458.308 Process Control & Design

Lecture 3: Dynamic Simulation and Analysis

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Standard Form of the Model

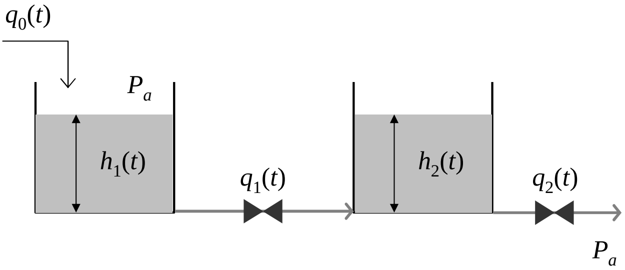
2 dependent variables and 2 independent variables

$$\dot{x}_1 = f_1(x_1,\ x_2,\ u_1,\ u_2)$$
$$\dot{x}_2 = f_2(x_1,\ x_2,\ u_1,\ u_2)$$

$f_1,\ f_2$: some (nonlinear) functions of $x_1,\ x_2,\ u_1,\ u_2$

- There can be many more dependent variables and many more independent variables. What would be the form with n dependent variables and m independent variables?

Example: Interacting Tanks



$$q_1 = C'_{v1}\sqrt{(P_a + \rho g h_1) - (P_a + \rho g h_2)} = C'_{v1}\sqrt{\rho g (h_1 - h_2)} = C_{v1}\sqrt{h_1 - h_2}$$

$$q_2 = C'_{v2}\sqrt{(P_a + \rho g h_2) - P_a} = C'_{v2}\sqrt{\rho g h_2} = C_{v2}\sqrt{h_2}$$

Standard Form

$$\frac{d(A_1 h_1 \rho)}{dt} = \rho q_0 - \rho \underbrace{C_{v1}\sqrt{h_1 - h_2}}_{q_1}$$

$$\frac{d(A_2 h_2 \rho)}{dt} = \rho \underbrace{C_{v1}\sqrt{h_1 - h_2}}_{q_1} - \rho \underbrace{C_{v2}\sqrt{h_2}}_{q_2}$$

$$x_1 \triangleq h_1,\ x_2 \triangleq h_2,\ u_1 \triangleq q_0$$

$$\underbrace{\frac{dh_1}{dt}}_{\dot{x}_1} = \underbrace{\frac{q_0 - C_{v1}\sqrt{h_1 - h_2)}}{A_1}}_{f_1(x_1,\, x_2,\, u_1)}$$

$$\underbrace{\frac{dh_2}{dt}}_{\dot{x}_2} = \underbrace{\frac{C_{v1}\sqrt{h_1 - h_2} - C_{v2}\sqrt{h_2}}{A_2}}_{f_2(x_1,\, x_2,\, u_1)}$$

What can you do with the model?

- Numerical integration (``simulation") to investigate the time behaviour of the dependent variables to a particular $x(0)$ (initial condition) and $u(t),\ t \geq 0$ (independent variable).
  - What does ``numerical solving" the ODEs mean?
    - Given: $x(0)$ and $u(t),\ t \geq 0$
    - Obtain: $x(t),\ t \geq 0$
- Analysis
  - Linearization
  - Analytical solution via Laplace Transform

