## Chapter 4. Statics of Plate

o Classical Linear Theory of Plate
o Plane stress and Plane strain
o Equations of Equilibrium and Boundary Conditions

- Equilibrium Approach
- Minimum Potential Energy Approach-Thin plate
o Solution methods
etc


## Basic Assumptions

~In the initial state, the surfaces of a flat plate are parallel planes, called the faces of the plate
~The distance between the faces is called the thickness of the plate.
~A plate is distinguished from a block by the fact that the thickness is small compared to the dimensions paralled to the faces.
~The plane midway between the faces is called the middlle plane of the plates
~The bending properties of plate depend greatly on its thickness as compared with its other dimensions.
:We will discuss the linear static theory of thin elastic plate in this chapter.
$\sim$ shell ? membrane ? plate?

### 4.1 Classical linear theory of plate

Definition of small deformation : $\left(\frac{w}{h}\right) \leq 1$ :
Decoupled set of Bending and stretching ~ Isotropic material
~ Homogeneous materials
~ Isotropic or Anisotropic(Composite)
$\sim$ Thick plate : $h / l>\frac{1}{20} \sim \frac{1}{10}$ Thin plate $h / l \ll \frac{1}{20}$
$\sim$ Loading :
o Bending action ~ deflection
o Membrane action~stretching
Configuration of curved surface ~

Displacements :

$$
u(x, y) y, x\left(y, w x, y \quad\left(\sim u_{\alpha \alpha}(\partial x=1, \lambda v)\right.\right.
$$

: In-plane displacements $u, v \sim$ deflection $w$

## 3-Dimensional linear elasticity

$$
\sigma_{i j} \bar{j} q
$$

For plate bending problem, let $q_{\alpha}(\alpha=1,2)=0$

$$
u_{\alpha}: \sigma_{\alpha \beta, \beta}+\sigma_{\alpha 3,3}=0
$$

$\sim w: \sigma_{3 \beta, \beta}+\sigma_{33,3}=F_{3}$
with boundary conditions !

## 4-1-1 Kinematics of Deformation

The fundamental assumptions of the small deflection theory of bending or so called classical theory of isotropic homogeneous elastic thin plate
; Kirchhoff - Love hypothesis material property change

Kinematic Assumptions

1. Deflection of mid-surface is small compared with the thickness of the plate.
: Slope of the deflected surface is very small and the square of the slope is a negligible quantity in comparison with unity
2.The mid-plane remains unstrained subsequent to bending
\# Large deflection case ?
2. Plane sections initially normal to surface remains plane and normal to that surface after the bending : $\gamma_{\alpha z}=0(\alpha=1,2) \sim G \sim \succ \infty$

Before bending ~> After bending?
? Higher-order theory
? Thick plate theory ~Mindlin plate theory
4. No stretching or contraction of normal : $\varepsilon_{z z}=0$
5.The stress normal to the mid-plane, $\sigma_{z z}$ :small compared with the other stress components and may be neglected
: This assumption becomes unreliable in the vicinity of highly concentracted transverse loads.

$$
\begin{aligned}
& u_{\alpha}\left(x, y z \Rightarrow \sum_{m=0} u_{\alpha}^{m} x\left(y z^{m} \neq u_{\alpha}^{m}(x y z)^{m}=u_{\alpha}^{0} x y+u_{\alpha}\right) x y z+u_{\alpha}\right\} y z+t^{2} \\
& w\left(x, y z \Rightarrow \sum_{m=0} u_{3}^{m} x\left(y z^{m} \text { 丰 } u_{3}^{m}(x y z)^{m}=u_{3}^{0} x y+u\right)_{x} y z x+u y_{3} y z+t^{2},\right.
\end{aligned}
$$

~ order of thickness effects !

$$
\varepsilon_{z z}=\frac{\partial w(x, y, z)}{\partial z}=0: u_{3}^{o}(x, y, z)=w(x, y)
$$

## Shear modulus :

Composite layered plate?
6. Developable surface

This type of surface can be bent back to a plane without any variation in the distance between any two points on the surface.

Under these assumptions for thin plate,

$$
\begin{aligned}
& \varepsilon_{\alpha \beta}=\frac{1}{2}\left(u_{\alpha, \beta}+u_{\beta, \alpha}\right) \\
& \varepsilon_{\alpha 3}=0=\frac{1}{2}\left(u_{\alpha, 3}+u_{3, \alpha}\right)=\frac{1}{2}\left(u_{\alpha, 3}+w_{, \alpha}\right)=\gamma_{\alpha 3} \\
& \varepsilon_{33}=0
\end{aligned}
$$

$$
\begin{aligned}
& \varepsilon_{\alpha 3}=\frac{1}{2}\left(u_{\alpha, 3}+w_{, \alpha}\right)=0, \\
& :->u_{\alpha, 3}=-w_{, \alpha} \\
& \sim \\
& u_{\alpha}(x, y, z)=u^{\alpha}(x, y)-z w_{, \alpha} \\
& : \\
& \varepsilon_{\alpha \beta}=?
\end{aligned}
$$

## 4-1.2. Plane stress and Plane strain

For 3 D elasticity theory:

$$
\begin{aligned}
& \sigma_{i j}=\lambda \delta_{l l}+2 G \varepsilon_{i j} \\
& \text { with }: G=\frac{E}{2(1+v)}, \lambda=\frac{E v}{(1+v)(1-2 v)}
\end{aligned}
$$

Plane stress field for kinematic assumptions of

$$
\mathcal{E}_{\alpha 3}=\varepsilon_{33}=0
$$

$$
\begin{aligned}
& \sigma_{\alpha \beta}=\frac{E}{1+v}\left(\varepsilon_{\alpha \beta}+\frac{v}{1-v} \delta_{\alpha \beta} \mathcal{E}_{\varsigma \varsigma}\right) \\
& \sim \quad \frac{\partial}{\partial z}(\text { Load }, \text { Geometry }) \approx 0
\end{aligned}
$$

(i) Thin flat plate
(ii) Applied load are acting on the $\pm x y$ surfaces and do not vary across the thickness.
(iii) Upper and lower surface are stress free

$$
\sigma_{z z}=\sigma_{\alpha z}=0
$$

$\sim \sigma_{i z}$ through thickness.
(iv) $\sigma_{\alpha \beta}$ do not vary across the thickness

## Plane strain

Dimension of $z \gg$ dimension of $x, y$

$$
\mathcal{E}_{i z} \approx 0
$$

4-1.3. Equation of Equilibrium \& Boundary Conditions

3D ~> plate theory?
$x_{3}:$ Elimination of explicit dependency !

Stress, Moment resultants

$$
\begin{gathered}
\left(N_{\alpha \beta}, M_{\alpha \beta}\right)=\int_{-\frac{h}{2}}^{\frac{1}{2}}(1, z) \sigma_{\alpha \beta} d x_{3} \\
M_{x x}, M_{y y}: ? \\
M_{x y}: ?
\end{gathered}
$$

Shear force intensity

$$
Q_{\alpha}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3} d x_{3}
$$

According to basic assumptions and plane stress assumption,

$$
\varepsilon_{i 3}=0
$$

$$
\begin{aligned}
& u_{\alpha}\left(x, y, z{ }_{\alpha}^{0} u(x-y), z_{\alpha}^{z} w( \right. \\
& u_{3}(x, y, z) w(x, y)
\end{aligned}
$$

Total number of independent displacement components?

$$
\begin{aligned}
\varepsilon_{\alpha \beta} & =\frac{1}{2}\left(u_{\alpha, \beta}^{0}+u_{\beta, \alpha}^{0}\right)-\frac{z}{2}\left(w_{, \alpha \beta}+w_{, \beta \alpha}\right) \\
& \sim(1, z) \sim \sigma_{\alpha \beta}:(1, z)
\end{aligned}
$$

A. Equilibrium Approach

3-D elasticity (without body force)

$$
\begin{aligned}
& \sigma_{\alpha \beta}+\sigma_{3 x}=0 \\
& \sigma_{3 \beta \beta}+\sigma_{33}=q
\end{aligned}
$$

## Model !

(a): $\left.\left.\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{\alpha \beta, \beta}+\sigma_{\alpha 3,3}\right) d x_{3}=N_{\alpha \beta, \beta}+\sigma_{\alpha 3}\right]_{-\frac{h}{2}}^{\frac{h}{2}}=N_{\alpha \beta, \beta}+\sigma_{\alpha 3}\right]_{-\frac{h}{2}}^{\frac{h}{2}}=0$
(b): $\left.\int_{-\frac{h}{2}}^{\frac{h}{2}} z\left(\sigma_{\alpha \beta, \beta}+\sigma_{\alpha 3,3}\right) d x_{3}=M_{\alpha \beta, \beta}+z \sigma_{\alpha 3}\right]_{-\frac{h}{2}}^{\frac{h}{2}}-\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3} d x_{3}=0$
$\sim M_{\alpha \beta, \beta}-Q_{\alpha}=0$
(c)

$$
\begin{aligned}
& \left.\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{3 \beta, \beta}+\sigma_{33,3}\right) d x_{3}=Q_{\beta, \beta}+\sigma_{33}\right]_{-\frac{h}{2}}^{\frac{h}{2}}=0 \\
& \sim Q_{\beta, \beta}+\sigma_{33}\left(\frac{h}{2}\right)-\sigma_{33}\left(-\frac{h}{2}\right)=0 \\
& : Q_{\beta, \beta}-q=0
\end{aligned}
$$

Substitute Eq.(b) into Eq.(c)

$$
M_{\beta \alpha, \beta \alpha}-q=0
$$

Finally,

$$
\begin{aligned}
& M_{x x, x x}+2 M_{x y, x y}+M_{y y, y y}= \pm q \\
& N_{x x, x}+N_{x y, y}=0 \\
& N_{y x, x}+N_{y y, y}=0
\end{aligned}
$$

Equations may be coupled set.

