Aircraft Structural Analysis

Chapter 6

Work-Energy Principles

6.1 Introduction

Virtual work methods, now widely implemented on computers, are the practical means of solving for the loads and deflections in complex structures. Virtual work principles convey the requirements of equilibrium and compatibility as integral equations, instead of the partial differential equations presented in Chapter 3(Equations 3.3.4 and 3.9.6). The partial work principles are mathematical alternatives to – not approximations of – the differential equations of elasticity.

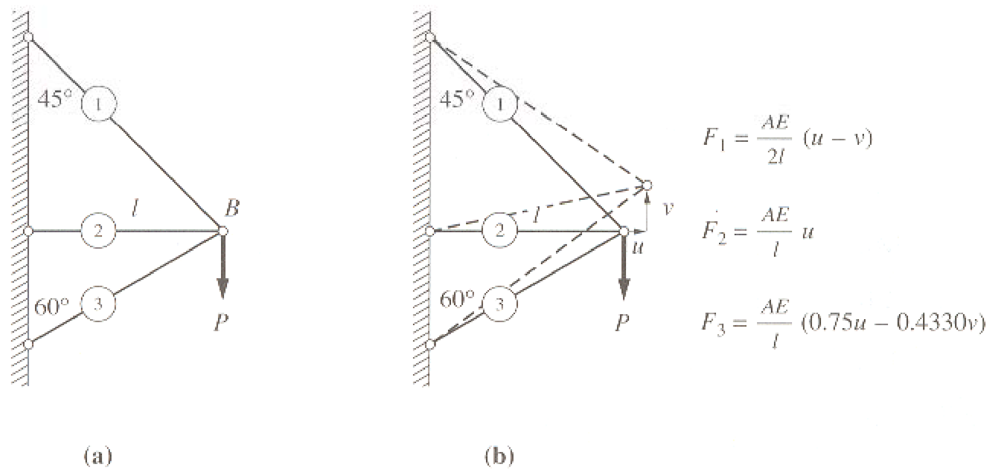

Figure 6.1.1 (a) Truss with an applied load P at joint A. (b) Member forces as a function of the displacements of point A.

All members have the same axial rigidity AE.

6.1 Introduction

This chapter develops the principles of virtual work and the principle of complementary virtual work from the basic concepts of vector statics. These principles are then extended to deformable continua, cast in general terms from which formulas for specific structural elements will be obtained in subsequent chapters. Castigliano's theorems and the theorems of minimum potential energy are consequences of applying the principles of virtual work to linear elastic structures. Since they underlie the Castigliano and minimum energy methods, virtual work methods will be used nearly exclusively in this text.

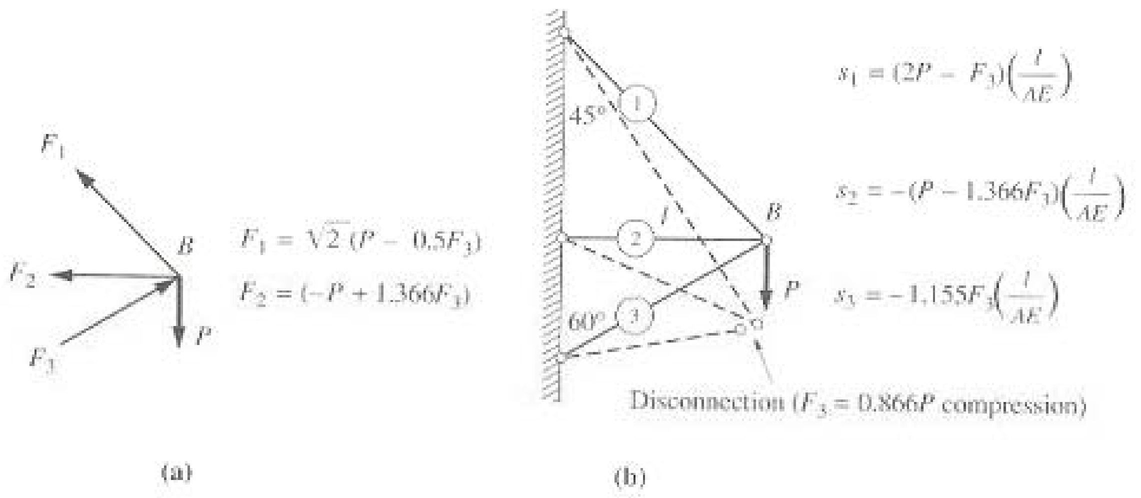

Figure 6.1.2 (a) Free-body diagram and the equilibrium equations for joint B of the truss in Figure 6.1.1a. (b) Example of a failed attempt to select F_3 such that displacement compatibility is maintained at B.

6.2 System Equilibrium and Compatibility

Consider a system of N particles.

$$\mathbf{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{f}_{ij} \qquad [6.2.1]$$

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji} \qquad [6.2.2]$$

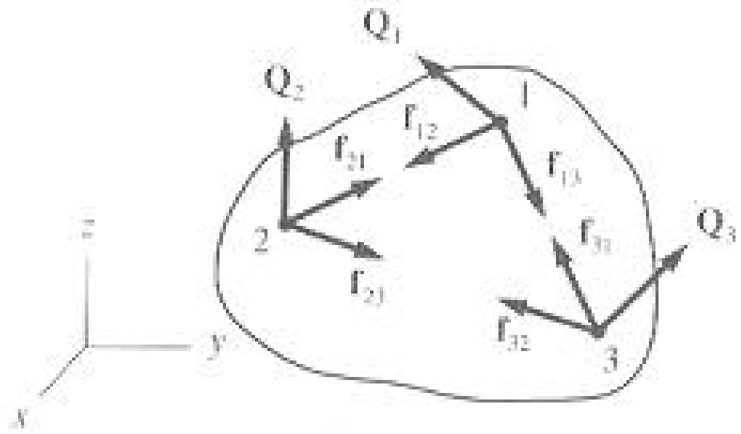

Figure 6.2.1 System of particles, showing external and internal forces.