Numerical Integration

1. Start with $x_1(0)$ and $x_2(0)$. Set $t = 0$.
2. Take an incremental step (of size Δt, which cannot be large, why?) forward in time by solving
   - Forward Euler: Explicit integration
   $$\begin{aligned} x_1(t+\Delta t) &= x_1(t) + \Delta t \cdot f_1(x_1(t), \ \cdots) \\ x_2(t+\Delta t) &= x_2(t) + \Delta t \cdot f_2(x_1(t), \ \cdots) \end{aligned}$$
   - Backward Euler: Implicit Integration
   $$\begin{aligned} x_1(t+\Delta t) &= x_1(t) + \Delta t \cdot f_1(x_1(t+\Delta t), \ \cdots) \\ x_2(t+\Delta t) &= x_2(t) + \Delta t \cdot f_2(x_1(t+\Delta t), \ \cdots) \end{aligned}$$
   - Trapezoidal (2nd order R-K): Implicit Integration
   $$\begin{aligned} x_1(t+\Delta t) &= x_1(t) + \Delta t \cdot \frac{f_1(x_1(t), \ \cdots) + f_1(x_1(t+\Delta t), \ \cdots)}{2} \\ x_2(t+\Delta t) &= x_2(t) + \Delta t \cdot \frac{f_2(x_1(t), \ \cdots) + f_2(x_1(t+\Delta t), \ \cdots)}{2} \end{aligned}$$
3. Repeat this until you reach the desired end time.

Equilibrium Calculation

- At steady state, $d/dt = 0$

$$\begin{aligned} 0 &= f_1(\bar{x}_1,\ \bar{x}_2,\ \bar{u}_1,\ \bar{u}_2) \\ 0 &= f_2(\bar{x}_1,\ \bar{x}_2,\ \bar{u}_1,\ \bar{u}_2) \end{aligned}$$

- Given the steady state values of the independent variables, one can calculate the corresponding steady-state values of the dependent values by solving the above equation.

Solving Algebraic Equations Numerically

Newton Iteration

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(\bar{x}_1,\ \bar{x}_2) \\ f_2(\bar{x}_1,\ \bar{x}_2) \end{bmatrix}$$
$$\approx \begin{bmatrix} f_1(\bar{x}_1^i,\ \bar{x}_2^i) \\ f_2(\bar{x}_1^i,\ \bar{x}_2^i) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1^i,\ \bar{x}_2^i)} \begin{bmatrix} \bar{x}_1 - \bar{x}_1^i \\ \bar{x}_2 - \bar{x}_2^i \end{bmatrix}$$

$\bar{x}^i$: current guess of the solutions

By solving the above approximate equation, one gets iterative formula:

$$\begin{bmatrix} \bar{x}_1^{i+1} \\ \bar{x}_2^{i+1} \end{bmatrix} = \begin{bmatrix} \bar{x}_1^i \\ \bar{x}_2^i \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}^{-1}_{(\bar{x}_1^i,\bar{x}_2^i)} \begin{bmatrix} f_1(\bar{x}_1^i, \bar{x}_2^i) \\ f_2(\bar{x}_1^i, \bar{x}_2^i) \end{bmatrix}$$

Example

$$\begin{aligned} 0 &= \bar{q}_0 - C_{v1}\sqrt{\rho g(\bar{h}_1 - \bar{h}_2)} \\ 0 &= C_{v1}\sqrt{\rho g(\bar{h}_1 - \bar{h}_2)} - C_{v2}\sqrt{\rho g \bar{h}_2} \end{aligned}$$

$$\begin{bmatrix} \bar{h}_1^{i+1} \\ \bar{h}_2^{i+1} \end{bmatrix} = \begin{bmatrix} \bar{h}_1^{i} \\ \bar{h}_2^{i} \end{bmatrix} - M_i^{-1} \begin{bmatrix} \bar{q}_0 - C_{v1}\sqrt{\rho g(\bar{h}_1^i - \bar{h}_2^i)} \\ C_{v1}(\bar{h}_1^i - \bar{h}_2^i) - C_{v2}\sqrt{\rho g \bar{h}_2^i} \end{bmatrix}$$

$$M_i = \begin{bmatrix} -\frac{C_{v1}\sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} & \frac{C_{v1}\sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} \\ \frac{C_{v1}\sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} & -\frac{C_{v1}\sqrt{\rho g}}{2\sqrt{(\bar{h}_1^i - \bar{h}_2^i)}} - \frac{C_{v2}\sqrt{\rho g}}{2\sqrt{\bar{h}_2^i}} \end{bmatrix}$$