For isotropic plate,

## Membrane stretching effect and Bending effect: Decoupled

## Mid-plane?

For rectangular plate:
Obtain stress field in terms of strain components
Obtain $\mathbf{N}, \mathbf{M}$ in terms of $\mathbf{u}, \mathbf{v}, \mathbf{w}$
then, ;

Bending problem; $4^{\text {th }}$ order eqn.( Hermite approximation: $C^{1}$ continuity)

$$
D\left(w,_{x x x x}+2 w,_{x y x y}+w_{, y y y y}\right) \equiv D \nabla^{4} w= \pm q ;
$$

and
Stretching equations

$$
\begin{aligned}
& u_{, x x}+\frac{1-v}{2} u,_{y y}+\frac{1+v}{2} v,_{x y}=0 \\
& v_{, y y}+\frac{1-v}{2} v,_{y y}+\frac{1+v}{2} u,_{x y}=0
\end{aligned}
$$

$\% \quad M_{\alpha \beta, \beta}-Q_{\alpha}=0$

$$
\begin{aligned}
& Q_{x}=M_{x x, x}+M_{x y, y}=-D\left(w,_{x x x}+v w,_{y y x}+(1-v) w, x y y\right)= \\
& -D(w, x x x+w, x y y)=-D(w, x x+w, y y), x=-D\left(\nabla^{2} w\right)_{, x} \\
& \tilde{Q_{y}}=-D\left(\nabla^{2} w\right), y
\end{aligned}
$$

B. Minimum Potential Energy Approach
~ Strain energy of entire plate with linear elastic behavior

$$
U_{(i)}=\frac{1}{2} \int_{A} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{i j} \varepsilon_{i j} d z d A
$$

Replacing $\sigma_{i j}$ in terms of $\varepsilon_{i j}$ in conjunction with plane stress state

$$
\begin{aligned}
& U_{(i)}=\frac{1}{2\left(1-v^{2}\right)} \int_{A} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha \beta} \varepsilon_{\alpha \beta} d z d A= \\
& \frac{1}{2\left(1-v^{2}\right)} \int_{A} \int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\varepsilon_{x x}^{2}+2 v \varepsilon_{x x} \varepsilon_{y y}+\varepsilon_{y y}^{2}+2(1-v) \varepsilon_{x y}^{2}\right\} d z d A
\end{aligned}
$$

Potential energy for the external load

$$
U_{(e)}=-\int_{A} q(x, y) w(x, y) d A
$$

$\underline{\text { Remembering, }} u_{\alpha}=u_{\alpha}^{0}-z w,{ }_{\alpha}$ and $\varepsilon_{\alpha \beta}=\frac{1}{2}\left(u_{\alpha, \beta}+u_{\beta, \alpha}\right)$

$$
U_{(i)}=\frac{1}{2\left(1-v^{2}\right)} \int_{A} \int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\left(u_{, x}^{0}-z w_{, x x}\right)^{2}+2 v \varepsilon_{x x} \varepsilon_{y y}+\varepsilon_{y y}^{2}+2(1-v) \varepsilon_{x y}^{2}\right\} d z d A
$$

Total potential energy

$$
U=U_{(i)}+U_{(e)}=U_{(i)}-\int_{A} q(x, y) w(x, y) d A
$$

## Integrating by part with respect to z:

( $\sim$ Extensional energy + Bending Energy ! )

$$
U_{(i)}=U_{i( }^{m}+U_{i}^{B}
$$

where,

$$
U_{(i)}^{m}\left(u^{0}, v^{0}\right), U_{(i)}^{B}(w)
$$

: Decoupling ! ?
Obtain the expression $\&$ govering eqn for (Report !)

We shall be concerned with the stresses and moments in plates as a result of transverse loadings of bending part

$$
\begin{aligned}
& U=U_{(i)}^{m}+U_{(i)}^{B} \\
& \sim U_{(i)}^{B}=\frac{D}{2} \int_{A} \int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\left(\nabla^{2} w\right)^{2}+2(1-v)\left[w_{, x y}^{2}-w,{ }_{x x} w,_{y y}\right]\right\} d A-\int_{A} q w d A
\end{aligned}
$$

To extremize this functional, we get

$$
\delta U=0
$$

$$
\delta\left[\frac{D}{2} \int_{A} \int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\left(\nabla^{2} w\right)^{2}+2(1-v)\left[w_{, x y}^{2}-w,{ }_{x x} w,_{y y}\right]\right\} d A-\int_{A} q w d A\right]
$$

$$
\frac{D}{2} \int 2\left(\nabla^{2} w\right) \delta \nabla^{2} w+2(1-v)\left[2 w_{x y} \delta w_{x y}-\delta w,_{x x} w,_{y y}-w,_{x x} \delta w,_{y y}\right] d A
$$

$$
=\frac{D}{2} \int 2\left(\nabla^{2} w\right) \delta \nabla^{2} w+(1-v)\left[2\left(w_{, x y} \delta w_{, x y}+w_{, y x} \delta w,_{y x}\right)\right.
$$

$$
\left.-2 \delta w_{x x} w_{, y y}-2 w_{, x x} \delta w_{, y y}\right] d A
$$

$$
=D \int\left(\nabla^{2} w\right) \delta \nabla^{2} w+(1-v)\left[2\left(w_{x y} \delta w,_{x y}+w_{, y x} \delta w,_{, y x}\right)\right.
$$

$$
\left.-2 \delta w_{x x} w_{, y y}-2 w_{,_{x x}} \delta w_{, y y}\right] d A
$$

$$
=D \int\left(w_{, x x} \delta w,_{x x}+w_{, y y} \delta w,_{y y}\right)+v\left(w,_{y y} \delta w,_{x x}+w,_{x x} \delta w,_{y y}\right.
$$

$$
\left.+(1-v)\left(w_{, x y} \delta w_{, x y}+w_{, y x} \delta w_{, y x}\right)\right] d A
$$

=>

## Using Green's formula(Report!)

$$
\int\left(D \nabla^{4} w-q\right) \delta w d A+D \oint \ldots .=0
$$

Introducing the moment intensity, then the above equation becomes(Report)

$$
\int\left(D \nabla^{4} w-q\right) \delta w d A-\oint M_{x x} \delta w,_{x} d y \ldots \ldots \ldots+\oint\left(M_{x x, x}+M_{x y, y}\right) \delta w d y=0
$$

We may rewrite the last two integrals in terms of shear force intensities.

[^0]$$
\int\left(D \nabla^{4} w-q\right) \delta w d A-\oint M_{x x} \delta w,_{x} d y \ldots . . . . .+\oint Q_{x} \delta w d y=0
$$

To simplify this functional further, let introduce the coordinate $n, s$ which are the normal and tangential directions to contour, respectively.

Then ( Report )...

$$
\int\left(D \nabla^{4} w-q\right) \delta w d x d y-\oint M_{n n} \frac{\partial \delta w}{\partial n} d s-\oint M_{n s} \frac{\partial \delta w}{\partial s} d s .+\oint Q_{n} \delta w d s=0
$$

The $3^{\text {rd }}$ integral becomes, by integration by parts

$$
\oint M_{n s} \frac{\partial \delta w}{\partial s} d s=-\oint \frac{\partial M_{n s}}{\partial s} \delta w d s \ldots(?)
$$

We then get for the variation of the total potential

$$
\int\left(D \nabla^{4} w-q\right) \delta w d A-\oint M_{n n} \frac{\partial \delta w}{\partial n} d s+\oint\left(Q_{n}+\frac{\partial M_{n s}}{\partial s}\right) \delta w d s=0
$$

Euler-Lagrange equation for this problem is

$$
D \nabla^{4} w=
$$

Report :Derive the governing equation with with variable bending rigidity
which have been derived in previous section. Now we get as a result of the variational process two sets of boudary conditions, namely, the natural and the kinematic bound conditions.

Then on the boundary we require that
(a) $M_{n n}=0$..or.. $\frac{\partial \delta w}{\partial n}$..is..prescribed
(b) $Q_{n}+\frac{\partial M_{n s}}{\partial s}=0 . . o r .$. w..is..prescribed

There are only 2 conditions, in spite of the fact the there $\mathbf{3}$ variables
$M_{n n}, Q_{n}, M_{n s}$
The $1^{\text {st }}$ condition is acceptable by physical consideration and needs no further comment.

We shall now examine the $2^{\text {nd }}$ condition with the view reaching some physical explanation.:

We have shown part of the edge of the plate when two panels of length $\Delta s$ have been identified.
_ Figure_

The twisting moments have been expressed in the first and second panel. A third panel of length $\Delta S$ may be imagined the center of two panels, which is shown in Fig AB. The shear force for thin panel is shown as $Q_{n} \Delta S$.

Now we make use of the Saint-Venant principle by replacing the twisting moment distribution of the original two panels by two couples having forces value $M_{n s}$ and $M_{n s}+\frac{\partial M_{n s}}{\partial s}$ respectively with a distance of $\Delta S$ between the forces as shown in the figure.

Clearly, any conclusion arising from the new arrangement valid away from the edege. Now we focus on attention on the central panel AB. The effective shear force intensity, $Q_{\text {eff }}$ for this panel is then seen to be

$$
Q_{e f f}=\frac{1}{\Delta S}\left[Q_{n} \Delta S+\left(M_{n s}+\frac{\partial M_{n s}}{\partial s} \Delta S\right)-M_{n s}\right]=Q_{n}+\frac{\partial M_{n s}}{\partial s}
$$

We may now say that the $2^{\text {nd }}$ of our natural boundary conditions renders the effective shear force intensity equation to zero. From the above equations we can see that the difference $\frac{\partial M_{n s}}{\partial s}$ does work in the same manner as $Q_{n}$ and these two effect cannot distinguished.

The force per unit length, $\frac{\partial M_{n s}}{\partial s}$ is called Kirchhoff supplementary force.
This explanation is by Thomson and Tait.
Because we employed a constitutive equation, Hook law, in the formulation it might seem that the natural boundary conditions are restricted to Hookean material.

