Theory of Beam, Plate and Shell:

Physical point of view of a structure:

• **Geometrical point**	• **Material point**
~ Flat or Curved shape	**~ Elasticity : Isotropic or Composite**
~ Dimension ?	**~ Functionally Graded Material**
~ Thickness ?	**~ Plasticity, Viscoelasticity, Hygrothermal…**

What is Modelling ?

~ Closely related in geometric shape !

v What is mechanics ? Concern on the causes and effects

Newton's law : $F \propto a => ma$

Cause ~ Force : Effect ~ Deformation(Solid mechanics)

or

Motion(Dynamics)

v Special simplification from 3-D body :

How can we logically follow 1D or 2D approximation ?

i) Stress field ii) Strain field iii) Deformation field

Signification and limitation of their use can best be understood in terms of the general theory !

->

Final goal:

Systematic and concise derivation of engineering theory of plate and shell.

: Linear, Nonlinear theory…

1. Preliminary mathematics

1-1 Indicial Notation

Compact notation --- Meaning ?

: Generalization of formulation

Ex : Circular, rectangular, elliptic plate models

Geometric or boundary shapes are different,

but the same assumptions are used.

: Mathematical expressions are different,

but the meanings are same !

~ Invariant !

Why we choose a special coordinate system?

Just for convenience !

~ FEM? Analytical approach?

Right-hand coordinate system

Vector :

$$x_i\,(i = 1, 2, 3)$$
or
$$x_\alpha\,(\alpha = 1, 2), x_3$$

Summation Convention (Dummy or repeated index)

: $\vec{V} = V_x \vec{i} + V_y \vec{j} + V_z \vec{k} = V_\ell \vec{i}_\ell$

$$\int_a^b f(x)dx \equiv \int_a^b f(y)dy$$

Orthogonality of unit vector : $(\vec{i}, \vec{j}, \vec{k})$

$$\vec{i} \bullet \vec{j} = 0, \vec{i} \bullet \vec{i} = 1 \ldots$$

Kronecka Delta : δ_{ij} -> $\begin{cases} i \neq j : 0 \\ i = j : 1 \end{cases}$

$$: \begin{bmatrix} 1,0,0 \\ 0,1,0 \\ 0,0,1 \end{bmatrix}$$

Ex: $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3!$

*** Einstein : <u>Summation Convention</u>**

$$a_i + b_{ij} c_j + d_{kl} q_{kli} = 0..(i, j, k, l = 1, ...N)$$

Stands for the N separate equations :

~ free index $\boldsymbol{i}$ **.**

o <u>Live script</u> must occur only once in each term of eqn.

o Dummy script occurs twice in a term and is to be summed <u>over the range</u>

If N=3 : $b_i => b_1, b_2, b_3$

~ 3 Components of <u>a vector</u>

$$A_{ij} =\rangle \begin{bmatrix} A_{11}, A_{12}, A_{13} \\ A_{21}, A_{22}, A_{23} \\ A_{31}, A_{32}, A_{33} \end{bmatrix}$$

~ 3×3=9 Components of <u>2nd order Tensor</u>

*** Contracted product :**

$b_i A_{ij}$: **Summations on index** i **: Free index** j

\- ***Simply contracted product with 2 live script***

$$C_{ik} = A_{ij} B_{jk}$$

- ***Doubly contracted product without live script : 1 component : <u>Scalar</u> !***

$$S = A_{ij} B_{ji}$$

Ex : b= (b_1, b_2, b_3)

3 components of a vector : b_i

Scalar : $b_i b_i$

Tensor product or open product of 2 vectors : $b_i b_j$

<u>***Kronecker delta and alternating symbol*** ε</u>

$$\delta_{ij} = \begin{cases} 1 : i = j \\ 0 : i \neq j \end{cases} \Rightarrow \begin{bmatrix} 1,0,0 \\ 0,1,0 \\ 0,0,0 \end{bmatrix}$$

$$\delta_{ii} = ?$$

$$A_i \delta_{ij} = A_j$$

$$\varepsilon_{ijk} = \begin{cases} 1 : Even.transposition.of.1,2,3 \\ -1 : Odd.transposition.of.1,2,3 \\ 0 : At.least.2.scripts.are.the.same! \\ \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{321} = 1$$

$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = 1$, ***All other terms = 0***

<u>Symmetry tensor and Anti-symmtry tensor</u>

$\alpha_{ij} = \alpha_{ji}, A_{lmn} = A_{lnm}$ ***or*** $\alpha_{ij} = -\alpha_{ji}, A_{lmn} = -A_{lnm}$

Index notations

$$\frac{\partial \phi}{\partial x_i} = \phi_{,i}$$

$$\nabla \bullet \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \frac{\partial V_i}{\partial x_i} = V_{i,i}$$

$$\nabla^2 \phi = \phi_{,ii}$$

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}(\varepsilon_{\alpha\beta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\varepsilon_{\varsigma\varsigma}) : plane.stress.problem!$$

$$\vec{C} = \vec{A} \times \vec{B} -> C_i = \varepsilon_{ijk} A_j B_k$$

Practice !!