Linearization: 1^{st} Order Approximation of ODEs Around an Equilibrium

$$\dot{x}_1 = f_1(x_1,\ x_2,\ u_1,\ u_2)$$
$$\dot{x}_2 = f_2(x_1,\ x_2,\ u_1,\ u_2)$$

1st-order Taylor series expansion at the equilibrium $(\bar{x}_1,\ \bar{x}_2,\ \bar{u}_1,\ \bar{u}_2)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} f_1(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2) \\ f_2(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)} \begin{bmatrix} u_1 - \bar{u}_1 \\ u_2 - \bar{u}_2 \end{bmatrix}$$

Linearized Model

Standard Form: 2 × 2 System

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1,\bar{x}_2,\bar{u}_1,\bar{u}_2)} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}_{(\bar{x}_1,\bar{x}_2,\bar{u}_1,\bar{u}_2)} \begin{bmatrix} u_1 - \bar{u}_1 \\ u_2 - \bar{u}_2 \end{bmatrix}
$$

Deviation variables: $x_1' \equiv x_1 - \bar{x}_1$, etc.

$$
\begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = A \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + B \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}
$$

Example

$$\underbrace{\frac{dh_1}{dt}}_{\dot{x}_1} = \underbrace{\frac{q_0 - C_{v1}\sqrt{h_1 - h_2}}{A_1}}_{f_1(x_1,\, x_2,\, u_1)}$$

$$\underbrace{\frac{dh_2}{dt}}_{\dot{x}_2} = \underbrace{\frac{C_{v1}\sqrt{h_1 - h_2} - C_{v2}\sqrt{h_2}}{A_2}}_{f_2(x_1,\, x_2,\, u_1)}$$

Linearization at $\bar{h}_1, \bar{h}_2, \bar{q}_0$

$$\begin{bmatrix} \frac{dh_1'}{dt} \\ \frac{dh_2'}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{C_{v1}}{2A_1\sqrt{\bar{h}_1 - \bar{h}_2}} & \frac{C_{v1}}{2A_1\sqrt{\bar{h}_1 - \bar{h}_2}} \\ \frac{C_{v1}}{2A_2\sqrt{\bar{h}_1 - \bar{h}_2}} & -\frac{C_{v1}}{2A_2\sqrt{\bar{h}_1 - \bar{h}_2}} - \frac{C_{v2}}{2A_2\sqrt{\bar{h}_2}} \end{bmatrix} \begin{bmatrix} h_1' \\ h_2' \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} q_0'$$

Laplace Transform -- Main Idea

Linear ODEs

$$\tau \frac{dy}{dt} + y(t) = u(t)$$

Integration →

Solution

$$y(t) = \exp\left(-\tfrac{t}{\tau}\right) y(0) + \int_0^t \exp\left(-\tfrac{t-\zeta}{\tau}\right) u(\zeta)\, d\zeta$$

Laplace Transform ↓

$$(\tau s + 1) Y(s) = U(s)$$

Algebraic Manipulation →

$$Y(s) = \frac{U(s)}{\tau s + 1}$$

Inverse Laplace Transform ↑

Laplace Transform -- Key Points

Definition

$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$

- Lapalace Transform for simple signals
  - Steps, ramps, exponential decay or rise, pulse, impulse, etc.
  - Can be found by evaluating the integral
  - See Table 3.1
  - Must memorize the simple ones

Important Properties

$$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0)$$

$$\mathcal{L}\left(\frac{d^2 f}{dt^2}\right) = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\left(\int_0^t f(\zeta) d\zeta\right) = \frac{1}{s} F(s)$$

$$\mathcal{L}\left(f(t-\delta)\right) = F(s) e^{-\delta s}$$

Key Points

Warning: LT is a linear operation!