In the next section, no constitutive law is used and we will derive the general moment intensity and shear force intensity equations present in Sec.3.1.3.a

### 4.1.3.c.(Thin Plate) Virtual work Approach (Rectangular plate)

In using the principle of virtual work, we shall consider a rectangular plate.
In particular we shall examine closely the corner condition for such a problem. We now apply the principle of virtual work, under the assumption of plane stress, to a rectangular plane having dimensions $a \times b \times h$ and loaded normal to the middle-plane of the plate by a loading intensity $q(x, y)$.

If body forces are absent,

$$
\int_{0}^{a} \int_{0}^{b} q \delta w d x d y=\int_{0}^{a} \int_{0}^{b} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha \beta} \varepsilon_{\beta \alpha} d x d y
$$

Considering only bending contribution, then by $\varepsilon_{\alpha \beta}=u_{, \alpha}^{o}-z w_{, \alpha \beta}$ ( thin plate assumption)
we got

$$
\int_{0}^{a} \int_{0}^{b} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{\alpha \beta} \delta w,_{\beta \alpha} d x d y+\int_{0}^{a} \int_{0}^{b} q \delta w d x d y=0
$$

Integrate the $1^{\text {st }}$ integral w,r,t, $z$ and use $M_{\alpha \beta} \sim z \sigma_{\alpha \beta}$, then we get

$$
\int_{0}^{a} \int_{0}^{b} \int_{-\frac{h}{2}}^{\frac{h}{2}} M_{\alpha \beta} \delta w{ }_{\beta \alpha} d x d y+\int_{0}^{a} \int_{0}^{b} q \delta w d x d y=0
$$

As in previous section, we decompose $2(\delta w)_{, x y}=(\delta w)_{, x y}+(\delta w)_{, y x}$ and apply Green' theorem and integration by parts.

$$
\begin{aligned}
& \left.\left.\int_{0}^{a} \int_{0}^{b} \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(M_{\alpha \beta, \beta \alpha}+q\right) \delta w d x d y+\int_{0}^{b} M_{x x}(\delta w)_{x,}\right]^{a} d y+\int_{0}^{a} M_{y y}(\delta w)_{y}\right]^{b} d x \\
& \left.\left.-\int_{0}^{b}\left(M_{x x, x}+M_{x y y}\right) \delta w\right]^{a} d y-\int_{0}^{a}\left(M_{y y y}+M_{x y x}\right) \delta w\right]^{b} d x{ }_{0} \\
& +\int_{0}^{a}\left(M_{x y}(\delta w)_{, x}\right]_{0}^{b} d x+\int_{0}^{b}\left(M_{x y}(\delta w)_{, y}\right]_{0}^{a} d y=0
\end{aligned}
$$

Now, integrate the last two integrate by part, then we obtain

$$
\begin{aligned}
& \left.\int_{0}^{a}\left(M_{x y}(\delta w)_{, x}\right]_{0}^{b} d x=\left[\left(M_{x y} \delta w\right]_{y=0}^{y=b}\right]_{x=0}^{x=a}-\int_{0}^{a} M_{x y, x} \delta w\right]_{0}^{b} d x \\
& \int_{0}^{b}\left(M_{x y}(\delta w)_{, y}\right]_{0}^{a} d y=?
\end{aligned}
$$

Inserting these results and noting $Q_{\alpha} \sim$ derivative..of..$M_{\alpha \beta}$, we get

$$
\begin{aligned}
& \left.\left.\int_{0}^{a} \int_{0}^{b} \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(M_{\alpha \beta, \beta \alpha}+q\right) \delta w d x d y+\int_{0}^{b} M_{x x}(\delta w)_{, x}\right]_{0}^{a} d y+\int_{0}^{a} M_{y y}(\delta w)_{, y}\right]_{0}^{b} d x \\
& \left.\left.-\int_{0}^{b}\left(Q_{x}+M_{x y, y}\right) \delta w\right]_{0}^{a} d y-\int_{0}^{a}\left(Q_{y}+M_{x y, x}\right) \delta w\right]_{0}^{b} d x \\
& +2\left(M_{x y} \delta w\right)_{a, b}-2\left(M_{x y} \delta w\right)_{0, b}-2\left(M_{x y} \delta w\right)_{a, 0}+2\left(M_{x y} \delta w\right)_{0,0} \\
& =0
\end{aligned}
$$

$$
\oint M_{n s} \frac{\partial \delta w}{\partial s} d s=-\oint \frac{\partial M_{n s}}{\partial s} \delta w d s \ldots(?)
$$

We then get for the variation of the total potential

$$
\int\left(D \nabla^{4} w-q\right) \delta w d A-\oint M_{n n} \frac{\partial \delta w}{\partial n} d s+\oint\left(Q_{n}+\frac{\partial M_{n s}}{\partial s}\right) \delta w d s=0
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There are only $\mathbf{2}$ conditions, in spite of the fact the there $\mathbf{3}$ variables

$$
M_{n n}, Q_{n}, M_{n s}
$$

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we got

$$
\int_{0}^{a} \int_{0}^{b} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{\alpha \beta} \delta w,{ }_{\beta \alpha} d x d y d z+\int_{0}^{a} \int_{0}^{b} q \delta w d x d y=0
$$

Integrate the $1^{\text {st }}$ integral $w, r, t, z$ and use $M_{\alpha \beta} \sim z \sigma_{\alpha \beta}$, then we get

$$
\int_{0}^{a} \int_{0}^{b} M_{\alpha \beta} \delta w,_{\beta \alpha} d x d y+\int_{0}^{a} \int_{0}^{b} q \delta w d x d y=0
$$

As in previous section, we decompose $2(\delta w)_{, x y}=(\delta w)_{, x y}+(\delta w)_{, y x}$ and apply Green' theorem and integration by parts.

$$
\begin{aligned}
& \left.\left.\int_{0}^{a} \int_{0}^{b}\left(M_{\alpha \beta, \beta \alpha}+q\right) \delta w d x d y+\int_{0}^{b} M_{x x}(\delta w)_{x,}\right]^{a} d_{0} y+\int_{0}^{a} M_{y y}(\delta w)_{y}\right]^{b} d x \\
& \left.\left.-\int_{0}^{b}\left(M_{x x, x}+M_{x y}\right) \delta w\right]^{a} d y-\int_{0}^{a}\left(M_{y y y}+M_{x y}\right) \delta w\right]^{b} d x \\
& +\int_{0}^{a}\left(M_{x y}(\delta w)_{, x}\right]_{0}^{b} d x+\int_{0}^{b}\left(M_{x y}(\delta w)_{, y}\right]_{0}^{a} d y=0
\end{aligned}
$$

Now, integrate the last two integrate by part, then we obtain

$$
\begin{aligned}
& \left.\int_{0}^{a}\left(M_{x y}(\delta w)_{, x}\right]_{0}^{b} d x=\left[\left(M_{x y} \delta w\right]_{y=0}^{y=b}\right]_{x=0}^{x=a}-\int_{0}^{a} M_{x y, x} \delta w\right]_{0}^{b} d x \\
& \int_{0}^{b}\left(M_{x y}(\delta w)_{, y}\right]_{0}^{a} d y=?
\end{aligned}
$$

Inserting these results and noting $Q_{\alpha} \sim$ derivative..of..$M_{\alpha \beta}$, we get

$$
\begin{aligned}
& \left.\left.\int_{0}^{a} \int_{0}^{b}\left(M_{\alpha \beta, \beta \alpha}+q\right) \delta w d x d y+\int_{0}^{b} M_{x x}(\delta w)_{, x}\right]_{0}^{a} d y+\int_{0}^{a} M_{y y}(\delta w)_{, y}\right]_{0}^{b} d x \\
& \left.\left.-\int_{0}^{b}\left(Q_{x}+M_{x y, y}\right) \delta w\right]_{0}^{a} d y-\int_{0}^{a}\left(Q_{y}+M_{x y, x}\right) \delta w\right]_{0}^{b} d x \\
& +2\left(M_{x y} \delta w\right)_{a, b}-2\left(M_{x y} \delta w\right)_{0, b}-2\left(M_{x y} \delta w\right)_{a, 0}+2\left(M_{x y} \delta w\right)_{0,0} \\
& =0
\end{aligned}
$$

Now we get from the above eqn,

$$
M_{\alpha \beta, \beta \alpha}+q=0
$$

~ Identical to the eqn by equilibrium approach ! valid for all mterials

On the boundary,

1) $0 \leq x \leq a$

$$
\begin{aligned}
& M_{x x}=0 . o r . w_{\bar{x}} \quad p \text { rescr } \\
& Q_{x}+M_{x, y} \overline{\bar{y}} . \text { or r. pres }
\end{aligned}
$$

2) $0 \leq y \leq b$

Results: Same boundary conditions are appears as in Minimum total potential energy.

Corner conditions?

## A Note on the Validity of the Classical Plate Theory

Now we have discussed fully for the classical theory of plate.
It seems to be the time to note the validity of the theory which we have used the assumption of plane stress in the plate. By establishing the order of magnitude of the stresses we can make comparisons as follows. The transverse shears $\sigma_{z \alpha}$ are smaller by an order of $\mathbf{h} / \mathbf{L}$ ( L : characteristic length of the middle plane) than the middle plane shear $\sigma_{\alpha \beta}(\alpha \neq \beta)$. Also the transverse normal stress $\sigma_{z z}$ is smaller by an order $(\mathbf{h} / \mathbf{L})^{2}$ than the middle plane normal stresses $\sigma_{(\alpha)(\alpha)}$ Thus
for a very thin plate $h / L \ll 1 / 10$, we have inner consistency and the classical plate theory may be expected to give good results.

For a moderately thin plate, especially in the vicinity of concentrated loads it might be well to account for the effects of transverse shear deformation.