For the three particle system of Figure 6.2.1, Equations 6.2.1 and 6.2.2 imply that

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{f}_{12} + \mathbf{f}_{13} \\ \mathbf{f}_2 &= \mathbf{f}_{21} + \mathbf{f}_{23} = -\mathbf{f}_{12} + \mathbf{f}_{23} \\ \mathbf{f}_3 &= \mathbf{f}_{31} + \mathbf{f}_{32} = -\mathbf{f}_{13} - \mathbf{f}_{23} \end{aligned} \qquad [6.2.3]$$

6.2 System Equilibrium and Compatibility

$$\mathbf{Q}_i + \mathbf{f}_i = \mathbf{0}, \; i = 1, \cdots, N \qquad [6.2.4]$$

$$\mathbf{Q}_1 = -\mathbf{f}_{12} - \mathbf{f}_{13}$$
$$\mathbf{Q}_2 = \mathbf{f}_{12} - \mathbf{f}_{23} \qquad [6.2.5]$$
$$\mathbf{Q}_3 = \mathbf{f}_{13} + \mathbf{f}_{23}$$

The position vector $\mathbf{r}_{ij}$ of point j relative to point i before deformation is

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \qquad [6.2.6]$$

The relative position vector,

$$\mathbf{r}'_{ij} = (\mathbf{r}_j + \mathbf{q}_j) - (\mathbf{r}_i + \mathbf{q}_i) = \mathbf{r}_{ij} + \mathbf{q}_j - \mathbf{q}_i$$

$$\Rightarrow \quad \Delta\mathbf{r}_{ij} = \mathbf{q}_j - \mathbf{q}_i \qquad [6.2.7]$$

This equation is a *compatibility condition* which states that: *the change in relative position between two particles of the system cannot be prescribed independently of the displacements of those points.*

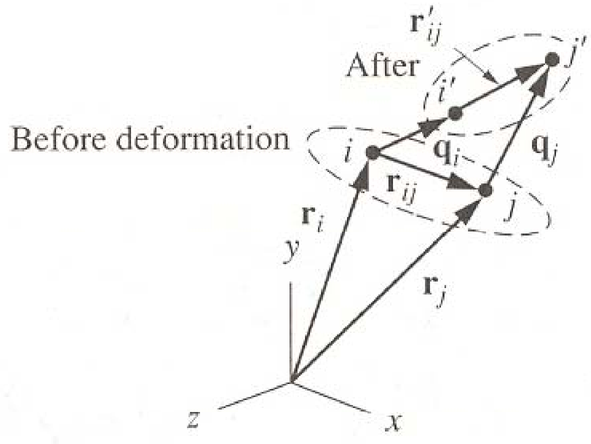

Figure 6.2.2 Relative position vectors of a pair of particles i and j before and after deformation.

6.2 System Equilibrium and Compatibility

Let us take the dot product of the equilibrium equation with the particle's displacement vector q_i,

$$\mathbf{Q}_i \cdot \mathbf{q}_i = -\mathbf{f}_i \cdot \mathbf{q}_i$$

Summing this equation over all the particles of the system,

$$\sum_{i=1}^{N} \mathbf{Q}_i \cdot \mathbf{q}_i = -\sum_{i=1}^{N} \mathbf{f}_i \cdot \mathbf{q}_i \qquad [6.2.8]$$

For the special case of the three-particle system,

$$\sum_{i=1}^{N} \mathbf{f}_i \cdot \mathbf{q}_i = \mathbf{f}_1 \cdot \mathbf{q}_1 + \mathbf{f}_2 \cdot \mathbf{q}_2 + \mathbf{f}_3 \cdot \mathbf{q}_3 \qquad (N = 3)$$

$$\sum_{i=1}^{N} \mathbf{f}_i \cdot \mathbf{q}_i = (\mathbf{f}_{12} + \mathbf{f}_{13}) \cdot \mathbf{q}_1 + (-\mathbf{f}_{12} + \mathbf{f}_{23}) \cdot \mathbf{q}_2 + (-\mathbf{f}_{13} - \mathbf{f}_{23}) \cdot \mathbf{q}_3$$

$$= \mathbf{f}_{12} \cdot (\mathbf{q}_1 - \mathbf{q}_2) + \mathbf{f}_{13} \cdot (\mathbf{q}_1 - \mathbf{q}_3) + \mathbf{f}_{23} \cdot (\mathbf{q}_2 - \mathbf{q}_3)$$

Using the compatibility relation,

$$\sum_{i=1}^{N} \mathbf{f}_i \cdot \mathbf{q}_i = \mathbf{f}_{12} \cdot (-\Delta\mathbf{r}_{12}) + \mathbf{f}_{13} \cdot (-\Delta\mathbf{r}_{13}) + \mathbf{f}_{23} \cdot (-\Delta\mathbf{r}_{23})$$

6.2 System Equilibrium and Compatibility

If a system is in equilibrium and the relative displacements are compatible, then

$$\sum_{i=1}^{N} \mathbf{Q}_i \cdot \mathbf{q}_i = \sum_{\substack{i,j=1 \\ i<j}}^{N} \mathbf{f}_{ij} \cdot \Delta\mathbf{r}_{ij} \qquad [6.2.9]$$

For the three-particle system,

$$\mathbf{Q}_1 \cdot \mathbf{q}_1 + \mathbf{Q}_2 \cdot \mathbf{q}_2 + \mathbf{Q}_3 \cdot \mathbf{q}_3 = \mathbf{f}_{12} \cdot \Delta\mathbf{r}_{12} + \mathbf{f}_{13} \cdot \Delta\mathbf{r}_{13} + \mathbf{f}_{23} \cdot \Delta\mathbf{r}_{23}$$

$$\mathbf{Q}_1 \cdot \mathbf{q}_1 + \mathbf{Q}_2 \cdot \mathbf{q}_2 + \mathbf{Q}_3 \cdot \mathbf{q}_3 = \mathbf{f}_{12} \cdot (\mathbf{q}_2 - \mathbf{q}_1) + \mathbf{f}_{13} \cdot (\mathbf{q}_3 - \mathbf{q}_1) + \mathbf{f}_{23} \cdot (\mathbf{q}_3 - \mathbf{q}_2)$$

$$(\mathbf{Q}_1 + \mathbf{f}_1) \cdot \mathbf{q}_1 + (\mathbf{Q}_2 + \mathbf{f}_2) \cdot \mathbf{q}_2 + (\mathbf{Q}_3 + \mathbf{f}_3) \cdot \mathbf{q}_3 = 0 \qquad [6.2.10]$$

$$(\mathbf{Q}_1 + \mathbf{f}_1) \cdot \mathbf{q}_1 = 0 \qquad \text{for any } \mathbf{q}_1 \neq 0$$

Repeating this argument for each of the N particles,

$$\mathbf{Q}_i + \mathbf{f}_i = 0 \quad i = 1, \cdots, N$$

A system is in equilibrium if and only if Equation 6.2.9 is valid for any compatible deformation.