$\mathcal{L}\left(af_1(t) + bf_2(t)\right) = aF_1(s) + bF_2(s)$
$\mathcal{L}\left(y^2(t)\right) \neq Y^2(s)$
$\mathcal{L}\left(y(t)u(t)\right) \neq Y(s)\,U(s)$

Final and Initial Value Theorem

$$\begin{aligned} \lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s) \\ f(0) &= \lim_{s\to\infty} sF(s) \end{aligned}$$

Inverse Laplace Transform

- Needed to take the solution obtained through Laplace transform back to the time domain.
- The formula involves complex contour integral.
- Use partial fraction expansion to break the solution down to small pieces and use the table (or your memory) to invert.
  1. Break down $X(s)$ into sum of fractions

$$X(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \cdots + \frac{A_m}{s - p_m}$$

  2. Using linearity property, write

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{A_1}{s-p_1}\right\} + \mathcal{L}^{-1}\left\{\frac{A_2}{s-p_2}\right\} + \cdots + \mathcal{L}^{-1}\left\{\frac{A_m}{s-p_m}\right\} \\ &= A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_m e^{p_m t} \end{aligned}$$

Transfer Function

- Linear differential equation with a general forcing function (input)

  1^{st} order $\quad a\frac{dy}{dt} + y(t) = Ku(t)$

  2^{nd} order $\quad b\frac{d^2y}{dt^2} + a\frac{dy}{dt} + y(t) = Ku(t)$

  $\dots$

- We can solve the eqn. for a specific forcing function, but we can also leave it general and take Laplace transform (with the initial condition zero!) to arrive at a general relationship between output and input.

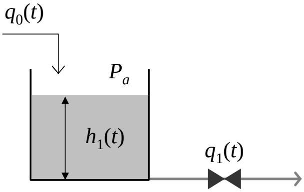
$$Y(s) = \frac{K}{as+1}U(s); \qquad Y(s) = \frac{K}{bs^2+as+1}U(s)$$

- With the transfer function, one can conveniently calculate the response of the output to any input by multiplication.

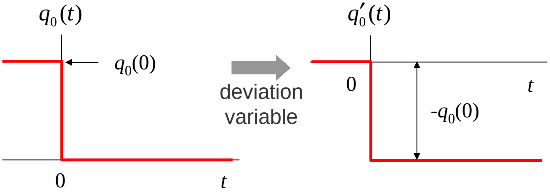
$$Y(s) = G(s)\,U(s)$$

Single-Tank Draining

Assume that you start at the steady state: $q_0(0) = \bar{q}_0, \; h_1(0) = \bar{h}_1$



$$q_1 = C_{v1}\sqrt{P_a + \rho g h_1 - P_a} = C'_{v1}\sqrt{\rho g h_1} = C_{v1}\sqrt{h_1}$$



Mass Balances

$$\frac{d(A_1 h_1 \rho)}{dt} = \rho q_0 - \rho \underbrace{C_{v1}\sqrt{h_1}}_{q_1}$$

Q. Do you think you can solve the above using L.T.?
A. No.
Linearize!

$$A_1 \frac{dh_1'}{dt} = q_0' - \underbrace{\left(\frac{C_{v1}}{2\sqrt{\bar{h}_1}}\right)}_{1/R_1} h_1'$$

Note: The above can be solved (using L.T.) but will the linear model be valid throughout the entire draining experiment?

Solution Based on the Linearized Model

$$A_1 \frac{dh_1'}{dt} = q_0' - \frac{1}{R_1} h_1' \text{ with } h_1'(0) = 0,\ q_0'(s) = -\frac{\bar{q}_0}{s}$$

$$\Downarrow\ \mathcal{L}$$

$$\left(A_1 s + \frac{1}{R_1}\right) H_1(s) = -\frac{\bar{q}_0}{s},\ H_1(s) = \frac{-\bar{q}_0 R_1}{s} + \frac{\bar{q}_0 R_1}{s + \frac{1}{A_1 R_1}}$$

$$\Downarrow\ \mathcal{L}^{-\infty}$$

$$h_1'(t) = -\bar{q}_0 R_1 \left(1 - \exp\left(-\frac{t}{A_1 R_1}\right)\right)$$

†Note: We can see that $h_1(t)$ is an exponentially decaying function of t.