It is obvious from the equilibrium considerations that such stresses will seldom be zero. Accordingly, we used assurance that the classical plate theory, under proper conditions, yield meaningful results.

There are two approaches:

1. Compare with the results from a more exact theory
~ Theory of elasticity
2. Determine the bounds on the magnitude of stresses $\sigma_{i 3}$ that have been deleted from the theory to show that these stresses, although not zero, are nevertheless small compared to the stressesw.

In view of the scarcity of the exact solutions for plate bending problem from 3-dimensional theory of elasticity, we shall follow the latter procedure.

For this purpose,

$$
\begin{aligned}
\tau_{x x} & =-\frac{E z}{1-v^{2}}\left(w_{, x x}+v w_{, y y}\right), \\
\tau_{y y} & =? \\
\tau_{x y} & =-\frac{E z}{1+v} w_{, x y}
\end{aligned}
$$

Remember : $\quad \sigma_{\alpha \beta} \sim \varepsilon_{\alpha \beta} \sim \frac{1}{2}\left(u_{, \alpha}+u,_{\beta}\right) \sim \frac{1}{2}\left(u_{, \alpha}^{o}+u,{ }_{,}{ }_{\beta}\right)-z w_{, \alpha \beta}$

$$
M_{x x}=\int \tau_{x x} z d z . . e t c
$$

$\sim \tau_{x x}=\frac{M_{x x} z}{\left(\frac{h^{3}}{12}\right)}, . . \tau_{y y}=?, . . \tau_{x y}=?$
Now we use the stresses in the 3-D equilibrium equations

$$
\sigma_{i j, j}=q_{3}
$$

$$
\frac{\partial \tau_{x x}}{\partial x}+\frac{\partial \tau_{x y}}{\partial y}+\frac{\partial \tau_{x z}}{\partial z}=0 \sim \tau_{x x, x}+\tau_{x y, y}+\tau_{x z, z}=0
$$

$$
\tau_{x z, z}=-\frac{z}{\left(\frac{h^{3}}{12}\right)}\left(M_{x x, x}+M_{x y, y}\right)=-\frac{z}{\left(\frac{h^{3}}{12}\right)} Q_{x}
$$

## Integrating w.r.t z

$$
\begin{aligned}
& \tau_{x z}=-\frac{\left(\frac{z^{2}}{2}\right)}{\left(\frac{h^{3}}{12}\right)} Q_{x}+f(x, y) \sim \text { for }: z= \pm \frac{h}{2}: \tau_{x z}=0 \\
& \sim>f(x, y)=\frac{3}{2} Q_{x} / h \\
& \sim>\tau_{x z}=\ldots \text { Similarly } \sim>\tau_{y z}=\ldots
\end{aligned}
$$

Inserting $\tau_{x z}, \tau_{y z}$ into the third equation

$$
\begin{aligned}
& \frac{\partial \tau_{z z}}{\partial z}=-\frac{\partial \tau_{x z}}{\partial x}-\frac{\partial \tau_{y z}}{\partial y} \sim \tau_{z z, z}=-\tau_{x z, x}+\tau_{y z, y} \\
& =-\frac{3}{2 h}\left(1-\frac{4 z^{2}}{h^{2}}\right)\left(Q_{x, x}+Q_{y, y}\right)=-\frac{3 q}{2 h}\left(1-\frac{4 z^{2}}{h^{2}}\right)
\end{aligned}
$$

$$
\tau_{z z}=-\frac{3 q}{2 h}\left(z-\frac{4 z^{3}}{3 h^{2}}\right)+g(x, y)
$$

considering

$$
\begin{aligned}
& z=-\frac{h}{2} ;-q \\
& z=\frac{h}{2}: 0
\end{aligned}
$$

$$
-q\left(\sim \tau_{z z}\right)=-\frac{3 q}{2 h}\left(-\frac{h}{2}+\frac{4\left(\frac{h}{2}\right)^{3}}{3 h^{2}}\right)+g(x, y): g(x, y)=-\frac{q}{2}
$$

Finally,

$$
\begin{aligned}
& \tau_{x z}=\frac{3 Q_{x}}{2 h}\left(1-\frac{4 z^{2}}{h^{2}}\right) \\
& \tau_{y z}=? \\
& \tau_{z z}=-\frac{3 q}{2 h}\left(z-\frac{4 z^{3}}{3 h^{2}}\right)-\frac{q}{2}
\end{aligned}
$$

We have $\tau_{i z}$ which are computed from equilibrium consideration
using results from a theory that neglected these very stresses.

If it turns out that the above computed stresses are small, we have inner consistency in the theory.

To do so, we shall next make an order of magnitude study

Consider a portion of plate !

Figure ~
$O\left(F \equiv\left[O(][Q)^{2} O\right\rceil \nexists[) \quad, \quad\right] Q(\lceil Q)\rfloor$

Definition of shear force intensity?

$$
\left[O\left(Q_{v}\right)\right]=[O(q)][O(L)]=[O(q L)]
$$

Equate moments about an axis $\mathbf{C - C}$ ' going through the line of the action of the resultant of the force distribution:

$$
\left[O\left(Q_{v}\right)\right][O(L)][O(L)]+\left[O\left(M_{v}\right)\right][O(L)]+\left[O\left(M_{v s}\right)\right][O(L)]=0
$$

Replacing $O\left(Q_{v}\right)$ by using $O\left(Q_{v}\right)=O(q L)$.

Assuming next that shear effects are as significant as either the bending or the twisting effects.

If the resulting shear stresses turn out still to be small in our order of magnitude studies.

We then equate the order of magnitude of bending and twisting with
That of the shear effect:

$$
O\left(M_{v}\right)=O\left(q L^{2}\right), O\left(M_{v s}\right)=O\left(q L^{2}\right)
$$

We thus have order of magnitude formulations for the stress
resultant intensity functions, $Q_{v}, M_{v} M_{v s}$ which may now be used in
conjunction with stresses for comparison purposes.

$$
\begin{aligned}
& O(z)=O(h) \\
& O\left(\tau_{x x}\right)=O\left(q L^{2} / h^{2}\right), O\left(\tau_{x y}\right)=\ldots, O\left(\tau_{y y}\right)=\ldots \\
& O\left(\tau_{x z}\right)=O\left(\frac{3}{2 h} Q_{x}\left(1-4 \frac{z^{2}}{h^{2}}\right)\right)=O\left(\frac{1}{h}\right) O(q L)=O\left(q \frac{L}{h}\right) \\
& O\left(\tau_{y z}\right)=\ldots \\
& O\left(\tau_{z z}\right)=O(q)
\end{aligned}
$$

We can now make comparisons :

Methods of Solution

Governing equations(A) ? + Boundary conditions(B) ?

Exact solution $=A+B$ : Very special sets of Problems
Approximate solutions based on : Shape function:
~ Complete sets
Full domain :
Ritz method: Minimization of Potential Energy
Levy method
Galerkin's method: Governing equations

Kantorovich method

Sub-domain~

FEM ? BEM ,XFEM ...

### 4.2 Shear deformable plate theory

So far, theory of thin plate have been established based on Kirchhoff-Love assumptions

In this section we shall consider the small deflection theory of thick plates
including the effect of transverse shear deformation which was neglected despite of the fact that we know from equilibrium considerations that it was not zero. This addition in our treatment of plates is analogous to the Timoshenko beam theory.

In accordance with Reissner' theory we assume a linear law for distribution of stresses $\sigma_{x x}, \sigma_{y y}$ and $\sigma_{x y}$ through the thickness of the plate. It may be seen by replacing the bracked expression of the bending part in
$\sigma_{x x}=\frac{E}{1-\nu^{2}}\left(\varepsilon_{x x}+v \varepsilon_{y y}\right)=\frac{E}{1-\nu^{2}}\left[u_{x}+\nu v_{y}-z\left(\omega_{x x}+v \omega_{y y}\right)\right]$
$\sigma_{y y}=\frac{E}{1-\nu^{2}}\left(\varepsilon_{y y}+v \varepsilon_{x y}\right)=\frac{E}{1-\nu^{2}}\left[v_{y}+\nu u_{x}-z\left(\omega_{y y}+v \omega_{x x}\right)\right]$
$\sigma_{x y}=2 G \varepsilon_{y y}=2 G\left[\frac{1}{2}\left(u_{y}+v_{x}\right)-z \omega_{x y}\right]$
by bending moment and twisting moment intensities with the aid of
$M_{x}=-D\left(w_{x+x}+\nu w_{w, p}\right)$
$M_{y}=-D\left(w_{, y y}+v w_{, x x}\right)$
$M_{x y}=M_{y x}=-(1-v) D w_{, x y}$
Where $D=\frac{E h^{3}}{12\left(1-v^{3}\right)}$ : bending rigidity of plates.