6.2 System Equilibrium and Compatibility

Next, let us place no restrictions on the displacements,

$$\mathbf{Q}_1 \cdot \mathbf{q}_1 + \mathbf{Q}_2 \cdot \mathbf{q}_2 + \mathbf{Q}_3 \cdot \mathbf{q}_3 = \mathbf{f}_{12} \cdot \Delta\mathbf{r}_{12} + \mathbf{f}_{13} \cdot \Delta\mathbf{r}_{13} + \mathbf{f}_{23} \cdot \Delta\mathbf{r}_{23}$$

Since the loads must be in equilibrium,

$$[\Delta\mathbf{r}_{12} - (\mathbf{q}_2 - \mathbf{q}_1)] \cdot \mathbf{f}_{12} + [\Delta\mathbf{r}_{13} - (\mathbf{q}_3 - \mathbf{q}_1)] \cdot \mathbf{f}_{13} + [\Delta\mathbf{r}_{23} - (\mathbf{q}_3 - \mathbf{q}_2)] \cdot \mathbf{f}_{23} = 0$$

The forces are arbitrary and independent, so we can set $f_{13}=f_{23}=0$

$$|\Delta\mathbf{r}_{12} - (\mathbf{q}_2 - \mathbf{q}_1)||\mathbf{f}_{12}| = 0 \qquad \text{for any value of } |\mathbf{f}_{12}|$$

$$\Delta\mathbf{r}_{12} - (\mathbf{q}_2 - \mathbf{q}_1) = 0 \quad \text{or} \quad \Delta\mathbf{r}_{12} = \mathbf{q}_2 - \mathbf{q}_1$$

The deformation of a system is compatible if and only if Equation 6.2.9 is valid for any self-equilibrating load system.

6.3 The Virtual Work Principle

Consider a particle acted on by N forces, Q_i, i=1, … , N. The resultant force on the particle is Q_R, where $\mathbf{Q}_R = \sum_{i=1}^{N} \mathbf{Q}_i$. If the particle undergoes a real, infinitesimal displacement d**q**, then the incremental work done by the forces on the particle is $dW = \mathbf{Q}_R \cdot d\mathbf{q}$. If instead we imagine that the particle is given a small but fictitious, or virtual, displacement $\delta\mathbf{q}$, while the forces are held constant, then the total virtual work δW done on the particle is

$$\delta W = \mathbf{Q}_R \cdot \delta\mathbf{q}$$

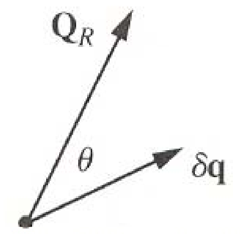

Figure 6.3.1 Particle undergoing a virtual displacement while acted on by the net force $\mathbf{Q}_R$.

A particle is in equilibrium if and only if the virtual work done on the particle is zero for any virtual displacement.

6.3 The Virtual Work Principle

$$\delta W_{\text{ext}} = \sum_{i=1}^{N} \mathbf{Q}_i \cdot \delta \mathbf{q}_i \qquad [6.3.1]$$

$$\delta W_{\text{int}} = \sum_{\substack{i,j=1 \\ i<j}}^{N} \mathbf{f}_{ij} \cdot \delta \mathbf{r}_{ij} \qquad [6.3.2]$$

If the actual displacements $\mathbf{q}_i$ and $\Delta \mathbf{r}_{ij}$ in Equation 6.2.9 are replaced by the virtual displacements $\delta \mathbf{q}_i$ and $\delta \mathbf{r}_{ij}$, where $\delta \mathbf{r}_{ij} = \delta \mathbf{q}_j - \delta \mathbf{q}_i$ (cf. Equation 6.2.7), we obtain

$$\sum_{i=1}^{N} \mathbf{Q}_i \cdot \delta \mathbf{q}_i = \sum_{\substack{i,j=1 \\ i<j}}^{N} \mathbf{f}_{ij} \cdot \delta \mathbf{r}_{ij}$$

$$\therefore \; \delta W_{\text{ext}} = \delta W_{\text{int}}$$

A system is in equilibrium if and only if $\delta W_{\text{ext}} = \delta W_{\text{int}}$ *for any compatible virtual deformation.*

6.3 The Virtual Work Principle

Example 6.3.1

Figure 6.3.2 shows a weight W supported by two cables. Use the principle of virtual work to find the tension in each cable.

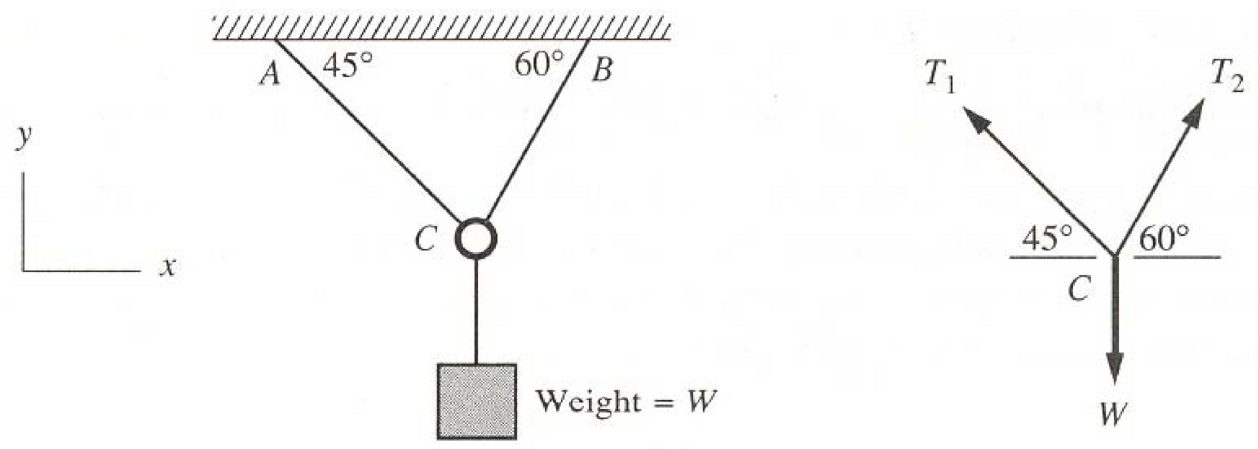

Figure 6.3.2 Supported weight and the free-body diagram.

6.3 The Virtual Work Principle

Example 6.3.1

Resolving the forces on C into components, the total virtual work of the weight and the cable tension is,

$$\delta W = (-W\mathbf{j}) \cdot (\delta u\mathbf{i} + \delta v\mathbf{j}) + (-T_1 \cos 45^\circ\mathbf{i} + T_1 \sin 45^\circ\mathbf{j}) \cdot (\delta u\mathbf{i} + \delta v\mathbf{j}) + (T_2 \cos 60^\circ\mathbf{i} + T_2 \sin 60^\circ\mathbf{j}) \cdot (\delta u\mathbf{i} + \delta v\mathbf{j})$$

Setting $\delta W = 0$,

$$(-T_1 \cos 45^\circ + T_2 \cos 60^\circ)\delta u + (-W + T_1 \sin 45^\circ + T_2 \sin 60^\circ)\delta v = 0$$

According to the principle of virtual work, this equality must hold for arbitrary values of δu and δv

$$-T_1 \cos 45^\circ + T_2 \cos 60^\circ = 0$$

$$T_1 \sin 45^\circ + T_2 \sin 60^\circ = W$$

The solution of these equations is $T_1 = 0.518W$ and $T_2 = 0.732W$.

6.3 The Virtual Work Principle

Example 6.3.2

The figure shows three springs rigidly attached to the wall at points B, C, and D, and attached to each other at point A, where the external load P is applied. Each spring has a unique spring constant. Use the principle of virtual work to find the spring forces F_1, F_2, and F_3.

