Chapter 5

Deep Shell Equations

The term *deep* is used to distinguish the set of equations used in this chapter from the

shallow shell equations discussed later. The equations are based on the assumptions

that the shells are thin with respect to their radii of curvature and that deflections are

reasonably small. On these two basic assumptions, secondary assumptions rest.

They discussed as the development warrants it.

The basic theoretical approach is due to Love, who published the equations in their

essential form toward the end of the nineteenth century. Essentially, he extended

work on shell vibrations by Rayleigh, who divided shells into two classes: one where

the middle surface does not stretch and bending effects are the only important ones,

and one where only the stretching of the middle is important and the bending stiffness can be neglected. Love allowed the coexistence of these two classes.

He used the principle of virtual work to derive his equations, following Kirchhoff,

who had used it where deriving the plate equation. The derivation given here uses

Hamilton's principles, following Reissner's derivation.

5.1 Shell Coordinates and Infinitesimal Distances in Shell Layers

We assume that thin, isotropic, and homogeneous shell of constant thickness have

neutral surfaces, just as beams in transverse deflection have neutral fibers. That is

true will become evident later. Stresses in such a neutral surface can be of the

membrane type but cannot be bending stresses. Locations on the neutral surface, placed into a three-dimensional Cartesian coordinate system, can also defined by two-dimensional curvilinear surface coordinate α_1 and α_2.

The location of a point P on the neutral surface (Fig.) in Cartesian coordinates is related to the location of the point in the surface coordinates by

$$x_1 = f_1(\alpha_1, \alpha_2), x_2 = f_2(\alpha_1, \alpha_2), x_3 = f_3(\alpha_1, \alpha_2) \qquad (5.1.1)$$

The location of P on the neutral surface can also be expressed by a vector :

$$\bar{r}(\alpha_1, \alpha_2) = f_1(\alpha_1, \alpha_2)\bar{e}_1 + f_2(\alpha_1, \alpha_2)\bar{e}_2 + f_3(\alpha_1, \alpha_2)\bar{e}_3 \qquad (5.1.2)$$

Now let us define the infinitesimal distance between point P and P' on the neutral surface. The differential change $d\bar{r}$ of the vector $\bar{r}$ as we move from P to P' is

$$d\bar{r}(\alpha_1,\alpha_2)=\frac{\partial\bar{r}(\alpha_1,\alpha_2)}{\partial\alpha_1}d\alpha_1+\frac{\partial\bar{r}(\alpha_1,\alpha_2)}{\partial\alpha_2}d\alpha_2 \tag{5.1.3}$$

The magnitude ds of $d\bar{r}$ is obtained by

$$ds^2=d\bar{r}\bullet d\bar{r} \tag{5.1.4}$$

Or

$$ds^2=\left(\frac{\partial\bar{r}}{\partial\alpha_1}d\alpha_1+\frac{\partial\bar{r}}{\partial\alpha_2}d\alpha_2\right)\bullet\left(\frac{\partial\bar{r}}{\partial\alpha_1}d\alpha_1+\frac{\partial\bar{r}}{\partial\alpha_2}d\alpha_2\right)$$

$$\sim\ \frac{\partial\bar{r}}{\partial\alpha_1}\bullet\frac{\partial\bar{r}}{\partial\alpha_1}(d\alpha_1)^2+\frac{\partial\bar{r}}{\partial\alpha_2}\bullet\frac{\partial\bar{r}}{\partial\alpha_2}(d\alpha_2)^2+2\frac{\partial\bar{r}}{\partial\alpha_1}\bullet\frac{\partial\bar{r}}{\partial\alpha_2}d\alpha_1 d\alpha_2 \tag{5.1.5}$$

Practice : Index notations !

In the following, we limit ourselves to orthogonal curvilinear coordinates which coincide with the lines of principal curvature of the neutral surface

The third term of in Eq.(5.1.5) this becomes

$$2\frac{\partial \bar{r}}{\partial \alpha_1}\bullet\frac{\partial \bar{r}}{\partial \alpha_2}d\alpha_1 d\alpha_2 = 2\left|\frac{\partial \bar{r}}{\partial \alpha_1}\right|\left|\frac{\partial \bar{r}}{\partial \alpha_2}\right|\cos\frac{\pi}{2}d\alpha_1 d\alpha_2 = 0 \qquad (5.1.6)$$

When we define

$$\frac{\partial \bar{r}}{\partial \alpha_1} \bullet \frac{\partial \bar{r}}{\partial \alpha_1} = \left| \frac{\partial \bar{r}}{\partial \alpha_1} \right|^2 = A_1^2$$
$$\frac{\partial \bar{r}}{\partial \alpha_2} \bullet \frac{\partial \bar{r}}{\partial \alpha_2} = \left| \frac{\partial \bar{r}}{\partial \alpha_2} \right|^2 = A_2^2 \qquad (5.1.7)$$

Eq.(5.1.5) becomes

$$ds^2 = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2 \qquad (5.1.8)$$

This equation is called the *fundamental form* and A_1 and A_2 are the *fundamental form parameters* or *Lame parameters.*

As an example, let us look at the circular cylindrical shell shown in Fig.

The lines of principal curvature (for each shell surface point there exists a

maximum

and a minimum radius of curvature, whose directions are at the angle $\pi/2$) are in this case parallel to the axis of revolution, where the radius of curvature $R_x = \infty$ or the curvature $1/R_x = 0$, and along circles, where the radius of curvature $R_\theta = a$ or the curvature $1/R_\theta = 1/a$.

