458.308 Process Control & Design

Lecture 8: Frequency Response

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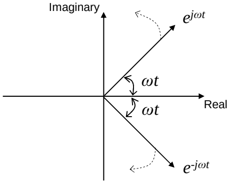
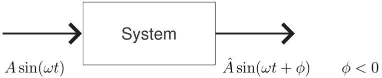
$e^{j\omega t}$: Review

- a vector in the complex plane with constant length one, making angle ωt with the positive real axis
- As time t increases the angle ωt increases, and so the vector rotates around the origin in the complex plane, in the counter-clockwise sense. It makes one complete rotation when $\omega t = 2\pi$ radians. If that occurs when $t = T$, then $\omega = 2\pi/T$, which has the units of radians per time.
- ω is called the angular frequency of the vector's rotation.
- If we measure frequency in units of rotations (or cycles) per second, and if we denote that number by ν, then $\nu = 1/T$ and $\omega = 2\pi\nu$.
- The only difference between $e^{j\omega t}$ and $e^{-j\omega t}$ is ωt, that is, e^{-jwt} rotates in the opposite direction.



Definition of Frequency Response

- Linear System:
  Sinusoidal forcing $\rightarrow$ Sinusoidal Response



Defined by two quantities that depend on the frequency ω

$\frac{\hat{A}}{A}$: Amplitude Ratio (``gain" for sinusoidal change)

ϕ: Phase Angle (``delay" in terms of radians)

Bode Plots

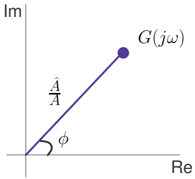
- Since Amplitude Ratio(A.R.) and Phase Angle (P.A.) depend on the frequency of ω, a continuous parameter, it is convenient to display as plots
  - ω vs. A.R. on a log-log scale
  - ω vs. P.A. on a log-linear scale

How to Compute Frequency Response?

Quickly from transfer function

$$\frac{Y'(s)}{U'(s)} = G(s) \quad \xrightarrow{s=j\omega} \quad G(j\omega) = \frac{\hat{A}}{A} e^{j\phi}$$

$$\begin{aligned}
\frac{\hat{A}}{A}(\omega) &= \sqrt{\mathrm{Re}^2[G(j\omega)] + \mathrm{Im}^2[G(j\omega)]} = \mod [G(j\omega)] \\
\phi(\omega) &= \tan^{-1}\left(\frac{\mathrm{Im}[G(j\omega)]}{\mathrm{Re}[G(j\omega)]}\right) = \arg[G(j\omega)]
\end{aligned}$$



Example

$$\frac{Y'(s)}{U'(s)} = G(s) = \frac{K}{\tau s + 1}$$

$$G(j\omega) = \frac{K}{\tau(j\omega) + 1} = \frac{K}{1 + \omega^2\tau^2} - j\frac{K\omega\tau}{1 + \omega^2\tau^2}$$

$$\frac{\hat{A}}{A} = \frac{K}{\sqrt{1 + \omega^2\tau^2}}$$

$$\phi = \tan^{-1}(-\omega\tau)$$

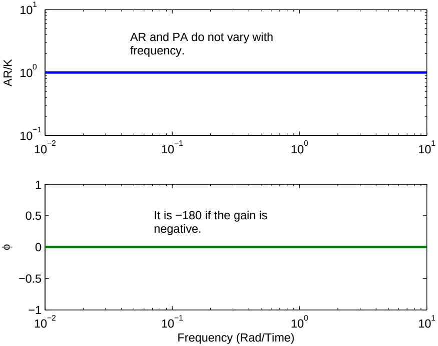
Notice AR and PA are functions of ω

Static (Pure Gain) System

$$\frac{Y'(s)}{U'(s)} = K$$

$$\downarrow$$

$$\frac{\hat{A}}{A} = K$$
$$\phi = 0$$



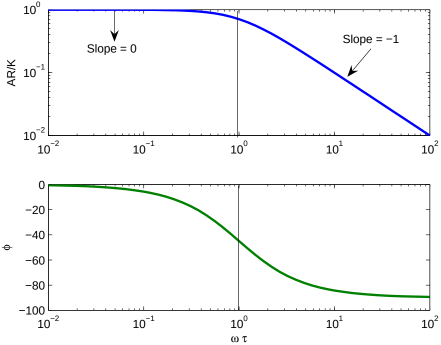
1^{st} Order System (Lag)

$$\frac{Y'(s)}{U'(s)} = \frac{K}{\tau s+1}$$

$$\downarrow \quad s = j\omega$$

$$\frac{\hat{A}}{A} = \frac{K}{\sqrt{1+\omega^2\tau^2}}$$
$$\phi = \tan^{-1}(-\omega\tau)$$

