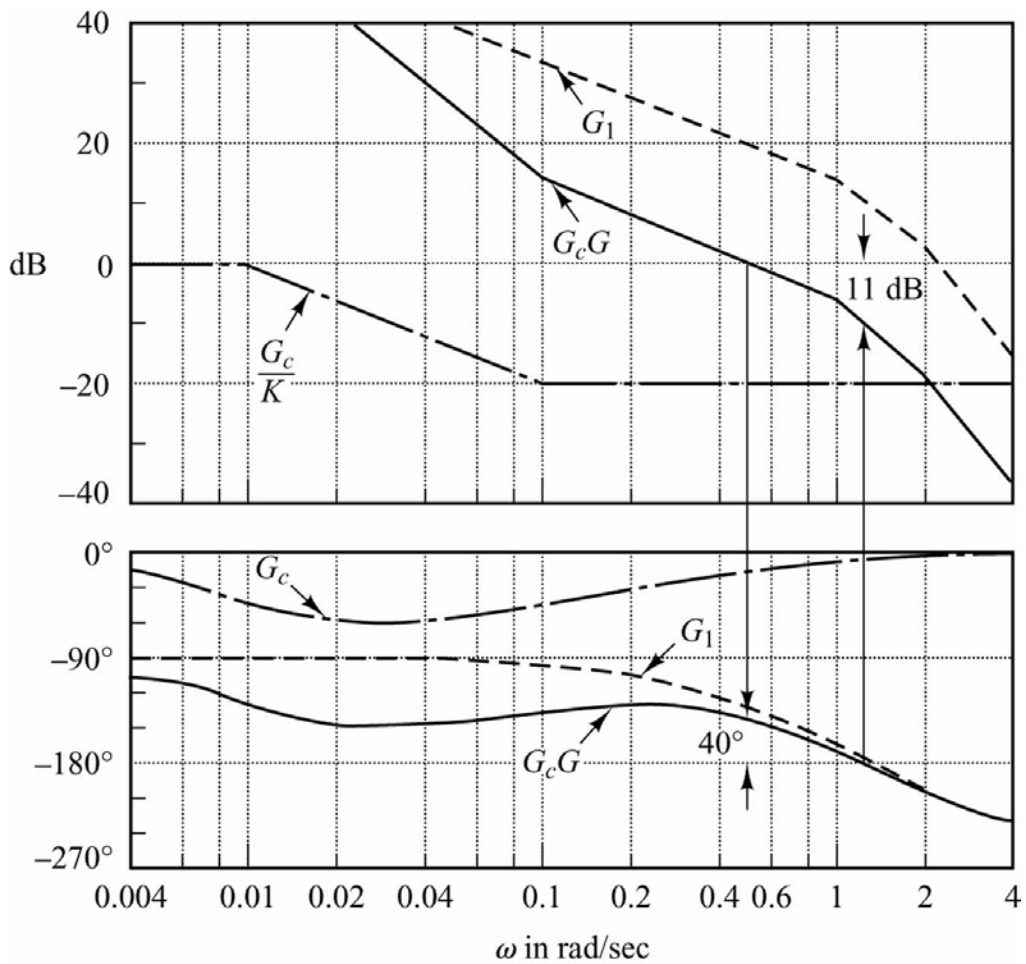
Bode diagrams for G_1 (gain-adjusted but uncompensated open-loop transfer function),G_c (compensator), and G_cG (compensated open-loop transfer function).



Lag Compensation – Frequency Response Approach

3) T **; let** $\omega_c = 0.5$

$$\omega = \frac{1}{T} = 0.1 rad/\sec \quad \Rightarrow \quad T = 10$$

4) β **: at** $\omega_c = 0.5 \qquad |G(j\omega)| \simeq 20dB$

Hence $20\log\frac{1}{\beta} = -20$

$$\beta = 10$$

5) $K_c \cdot \beta = K = 5 \qquad K_c = \frac{5}{\beta} = 0.5, \quad \frac{1}{\beta T} = \frac{1}{100}$

$$G_c(s) = K_c \cdot 10 \cdot \frac{10s+1}{100s+1} = K_c \cdot \frac{s+\frac{1}{10}}{s+\frac{1}{100}}$$