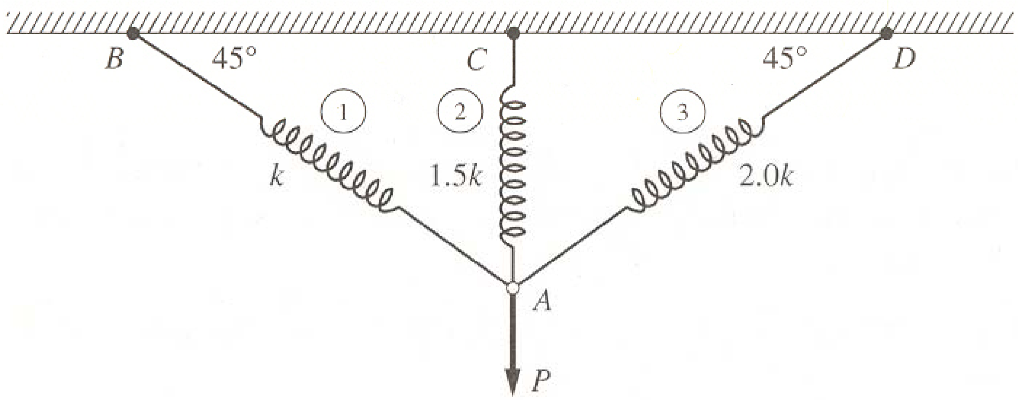

Figure 6.3.3 Three springs in equilibrium under the point load *P*.

6.3 The Virtual Work Principle

Example 6.3.2

The virtual displacement of point A is

$$\delta \mathbf{q}_A = \delta u_A \mathbf{i} + \delta v_A \mathbf{j}$$

Let the extension or stretch of each spring due to the load be s_i, i=1,2,3

$$s_1 = \mathbf{q}_A \cdot \mathbf{n}_{BA} = u_A \cos 45^\circ - v_A \sin 45^\circ$$

$$s_2 = \mathbf{q}_A \cdot \mathbf{n}_{CA} = -v_A$$

$$s_3 = \mathbf{q}_A \cdot \mathbf{n}_{DA} = -u_A \cos 45^\circ - v_A \sin 45^\circ$$

And the virtual stretches are,

$$\delta s_1 = \delta \mathbf{q}_A \cdot \mathbf{n}_{BA} = \delta u_A \cos 45^\circ - \delta v_A \sin 45^\circ$$

$$\delta s_2 = \delta \mathbf{q}_A \cdot \mathbf{n}_{CA} = -\delta v_A$$

$$\delta s_3 = \delta \mathbf{q}_A \cdot \mathbf{n}_{DA} = -\delta u_A \cos 45^\circ - \delta v_A \sin 45^\circ$$

6.3 The Virtual Work Principle

Example 6.3.2

Thus, the internal virtual work is,

$$\delta W_{\text{int}} = \delta W_{\text{int, spring 1}} + \delta W_{\text{int, spring 2}} + \delta W_{\text{int, spring 3}}$$
$$= k s_1 \delta s_1 + (1.5k)\, s_2 \delta s_2 + (2k)\, s_3 \delta s_3$$
$$= k(u_A \cos 45^\circ - v_A \sin 45^\circ)(\delta u_A \cos 45^\circ - \delta v_A \sin 45^\circ) + (1.5k)(-v_A)(-\delta v_A)$$
$$+ (2k)\,(-u_A \cos 45^\circ - v_A \sin 45^\circ)(-\delta u_A \cos 45^\circ - \delta v_A \sin 45^\circ)$$

or
$$\delta W_{\text{int}} = (1.5ku_A + 0.5kv_A)\delta u_A + (0.5ku_A + 3.0kv_A)\delta v_A$$

Next, the external virtual work of the applied load is,

$$\delta W_{\text{ext}} = (-P\mathbf{j}) \cdot \delta \mathbf{q}_A = -P\delta v_A$$

$$\delta W_{\text{ext}} = \delta W_{\text{int}} \quad \Rightarrow \quad \begin{aligned} 1.5ku_A + 0.5kv_A &= 0 \\ 0.5ku_A + 3.0kv_A &= -P \end{aligned}$$

$$s_1 = 0.3328(P/k) \qquad s_2 = 0.3529(P/k) \qquad s_3 = 0.1664(P/k)$$

$$F_1 = k\left[0.3328\left(\frac{P}{k}\right)\right] = 0.3328\,P \qquad F_2 = 1.5k\left[0.3529\left(\frac{P}{k}\right)\right] = 0.5294\,P \qquad F_3 = 2.0k\left[0.1664\left(\frac{P}{k}\right)\right] = 0.3328\,P$$

6.3 The Virtual Work Principle

From Equation 3.14.1, we know that in a quasistatic loading process, the work done within a solid by the true stresses during an increment of the true strains is

$$dW = \iiint_V dw_o dV = \iiint_V \left(\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{xz} d\gamma_{xz} + \tau_{yz} d\gamma_{yz}\right) dV \quad [6.3.3]$$

$$\delta W_{\text{int}} = \iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz}) dV \quad [6.3.4]$$

$$\delta W_{\text{ext}} = \sum_{i=1}^{n} \mathbf{Q}_i \cdot \delta \mathbf{q}_i + \iint_S \mathbf{T}^{(n)} \cdot \delta \mathbf{u} dS + \iiint_V \mathbf{b} \cdot \delta \mathbf{u} d \quad [6.3.5]$$

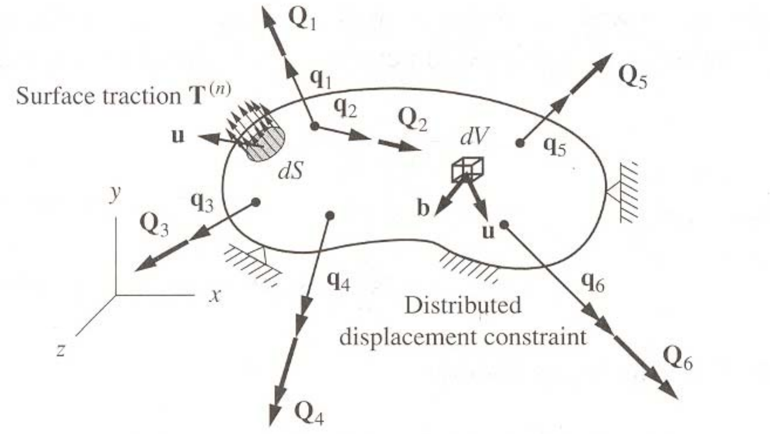

Figure 6.3.4 A solid body, constrained in an arbitrary fashion and acted upon by generalized point loads (**Q**) plus surface ($\mathbf{T}^{(n)}$) and volume (**b**) force fields.