We then proceed to obtain the fundamental form parameters from definition (5.1.7)

The curvature coordinates are

$$\alpha_1 = x, \alpha_2 = \theta \qquad (5.1.9)$$

And Eq.(5.1.3) becomes

$$\bar{r}(x,\theta) = x\bar{e}_1 + a\cos\theta\bar{e}_2 + a\sin\theta\bar{e}_3 \qquad (5.1.10)$$

Fig 5.1.12—

Thus

$$\frac{\partial\bar{r}}{\partial\alpha_1} = \frac{\partial\bar{r}}{\partial x} = \bar{e}_1$$

Or

$$\left|\frac{\partial\bar{r}}{\partial\alpha_1}\right| = A_1 = 1$$

and

$$\frac{\partial \bar{r}}{\partial \alpha_2} = \frac{\partial \bar{r}}{\partial \theta} = -a\sin\theta \bar{e}_2 + a\cos\theta \bar{e}_3$$

Or

$$\left|\frac{\partial \bar{r}}{\partial \theta}\right| = A_2 = a\sqrt{\sin^2\theta + \cos^2\theta} = a$$

The fundamental form is therefore

$$ds^2 = dx^2 + a^2 d\theta^2$$

Recognizing that the fundamental form can be interpreted as defining the hypotenuse *ds* of a right triangle whose sides are infinitesimal distances along the surface coordinates of the shell, we may obtain A_1 and A_2 in a simpler fashion by expressing *ds* directly using inspection:

$$ds^2 = dx^2 + a^2 d\theta^2$$

By comparison with Eq.(5.1.8), we obtain A_1=1 and A_2=a

For the general case, let us now define the infinitesimal distance between a point P_1 that is normal to P and a point P_1' which is normal to P' (see Fig.5.1.3) P_1 is located at a distance α_3 from the neutral surface (α_3 is defined to be along a normal straight line to the neutral surface).

P_1' is located at a distance $\alpha_3 + d\alpha_3$ from the neutral surface.

We may therefore express the location of P_1' as

$$\bar{R}(\alpha_1, \alpha_2, \alpha_3) = \bar{r}\,(\alpha_1, \alpha_2) + \alpha_3 \bar{n}\,(\alpha_1, \alpha_2) \qquad (5.1.16)$$

where

$\bar{n}$: unit vector normal to the neutral surface.

The differential change dR as we move from P_1 to P_1' is

$$d\bar{R}(\alpha_1,\alpha_2,\alpha_3) = d\bar{r}\,(\alpha_1,\alpha_2) + \alpha_3 d\bar{n}\,(\alpha_1,\alpha_2) + \bar{n}\,(\alpha_1,\alpha_2)\,d\alpha_3 \qquad (5.1.17)$$

where

$$d\bar{n}(\alpha_1,\alpha_2) = \frac{\partial \bar{n}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{n}}{\partial \alpha_2} d\alpha_2 \qquad (5.1.18)$$

The magnitude ds of $d\bar{R}$ is obtained by

$$ds^2 = d\bar{R} \bullet d\bar{R} \qquad (5.1.19)$$

Or

$$dS^2 = d\bar{R}(\alpha_1,\alpha_2,\alpha_3)\bullet d\bar{R}(\alpha_1,\alpha_2,\alpha_3) =$$

$$\left[d\bar{r}\ (\alpha_1,\alpha_2) + \alpha_3 d\bar{n}\ (\alpha_1,\alpha_2) + \bar{n}\ (\alpha_1,\alpha_2) d\alpha_3\right]^2 \qquad (5.1.20)$$

$$= d\bar{r}\bullet d\bar{r} + \alpha_3^2 d\bar{n}\bullet d\bar{n} + \bar{n}\bullet\bar{n}(d\alpha_3)^2$$

$$+2\alpha_3 d\bar{r}\bullet d\bar{n} + 2\alpha_3 d\bar{r}\bullet\bar{n} + 2\alpha_3 d\alpha_3 d\bar{n}\bullet\bar{n}$$

$$= d\bar{r}\bullet d\bar{r} + \alpha_3^2 d\bar{n}\bullet d\bar{n} + (d\alpha_3)^2 + 2\alpha_3 d\bar{r}\bullet d\bar{n}$$

Why ?

We have already seen that

$$d\bar{r} \bullet d\bar{r} = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2 \tag{5.1.21}$$

Next

$$\alpha_3^2 d\bar{n} \bullet d\bar{n} = \alpha_3^2 \left[\frac{\partial \bar{n}}{\partial \alpha_1} \bullet \frac{\partial \bar{n}}{\partial \alpha_1} d\alpha_1^2 + \frac{\partial \bar{n}}{\partial \alpha_2} \bullet \frac{\partial \bar{n}}{\partial \alpha_2} d\alpha_2^2 + 2 \frac{\partial \bar{n}}{\partial \alpha_1} \bullet \frac{\partial \bar{n}}{\partial \alpha_2} d\alpha_1 d\alpha_2 \right] \tag{5.1.22}$$

The third term of this expression is zero because of orthogonality (see also Fig.5.1.3).

The second term may be written

$$\alpha_3^2 \frac{\partial \bar{n}}{\partial \alpha_2} \bullet \frac{\partial \bar{n}}{\partial \alpha_2} \bullet d\alpha_2^2 = \left| \alpha_3^2 \frac{\partial \bar{n}}{\partial \alpha_2} \right|^2 \bullet d\alpha_2^2 \tag{5.1.23}$$

from Fig.2.1.3 we recognize the following relationship to the radius of curvature R_2

$$\frac{\left|\partial \overline{r} / \partial \alpha_2\right|}{R_2} = \frac{\left|\alpha_3 \partial \overline{n} / \partial \alpha_2\right|}{\alpha_3} \tag{5.1.24}$$

Since

$$\left|\frac{\partial \overline{r}}{\partial \alpha_2}\right| = A_2 \tag{5.1.25}$$

We get

$$\left|\alpha_3 \frac{\partial \overline{n}}{\partial \alpha_2}\right| = \frac{\alpha_3 A_2}{R_2} \tag{5.1.26}$$

And therefore

$$\alpha_3^2 \frac{\partial \overline{n}}{\partial \alpha_2} \bullet \frac{\partial \overline{n}}{\partial \alpha_2} d\alpha_2^2 = \alpha_3^2 \frac{A_2^2}{R_2} d\alpha_2^2 \tag{5.1.27}$$

Similarly, the first term becomes

$$\alpha_3^2 \frac{\partial \overline{n}}{\partial \alpha_1} \bullet \frac{\partial \overline{n}}{\partial \alpha_1} d\alpha_1^2 = \alpha_3^2 \frac{A_1^2}{R_1} d\alpha_1^2 \tag{5.1.28}$$

And expressions (5..1.22) becomes

$$\alpha_3^2 d\bar{n} \bullet d\bar{n} = \alpha_3^2 \left[\frac{A_1^2}{R_1} d\alpha_1^2 + \frac{A_2^2}{R_2} d\alpha_2^2 \right] \qquad (5.1.29)$$