Substituting these expression into the first of the equilibrium equation $\sigma_{i, j}=0$ and then integrating $w, r, t, z$, we get by using the first of
$M_{\alpha \beta, \beta}-Q_{\alpha}=0$
$\frac{\gamma \sigma_{x z}}{\gamma Z}=-\frac{Z}{h^{3} / 12}\left(M_{x, x}+M_{x y, y}\right)=-\frac{Z}{h^{3} / 12} Q_{x}$
Now integrating $w, r, t, z$, we get
$\sigma_{x z}=-\frac{6 z^{2}}{h^{3}} Q_{x}+f(x, y)$
Using the boundary condition $\sigma_{\mathrm{z}}=0$ at $z= \pm \frac{h}{2}$

We get $f(x, y)=\frac{3 h}{2} Q_{s}$
Finally, $\sigma_{x=}=\frac{3 Q_{x}}{2 h}\left(1-4 \frac{Z^{2}}{h^{2}}\right)$
Next using the second equation of equilibrium
$\sigma_{\mathrm{r}}=\frac{3 Q_{y}}{2 h}\left(1-4 \frac{Z^{2}}{h^{2}}\right)$
Finally, we consider the last of equilibrium equation,

$$
\begin{gathered}
\sigma_{z z,}=-\sigma_{x, x}-\sigma_{x y, y} \\
=-\frac{3}{2 h}\left(1-4 \frac{z^{2}}{h^{2}}\right)\left(Q_{x, x}+Q_{y, y}\right) \\
\prod V
\end{gathered}
$$

$$
Q_{\alpha, \alpha}=q \pm
$$

It becomes
$\sigma_{z, z}=\frac{3 g}{2 h}\left(1-4 \frac{z^{2}}{h^{2}}\right)$
Integrating and then using the boundary conditions
$\sigma_{z z=}=-q$ at $\quad z=-\frac{h}{2}$
$\sigma_{z z, z}=0 \quad$ at $\quad z=\frac{h}{2}$
We get
$\sigma_{z=}=-\frac{q}{4}\left[2-3 \frac{2 z}{h}+\left(\frac{2 z}{h}\right)^{3}\right]$
We thus have a set of stresses $\sigma_{x}, \sigma_{x}, \sigma_{z z}$
which are computed from nigorous// equilibrium considerations
Next we introduce some average value $\bar{\omega}^{*} q$ the transverse displacement, taken over the thickness q the plate, as well as some average values $\tau_{x}, \tau_{y} \mathrm{q}$
the rotation q the sections $x=$ conss and $y=$ const , respectively. We define these quantities ???? by equating the work q the resultant moments on the average rotation and the work q the corresponding stresses, actual displacement for bending ${ }_{u, v}$ and ${ }_{w}$ in the same section, i,e., we put

$$
\begin{aligned}
& \int_{\frac{-}{2}}^{\frac{h}{2}} \sigma_{x=} u d z=M_{x} \tau_{x} \quad \int_{\frac{1}{2}}^{\frac{2}{2}} \sigma_{x} v d z=M_{\infty} \tau_{y} \\
& \int_{-\frac{1}{2}}^{\frac{n}{2}} \sigma_{v} v d z=M_{v} \tau_{y} \quad \int_{\frac{-2}{2}}^{\frac{h}{2}} \sigma_{w} u d z=M_{w_{y}} \tau_{x} \\
& \int_{\frac{-}{2}}^{\frac{h}{2}} \sigma_{x} w d z=Q_{x}^{\bar{w}} \quad \int_{\frac{-1}{2}}^{\frac{h}{2}} \sigma_{\bar{x}} w d z=Q_{y} \bar{w}
\end{aligned}
$$

Substituting previously obtained stresses in the above equation, we obtain the following relations between the average ?? and the actual displacements, $\bar{w}=\frac{1}{Q_{v}} \int_{\frac{-1}{2}}^{\frac{n}{2}} \frac{2}{2} ? \bar{w}=\frac{3}{2 h} \frac{3}{2} \frac{n}{2} w\left[1-\left(\frac{2 z}{h}\right)^{2}\right] d z \mathrm{p} .127$
$\tau_{x}=\frac{12}{h^{3}} \frac{n_{h}^{2}}{\frac{n}{2}} u z d z$
$\tau_{y}=\frac{12}{h^{3}} \int_{-\frac{h}{2}}^{\frac{h}{2}} v z d z$
We have the strain-displacement and Hook's law
$\varepsilon_{x x}=\frac{\partial u}{\partial x}=\frac{1}{E}\left[\sigma_{x x}-v\left(\sigma_{y y}+\sigma_{z z}\right)\right]$
$\varepsilon_{y y}=\frac{\partial v}{\partial y}=\frac{1}{E}\left[\sigma_{y y}-v\left(\sigma_{x x}+\sigma_{z z}\right)\right]$
$2 \varepsilon_{x y}\left(=\gamma_{x y}\right)=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{1}{G} \sigma_{x y}$
$2 \varepsilon_{x z}\left(=\gamma_{x z}\right)=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{1}{G} \sigma_{x z}$
$2 \varepsilon_{y z}\left(=\gamma_{y z}\right)=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\frac{1}{G} \sigma_{y z} \backslash$
With $G=\frac{E}{2(1+v)}$
$\sigma_{x x}=\frac{E}{1-v^{2}}\left(u_{, x}+v v_{, y}\right)-\frac{3 q v}{4(1-v)}\left[\frac{2}{3}-\frac{2 z}{h}+\frac{1}{3}\left(\frac{2 z}{h}\right)^{3}\right]$
$\sigma_{y y}=\frac{E}{1-v^{2}}\left(v_{, y}+v u_{, x}\right)-\frac{3 q v}{4(1-v)}\left[\frac{2}{3}-\frac{2 z}{h}+\frac{1}{3}\left(\frac{2 z}{h}\right)^{3}\right]$

Substituting these into
$\sigma_{x x}=\frac{12 M_{x}}{h^{3}} Z, \sigma_{w y}=\frac{12 M_{y}}{h^{3}} Z, \sigma_{y y}=\frac{12 M_{v y}}{h^{s}} Z$
and multiplying the obtained equations by $\frac{12}{h^{3}} z d z$, integrating $w, r, t, z\left(-\frac{h}{2}, \frac{h}{2}\right)$, then we obtain

$$
\begin{aligned}
& M_{x}=D\left[\tau_{x, x}+v \tau_{y, y}+\frac{6 v(1+v)}{5 E h} q\right] \\
& M_{y}=D\left[\tau_{v, y}+v \tau_{x, x}+\frac{6 v(1+v)}{5 E h} q\right] \\
& M_{x y}=\frac{D(1-v)}{2}\left(\tau_{x, y}+\tau_{y, x}\right)
\end{aligned}
$$

Substituting eqns
$\sigma_{x=}=\frac{3}{2} \frac{Q_{x}}{h}\left(1-4 \frac{z^{2}}{h^{2}}\right)$
$\sigma_{y z}=\frac{3}{2} \frac{Q_{v}}{h}\left(1-4 \frac{z^{2}}{h^{2}}\right)$

Into the $2 \varepsilon_{x z}=\frac{1}{G} \sigma_{x}$
$2 \varepsilon_{y_{z}}=\frac{1}{G} \sigma_{x z}$
And multiplying the results by
$\frac{3}{2 h}\left[1-\left(\frac{2 z}{h}\right)^{2}\right] d z \quad$ and integrating between
The limits $z= \pm \frac{h}{2}$, we obtain
$\tau_{x}=-\bar{w}_{x}+\frac{121+v}{5} \frac{1}{E h} Q_{x}=-\bar{w}_{x}+\frac{6}{5 G h} Q_{x}$
$\tau_{y}=-\bar{w}_{y}, \frac{12}{5} \frac{1+v}{E h} Q_{y}=-\bar{w}_{, y}+\frac{6}{5 G h} Q_{v}$
Now eight unknowns, $M_{x}, M_{y}, M_{v}, Q_{v}, Q_{y}, \bar{w}, \tau_{x}$ and $\tau_{y}$ are connected by 2 equations of (*), 3 equations of (?), and three equations $m_{\alpha \beta, \beta}-Q_{\alpha}=0$
$\varrho_{\alpha, a}+q=0$

Eliminating $\tau_{x}, \tau_{y}$ from above, and taking into account $\varrho_{\alpha, \alpha+q=0}$, we obtain $M_{x}=-D\left(\bar{w}_{w x}+\nu \bar{w}_{w, y}+\frac{h^{2}}{5} Q_{x x-} \frac{q h^{2}}{10} \frac{v}{1-v}\right.$
$M_{y}=-D\left(\bar{w}_{, w}+\nu \bar{w}_{w, x}+\frac{h^{2}}{5} Q_{y, y}-\frac{q h^{2}}{10} \frac{v}{1-v}\right.$
$M_{w}=-(1-v) D_{w_{w, w}} \frac{h^{2}}{10}\left(Q_{x, y}+Q_{y, x}\right)$
Substituting this equation into $M_{a p, \beta}+Q_{a}=0$, we get
$Q_{x}=-\frac{h^{2}}{10} \nabla^{2} Q_{x}=-D \frac{\partial\left(\nabla^{2} \bar{w}\right)}{\partial x}-\frac{h^{2}}{10(1-v)} \frac{\partial q}{\partial x}$
$Q_{y}=-\frac{h^{2}}{10} \nabla^{2} Q_{y}=-D \frac{\partial\left(\nabla^{2} \bar{w}\right)}{\partial y}-\frac{h^{2}}{10(1-v)} \frac{\partial q}{\partial y}$
To obtain the more complete deflection equation for plate we only have to substitute the above equations into $Q_{\alpha, \alpha}+q=0$, we then get $D \nabla^{4} w=q-\frac{h^{2}}{10} \frac{2-v}{1-\nu} \nabla^{2} q$.

Reissner's theory

## Mindlin's theory

## Shear deformation 고려(thickness direction)

```
\mp@subsup{\varepsilon}{\alpha\beta}{}->\mathrm{ classical theory}
V
\varepsilon}\mp@subsup{\alpha}{3}{}->\mathrm{ shear deformation
V
```

$\varepsilon_{33} \rightarrow$ stretching effect in the thickness direction
$\varepsilon_{\alpha 3}=\frac{1}{2}\left(u_{\alpha, 3}+u_{3, \alpha}\right)=0 \quad: \quad u_{\alpha, 3}=-u_{3, \alpha}=-w_{, \alpha}$
$u_{\alpha, 3}+u_{3, \alpha} \equiv \phi_{\alpha}$
;
$u_{\alpha, 3}=\phi_{\alpha}-u_{3, \alpha} \equiv-\psi_{\alpha}$
(constant shear ~)

$$
=\phi_{\alpha}-w_{, \alpha} \equiv-\psi_{\alpha}(x, y)
$$

1st order theory -> higher order theory
$\varepsilon_{\alpha \beta}=\frac{1}{2}\left(u_{\alpha, \beta}^{\circ}+u_{\beta, \alpha}^{\circ}\right)-\frac{z}{2}\left(\psi_{\alpha, \beta}+\psi_{\beta, \alpha}\right)$
$\gamma_{\alpha 3}=2 \varepsilon_{\alpha 3}=\phi_{\alpha}=w_{, \alpha}-\psi_{\alpha}$
$\left(N_{\alpha \beta}, M_{\alpha \beta}\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}(1, z) \sigma_{\alpha \beta} d z=>(-) D\left(\psi_{x, x}+\nu \psi_{y, y}\right)=M_{x}$
$(-) D\left(\psi_{y, y}+v \psi_{x, x}\right)=M_{y}$
$=M_{x y}$
$Q_{\alpha}=\int \sigma_{\alpha \alpha} d z=k h \sigma_{\alpha z} \quad ; Q_{\alpha}=k h G\left(w_{\alpha}-\psi_{\alpha}\right)$
Shear correction factor steming?? from the fact that $\sigma_{\alpha z}$ is constant through the thickness.