$\omega = 1/\tau$: corner frequency



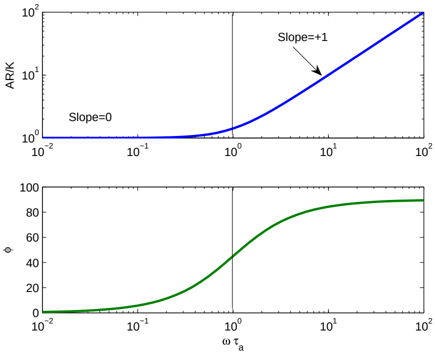
1^{st} Order Lead

$$\frac{Y'(s)}{U'(s)} = K(\tau_a s + 1) \;\; \tau_a > 0$$

$$\downarrow \quad s = j\omega$$

$$\frac{\hat{A}}{A} = K\sqrt{1+\omega^2\tau_a^2}$$
$$\phi = \tan^{-1}(\omega\tau_a)$$

$\omega = 1/\tau_a$: corner frequency



Pure Capacity (Integrating) System

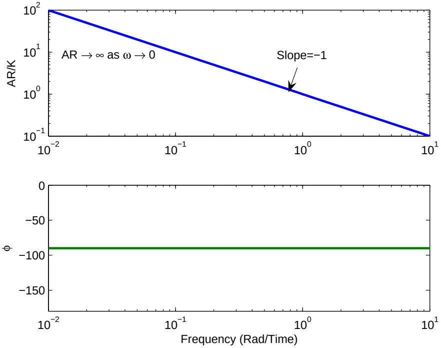
$$\frac{Y'(s)}{U'(s)} = \frac{K}{s}$$

$$\downarrow \quad s = j\omega$$

$$\frac{\hat{A}}{A} = \frac{K}{\omega}$$
$$\phi = -\frac{\pi}{2}$$

Pure differentiator?

$G(s) = K \cdot s$



2^{nd} Order System

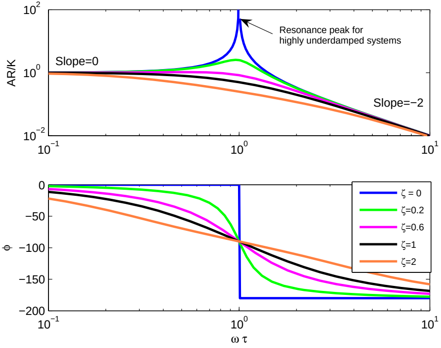
$$\frac{Y'(s)}{U'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\downarrow \qquad s = j\omega$$

$$\frac{\hat{A}}{A} = \frac{K}{\sqrt{(1-\omega^2\tau^2)^2 + (2\omega\tau\zeta)^2}}$$

$$\phi = \tan^{-1}\left(\frac{-\omega\tau\zeta}{1-\omega^2\tau^2}\right)$$

$\omega = 1/\tau$: corner frequency

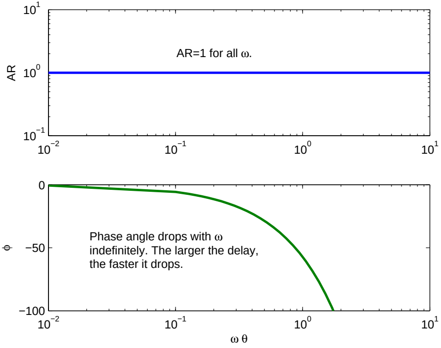


Pure Delay System

$$\frac{Y'(s)}{U'(s)} = e^{-\theta s}$$

$$\downarrow \quad s = j\omega$$

$$\frac{\hat{A}}{A} = 1$$
$$\phi = -\theta\omega$$

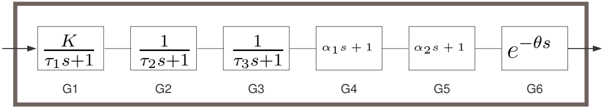


Sketching Bode Plots: Complex Systems

Main Idea

For systems connected in series, **the A.R.s multiply and the phase angle adds.**(Why? Euler's identity!) This rule can be used to obtain frequency response formula and sketch bode plots very quickly.