Finally, the last expression of Eq.(5.1.20) becomes

$$2\alpha_3 d\bar{r} \bullet d\bar{n} = 2\alpha_3 \left[\frac{\partial \bar{r}}{\partial \alpha_1} \bullet \frac{\partial \bar{n}}{\partial \alpha_1} d\alpha_1^2 + \frac{\partial \bar{r}}{\partial \alpha_2} \bullet \frac{\partial \bar{n}}{\partial \alpha_2} d\alpha_2^2 + \frac{\partial \bar{r}}{\partial \alpha_1} \bullet \frac{\partial \bar{n}}{\partial \alpha_2} d\alpha_1 d\alpha_2 + \frac{\partial \bar{r}}{\partial \alpha_2} \bullet \frac{\partial \bar{n}}{\partial \alpha_1} d\alpha_1 d\alpha_2 \right] \qquad (5.1.30)$$

The last two terms are zero because of the orthogonality. The first term may be written

$$\frac{\partial \bar{r}}{\partial \alpha_1} \bullet \frac{\partial \bar{n}}{\partial \alpha_1} \bullet d\alpha_1^2 = \left| \frac{\partial \bar{r}}{\partial \alpha_1} \right| \bullet \left| \frac{\partial \bar{n}}{\partial \alpha_1} \right| d\alpha_1^2 = \frac{A_1^2}{R_1} \bullet d\alpha_1^2 \qquad (5.1.31)$$

Similarly

$$\frac{\partial \bar{r}}{\partial \alpha_2} \bullet \frac{\partial \bar{n}}{\partial \alpha_2} \bullet d\alpha_2^2 = \frac{A_2^2}{R_2} \bullet d\alpha_1^2 \qquad (5.1.32)$$

Expression (5.1.30) therefore becomes

$$2\alpha_3 d\bar{r} \bullet d\bar{n} = 2\alpha_3 \left[\frac{A_1^2}{R_1} d\alpha_1^2 + \frac{A_2^2}{R_2} d\alpha_2^2 \right] \qquad (5.1.33)$$

Substituting expressions (5.1.33),(5.1.29), and (5.1.21) in Eq.(5.1.20) gives

$$ds^2 = A_1^2(1+\frac{\alpha_3}{R_1})^2 d\alpha_1^2 + A_2^2(1+\frac{\alpha_3}{R_2})^2 d\alpha_2^2 + d\alpha_3^2 \qquad (5.1.34)$$

~ What is ε ?

5.2 Stress-Strain Relations

Having chosen the mutually perpendicular lines of principal curvature as coordinates, plus the normal to the neutral surface as the third coordinate, we have three mutually perpendicular plane of strain and three shear strains.

Assuming that Hook's law applies, we have a three dimensional element

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \mu(\sigma_{22} + \sigma_{33})\right] \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \mu(\sigma_{11} + \sigma_{33})\right] \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \mu(\sigma_{11} + \sigma_{22})\right] \\
\varepsilon_{12} &= \frac{G_{12}}{E} \\
\varepsilon_{13} &= \frac{G_{13}}{E} \\
\varepsilon_{23} &= \frac{G_{23}}{E}
\end{aligned}
\qquad (5.2.1\sim6)
$$

where $\sigma_{(i)(i)}$: Normal stresses,

$\sigma_{ij}(i \neq j)$: Shear stresses as shown in Fig.2.2.1

$$\sigma_{ij} = \sigma_{ji} \qquad (5.2.7)$$

We will later assume that transverse shear deformations can be neglected

This implies that

$$\sigma_{\alpha 3} = 0 \qquad (5.2.8)$$

However, we will not neglect integrated effect of the transverse stresses $\sigma_{\alpha 3}$

This is discussed later.

The normal stress σ_{33} which is normal direction to the neutral surface will be neglected.

$$\sigma_{33} = 0 \qquad (5.2.9)$$

This is based on the argument that on an unloaded outer shell surface it I zero, or if a load acts on the shell, it is equivalent in magnitude to external load on the shell, which is relatively small value in most cases. Only in the close vicinity of a concentrated load do we reach magnitudes that would make the consideration of σ_{33} worthwhile.

Our equation system therefore reduces to

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \mu\sigma_{22})$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \mu\sigma_{11})$$

$$\varepsilon_{12} = \frac{G_{12}}{E} \qquad (5.2.10\sim5.2.12)$$

and $$\varepsilon_{33} = -\frac{\mu}{E}(\sigma_{11} + \sigma_{22}) \qquad (5.2.13)$$

Only the first three relationships will be importance in the following.

Equation (5.2.13) can later be used to calculate the constriction of the shell thickness during vibration, which is of some interest to acousticians since it is an additional noise generating mechanism, along with transverse deflection.

2.3 Strain-displacement Relations

We have seen that the infinitesimal distance between two points P_1 and P_1' of an undeformed shell is given by Eq.(5.1.34).

Defining, for the purpose of a short notation,

$$A_1^2(1+\frac{\alpha_3}{R_1})^2 = g_{11}(\alpha_1,\alpha_2,\alpha_3) \tag{5.3.1}$$

$$A_2^2(1+\frac{\alpha_3}{R_2})^2 = g_{22}(\alpha_1,\alpha_2,\alpha_3) \tag{5.3.2}$$

$$1 = g_{33}(\alpha_1,\alpha_2,\alpha_3) \tag{5.3.3}$$

$\sim$ $A_{(i)}^2(1+\frac{\alpha_3}{R_{(i)}})^2 = g_{(i)(i)}(\alpha_1,\alpha_2,\alpha_3)$ for $i=1,2$ and $i=3$ then left hand side equals to 1

We may write Eq.(5.1.34) as

$$ds^2 = \sum_1^3 g_{ii}(\alpha_1,\alpha_2,\alpha_3)d\alpha_i^2 \tag{5.3.4}$$

If point P_1 originally located at $(\alpha_1,\alpha_2,\alpha_3)$, is deflected in the α_1 direction by U_1,

in the α_2 direction by U_2,and in the α_3 (normal) direction by U_3, it will be located at $(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)$. Deflection U_i and coordinate changes ξ_i are related by

$$U_i=\sqrt{g_{ii}(\alpha_1,\alpha_2,\alpha_3)}\xi_i \qquad (5.3.5)$$

Point $P_1^{'}$, originally at $(\alpha_1+d\alpha_1,\alpha_2+d\alpha_2,\alpha_3+d\alpha_3)$ will be located at $(\alpha_1+d\alpha_1+\xi_1+d\xi_1,\alpha_2+d\alpha_2+\xi_2+d\xi_2,\alpha_3+d\alpha_3+\xi_3+d\xi_3)$ after deflection(Fig.5.3.1).