6.4 Minimum Potential Energy and Castigliano's First Theorem

If a solid is linearly elastic, then according to section 3.14, the internal work associated with a quasistatic loading process equals the strain energy U, or

$$W_{\text{int}} = U$$

$$U = U(q_1, q_2, \cdots, q_n) \qquad [6.4.1]$$

For a virtual deformation, $\delta W_{\text{int}} = \delta U$ [6.4.2]

$$\delta U = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} \delta q_i$$

$$\delta W_{\text{ext}} = \sum_{i=1}^{n} Q_i \delta q_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} Q_i \delta q_i = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} \delta q_i \qquad [6.4.4]$$

6.4 Minimum Potential Energy and Castigliano's First Theorem

If we set all of the virtual displacements except δq_1 equal to zero,

$$Q_1 \delta q_1 = \left(\frac{\partial U}{\partial q_1} \right) \delta q_1 , \qquad \text{or} \qquad Q_1 = \frac{\partial U}{\partial q_1}$$

Repeating the argument for the remaining virtual displacements,

$$Q_i = \frac{\partial U}{\partial q_i}, \quad i = 1, 2, \cdots, n \qquad [6.4.5]$$

Castigliano's first theorem

The potential energy V of the external loads is,

$$V = -\sum_{i=1}^{n} Q_i q_i \qquad [6.4.6]$$

$$\frac{\partial V}{\partial q_i} = -Q_i \quad i = 1, \cdots, n \qquad [6.4.7]$$

Castigliano's first theorem may thus be written

$$\frac{\partial \Pi}{\partial q_i} = 0 \quad i = 1, \cdots, n \qquad [6.4.8]$$

(Eq 6.4.8 is a statement of the *theorem of minimum potential energy*)

(where $\Pi = U + V$ is the total potential energy of the structure)

6.4 Minimum Potential Energy and Castigliano's First Theorem

Example 6.4.1

Solve the problem in Example 6.3.2 using the principle of minimum potential energy.

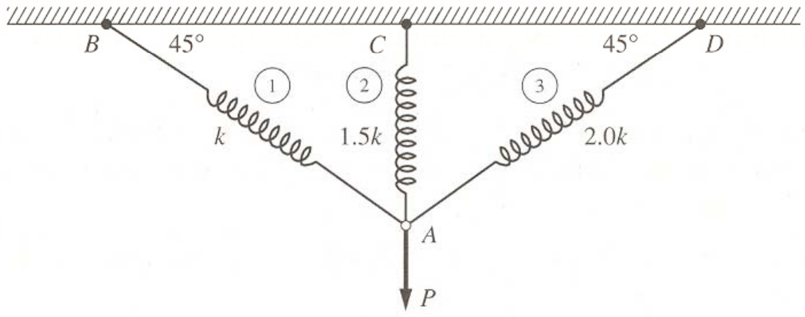

Figure 6.3.3 Three springs in equilibrium under the point load *P*.

The total strain energy of the three-spring assembly is,

$$U = \frac{1}{2}k_1 s_1^2 + \frac{1}{2}k_2 s_2^2 + \frac{1}{2}k_3 s_3^2 = \frac{1}{2}ks_1^2 + \frac{1}{2}(1.5k)s_2^2 + \frac{1}{2}(2k)s_3^2$$

$$U = \frac{1}{2}k(u_A \cos 45^\circ - v_A \sin 45^\circ)^2 + \frac{1}{2}(1.5k)(-v_A)^2 + \frac{1}{2}(2k)(-u_A \cos 45^\circ - v_A \sin 45^\circ)^2$$

$$= \frac{3}{4}ku_A^2 + \frac{3}{2}kv_A^2 + \frac{1}{2}ku_A v_A \qquad [a]$$

From Equation 6.4.6, the potential energy of the load P is

$$V = -(-P)v_A \qquad [b]$$

6.4 Minimum Potential Energy and Castigliano's First Theorem

The total potential energy is Π = U + V,

$$\Pi = \frac{3}{4}ku_A^2 + \frac{3}{2}kv_A^2 + \frac{1}{2}ku_Av_A + Pv_A \qquad [c]$$

$$\begin{aligned} \frac{\partial \Pi}{\partial u_A} = 0: &\quad \frac{3}{2}ku_A + \frac{1}{2}kv_A = 0 \\ \frac{\partial \Pi}{\partial v_A} = 0: &\quad \frac{1}{2}ku_A + 3kv_A + P = 0 \end{aligned} \qquad [d]$$

$$u_A = 0.1176\left(\frac{P}{k}\right) \qquad v_A = -0.3529\left(\frac{P}{k}\right) \qquad [e]$$

Substituting the displacements in Equation [e] into Equation [c],

$$\frac{\partial^2\Pi}{\partial u_A^2}\cdot\frac{\partial^2\Pi}{\partial v_A^2} - \left(\frac{\partial^2\Pi}{\partial u_A \partial v_A}\right)^2 > 0 \qquad \frac{\partial^2\Pi}{\partial u_A^2} > 0 \qquad [g]$$

6.4 Minimum Potential Energy and Castigliano's First Theorem

Calculating the second partial derivatives of Π yields,

$$\frac{\partial^2 \Pi}{\partial u_A^2} = \frac{3}{2}k \qquad \frac{\partial^2 \Pi}{\partial v_A^2} = 3k \qquad \frac{\partial^2 \Pi}{\partial u_A \partial v_A} = \frac{1}{2}k$$

Since k is positive, both conditions in Equation [g] are satisfied: Π is indeed a minimum.

6.5 Stiffness Matrix

If a structure is not only elastic but *linearly elastic*, then by definition, the generalized loads Q_i and the generalized displacements q_i in the direction of the loads are directly proportional to each other.

$$Q_i = \sum_{l=1}^{n} k_{il} q_l, \qquad i = 1, 2, \cdots, n \qquad [6.5.1]$$

$$\begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot & \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{n1} & k_{n2} & \cdot & \cdot & \cdot & k_{nn} \end{bmatrix}$$