Straight line remains straight but conseration?? Of normal discarded.
Meaning of k : surface condition
$M_{\alpha \beta, \beta}+Q_{\alpha}=0$
$Q_{\alpha, \alpha}+q=0 \longrightarrow$

$$
Q_{\alpha, \alpha}=q
$$

$D\left[\psi_{x, x x}+\frac{1-v}{2} \psi_{x, y+y}+\frac{1+v}{2} \psi_{y, x y}\right]+k G h\left(w_{x, x}-\psi_{x}\right)=0$
$D\left[\psi_{y, y}+\frac{1-v}{2} \psi_{y, x+}+\frac{1+v}{2} \psi_{x, x y}\right]+k G h\left(w_{y, y}-\psi_{y}\right)=0$
$-k G h\left(w_{x x}+w_{y y}-\psi_{x, x}-\psi_{y, y}\right)=q$
Eliminate $\psi_{\alpha}: D \nabla^{4} w=q-\frac{D}{k G h} \nabla^{2} q$; verity
$\left(k=\frac{5}{6}\right)$

Constant bending Mơments and constant twisting moment . It is only for convenience that we examine stress states rather than strain states : typically a computer program outputs stresses rather than strains. Computed n nodal d.o.f can also be examined; if they are incorrect strains and stresses will also be incorrect. If displacements are correct but stresses are incorrect, one suspects that the stress calculation subroutine must be incorrect. Support conditions must not prevent the constant state from occurring.
~ Effect of locking


$$
\begin{aligned}
& \theta=a_{1}+a_{2} x \\
& \omega=b_{1}+b_{2} x \\
& \varepsilon_{x 2}=\left(\omega_{x}+\theta\right)=a_{1}+a_{2} x+b_{2} \\
& \varepsilon_{x z}^{2}:
\end{aligned}
$$

2 point rule to integrate exactly

$$
\varepsilon_{x z}=0 ; a_{1}+b_{2}=0, a_{2}=0
$$

$$
\rightarrow \theta_{1}=a_{1}, \omega=b_{1}-a_{1} x
$$

$$
U_{b} \sim\left(\frac{\partial \theta}{\partial x}\right)^{2}=0 \rightarrow \text { Locking }
$$

reduced integration : 1 point rule
$\varepsilon_{x z}=0, a_{1}+b_{2}=0 \rightarrow \theta=a_{1}+a_{2} x$

$$
\begin{array}{r}
\omega=b_{1}-a_{1} x \\
\rightarrow \frac{\partial \theta}{\partial x}=a_{2} \neq 0 \rightarrow U_{b} \sim\left(\frac{\partial \theta}{\partial x}\right) \neq 0
\end{array}
$$

Classical(~Thin) plate theory

$$
\left\{\begin{array}{l}
u(x, y, z) \\
v(x, y, z) \\
w(x, y, z)
\end{array}\right\}=\left\{\begin{array}{l}
u_{0}(x, y) \\
v_{0}(x, y) \\
w_{0}(x, y)
\end{array}\right\}+z\left\{\begin{array}{l}
w_{, x}(x, y) \\
w_{, y}(x, y) \\
0
\end{array}\right\}
$$

## First-order shear deformation theory

$$
\left\{\begin{array}{l}
u(x, y, z) \\
v(x, y, z) \\
w(x, y, z)
\end{array}\right\}=\left\{\begin{array}{l}
u_{0}(x, y) \\
v_{0}(x, y) \\
w_{0}(x, y)
\end{array}\right\}+z\left\{\begin{array}{l}
u_{1}(x, y) \\
v_{1}(x, y) \\
0
\end{array}\right\}:
$$

Shear correction factor: $\sigma_{\alpha 3} \sim \varepsilon_{\alpha 3}=0$ ?

## 2nd-order(Higher-order)shear deformation theory

$$
\left\{\begin{array}{l}
u(x, y, z) \\
v(x, y, z) \\
w(x, y, z)
\end{array}\right\}=\left\{\begin{array}{l}
u_{0}(x, y) \\
v_{0}(x, y) \\
w_{0}(x, y)
\end{array}\right\}+z\left\{\begin{array}{l}
u_{1}(x, y) \\
v_{1}(x, y) \\
0
\end{array}\right\}+z^{2}\left\{\begin{array}{l}
u_{2}(x, y) \\
v_{2}(x, y) \\
0
\end{array}\right\}
$$

Shear correction?

Generally, n-th order

$$
\left\{\begin{array}{l}
u(x, y, z) \\
v(x, y, z) \\
w(x, y, z)
\end{array}\right\}=\left\{\begin{array}{l}
u_{0}(x, y) \\
v_{0}(x, y) \\
w_{0}(x, y)
\end{array}\right\}+z\left\{\begin{array}{l}
u_{1}(x, y) \\
v_{1}(x, y) \\
0
\end{array}\right\}+z^{2}\left\{\begin{array}{l}
u_{2}(x, y) \\
v_{2}(x, y) \\
0
\end{array}\right\}+\ldots \ldots+z^{n}\left\{\begin{array}{l}
u_{n}(x, y) \\
v_{n}(x, y) \\
0
\end{array}\right\}
$$

## what is the merit?

$$
\varepsilon_{\alpha \beta}, \varepsilon_{\alpha 3}, \varepsilon_{33}: ?
$$

$$
\varepsilon_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}+u_{m i} u_{m j}\right)!
$$

## 4.3 von Karman Plate theory

The theory of small deflection of thin plate was derived in the previous sections under the assumption of infinitesimal displacement. For small deflection ( $w_{\max } \leq 0.2 h$ ) the theory gives sufficiently accurate results. By increasing the magnitude of deflection beyond a certain level $\left(w_{\text {max }} \geq 0.3 h\right)$ however, the lateral deflections are accompanied by stretching of the middle surface, provided that the edges, or at least the corners, of the plate are restrained again in-plane motion. Membrane forces produced by such stretching can help appreciably in carrying the lateral load. If the plate, for instance, is permitted to deform beyond its thickness, its load-carrying capacity is already significantly increased.

When $w_{\max } \approx h$, the membrane action becomes comparable to that of bending. This is in sharp contrast with the theory of beams, for which the linear theory is valid
as long as the slope of the deflection curve is everywhere small in comparison to unity.

Figure : Deflection of simply-supported square plate under constant lateral load

A well known theory of large deflection of thin pates is due to von Karman.
In this theory, the following assumptions are made.
$\sim|w|=O(h)$ and $|\omega| \ll L ; L$ : typical plate dimension
$\sim$ The typical displacements $u, v$ are small, and in the strain-displacement relations only the quadratic terms in $w_{x}$ and $w_{y}$ are retained. (or strains are much smaller than rotations)
$\sim$ Kirchhoff's assumption that lines normal to the un-deformed middle surface
remain normal to this surface in the deformed state and are un-extended after deformation holds.
$\sim$ The slope is everywhere small, $\left|w_{x,}\right| \ll 1$ and $\left|w_{, y}\right| \ll 1$.
$\sim$ All strains are small and Hooke's law holds.

Thus von Karman theory differs from the linear theory only in retaining certain powers of derivatives $w_{x x}$ and $w_{y}$ in the strain-displacement relationship. Because of the Kirchhoff assumption we may write

$$
\begin{aligned}
& u_{1}(x, y, z)=u(x, y)-z w(x, y),_{x} \\
& u_{2}(x, y, z)=v(x, y)-z w(x, y),{ }_{y} \quad \text { or } \quad u_{\alpha}(x, y, z)=u_{\alpha}^{0}(x, y)-z w(x, y),_{\alpha} \\
& u_{3}(x, y, z)=w(x, y)
\end{aligned} \quad u_{3}(x, y, z)=w(x, y)
$$

\% Strain-displacement relation from the theory of elasticity :

$$
e_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right)+\frac{1}{2} u_{k, i} u_{k, j}
$$

### 3.3.1 Strain - Displacement relationship

$$
u_{i, j}=\varepsilon_{i j}+\omega_{i j}=\text { Stretching + rotation : }
$$

Note: $\omega_{\alpha 3} \gg \omega_{12}$

Model !:

$$
\begin{aligned}
& u_{1,2}=\frac{1}{2}\left(u_{1,2}+u_{2.1}\right)+\frac{1}{2}\left(u_{1,2}-u_{2,1}\right)=\varepsilon_{12}+\omega_{12} \\
& u_{1,3}=\frac{1}{2}\left(u_{1,3}+u_{3.1}\right)+\frac{1}{2}\left(u_{1,3}-u_{3,1}\right)=\varepsilon_{13}+\omega_{13} \\
& u_{1,1}=\frac{1}{2}\left(u_{1,1}+u_{1.1}\right)+\frac{1}{2}\left(u_{1,1}-u_{1,1}\right)=\varepsilon_{11}+\omega_{11}=\varepsilon_{11}+0!
\end{aligned}
$$