The distance $$(ds^{'})^2=\sum_1^1 g_{ii}\,(\alpha_1+d\alpha_1,\alpha_2+d\alpha_2,\alpha_3+d\alpha_3)(d\alpha_i+d\xi_i)^2 \qquad (5.3.6)$$

Since $g_{ii}(\alpha_1,\alpha_2,\alpha_3)$ varies in a continuous fashion as $\alpha_1,\alpha_2,$ and α_3 change, we

may utilize as an approximation the first few terms of a Taylor series expension of $g_{(ii}(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)$ about the point $(\alpha_1,\alpha_2,\alpha_3)$:

$$g_{ii}(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)=g_{ii}(\alpha_1,\alpha_2,\alpha_3)+\sum_{j=1}^{3}\frac{\partial g_{ii}(\alpha_1,\alpha_2,\alpha_3)}{\partial\alpha_j}\xi_j+\ldots \qquad (5.3.7)$$

For the special case of an arch that deflection only in the plane of its curvature, the Taylor series expansion is illustrated in Fig.5.3.2. in this example,

$$g_{22}(\alpha_1,\alpha_2,\alpha_3)=g_{33}(\alpha_1,\alpha_2,\alpha_3)=0, \quad \text{and} \quad g_{11}(\alpha_1,\alpha_2,\alpha_3)=g_{11}(\alpha_1),$$

Equation (2.3.7) becomes

$$g_{11}(\alpha_1+\xi_1)=g_{11}(\alpha_1)+\frac{\partial g_{11}(\alpha_1)}{\partial\alpha_1}\xi_1 \qquad (5.3.8)$$

Continuing with the general case, we may write

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i d\xi_i + (d\xi_i)^2 \qquad (5.3.9)$$

In the order of approximation consistent with linear theory, $(d\xi_i)^2$ can be neglected.

Thus

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i d\xi_i \qquad (5.3.10)$$

The differential $d\xi_i$ is

$$d\xi_i = \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \qquad (5.3.11)$$

Therefore,(Eqn(5.3.10) becomes

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \qquad (5.3.12)$$

Substituting Eqs.(5.3.12) and (5.3.7) in Eq.(5.3.6) gives

$$(ds')^2 = \sum_{i=1}^{3} [g_{ii}(\alpha_1, \alpha_2, \alpha_3) + \sum_{j=1}^{3} \frac{\partial g_{ii}(\alpha_1, \alpha_2, \alpha_3)}{\partial \alpha_j} \xi_j] \times [(d\alpha_i)^2 + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j] \qquad (5.3.13)$$

Expanding the expression and writing

$$g_{ii}(\alpha_1, \alpha_2, \alpha_3) = g_{ii} \qquad (5.3.14)$$

gives

$$(ds')^2 = \sum_{i=1}^{3} [(g_{ii} + \sum_{j=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_j} \xi_j)(d\alpha_i)^2] + 2d\alpha_i g_{ii} \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_j} \xi_j \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j] \qquad (5.3.15)$$

The last term is negligible except for cases where initial stresses exist in the shell.

We have ,therefore, replacing j by k in the first term,

$$(ds')^2 = \sum_{i=1}^{3}[(g_{ii} + \sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k)(d\alpha_i)^2] + \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}d\alpha_j d\alpha_i + \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}d\alpha_j d\alpha_i \qquad (5.3.16)$$

Utilizing the Kronecker delta notation

$$\delta_{ij} = \begin{cases} 1 : i = j \\ 0 : i \neq j \end{cases} \qquad (5.3.17)$$

we may write the first term of Eq.(2.3.16) as

$$(ds')^2 = \sum_{i=1}^{3}\sum_{i=1}^{3}(g_{ii} + \sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k)\delta_{ij} d\alpha_i d\alpha_j \qquad (5.3.18)$$

The last two terms of Eq.(5.3.16) , we may write in symmetric fashion by noting that

$$\sum_{i=1}^{3}\sum_{j=1}^{3} g_{ii}\frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j d\alpha_i = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ii}\frac{\partial \xi_j}{\partial \alpha_i} d\alpha_i d\alpha_j \qquad (5.3.19)$$

Thus

$$(ds')^2 = \sum_{i=1}^{3}\sum_{i=1}^{3}[(g_{ii} + \sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k)\delta_{ij} + g_{ij}\frac{\partial \xi_i}{\partial \alpha_j} + g_{ij}\frac{\partial \xi_j}{\partial \alpha_i}]d\alpha_i d\alpha_j \qquad (5.3.20)$$

Denoting

$$G_{ij} = (g_{ii} + \sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k)\delta_{ij} + g_{ij}\frac{\partial \xi_i}{\partial \alpha_j} + g_{ij}\frac{\partial \xi_j}{\partial \alpha_i} \qquad (5.3.21)$$

Gives

$$(ds')^2 = \sum_{i=1}^{3} \sum_{i=1}^{3} G_{ij} d\alpha_i d\alpha_j \tag{5.3.22}$$

Note that

$$G_{ij} = G_{ji} \tag{5.3.23}$$

Eqn(5.3.22) defines the distance between two points *P* and *P'* after deflection, where point *P* was originally located at $(\alpha_1, \alpha_2, \alpha_3)$ and point *P'* at $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3)$.