<Stiffness matrix>

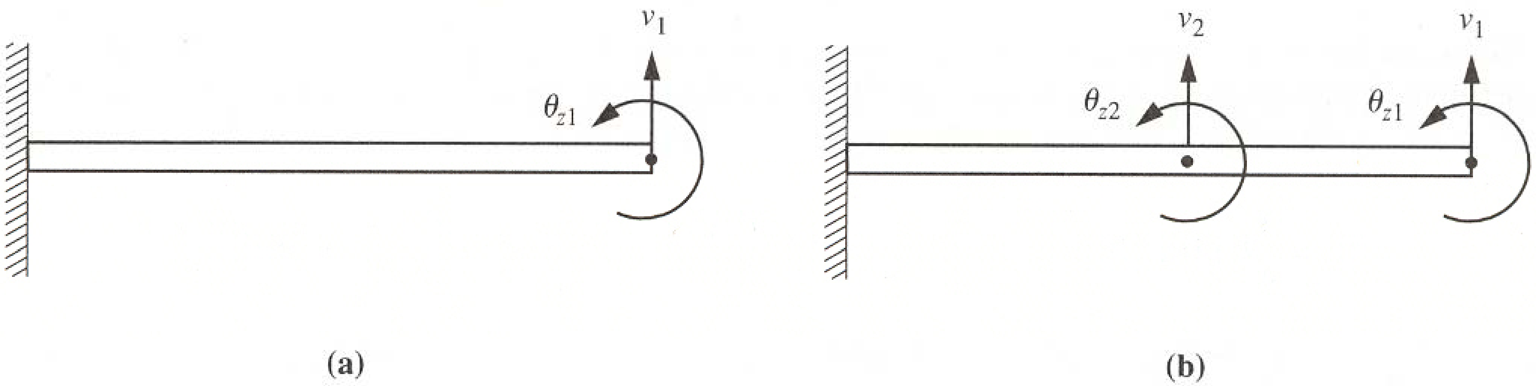

Figure 6.5.1 (a) Cantilever beam with two degrees of freedom. (b) The same beam with four degrees of freedom.

6.5 Stiffness Matrix

$$Y_1 = k_{11}v_1 + k_{12}\theta_{z_1}$$
$$M_{z_1} = k_{21}v_1 + k_{22}\theta_{z_1}$$

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial}{\partial q_j}\sum_{l=1}^{n} k_{il}q_l = \sum_{l=1}^{n} k_{il}\frac{\partial q_l}{\partial q_j} \qquad [6.5.2]$$

⇨ $\frac{\partial Q_i}{\partial q_j} = k_{ij}$ and $\frac{\partial Q_j}{\partial q_i} = k_{ji}$ [6.5.3] & [6.5.4]

From Castigliano's first theorem, $\frac{\partial Q_i}{\partial q_j} = \frac{\partial}{\partial q_j}\frac{\partial U}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{\partial U}{\partial q_j} = \frac{\partial Q_j}{\partial q_i}$ [6.5.5]

$$k_{ij} = k_{ji}, \qquad i, j = 1, 2, \cdots, n \qquad [6.5.6]$$

The stiffness matrix of an elastic structure is symmetric.

The number of independent components of an n by n symmetric matrix is $\frac{n(n+1)}{2}$.

[6.5.7]

6.6 The Complementary Virtual Work Principle

The internal complementary virtual work is defined as,

$$\delta W^*_{\text{int}} = \sum_{\substack{i,j=1 \\ i<j}}^{N} \Delta \mathbf{r}_{ij} \cdot \delta \mathbf{f}_{ij} \qquad [6.6.1]$$

The external complementary virtual work is defined as,

$$\delta W^*_{\text{ext}} = \sum_{i=1}^{N} \mathbf{q}_i \cdot \delta \mathbf{Q}_i \qquad [6.6.2]$$

(The virtual quantities are the loads instead of the displacements)

$$\sum_{i=1}^{N} \mathbf{q}_i \cdot \delta \mathbf{Q}_i = \sum_{\substack{i,j=1 \\ i<j}}^{N} \Delta \mathbf{r}_{ij} \cdot \delta \mathbf{f}_{ij}$$

$$\therefore \quad \delta W^*_{\text{ext}} = \delta W^*_{\text{int}} \, .$$

The displacements of a system satisfy compatibility if and only if $\delta W^*_{\text{ext}} = \delta W^*_{\text{int}}$ *for any self-equilibrating virtual loading.*

6.6 The Complementary Virtual Work Principle

Example 6.6.1

Figure 6.6.1a shows a load W supported by two springs with identical spring constants k. The picture is similar to that for Example 6.3.1, in which the spring loads were found to have the values shown. Find the horizontal displacement of point P, using the principle of complementary virtual work.

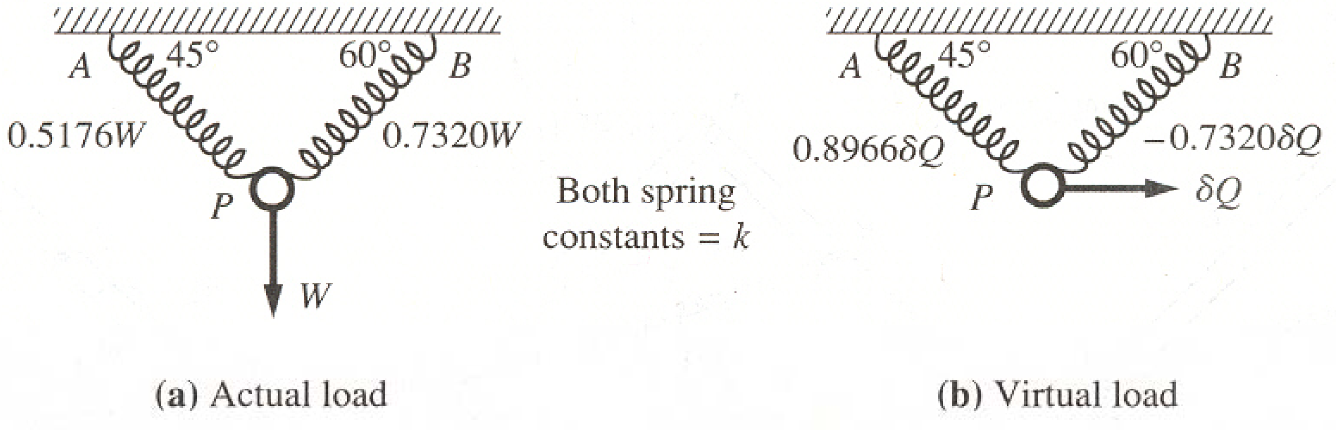

Figure 6.6.1 Point load supported by two springs.

6.6 The Complementary Virtual Work Principle

Example 6.6.1

The external virtual complementary work is,

$$\delta W^*_{\text{ext}} = u\delta Q$$

To calculate the internal complementary virtual work,

$$\Delta \mathbf{r}_{ij} \cdot \delta \mathbf{f}_{ij} = \Delta s_{ij} \delta f_{ij}$$

(where Δr_{ij} is the change in the distance between the points caused by the actual loading, δf_{ij} is the signed magnitude of the virtual force in the spring)

If the spring is elastic with spring rate k,

$$\Delta s_{ij} = \frac{f_{ij}}{k}$$

6.6 The Complementary Virtual Work Principle

Example 6.6.1

Then,

$$\begin{aligned}\delta W_{\text{int}}^* &= \left(\frac{f_{AP}}{k}\right)\delta f_{AP} + \left(\frac{f_{BP}}{k}\right)\delta f_{BP} \\ &= \left(\frac{0.5176W}{k}\right)(0.8966\delta Q) + \left(\frac{0.7320W}{k}\right)(-0.7320\delta Q) \\ &= \frac{0.4641W}{k}\delta Q - \frac{0.5358W}{k}\delta Q \\ &= -\frac{0.0717W}{k}\delta Q\end{aligned}$$

Setting $\delta W_{\text{ext}}^* = \delta W_{\text{int}}^*$, we have

$$u\delta Q = -0.0717\left(\frac{W}{k}\right)\delta Q \quad \Rightarrow \quad u = -0.0717\left(\frac{W}{k}\right)$$

6.6 The Complementary Virtual Work Principle

Example 6.6.2

The structure shown in Figure 6.6.2a supports a vertical at A. Use the principle of complementary virtual work to find (a) the horizontal displacement u of point C, and (b) the rotation θ_{AD} of member AD, due to the load P.