1-2 Calculus of Variation

$$U = \frac{1}{2}\int \sigma_{i}\,\varepsilon_{j} \quad d_{j}$$

There are at least 3 important reasons for taking up the 'Calculus of Variation' in the study of continuum mechanics

1. **Basic minimum principles exist**

(Minimum Total Potential Energy : M.T.P.E.)

2. **The field eqns & associate B.Cs of many problems can be derived from "Variational Principles".**

~ In formulating an approximate theory, the shortest and clearest derivation is usually obtained through Variational Calculus

3. Direct method of solution of variational problem is one of the most powerful tools for obtaining numerical results in practical problems of engineering importance.

In this section, we shall summarize the relation and properties derived from Variational Principles briefly, and then discuss their applications

[A] Euler’s Equations

To determine the function $y(x)$**, for** $x_0 \le x \le x_1$ **that minimize the definite integral**

$$V = \int_{x_1}^{x_2} F(x, y(x), y'(x), y''(x))dx$$

we call it the functional that depends on the unknown function $y(x)$

in which $y(x)$ **: continuous & differentiable with continuous** $y'(x), y''(x)$ **for** $x_0 \le x \le x_1$ **and satisfies forced function B.C (= admissible function)**

We assume that the function F **continuous with all its partial derivatives to order 3 for all real values of** $y(x), y'(x), y''(x)$ **and for all valves of** x **in** $x_0 \le x \le x_1$

Admissible variation ?

Euler- Lagrange's eqns

$$F_{,y} - (F_{,y'})_{,x} + (F_{,y''})_{,xx} = 0$$

with Boundary Conditions

$$\varepsilon\,\eta(x)\left[F_{,y'} - (F_{,y''})_{,x}\right]_{x1}^{x2} + \varepsilon\,\eta_{,x}(x)\left[F_{,y''}\right]_{x1}^{x2} = 0$$

=> Necessary condition for V to have a minimum value since $\delta V = 0$

~ Sufficient condition : $\delta^2 V = 0$: positive definite

[B] Variational notation

In the derivation of Euler-Lagrange eqn, $y(x)$ in argumented by an infinitesimal ftn $\varepsilon\eta(x)$ in which $\eta(x)$ is any admissible variation & ε is arbitrary constant

in practical application, $\varepsilon\eta(x) \equiv \delta y(x)$

=> $\delta y'(x) \equiv \varepsilon\eta'(x), \delta y''(x) \equiv \varepsilon\eta''(x).etc.$

=> $\delta y'(x) \equiv (\delta y(x))', \delta y''(x) \equiv (\delta y(x))''$

: $\delta\int_{x0}^{x1} y(x)dx = \int_{x0}^{x1} \delta y(x)dx$: <=> ?

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: $\delta\int_{x0}^{x1} y(x)dx = \int_{x0}^{x1} \delta y(x)dx$ **: <=> ?**

Energy($y(x)$**)** $\Leftarrow\!\!\Rightarrow$ **Equation of motion(**$\delta y(x)$**) : ?**

;Non-conservative forces:Damping , Follower force…

Elastic material:

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} : (i, j, k, l = 1..3)$$

$$U = \frac{1}{2}\int \sigma_{ij}\varepsilon_{ij} dV :$$

$$\delta U = \frac{1}{2}\int \sigma_{ij}\delta\varepsilon_{ij} dV + \frac{1}{2}\int \delta\sigma_{ij}\varepsilon_{ij} dV (?) :$$

: Displacement analysis ~ Stress analysis

$$\delta V = \int_{x0}^{x1} F(x, y(x)+\delta y(x), y'(x)+\delta y'(x), y''(x)+\delta y''(x))dx - \int_{x0}^{x1} F(x, y(x), y'(x), y''(x))dx$$

$$\equiv \int_{x0}^{x1} \delta F(...)dx$$

Expand around (x, y, y', y'') **!**

$$= \int_{x0}^{x1} (F(x, y(x), y'(x), y''(x)),_{y} \delta y + F(...),_{y'} \delta y' + F(...),_{y''} \delta y'')dx$$

Integration by part,

$$\int_{x0}^{x1} (...)\delta y dx + \left[(...)\delta y\right]_{x0}^{x1} + \left[(...)\delta y'\right]_{x0}^{x1}$$

Last two terms denote the Boundary Conditions for Forces and Displacements !

~ Dimensions ?