## \% In the matrix form :

Report : $\left[u_{i, j}\right]=$ ?

$$
e_{i j}=\frac{1}{2}\left(\varepsilon_{i j}+\omega_{i j}+\varepsilon_{j i}+\omega_{j i}\right)+\frac{1}{2}\left(\varepsilon_{k i}+\omega_{k i}\right)\left(\varepsilon_{k j}+\omega_{k j}\right)
$$

## Symmetric or skew-symmetric part !

$$
e_{i j}=\varepsilon_{i j}+\frac{1}{2}\left(\varepsilon_{k i}+\omega_{k i}\right)\left(\varepsilon_{k j}+\omega_{k j}\right)
$$

$$
\text { Assume : } \varepsilon_{i j}=O\left(\theta^{2}\right), \omega_{\alpha 3}=O(\theta), \omega_{\alpha \beta}=O\left(\theta^{2}\right)
$$

$$
e_{\alpha \beta}=C_{2}\left(\varepsilon_{\alpha \beta}+\frac{1}{2} \omega_{3 \alpha} \omega_{3 \beta}\right)+\frac{1}{2} C_{3}\left(\varepsilon_{3 \alpha} \omega_{3 \beta}+\omega_{3 \alpha} \varepsilon_{3 \beta}\right)+\frac{1}{2} C_{4}\left(\varepsilon_{\lambda \alpha} \omega_{\lambda \beta}+\omega_{\lambda \alpha} \varepsilon_{\lambda \beta}+\varepsilon_{3 \alpha} \varepsilon_{3 \beta}+\omega_{\lambda \alpha} \omega_{\alpha \beta}\right)
$$

$$
e_{\alpha 3}=C_{2}\left(\varepsilon_{\alpha 3}\right)+\frac{1}{2} C_{3}\left(\varepsilon_{\lambda \alpha} \omega_{\lambda 3}+\omega_{3 \alpha} \varepsilon_{33}+\omega_{\lambda \alpha} \omega_{\lambda 3}\right)+\frac{1}{2} C_{4}\left(\varepsilon_{\lambda 3} \omega_{\lambda \alpha}+\varepsilon_{\lambda \alpha} \varepsilon_{\lambda 3}+\varepsilon_{3 \alpha} \varepsilon_{33}\right)
$$

$$
e_{33}=C_{2}\left(\varepsilon_{33}+\frac{1}{2} \omega_{\lambda 3} \omega_{\lambda 3}\right)+\frac{1}{2} C_{3}\left(\varepsilon_{\lambda 3} \omega_{\lambda 3}+\omega_{\lambda 3} \varepsilon_{\lambda 3}\right)+\frac{1}{2} C_{4}\left(\varepsilon_{\lambda 3} \omega_{\lambda 3}+\varepsilon_{33} \varepsilon_{33}\right)
$$

## Take

$$
e_{\alpha \beta}=\varepsilon_{\alpha \beta}+\frac{1}{2} \omega_{3 \alpha} \omega_{3 \beta}
$$

## and assuming

$e_{i 3}=0:$ for.$(i=\alpha=1$ or $2:$ shear..deformation $)(i=3:$ normal..deformation $)$

## Then

$$
e_{\alpha \beta}=\varepsilon^{0}{ }_{\alpha \beta}-z w,_{\alpha \beta}+\frac{1}{2} w,_{\alpha} w,_{\beta}:
$$

$$
\text { Ex) } \quad e_{x x}=\varepsilon_{x x}^{0}-z w,_{x x}+\frac{1}{2} w_{,}^{2}=u_{, x}^{0}-z w,_{x x}+\frac{1}{2} w_{x}^{2} .
$$

Q : Non-linear first-order shear deformation theory ? Higher-order shear deformation theory?

## Report : Due Dec. 2

Newton-Raphson method, Arch-length method

- Numerical study for nonlinear system - example results


### 4.2 Equation of equilibrium \& Boundary conditions

For a constitutive law, we will employ Hooke's laws between the stresses and the strains for plane stress over the thickness of the plate.

Thus we shall consider here only with $e_{\alpha \beta}(\alpha, \beta=1,2)$.
Accordingly the total potential energy is written by simple form.
At first, the strain energy is written

$$
\delta U=\int_{V} \sigma_{i j} \delta e_{i j} d V=>\iint_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha \beta} \delta e_{\alpha \beta} d z d A
$$

So, we get

$$
\delta U_{i}=\iint_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\sigma_{11}\left[\frac{\partial \delta u}{\partial x}+\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x}-z \frac{\partial^{2} \delta w}{\partial x^{2}}\right]+\sigma_{12}[\ldots . .]+\sigma_{22}[\ldots . .]\right\} d z d A
$$

Now we use the notation, $M_{\alpha \beta}=\int_{\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha \beta} d z$ to get
(What is the difference between linear plate theory ?)
: Couple terms, non-linear terms

$$
\delta U_{i}=\iint_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\sigma_{11}\left[\frac{\partial \delta u}{\partial x}+\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x}-z \frac{\partial^{2} \delta w}{\partial x^{2}}\right]+\sigma_{12}[\ldots \ldots]+\sigma_{22}[\ldots \ldots]\right\} d z d A
$$

$$
\begin{aligned}
\delta U_{i} & \left.=\iint\left\{N_{x x}\left(\frac{\partial \delta u}{\partial x}+\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x}\right)-M_{x x} \frac{\partial^{2} \delta w}{\partial x^{2}}\right]+\ldots \ldots \ldots . .\right\} d A \\
& \sim \iint\left\{N_{x x}\left(\delta u_{, x}+w_{, x} \delta w_{, x}\right)-M_{x x} \delta w_{, x x}+\ldots \ldots \ldots . .\right\} d A
\end{aligned}
$$

where

$$
N_{\alpha \beta}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha \beta} d z
$$

For the applied forces

$$
\delta U_{e}=-\iint q \delta w d x d y+\oint \bar{N}_{v} \delta u_{v} d s-\oint \bar{N}_{v s} \delta u_{s} d s
$$

where $\bar{N}_{v} ;$ normal force distribution per unit length of the edge which is taken as positive in compression
$\bar{N}_{v s}$; tangential force distribution per unit length of the edge
$u_{v}, u_{s} ;$ in-plane displacements of the boundary in the directions of normal and

## tangential respectively to the boundary. $q(x, y)$; load distribution (lateral)

By using the above results for $\delta U_{i}$ and $\delta U_{e}$ we may form $\delta U$.
Since we used un-deformed states for stresses and external load, we are limited to small(~moderate) deformation as far as using this approximate functional for the total potential energy.

Thus we have for the total potential energy principle

$$
\delta U=0=\delta U_{i}+\delta U_{e}
$$

By using Green's Theorem and coordinate transformation from $x, y$
to $v, s$ and also $Q_{v}=M_{\alpha \beta, \beta}$
Then we get

$$
\begin{aligned}
& \iint\left\{\left(N_{x x, x}+N_{x y, y}\right) \delta u+\left(N_{x y, x}+N_{y y, y}\right) \delta v+\left[M_{x x, x x}+2 M_{x y, x y}+M_{y y, v y}+\right.\right. \\
& \left.\left.\left(N_{x x} w_{, x}\right)_{, x}+\left(N_{y y} w_{x x}\right)_{, y}+\left(N_{x y} w_{, y}\right)_{, x}+\left(N_{y y} w_{, y}\right)_{, y}+q\right] \delta w\right\} d x d y+ \\
& \left.\oint \ldots+\oint \ldots-M_{v s}\right]=0
\end{aligned}
$$

$$
\iint\left[N_{\alpha \beta, \beta} \delta u_{\alpha}+\left(M_{\beta \alpha, \beta \alpha}+. ?+q\right) \delta u_{, 3}\right] d A+
$$

$$
\left.\sim \oint \ldots .+\oint \ldots-M_{v s}\right]=0
$$

Report : Complete '? Portion' ( Next week)

The last term accounts for corners in the boundary as was discussed.
From this equation we can conclude that in the interior of the plate
???!!!

The first 2 equations are identical to the equations expressing the equilibrium for plane stresses. Using these equations to simplify the $3^{\text {rd }}$ equation.

We get

$$
M_{\alpha \beta, \beta \alpha}+N_{\alpha \beta} w_{\beta_{\alpha} \alpha}+q=0
$$

Now comparing this equation with the classical case

$$
M_{\alpha \beta, \beta \alpha}+q=0
$$

We see that it is introduced nonlinear term

$$
\left(\sim N_{\alpha \beta} w_{\beta \alpha}=N_{x x} w_{,_{x x}}+2 N_{x y} w_{x y}+N_{y y} w_{y y}\right)
$$

Involving the in-plane force intensities as additional "transverse loading" terms"
We also obtain the following boundary conditions as the text :
$\sim$ or $\sim$ is specified. etc.

The last three conditions are familiar from earlier work on plates. except that the effective shear force is now argumented by projections of the inplane forces at the edges.

If we use Hook's law to replace the resultant intensity functions by appropriate derivatives of the displacement field, and carrying out the finite integral w.r.t $\quad$ z, we obtain,

$$
\begin{aligned}
& M_{x x}=-D\left(w,_{x x}+\nu w,_{y y}\right) M \\
& . ? \\
& N_{x x}=C\left\{\left[u_{, x}+\frac{1}{2} w,_{x}^{2}\right]+v\left[v_{, y}+\frac{1}{2} w,_{y}^{2}\right]\right\} \\
& * \quad ? . . \\
& \text { can.be.. } \\
& M_{\alpha \beta}=?, N_{\alpha \beta}=?
\end{aligned}
$$

where

$$
\mathrm{C} \text { : Extensional rigidity }
$$

The first two equations will be satisfied if we use a function F defined by
\$

$$
N_{x x}=F_{, y, y} . N_{x x}=F_{, x,} . N_{w y}=-F_{, y}
$$

Replacing $M_{\alpha \beta}$ in terms of $w_{\alpha \beta}$ and $N_{\alpha \beta}$ in terms of derivatives of F above, then the last equation becomes

Then

$$
D \nabla^{4} w=F,_{y y} w_{x x}-2 F,_{x y} w_{x y}+F,_{x x} w,_{y y}+q
$$

We now have a single partial differential equation with two dependent variables w and F .

