458.308 Process Control & Design

Lecture 7: Dynamic Behavior and Stability of Closed-Loop Control Systems

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Overview: Closed-loop

To simplify the notation, the primes and s dependence have been omitted; thus, Y is used rather than $Y'(s)$

$$
\begin{aligned}
\frac{Y(s)}{Y_{sp}(s)} &= \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} = \frac{K_m G_c G_v G_p}{1 + G_{OL}} \\
\frac{Y(s)}{D(s)} &= \frac{G_d}{1 + G_c G_v G_p G_m} = \frac{G_d}{1 + G_{OL}}
\end{aligned}
$$

- $G_{OL} \triangleq G_c G_v G_p G_m$
- Different from open-loop!
- Depends on G_c

Analysis and Design Problems

- Analysis: Given particular G's and G_c
  - Are the closed-loop dynamics stable?
  - Speed of response? Damping?
  - Gains for Y/Y_{sp} and Y/D

- Design: Given particular G's, choose (``design") G_c so that
  - the closed-loop dynamics are stable
  - $\frac{Y}{Y_{sp}}$ has a gain of ~ 1 and $\frac{Y}{D}$ has a gain of ~ 0
  - the dynamics are sufficiently fast (but not too fast) and smooth (without excessive oscillation).

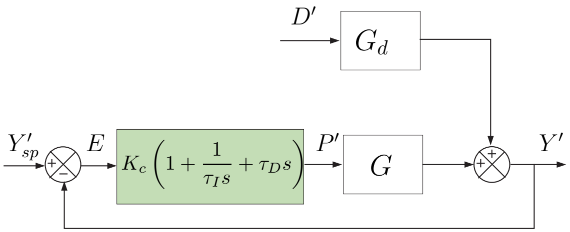
Model Used for Analysis and Design

- Case I (Less Frequent)
  - From a fundamental model, perform linearization and Laplace transform of the linearized ODEs to find $G_p(s)$ and $G_d(s)$
  - Find actuator and measurement dynamics G_v and G_m

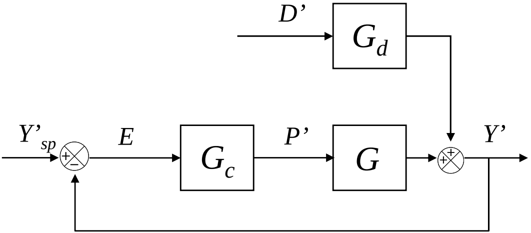
- Case II (More Frequent)
  - The composite model $G(= G_m G_p G_v)$ is fitted to data of y_m obtained by perturbing p (e.g., by making a step change).

PID Controller

$$
\begin{aligned}
p(t) &= \bar{p} + K_c\left(e(t) + \frac{1}{\tau_I}\int_0^t e(t^*)\,dt^* + \tau_D\frac{de}{dt}\right) \Rightarrow \\
P'(s) &= K_c\left(E(s) + \frac{1}{\tau_I s}E(s) + \tau_D s E(s)\right) \Rightarrow \\
\frac{P'(s)}{E(s)} &= K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right)
\end{aligned}
$$



Calculation of Closed-Loop Functions

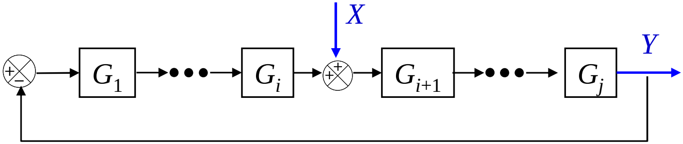


$$
\begin{aligned}
Y' &= G_d D' + G P' \qquad (P' = G_c(Y'_{sp} - Y')) \\
Y' &= G_d D' + G G_c(Y'_{sp} - Y')
\end{aligned}
$$

Convenience of L.T.

$$\frac{Y'(s)}{D'(s)} = \frac{G_d}{1 + GG_c} \text{ and } \frac{Y'(s)}{Y'_{sp}(s)} = \frac{GG_c}{1 + GG_c}$$

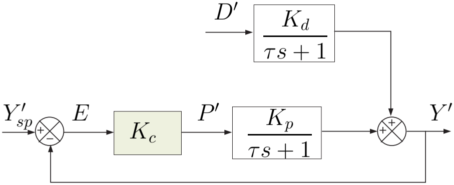
Calculation of Closed-Loop Functions: Generalization



$$\frac{Y(s)}{X(s)} = \frac{G_{i+1}G_{i+2}\cdots G_j}{1 + G_1 G_2 \cdots G_j} = \frac{\prod_f}{1+\prod_e}$$

- Assume negative feedback
- $\prod_f$: Product of the transfer functions in the forward path from X to Y.
- $\prod_e$: Product of every transfer function in the feedback loop.