For example, if *P'* was originally located at $(\alpha_1 + d\alpha_1, \alpha_2, \alpha_3)$, that is, $d\alpha_2 = 0$

$$(ds)^2 = g_{11}(d\alpha_1)^2 = (ds)_{11}^2 \tag{5.3.24}$$

$$(d's)^2 = G_{11}(d\alpha_1)^2 = (d's)_{11}^2 \tag{5.3.25}$$

If point P' was originally located at $(\alpha_1, \alpha_2 + d\alpha_2, \alpha_3)$,that is, $d\alpha_1 = 0$ and $d\alpha_3 = 0$,

$$(ds)^2 = g_{22}(d\alpha_2)^2 = (ds)_{22}^2 \tag{5.3.26}$$

$$(ds')^2 = G_{22}(d\alpha_2)^2 = (ds')_{22}^2 \tag{5.3.27}$$

Now let us investigate the case shown in Fig.5.3.3, where P was originally located at $(\alpha_1 + d\alpha_1, \alpha_2, \alpha_3)$ and P' was originally located at $(\alpha_1, \alpha_2 + d\alpha_2, \alpha_3)$. This is equivalent to saying that P was originally located at $(\alpha_1, \alpha_2, \alpha_3)$ and P' at $(\alpha_1 - d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3)$

We then get

$$(ds)^2 = g_{11}(d\alpha_1)^2 + g_{22}(d\alpha_2)^2 = (ds)_{12}^2 \tag{5.3.28}$$

$$(ds')^2 = G_{11}(d\alpha_1)^2 + G_{22}(d\alpha_2)^2 - 2G_{12}d\alpha_1 d\alpha_2 = (ds')_{12}^2 \tag{5.3.29}$$

In general,

$$(ds)_{ii}^2 = g_{ii}(d\alpha_i)^2 \tag{5.3.30}$$

$$(ds')^2 = G_{ii}(d\alpha_i)^2 \tag{5.3.31}$$

And

$$(ds)_{ij}^2 = g_{ii}(d\alpha_i)^2 + g_{jj}(d\alpha_j)^2 \tag{5.3.32}$$

$$(ds')_{ij}^2 = G_{ii}(d\alpha_i)^2 + G_{jj}(d\alpha_j)^2 - 2G_{ij}d\alpha_i d\alpha_j \tag{5.3.33}$$

We are really now to formulate strains. The normal strains are

$$\varepsilon_{ij} = \frac{(ds'_{ii}) - (ds_{ii})}{(ds)_{ii}} = \sqrt{\frac{G_{ii}}{g_{ii}}} - 1 = \sqrt{1 + \frac{G_{ii} - g_{ii}}{g_{ii}}} - \tag{5.3.34}$$

Noting that since

$$\frac{G_{ii} - g_{ii}}{g_{ii}} \ll 1 \tag{5.3.35}$$

We have the expansion

$$\sqrt{1+\frac{G_{ii}-g_{ii}}{g_{ii}}}=1+\frac{1}{2}\frac{G_{ii}-g_{ii}}{g_{ii}}-\ldots \qquad (5.3.36)$$

Thus

$$\varepsilon_{ii}=\frac{1}{2}\frac{G_{ii}-g_{ii}}{g_{ii}} \qquad (5.3.37)$$

Shear strains $\varepsilon_{ij}(i\neq j)$ are defined as the angular change of an infinitesimal element:

$$\varepsilon_{ij}=\frac{\pi}{2}-\theta_{ii} \qquad (5.3.38)$$

θ_{ii} for $i=1$ and $i=2$ is shown in Fig.2.3.3.

Utilizing the cosine law, we may compute this angle

$$(ds')_{ij}^2 = (ds')_{ii}^2 + (ds')_{jj}^2 - 2(ds')_{ii}(ds')_{jj}\cos\theta_{ij} \tag{5.3.39}$$

Substtuting Eqs.(5.3.31) and (5.3.33) and solving for $\cos\theta_{ij}$ gives

$$\cos\theta_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \tag{5.3.40}$$

Substtuting Eqs.(5.3.38) results in

$$\cos(\frac{\pi}{2} - \varepsilon_{ij}) = \sin\varepsilon_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \tag{5.3.41}$$

And since for reasonable shear strain magnitudes

$$\sin \varepsilon_{ij} \cong \varepsilon_{ij} \tag{5.3.42}$$

and

$$\frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \cong \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{5.343}$$

We may express the shear strain as

$$\varepsilon_{ij} = \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{5.3.44}$$

Substtuting Eqs.(5.3.21),(5.3.5), and (5.3.1) to (5.3.3) in Eq.(5.3.37) gives, for instance for $i=1,$

$$\varepsilon_{11} = \frac{1}{2A_1^2(1+\alpha_3/R_1)^2}\{\frac{\partial[A_1^2(1+\alpha_3/R_1)^2]}{\partial\alpha_1}\frac{U_1}{A_1(1+\alpha_3/R_1)} + \frac{\partial[A_1^2(1+\alpha_3/R_1)^2]}{\partial\alpha_2}\frac{U_2}{A_2(1+\alpha_3/R_2)} + \frac{\partial[A_1^2(1+\alpha_3/R_1)^2]}{\partial\alpha_3}U_3\}$$

$$+\frac{\partial}{\partial\alpha_1}\frac{U_1}{A_1(1+\alpha_3/R_1)}$$

$$= \frac{1}{A_1(1+\alpha_3/R_1)}\{\frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_1}\frac{U_1}{A_1(1+\alpha_3/R_1)} + \frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_2}\frac{U_2}{A_2(1+\alpha_3/R_2)} + \frac{A_1}{R_1}U_3\}$$

$$+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_1}{\partial\alpha_1} - \frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_1}\frac{U_1}{A_1^2(1+\alpha_3/R_1)^2}$$

(5.3.45)

Next, utilize the equalities

$$\frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_2} = (1+\frac{\alpha_3}{R_2})\frac{\partial A_1}{\partial\alpha_2}$$

and

$$\frac{\partial[A_2(1+\alpha_3/R_2)]}{\partial\alpha_1} = (1+\frac{\alpha_3}{R_1})\frac{\partial A_2}{\partial\alpha_1} \qquad (5.3.46), (5.3.47)$$

These relations are named after Codazzi

Substituting them in Eq.(5.3.45), we get

$$\varepsilon_{11} = \frac{1}{A_1(1+\alpha_3/R_1)}\left(\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} + U_3\frac{A_1}{R_1}\right) \qquad (5.3.48)$$