$$G(s) = \frac{K(\alpha_1 s + 1)(\alpha_2 s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)} e^{-\theta s}$$



$$\begin{aligned}
AR_G(\omega) &= AR_{G1}(\omega) \times AR_{G2}(\omega) \times \cdots \times AR_{G6}(\omega) \\
\phi_G(\omega) &= \phi_{G1}(\omega) + \phi_{G1}(\omega) + \cdots + \phi_{G6}(\omega)
\end{aligned}$$

Example

$$\begin{aligned} G(s) &= \tfrac{-5(3s+1)}{(5s+1)(10s+1)}e^{-2s} \\ &= (-5) \times \tfrac{1}{5s+1} \times \tfrac{1}{10s+1} \times (3s+1) \times e^{-2s} \end{aligned}$$

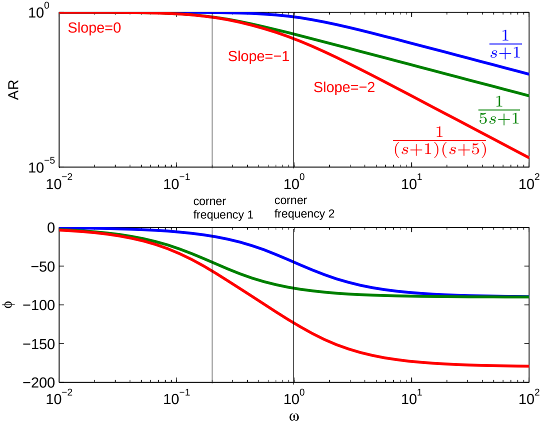
$$\begin{aligned} AR &= 5 \times \frac{1}{\sqrt{25\omega^2+1}} \times \frac{1}{\sqrt{100\omega^2+1}} \times \sqrt{9\omega^2+1} \times 1 \\ \phi &= -\pi + \tan^{-1}(-5\omega) + \tan^{-1}(-10\omega) + \tan^{-1}(3\omega) + (-2\omega) \end{aligned}$$

What if

$$G(s) = \frac{5(3s+1)}{(5s+1)(10s+1)}e^{-2s} \ \ ?$$

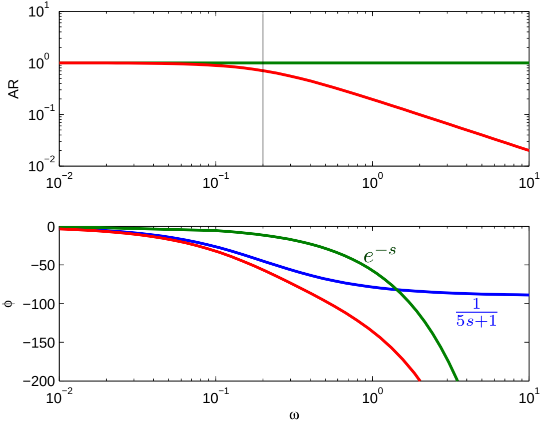
Example

$$G(s) = \frac{1}{(s+1)(5s+1)}$$



Example

$$G(s) = \frac{K}{\tau s + 1} e^{-\theta s}$$



PID Controller

$$G(s) = K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right)$$

- $\omega \to 0$: $G(s) \approx \frac{1}{\tau_I s}$.
- $\omega \to \infty$: $G(s) \approx \tau_D$.
- Corner frequency: $\frac{1}{\tau_I \omega} = \tau_D \omega \quad \longrightarrow \quad \omega = \frac{1}{\sqrt{\tau_I \tau_D}}$

Bode Plots using MATLAB

```
% 1st Order System (Lag)
gs = tf([1], [1 1]);
[mag, phase, w] = bode(gs); % magnitudes and phase
subplot(2, 1, 1), loglog(w, squeeze(mag))
ylabel('AR/K')
subplot(2, 1, 2), semilogx(w, squeeze(phase))
ylabel('\phi'), xlabel('\omega \tau')
```