Since we are studying in-plane effects using a stress approach, we must assure the compatibility of the in-plane displacements. This will give us a second companion equation to go with the above equation. To do this we have to eliminate $u, v$ from the
expression on F . It can be done by eliminating $u, v$ from $*$ and $\$$.
Then the compatibility is given by

$$
D \nabla^{4} F=E h\left(w,{ }_{x x}^{2}-w_{x_{x x}} w,_{, y y_{x}}\right)
$$

\% Gaussian Curvature
If the surface is developable, the Gaussian curvature vanishes. Hence the right-hand side of $D \nabla^{4} F=E h\left(w_{,}^{2}-w,_{x x} w,_{y y}\right)$ vanishes if the deflection surface is developable.

If a flat surface is deformed into a non-developable surface its middle plane will be stretched in some way and the right-hand side of above equation does not vanish. Thus, these nonlinear terms arise from the stretching of the middle plane of a flat plate due to bending into a non-developable surface.

When the nonlinear terms in the above equations are neglected, these equations
reduce to the corresponding equations to the small-deflection theory.

CF) Clamped 4 edges or Cantilever model?

The first two equations governing the membrane stresses $N_{x x}, N_{x y} N_{y y}$ are identical with the linear theory.

Substitution of $N_{\alpha \beta} \sim u, v, w$ into the equations yield the basic equations for stretching of plate as follows:

$$
e x) N_{x x, x}+N_{x y, y}=0: ? . .
$$

## 3-3-3 Methods of Analysis

(a) Berger's Method

$$
U_{b}=\frac{D}{2} \iint\left\{\left(\nabla^{2} w\right)^{2}+2(v-1)\left(w_{x x} w, y y-w_{x y}^{2}\right)+\frac{12}{h^{2}}\left[e^{2}+2(v-1) e_{2}\right]\right\} d x d y
$$

$e$ : First..in var iant..of ..middle..surface..strain
$=\varepsilon_{x x}+\varepsilon_{y y}:$ rec $\tan$ gular..coordinate
$=\varepsilon_{r r}+\varepsilon_{\theta \theta}:$ cylindrical..coordinate
:
$e_{2}$ :Second..in var iant..of ..middle..surface..strain
$=\varepsilon_{x x} \varepsilon_{y y}-\frac{1}{4} \gamma_{x y}^{2}:$ rec $\tan$ gular..coordinate
$=\varepsilon_{r r} \varepsilon_{\theta \theta}:$ cylindrical..coordinate..when..circular..symmetry
$\%$

$$
\begin{aligned}
& \varepsilon_{r r}=u,_{r}+\frac{1}{2} w_{r}^{2}, \varepsilon_{\theta \theta}=\frac{u}{r}: \text { for..symmetric..case } \\
& \varepsilon_{x x}=u,_{x}+\frac{1}{2} w_{x}^{2}, \varepsilon_{y y}=v_{, y}+\frac{1}{2} w_{, y}^{2}, \gamma_{x y}=u,_{y}+v_{x}+w_{x} w,_{y}
\end{aligned}
$$

If the deflection of a plate is of the order of magnitude of its thickness, the differential equations for the deflection and the displacement can be written in terms of one fourth order and two second order nonlinear equations. These three equations are coupled together.

The purpose of the present investigation is to develop a simple and yet sufficiently accurate method for determining the deflection of plates when that deflection is of the order of magnitude of the thickness. The approach used in the following analysis is to investigate the effect of neglecting the strain energy due to the $2^{\text {nd }}$ invariant of the strains in the middle surface of the plate when deriving the differential equation
by the energy methods. The resulting differential equations are still nonlinear, but they can be decoupled in such a manner that they may be solved readily.

These equations hereafter will be referred to as the approximate equations. To solve the problem of large deflections of plates completely, an estimate of the membrane stresses must be made. This can be done by assuming that the deflection is equal to that given by the solution to the approximate equations and substituting this deflection into the strain energy integral. The strain energy is then a function of the displacements only and, by the principle of virtual displacements, differential equations can be derived for these displacements..

It is the intent of this investigation to develop a simplified analysis for finding the deflection of plates when that deflection is large enough so that the strain of the middle surface cannot be neglected.
H.M. Berger " A New Approach to the analysis of Large Deflection of Plates"
(b) Banerjee's Method

For thin plates undergoing large deflections, a modified energy expressions has been suggested and a new set of differential equations has been obtained in a decoupled form. Accuracy of the equations has been tested for a circular and a square plate with immovable as well as movable edge conditions under a uniform static load. These new equations are more advantageous than Berger's decoupled equations which fail to give meaningful results for movable edge conditions.

An approximate method for solving the large deflections of plate has been proposed by Berger. This method is based on the neglect of $e_{2}$, the second invariant of the middle surface strains, in the expression corresponding to the total potential energy of the system. An advantage of Berger's method is that the coupled differential
equations are decoupled if $e_{2}$ is neglected.
But, some authors pointed out certain inaccuracies in Berger's equations and concluded that Berger's line of thought leads to meaningless results for movable edge conditions. This is due to the fact that the neglect of $e_{2}$ for movable edges fails to simply freedom of rotation in the meridian planes where the membrane stress

$$
\sigma_{r r}=\frac{E}{1-v^{2}}\left(u_{r}+\frac{1}{2} w_{r}^{2}+v \frac{u}{r}\right) \quad \text { exist. }
$$

For movable edges the in-plane displacement $u$ is never zero and thus Berger's equations lead to absurd results.

On the other hand, for immovable clamped edge, $u=0$ and $\frac{d w}{d r}=0$ and therefore, Berger's equations give sufficiently accurate results. For simply supported immovable edges, $u=0$, but $\frac{d w}{d r} \neq 0$. Thus Berger's equations give fairly accurate result. A modified energy expression has been suggested by bringing directly the expression for $\sigma_{r v}$ in the total potential energy of the system. A new set of differential equations has been obtained in a decoupled form.
In polar coordinates, the total potential energy, $v$ of a thin isotropic circular plate of radius $a$, and of thickness $h$ is given by
$V=\frac{D}{2} \int_{0}^{a}\left[w_{, r r}^{2}+\frac{2 v}{r} w_{r} w_{w, m}+\frac{1}{r^{2}} w_{r}^{2}+\frac{12}{n^{2}}\left\{e_{1}^{2}+2(v-1) e_{2}\right]\right] r d r-\int_{0}^{a} q w r d r$
Where, $D$ is the flexural rigidity of the plate given by $D=\frac{E h^{3}}{12\left(1-v^{2}\right)}$, $w$ is deflection, $v$ is poisson's ratio, $e_{1}$ and $e_{2}$ the first and second invariant of the middle surface strains respectively given by
$e_{1}=u_{, r}+\frac{1}{2} w_{, r}^{2}+\frac{u}{r}$
$e_{2}=\frac{u}{r}\left[u_{r}+\frac{1}{2} w_{r, r}^{2}\right]$
Here, $u$ is the in-plane displacement and q is the uniform static load. Equation (1) may be rewritten in the following form
$V=\frac{D}{2} \int_{0}^{a}\left[w_{, r r}{ }^{2}+\frac{2 v}{r} w_{r,} w_{r r}+\frac{1}{r^{2}} w_{r r}{ }^{2}+\frac{12}{n^{2}}\left\{\bar{e}_{1}^{2}+\left(1-v^{2}\right) \times \frac{u^{2}}{r^{2}}\right\}\right] r d r-\int_{0}^{a} q w r d r$
Where $\bar{e}_{1}=u, r+v \frac{u}{r}+\frac{1}{2} w_{r r}^{2} \sim \sigma_{r r}$
If the term $\left(1-v^{2}\right) \frac{u^{2}}{r^{2}}$ in (2) is replaced by $\frac{\lambda}{4}\left(\frac{d w}{d r}\right)^{4}$ being a factor depending on the Poisson's ratio for the plate material, decoupling of (2) is possible. Introducing the term $\frac{\lambda}{4}\left(\frac{d w}{d r}\right)^{4}$ in place of $\left(1-v^{2}\right) \frac{u^{2}}{r^{2}}$
and applying Euler's variational method to (2), the following decoupled differential equation is obtained.

$$
\begin{equation*}
\nabla^{4} w-\frac{12}{h^{2}}\left(w_{, r r}+\frac{v}{r} w_{, r}\right) A r^{\nu-1}-\frac{6 \lambda}{h^{2}} w_{, r}^{2}\left(\nabla^{2} w+2 w_{, r r}\right)=\frac{q}{D} \tag{3}
\end{equation*}
$$

Where A is a constant of integration to be determined from $u_{r r}+v \frac{u}{r}+\frac{1}{2} w_{r r}^{2}=A r^{v-1}$
In the rectangular Cartesion coordinate system (3) is
$\nabla^{4} w-\frac{12}{h^{2}} A\left(w_{x x}+v w_{y y}\right)-\frac{6 \lambda}{h^{2}}\left[\nabla^{2} w\left\{w_{x}^{2}+w_{y,}^{2}\right\}+2\left\{w_{x x} w_{x x}^{2}+w_{y y} w_{y y}^{2}\right\}\right]=\frac{q}{D}$
When the constant of integration A is to be determined from
$A=u_{x, x}+v u_{y}+\frac{1}{2} w_{x x}^{2}+\frac{v}{2} w_{y}^{2}$
To these equations must be added a suitable set of boundary conditions.
Refer to the paper
B.Banerjee\& S. Datta "A New Approach to an analysis of Large Deflections of Thin Elastic Plates"


[^0]:    We then have ( Report)