Similarly,

$$\varepsilon_{22} = \frac{1}{A_2(1+\alpha_3/R_2)}\left(\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + U_3\frac{A_2}{R_2}\right) \qquad (5.3.49)$$

$$\varepsilon_{33} = \frac{\partial U_3}{\partial \alpha_3} \qquad (5.3.50)$$

Substituting Eqs.(5.3.21),(5.3.5), and (5.3.1) to (5.3.3) in Eq.(5.3.44) gives, for

instance for $i=1, j=2,$

$$\varepsilon_{12} = \frac{A_1(1+\alpha_3/R_1)}{A_2(1+\alpha_3/R_2)}\frac{\partial}{\partial\alpha_2}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right) + \frac{A_2(1+\alpha_3/R_2)}{A_1(1+\alpha_3/R_1)}\frac{\partial}{\partial\alpha_1}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right) \tag{5.3.51}$$

Similarly,

$$\varepsilon_{13} = A_1(1+\alpha_3/R_1)\frac{\partial}{\partial\alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right) + \frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_3}{\partial\alpha_1} \tag{5.3.52}$$

$$\varepsilon_{23} = A_2(1+\alpha_3/R_2)\frac{\partial}{\partial\alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right) + \frac{1}{A_2(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial\alpha_2} \tag{5.3.53}$$

2.4 Love Simplifications

If the shell is thin, we we may assume that the displacement in the α_1 and α_2 directions vary linearly through the shell thickness, whereas displacements in the α_3 direction are independent of α_3 :

$$U_1(\alpha_1,\alpha_2,\alpha_3) = u_1(\alpha_1,\alpha_2) + \alpha_3\beta_1(\alpha_1,\alpha_2) \tag{5.4.1}$$

$$U_2(\alpha_1,\alpha_2,\alpha_3) = u_2(\alpha_1,\alpha_2) + \alpha_3\beta_2(\alpha_1,\alpha_2) \tag{5.4.2}$$

$$U_3(\alpha_1,\alpha_2,\alpha_3) = u_3(\alpha_1,\alpha_2) \tag{5.4.3}$$

where β_1 and β_2 represent angles. If we assume that we may neglect shear

deflection, which implies that the normal shear strains are zero,

$$\varepsilon_{13} = 0 \qquad (5.4.4)$$

$$\varepsilon_{23} = 0 \qquad (5.4.5)$$

We obtain,for example, a definition of β_1 from Eq.(2.3.52),

$$0 = A_1(1+\alpha_3/R_1)\frac{\partial}{\partial\alpha_3}\left(\frac{u_1+\alpha_3\beta_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial u_3}{\partial\alpha_1} = \beta_1 - \frac{u_1}{R_1}+\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1} \qquad (5.4.6)$$

or

$$\beta_1 = \frac{u_1}{R_1} - \frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}$$

(2.4.7)

Similarly, we get

$$\beta_2 = \frac{u_2}{R_2} - \frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} \qquad (5.4.8)$$

Substituting Eqs.(2.4.1) to (2.4.3) into Eqs.(2.3.48) to (2.3.51), recognizing that

$$\frac{\alpha_3}{R_1}, \frac{\alpha_3}{R_2} \ll 1 \qquad (5.4.9)$$

we get

$$\varepsilon_{11} = \frac{1}{A_1}\frac{\partial}{\partial \alpha_1}(u_1 + \alpha_3\beta_1) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}(u_2 + \alpha_3\beta_2) + \frac{u_3}{R_1} \qquad (5.4.10)$$

$$\varepsilon_{22} = \frac{1}{A_2}\frac{\partial}{\partial \alpha_2}(u_2 + \alpha_3 \beta_2) + \frac{1}{A_2 A_1}\frac{\partial A_2}{\partial \alpha_1}(u_1 + \alpha_3 \beta_1) + \frac{u_3}{R_2} \tag{5.4.11}$$

$$\varepsilon_{33} = 0 \tag{5.4.12}$$

$$\varepsilon_{13} = 0 \tag{5.4.13}$$

$$\varepsilon_{23} = 0 \tag{5.4.14}$$

$$\varepsilon_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2 + \alpha_3 \beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}(u_1 + \alpha_3 \beta_1) \tag{5.4.15}$$

It is convenient to express Eqs.(5.4.10) to (5.4.15) in a form where membrane strains (independent of α_3) and bending strains (proportional to α_3) are separated:

$$\varepsilon_{11} = \varepsilon_{11}^{o} + \alpha_3 \kappa_{11} \tag{5.4.16}$$

$$\varepsilon_{22} = \varepsilon_{22}^{o} + \alpha_3 \kappa_{22} \tag{5.4.17}$$

$$\varepsilon_{12} = \varepsilon_{12}^{o} + \alpha_3 \kappa_{12} \tag{5.4.18}$$

where the membrane strains are

$$\varepsilon_{11}^{o} = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1} \tag{5.4.19}$$

$$\varepsilon_{22}^{o} = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2} \tag{5.4.20}$$

$$\varepsilon_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) \tag{5.4.21}$$

and where the change-in-curvature terms(bending strains) are

$$\varepsilon_{11} = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} \tag{5.4.22}$$

$$\varepsilon_{22} = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \tag{5.4.23}$$

$$\varepsilon_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta_1}{A_1}\right) \tag{5.4.24}$$

The displacement relations of Eqs.(5.4.1) and (5.4.7) are illustrated in Fig. 5.4.1.

2.5 Membrane forces and Bending Moments

In the following, we integrate all stresses acting on a shell element whose dimensions

are infinitesimal in the α_1 and α_2 directions and equal to the shell thickness in the normal direction. Solving Eqs.(5.2.10) to (5.2.12) for stresses yields

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}(\varepsilon_{\alpha\beta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\varepsilon_{\varsigma\varsigma}) : plane.stress.problem! \qquad (5.5.1) \sim (5.5.3)$$

Substituting Eqs.(5.4.16) to (5.4.18) gives

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}[(\varepsilon^{\circ}{}_{\alpha\beta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\varepsilon^{\circ}{}_{\varsigma\varsigma}) + \alpha_3(\kappa_{\alpha\beta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\kappa_{\varsigma\varsigma})] \qquad (5.5.4) \sim (5.5.6)$$

For instance, referring to Fig.2.5.1, the force in the α_1 direction acting on a strip of the element face of height $d\alpha_3$ and the width

$A_2(1+\frac{\alpha_3}{R_2})d\alpha_2$ is $\sigma_{11}A_2(1+\frac{\alpha_3}{R_2})d\alpha_2 d\alpha_3$

Thus the total force acting on the element in the α_1 direction is

$$\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{11} A_2 (1+\frac{\alpha_3}{R_2}) d\alpha_3$$