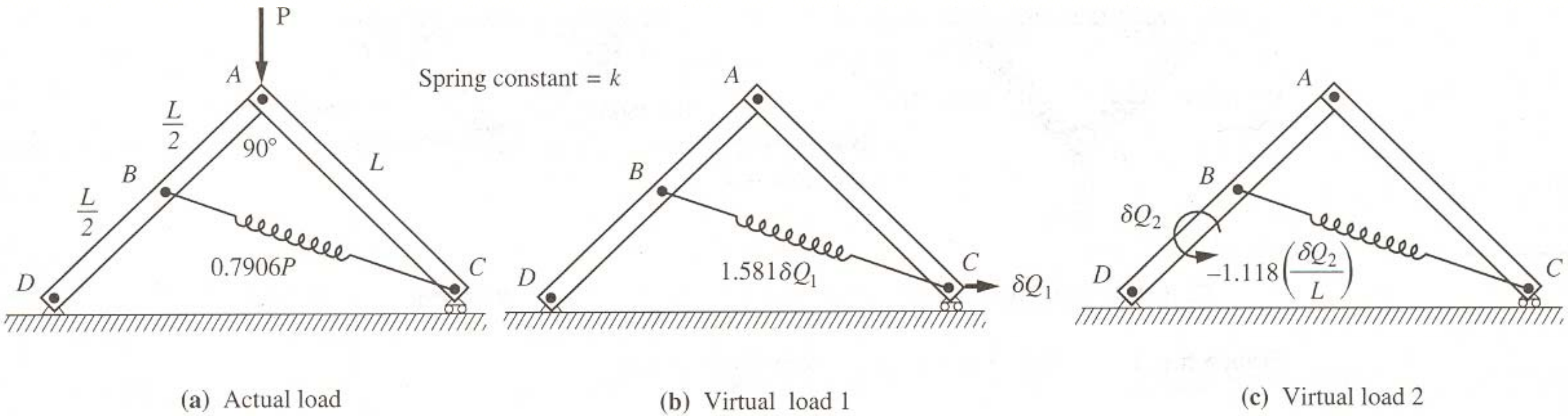

Figure 6.6.2 Two rigid pin-connected members joined by a spring.

6.6 The Complementary Virtual Work Principle

Example 6.6.2

(a)

$$\delta W^*_{\text{ext}} = u\delta Q_1$$

$$\delta W^*_{\text{int}} = \left(\frac{f_{BC}}{k}\right)\delta f_{BC}$$

$$= \left(\frac{0.7906P}{k}\right)(1.581\delta Q_1)$$

$$= 1.25\left(\frac{P}{k}\right)\delta Q_1$$

$$u = 1.25\left(\frac{P}{k}\right) \qquad \text{(to the right)}$$

(b)

$$\delta W^*_{\text{ext}} = \theta_{AD}\delta Q_2$$

$$\delta W^*_{\text{int}} = \left(\frac{f_{BC}}{k}\right)\delta f_{BC}$$

$$= \left(\frac{0.7906P}{k}\right)\left(-1.118\frac{\delta Q_2}{L}\right)$$

$$= -0.884\left(\frac{P}{kL}\right)\delta Q_2$$

$$\theta_{AD} = -0.884\left(\frac{P}{kL}\right)$$

(The negative sign means that AD rotates clockwise.)

6.6 The Complementary Virtual Work Principle

For a continuous medium, the internal complementary virtual work of the true strains (held fixed) acting through the virtual stresses is inferred from Equation 6.3.4 by analogy :

$$\delta W_{\text{int}}^* = \iiint_V (\varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \varepsilon_z \delta\sigma_z + \gamma_{xy}\delta\tau_{xy} + \gamma_{xz}\delta\tau_{xz} + \gamma_{yz}\delta\tau_{yz})dV \qquad [6.6.4]$$

$$\delta W_{\text{ext}}^* = \sum_{i=1}^{n} \mathbf{q}_i \cdot \delta\mathbf{Q}_i + \iint_S \mathbf{u} \cdot \delta\mathbf{T}^{(n)} dS + \iiint_V \mathbf{u} \cdot \delta\mathbf{b} dV \qquad [6.6.5]$$

$\delta\mathbf{Q}_i$: virtual generalized load

q_i : the generalized displacement in the direction of $\delta\mathbf{Q}_i$

6.7 Minimum Complementary Potential Energy and Castigliano's Second Theorem

From section 3.14, the internal complementary work of a quasistatic process from the undeformed to the deformed state equals the complementary strain energy U^*,

$$W^*_{\text{int}} = U^*$$

The complementary strain energy U^* is a function of the applied loads,

$$U^* = U^*(Q_1, Q_2, \cdots, Q_n)$$

For a virtual deformation,

$$\delta W^*_{\text{int}} = \delta U^* = \sum_{i=1}^{n} \frac{\partial U^*}{\partial Q_i} \delta Q_i$$

$$\delta W^*_{\text{ext}} = \sum_{i=1}^{n} q_i \delta Q_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} q_i \delta Q_i = \sum_{i=1}^{n} \frac{\partial U^*}{\partial Q_i} \delta Q_i$$

6.7 Minimum Complementary Potential Energy and Castigliano's Second Theorem

The coefficients of δQ_i on each side of the equation must be the same

$$q_i = \frac{\partial U^*}{\partial Q_i} \qquad i = 1, 2, \cdots, n$$

[6.7.4] *Castigliano's second theorem*

$$\frac{\partial \Pi^*}{\partial Q_i} = 0 \qquad i = 1, \cdots, n$$

[6.7.5] *theorem of minimum complementary potential energy*

(where $\Pi^* = U^* + V^*$ is the total complementary potential energy)

And,

$$V^* = V = -\sum_{i=1}^{n} Q_i q_i$$

[6.7.6]

6.7 Minimum Complementary Potential Energy and Castigliano's Second Theorem

Example 6.7.1

Solve the problem of Example 6.3.2 using the principle of minimum complementary potential energy as an alternative to using the principle of minimum potential energy, which was done in Example 6.4.1. The sketch for that problem is reproduced in Figure 6.7.1a for convenience.