Real Solution

Integrate the differential equation from time 0 to t to obtain

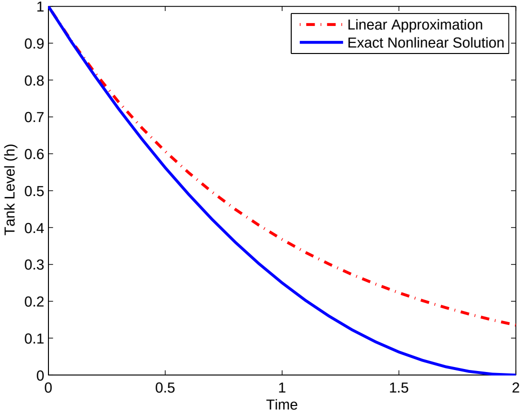
$$\begin{aligned} A_1 \frac{dh_1'}{dt} &= -C_{v1}\sqrt{h_1},\ h(0) = \bar{h} \\ \frac{1}{\sqrt{h_1}}\frac{dh_1}{dt} &= -\frac{C_{v1}}{A_1} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 2\left(\sqrt{h_1(t)} - \sqrt{\bar{h}_1}\right) &= \frac{-C_{v1}}{A_1}(t-0) \\ h_1(t) &= \left(\frac{-C_{v1}}{2A_1}t + \sqrt{\bar{h}_1}\right)^2 \end{aligned}$$

†Note: We can see that $h_1(t)$ is a quadratic function of t.

Comparison


Linear Approximation
Exact Nonlinear Solution
Tank Level (h)
Time
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2

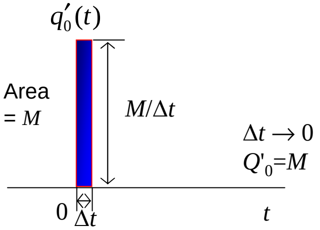
Interacting Tanks -- Revisited

$$\begin{bmatrix} \frac{dh_1'}{dt} \\ \frac{dh_2'}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{A_1 R_1} & \frac{1}{A_1 R_1} \\ \frac{1}{A_2 R_1} & -\frac{1}{A_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \end{bmatrix} \begin{bmatrix} h_1' \\ h_2' \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} q_0'$$

$$\Downarrow \quad \mathcal{L}$$

$$\begin{bmatrix} H_1'(s) \\ H_2'(s) \end{bmatrix} = \begin{bmatrix} \dfrac{\left(s+\frac{1}{A_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right)\frac{1}{A_1}}{\left(s+\frac{1}{A_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right)\left(s+\frac{1}{A_1 R_1}\right)-\frac{1}{A_1 A_2 R_1^2}} Q_0'(s) \\ \dfrac{\frac{1}{A_1 A_2 R_1}}{\left(s+\frac{1}{A_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right)\left(s+\frac{1}{A_1 R_1}\right)-\frac{1}{A_1 A_2 R_1^2}} Q_0'(s) \end{bmatrix}$$

Impulse Response



$$\begin{bmatrix} H_1'(s) \\ H_2'(s) \end{bmatrix} = \begin{bmatrix} G_1(s) \cdot M \\ G_2(s) \cdot M \end{bmatrix} \quad \overset{\text{Inverse L.T.}}{\Rightarrow} \quad \begin{bmatrix} h_1'(t) \\ h_2'(t) \end{bmatrix}$$

Properties of Transfer Functions

- Steady-state gain: $G(0)$ -- Output change under a unit step change in input as $t \to \infty$
- The order of the denominator polynomial = the order of the equivalent differential equation
- Physical realizability: $n \geq m$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

- Additive and multiplicative property