And the force per unit length of neutral surface $A_2 d\alpha_2$ is

$$N_{11} = \int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{11} \ (1+\frac{\alpha_3}{R_2}) d\alpha_3 \qquad (5.5.7)$$

Neglecting the second term in parentheses, we obtain ~

$$N_{\alpha\beta} = \int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{\alpha\beta} \ d\alpha_3 \qquad (5.5.8)$$

$$M_{\alpha\beta} = \int_{\alpha_3=-h/2}^{\alpha_3=h/2} \alpha_3 \sigma_{\alpha\beta} \ d\alpha_3$$

Substituting Eq.(5.5.4)~(5.5.6) results in: !!

$$N_{\alpha\beta}, M_{\alpha\beta} = ?$$

$$Q_{\alpha 3} = ?$$

$$\sigma_{\alpha\beta} = \frac{N_{\alpha\beta}}{h} + 12\frac{M_{\alpha\beta}}{h^3}\alpha_3$$

2.6 Energy expressions

The strain energy stored in one infinitesimal element that is acted on by stresses σ_{ij} is

$$dU = \frac{1}{2}(\sigma_{11}\varepsilon_{11} + \ldots\ldots + \sigma_{33}\varepsilon_{33})dV \qquad (5.6.1)$$

The last term is neglected in line with assumption (5.4.3). We do, however, have to keep the transverse shear terms, even though we have previously

assumed ε_{13} and ε_{23} to be negligible, to obtain expressions for β_1 and β_2.

The infinitesimal volume is given by

$$dV = \frac{1}{2} A_1 A_2 (1 + \tfrac{\alpha_3}{R_1})(1 + \tfrac{\alpha_3}{R_2}) d\alpha_1 d\alpha_2 d\alpha_3 \qquad (5.6.2)$$

Integrating Eq. (2.6.1) over the volume of the shell gives

$$U = \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} F \, dV \qquad (5.6.3)$$

where

$$F = \frac{1}{2} (\sigma_{11} \varepsilon_{11} + \ldots\ldots + \sigma_{23} \varepsilon_{23}) : ..delet..\sigma_{33} \varepsilon_{33} ...term \qquad (5.6.4)$$

The kinetic energy of one infinitesimal element is given by

$$dK = \frac{1}{2}\rho(\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2)dV \tag{5.6.5}$$

The dot indicates a time derivative.

Substituting Eqs. (2.4.1) to (2.4.3) and considering all the elements of the shell gives

$$K = \frac{\rho}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2 + \alpha_3^2(\dot{\beta}_1^2 + \dot{\beta}_2^2) + 2\alpha_3(\dot{u}_1\dot{\beta}_1 + \dot{u}_2\dot{\beta}_2)]$$

$$\times A_1 A_2 (1 + \frac{\alpha_3}{R_1})(1 + \frac{\alpha_3}{R_2})d\alpha_1 d\alpha_2 d\alpha_3 \tag{5.6.6}$$

Neglecting the α_3/R_1 and α_3/R_2 terms, we integrate over the thickness of the shell($\alpha_3=-h/2$ to $\alpha_3=h/2$). This gives

$$K=\frac{\rho h}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}[\dot{u}_1^2+\dot{u}_2^2+\dot{u}_3^2+\frac{h^2}{12}(\dot{\beta}_1^2+\dot{\beta}_2^2)]A_1A_2d\alpha_1d\alpha_2 \qquad (5.6.7)$$

The variation of energy put into the shell by possible applied boundary force resultants and moment resultants is, along typical $\alpha_2=\text{constant}$ and $\alpha_1=\text{constant lines}$,

$$\delta E_B=\int_{\alpha_1}(\delta u_2N_{22}^*+\delta u_1N_{21}^*+\delta u_3Q_{23}^*+\delta\beta_2M_{22}^*+\delta\beta_1M_{21}^*)A_1d\alpha_1+$$
$$\int_{\alpha_2}(\delta u_1N_{11}^*+\delta u_2N_{12}^*+\delta u_3Q_{13}^*+\delta\beta_1M_{11}^*+\delta\beta_2M_{12}^*)A_2d\alpha_2 \qquad (5.6.8)$$

Taking, for example, the $\alpha_2 = \text{constant edge}$, N_{22}^* is the boundary force normal to the boundary in the tangent plane to the neutral surface. The units are newtons per meter. Q_{23}^* is a shear force acting on the boundary normal to the shell surface, and N_{21}^* is a shear force acting along the boundary in the tangent plane. M_{22}^* is a moment in the α_2 direction, and M_{21}^* is a twisting moment in the α_1 direction. (Figure 2.6.1 illustrates this.)

The variation of energy introduced into the shell by distributed load components in the α_1, α_2 and α_3 directions, namely q_1, q_2 and $q_3 (N/m^2)$ is

$$\delta E_L = \int_{\alpha_1} \int_{\alpha_2} (q_1 \delta u_1 + q_2 \delta u_2 + q_3 \delta u_3) A_1 A_2 d\alpha_1 d\alpha_2 \qquad (5.6.9)$$

All loads are assumed to act on the neutral surface of the shell. The components are shown in Fig.2.6.2

2.7 Love's Equations by way of Hamilton's Principle

Hamilton's principle is given as [note the discussion in Sec. 2.9 and that Eq. (2.9.13) is multiplied here by -1 for convenience]

$$\delta \int_{t_0}^{t_1} (U - K - W_{in}) dt = 0 \tag{5.7.1}$$

where W_{in} is the total input energy. In our case

$$W_{in} = E_B + E_L \tag{5.7.2}$$

The times t_1 and t_0 are arbitrary, except that at $t = t_1$ and $t = t_0$, all variations are zero. The symbol δ is the variational symbol and is treated mathematically like a differential symbol. Variational displacements are arbitrary.