Analysis of P-only Control



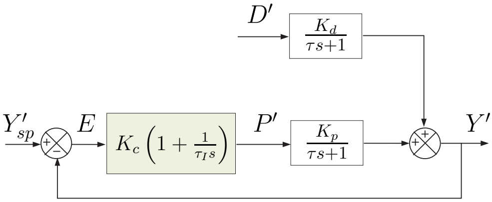
$$\frac{Y'(s)}{Y'_{sp}(s)} = \frac{\frac{K_cK_p}{\tau s+1}}{1+\frac{K_cK_p}{\tau s+1}} = \frac{K_cK_p}{\tau s+1+K_cK_p} = \frac{\frac{K_cK_p}{1+K_cK_p}}{\frac{\tau}{1+K_cK_p}s+1}$$

- Gain is not 1 unless $K_c = \infty$, Time constant decreases with increasing K_c.

$$\frac{Y'(s)}{D'(s)} = \frac{\frac{K_d}{\tau s+1}}{1+\frac{K_cK_p}{\tau s+1}} = \frac{K_d}{\tau s+1+K_cK_p} = \frac{\frac{K_d}{1+K_cK_p}}{\frac{\tau}{1+K_cK_p}s+1}$$

- Gain is not 0 unless $K_c = \infty$, Time constant decreases with increasing K_c.

Analysis of PI Control



$$\frac{Y'(s)}{Y'_{sp}(s)} = \frac{\frac{K_cK_p(\tau_I s+1)}{\tau_I s(\tau s+1)}}{1+\frac{K_cK_p(\tau_I s+1)}{\tau_I s(\tau s+1)}} = \frac{K_cK_p(\tau_I s+1)}{\tau_I s(\tau s+1)+K_cK_p(\tau_I s+1)}$$

$$= \frac{\tau_I s+1}{\frac{\tau_I \tau}{K_cK_p}s^2+\frac{1+K_cK_p}{K_cK_p}\tau_I s+1}$$

- Gain = 1 always! No offset
- 2nd order dynamics
- Underdamped dynamics for very small τ_I

Closed-Loop Stability

Characteristic Equation

$$1 + G_{OL} = 0$$

- Roots of the above equation are the poles of the closed-loop functions (important information for analyzing closed-loop dynamics)
- For stability, make sure all the roots are in the Left-Half-Plane (negative real parts)
  - Can be checked by Routh's test
  - Or by direct substitution

Example: Routh's Test

Main Idea: Form a Routh array to see if any roots are in the RHP

$$1 + \frac{6K_c}{(2s+1)(4s+1)(6s+1)} = 0$$

$$48s^3 + 44s^2 + 12s + (1 + 6K_c) = 0$$

$$\begin{array}{cc}
48 & 12 \\
44 & 1 + 6K_c \\
\hline
\frac{44 \times 12 - 48(1+6K_c)}{44} & 0 \\
\frac{\left(\frac{120}{11} - \frac{72}{11}K_c\right) \times (1+6K_c) - 44 \times 0}{\frac{120}{11} - \frac{72}{11}K_c} &
\end{array}$$

Must be all positive for closed-loop stability!

$\frac{120}{11} - \frac{72}{11}K_c > 0, \quad 1 + 6K_c > 0$

Example: Direct Substitution

Main Idea: At the limits of instability, the closed-loop poles will be on the imaginary axis (between LHP and RHP)

$$
\begin{aligned}
&48s^3 + 44s^2 + 12s + (1 + 6K_c) = 0 \xrightarrow{s=j\omega} \\
&-48j\omega^3 - 44\omega^2 + 12j\omega + (1 + 6K_c) = 0 \\
&(-48\omega^3 + 12\omega)j + \left(-44\omega^2 + (1 + 6K_c)\right) = 0 \\
&-48\omega^3 + 12\omega = 0 \qquad -44\omega^2 + (1 + 6K_c) = 0 \\
&\omega = 0,\ K_c = -1/6 \qquad \omega = \pm 1/2,\ K_c = 5/3
\end{aligned}
$$

Note

This method works with a system with time delay. Routh's method does not.