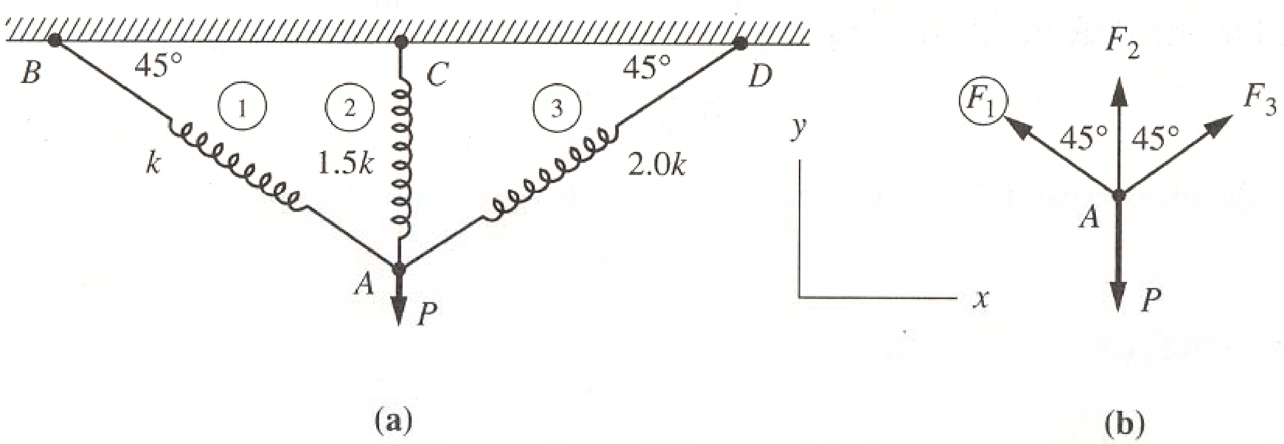

Figure 6.7.1 (a) The system of Example 6.3.2. (b) Free-body diagram of point A.

F_1 is circled to highlight its selection as the redundant force.

6.7 Minimum Complementary Potential Energy and Castigliano's Second Theorem

Example 6.7.1

The complementary strain energy of a spring is,

$$U_s^* = \frac{1}{2}Fs = \frac{1}{2}F\left(\frac{F}{k}\right) = \frac{F^2}{2k}$$

Therefore, the total complementary strain energy is,

$$U^* = \frac{F_1^2}{2k_1} + \frac{F_2^2}{2k_2} + \frac{F_3^2}{2k_3}$$

The potential energy of the applied load is,

$$V^* = -(-P)v_A = Pv_A$$

The total complementary potential energy is,

$$\Pi^* = \frac{F_1^2}{2k_1} + \frac{F_2^2}{2k_2} + \frac{F_3^2}{2k_3} + Pv_A$$

6.7 Minimum Complementary Potential Energy and Castigliano's Second Theorem

Example 6.7.1

The equations of equilibrium for point A

$$x: \quad -F_1 \cos 45^\circ + F_3 \cos 45^\circ = 0$$

$$y: \quad F_1 \sin 45^\circ + F_2 + F_3 \sin 45^\circ + P = 0$$

$\Rightarrow$

$$F_2 = -\sqrt{2}F_1 + P \quad F_3 = F_1$$

$$\Pi^* = 1.417\frac{F_1^2}{k} + 0.3333\frac{P^2}{k} - 0.9428\frac{F_1 P}{k} + P v_A$$

Since F_1 and P are independent variables,

$$\frac{\partial \Pi^*}{\partial F_1} = 2.833\frac{F_1}{k} - 0.9428\frac{P}{k} \quad \frac{\partial \Pi^*}{\partial P} = -0.9428\frac{F_1}{k} + 0.6667\frac{P}{k} + v_A$$

$$2.833\frac{F_1}{k} = 0.9428\frac{P}{k}$$

$$0.9428\frac{F_1}{k} + v_A = -0.6667\frac{P}{k}$$

$$F_1 = 0.3328P \quad v_A = -0.3529\frac{P}{k}$$

6.8 Flexibility Matrix

The *flexibility matrix* of a linearly elastic structure is the set of coefficients that relate the generalized displacements to the loads

$$q_i = \sum_{l=1}^{n} c_{il} Q_l \qquad i = 1, 2, \cdots, n$$ [6.8.1]

$$c_{ii} > 0 \quad i = 1, \cdots, n$$ [6.8.2]

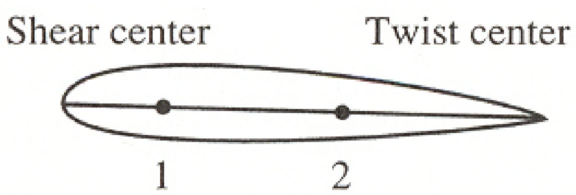

Figure 6.8.1 Shear center versus twist center.

$$u_1 = c_{11} P_1 + c_{12} M_2$$

$$\theta_2 = c_{21} P_1 + c_{22} M_2$$

6.8 Flexibility Matrix

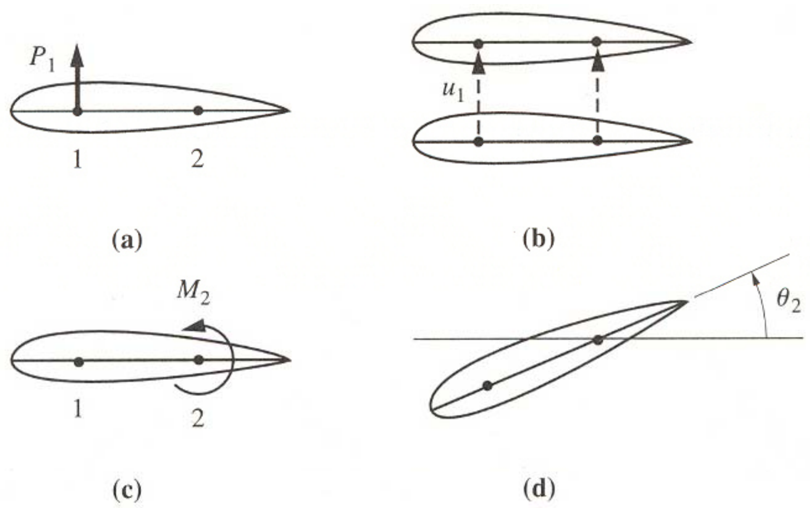

Figure 6.8.2: (a) and (b) A load through the shear center produces no twist. (c) and (d) A pure torque produces no displacement at the center of twist.

If we apply a load just to point 1, as in figure 6.8.2a, the displacements at point 1 and 2 are

$$u_1 = c_{11} P_1$$

$$\theta_2 = c_{21} P_1$$

Since point 1 is the shear center, $c_{21}=0$

If we apply just a point couple M_2 to point 2, as in figure 6.8.2c, then

$$u_1 = c_{12} M_2$$

$$\theta_2 = c_{22} M_2$$

However, since $c_{21}=c_{12}=0$, $u_1=0$

This means that point 1 is the center of twist

In an elastic structure, the shear center and the center of twist coincident.