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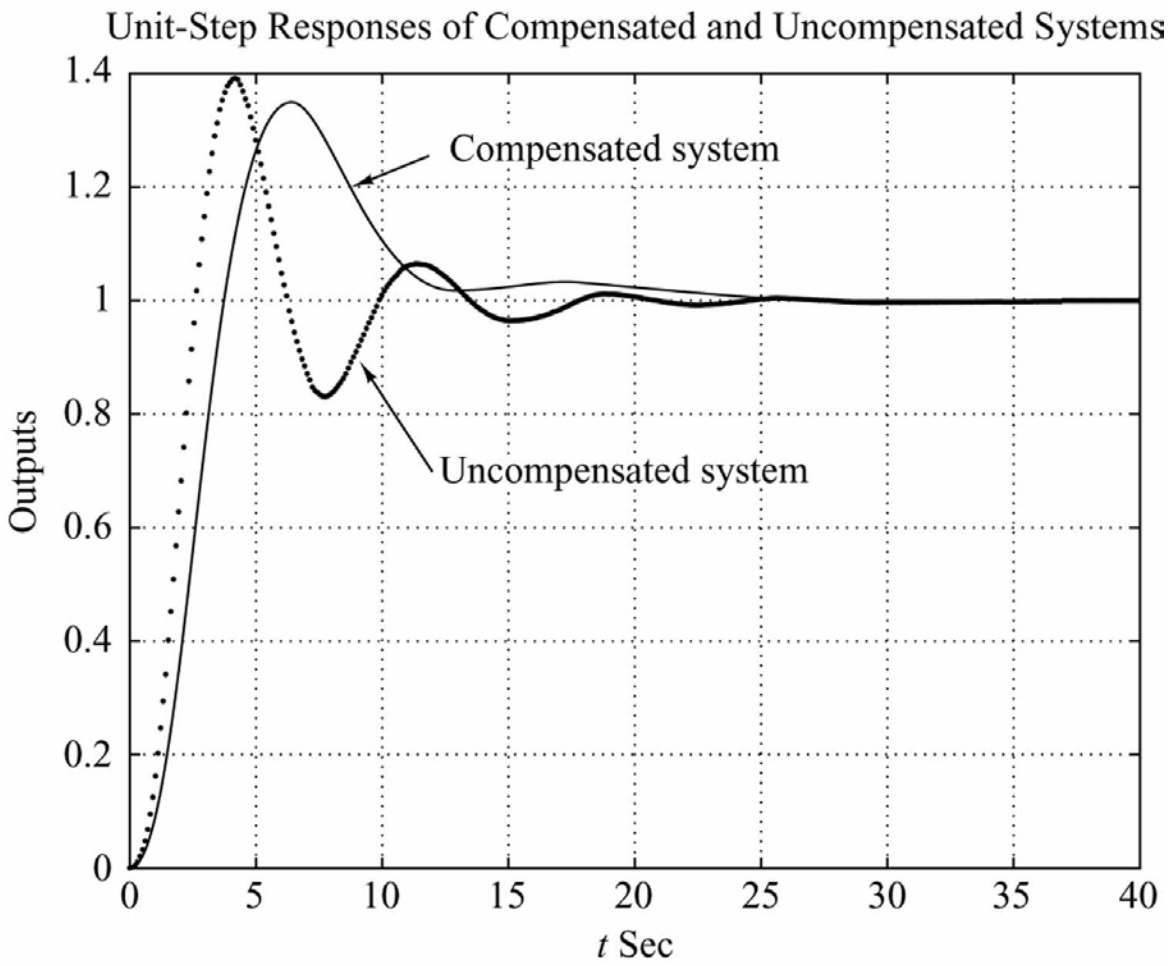
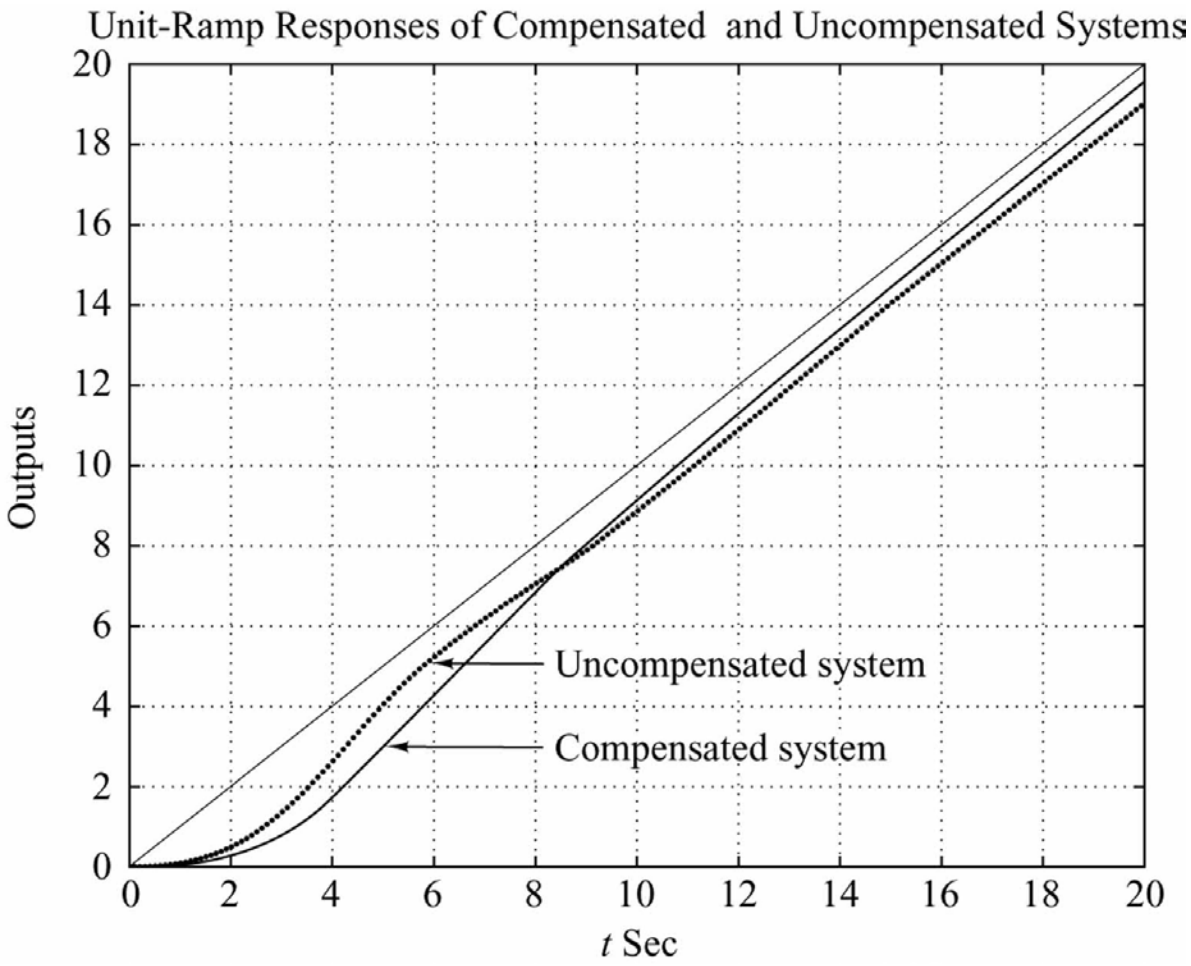

Unit-Step Responses of Compensated and Uncompensated Systems
Compensated system
Uncompensated system
Outputs
t Sec
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
5
10
15
20
25
30
35
40


Unit-Ramp Responses of Compensated and Uncompensated Systems
Outputs
t Sec
Uncompensated system
Compensated system
0
2
4
6
8
10
12
14
16
18
20

Lead-Lag Compensator

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \right) \left(\frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} \right) \qquad (\alpha < 1 \quad \beta > 1)$$

$$= K_c \left(\frac{s + \frac{1}{T}}{s + \frac{\gamma}{T}} \right) \left(\frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} \right) \qquad (\gamma > 1 \quad \beta > 1)$$

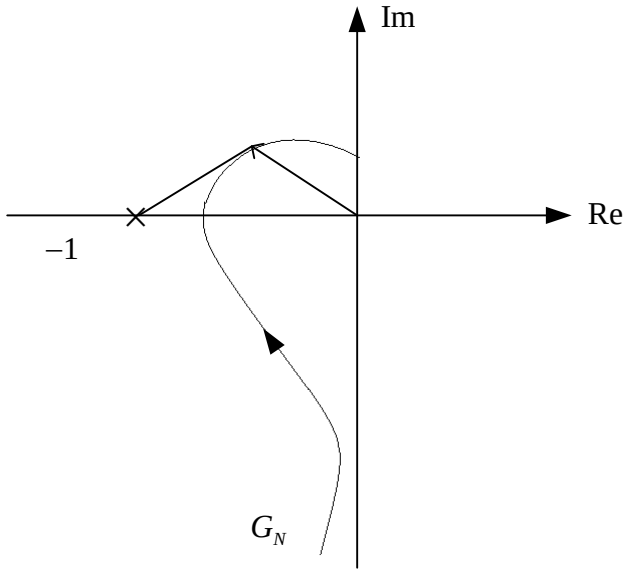
Concluding comments

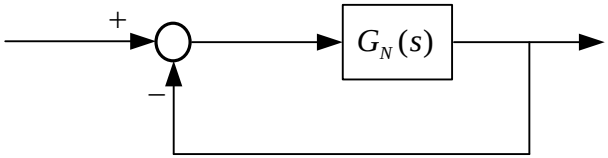
- PID Ziegler Nichols Tuning Rules
- Root locus method
- Frequency response method

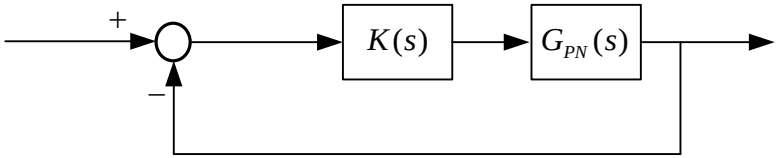
- State space method

End of frequency response method

Robustness to Modeling Errors (Stability Robustness)







<u>Additive Modeling Error</u>

$$G_A(j\omega) = G_N(j\omega) + \delta(j\omega)$$ **: additive modeling error**

At the verge of instability

$$1 + G_A(j\omega) = 0$$

Given $|\delta(j\omega)|$ **, will never happen if**

$$|1 + G_N(j\omega)| > |\delta(j\omega)|$$ **for all** ω

Robustness Condition

$$|1 + K(j\omega)G_N(j\omega)| > |\delta(j\omega)|$$

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Robustness to Modeling Errors (Stability Robustness)

Multiplicative Modeling error

$$G_{PA}(j\omega) = G_{PN}(j\omega)\left(1 + e_m(j\omega)\right)$$

critical condition

$$1 + K(j\omega)G_{PN}(j\omega)\left(1 + e_m(j\omega)\right) = 0$$

$$\left(KG_{PN}\right)^{-1} + 1 + e_m = 0$$

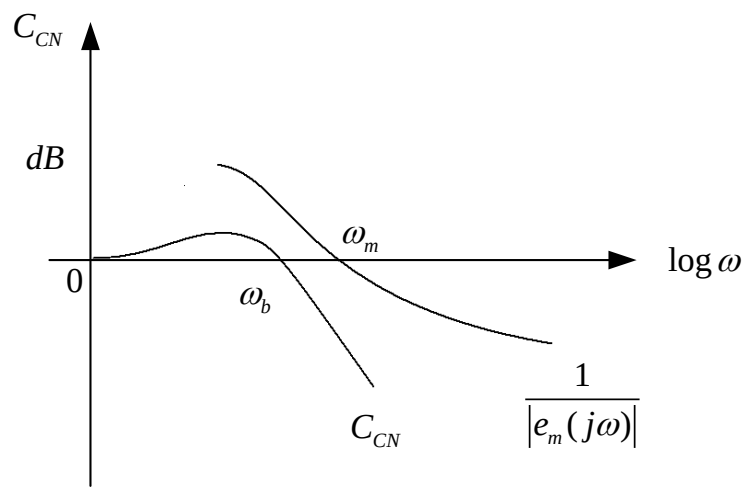
$$\left|1 + \left(KG_{PN}\right)^{-1}\right| > \left|e_m\right| \qquad \textbf{for all } \omega$$

Closed loop nominal

$$C_{CN} = \frac{y}{r} = \frac{K \cdot G_{PN}}{1 + K \cdot G_{PN}} = \frac{1}{1 + \left(K \cdot G_{PN}\right)^{-1}}$$

Robustness condition

$$\left|C_{CN}(j\omega)\right| < \frac{1}{\left|e_m(j\omega)\right|}$$



Simple statement of robustness $\underbrace{\omega_b}_{\textbf{limit closed loop bandwidth}} < \omega_m$

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