Substituting Eq. (5.7.2) for Eq. (5.7.1) and taking the variational operator

inside the integral, we obtain

$$\int_{t_0}^{t_1} (\delta U - \delta E_B - \delta E_L - \delta K) dt = 0 \qquad (5.7.3)$$

Let us examine these varations one by one. First, from Eq. (5.6.6)

$$\int_{t_0}^{t_1} \delta K dt =$$

$$\rho h \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} [\dot{u}_1 \delta \dot{u}_1 + \dot{u}_2 \delta \dot{u}_2 + \dot{u}_3 \delta \dot{u}_3 + \frac{h^2}{12} (\dot{\beta}_1 \delta \dot{\beta}_1 + \dot{\beta}_2 \delta \dot{\beta}_2)] A_1 A_2 d\alpha_1 d\alpha_2 dt \qquad (5.7.4)$$

Integrating by parts, for instance, the first term becomes

$$\int_{t_o}^{t_1} \dot{u}_1 \delta \dot{u}_1 dt = [\dot{u}_1 \delta u_1]_{t_o}^{t_1} - \int_{t_o}^{t_1} \ddot{u}_1 \delta u_1 dt \qquad (5.7.5)$$

Since the virtual displacement is zero at $t = t_0$ and $t = t_1$, we are left with

$$\int_{t_o}^{t_1} \dot{u}_1 \, \delta \dot{u}_1 \, dt = -\int_{t_o}^{t_1} \ddot{u}_1 \, \delta u_1 \, dt \tag{5.7.6}$$

Proceeding similarly with the other terms in the integral, we get

$$\int_{t_0}^{t_1} \delta K dt = -\rho h \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} (\ddot{u}_1 \, \delta u_1 + \ldots\ldots\ldots\ldots) A_1 A_2 d\alpha_1 d\alpha_2 dt \tag{5.7.7}$$

As in the classical Bernoulli-Euler beam theory, we neglect the influence of rotatory inertia, which we recognize as the terms involving $\ddot{\beta}_1$ and $\ddot{\beta}_2$. It will be shown later that rotatory inertia has to be considered only for very short wavelengths of vibration, and even then shear deformation is a more important effect.

$$\int_{t_0}^{t_1} \delta K dt = -\rho h \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} (\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{u}_3 \delta u_3) A_1 A_2 d\alpha_1 d\alpha_2 dt \quad (5.7.8)$$

Next, let us evaluate the variation of the energy due to the load. From Eq. (5.6.9),

$$\int_{t_0}^{t_1} \delta E_L dt = -\rho h \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} (q_1 \delta u_1 + q_2 \delta u_2 + q_3 \delta u_3) A_1 A_2 d\alpha_1 d\alpha_2 dt \quad (5.7.9)$$

The integral of the variation of boundary energy is, using Eq. (5.6.8),

$$\int_{t_0}^{t_1} \delta E_B = \int_{t0}^{t1}\int_{\alpha_1} (N_{22}^{*}\delta u_2 + N_{21}^{*}\delta u_1 + Q_{23}^{*}\delta u_3 + M_{22}^{*}\delta\beta_2 + M_{21}^{*}\delta\beta_1)A_1 d\alpha_1 dt +$$

$$\int_{t0}^{t1}\int_{\alpha_1} (N_{11}^{*}\delta u_1 + N_{12}^{*}\delta u_2 + Q_{13}^{*}\delta u_3 + M_{11}^{*}\delta\beta_1 + M_{12}^{*}\delta\beta_2)A_2 d\alpha_2 dt$$

(5.7.10)

It is more complicated to evaluate the integral of the variation in strain energy. Starting with Eq. (5.6.3), we have

$$\int_{t_0}^{t_1} \delta U dt = -\rho h \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha 3} \delta F dV dt$$

(5.7.11)

where

$$\delta F = \partial F,_{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} .....(i, j \neq 3)$$

(5.7.12)

Examining the first term of this equation, we see that

$$\delta F = \partial F,_{\partial \varepsilon_{11}} \delta \varepsilon_{11} = \frac{1}{2}\left(\frac{\partial \sigma_{11}}{\partial \varepsilon_{11}} \varepsilon_{11} + \sigma_{11} + \varepsilon_{22} \frac{\partial \sigma_{22}}{\partial \varepsilon_{11}} \right) \delta \varepsilon_{11}$$

(5.7.13)

Substituting Eqs. (5.5.1) and (5.5.2) gives

$$F,_{\varepsilon_{11}} \delta \varepsilon_{11} = \sigma_{11} \delta \varepsilon_{11}$$

(5.7.14)

Thus we can show that

$$\int_{t_0}^{t_1} \delta U dt = \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha 3} \sigma_{ij} \delta \varepsilon_{ij} A_1 A_2 (1+\frac{\alpha_3}{R_1})(1+\frac{\alpha_3}{R_2}) d\alpha_1 d\alpha_2 d\alpha_3 dt$$

$$(i, j \neq 3,3) \qquad (5.7.15)$$

We neglect the α_3 / R_1 and α_3 / R_2 terms as small. Substituting Eqs. (5.4.10) to (5.4.15) allows us to express the strain variations in terms of displacement variations. Integration with respect to α_3 allows us to introduce force resultants and moment resultants. Integration by parts will put the integral into a manageable form. Let us illustrate all this on the first term of Eq. (5.7.15):

$$\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha 3}\sigma_{11}\delta\varepsilon_{11}A_1A_2 d\alpha_1 d\alpha_2 d\alpha_3 dt$$

$$(i, j \neq 3,3)$$

$$=\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha 3}\Big[\ \sigma_{11}(A_2\delta u_{1,\alpha_1}+\delta u_2 A_{1,\alpha 2}+\frac{A_1A_2}{R_1}\delta u_3)$$

$$+\alpha_3\sigma_{11}(A_2\delta\beta_{1,\alpha_1}+\delta\beta_2 A_{1,\alpha 2})\Big]d\alpha_1 d\alpha_2 d\alpha_3 dt$$

$$=\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha 3}\Big[\ N_{11}(A_2\delta u_{1,\alpha_1}+\delta u_2 A_{1,\alpha 2}+\frac{A_1A_2}{R_1}\delta u_3)$$

$$+\alpha_3 M_{11}(A_2\delta\beta_{1,\alpha_1}+\delta\beta_2 A_{1,\alpha 2})\Big]d\alpha_1 d\alpha_2 dt \qquad (5.7.16)$$

Now we illustrate the integration by parts on the first term of Eq. (5.7.16):

$$\int_{\alpha_1} \int_{\alpha_2} N_{11} A_2 \delta u_{1,\alpha_1} d\alpha_1 d\alpha_2 =$$

$$= \int_{\alpha_2} N_{11} A_2 \delta u_1 d\alpha_2 - \int_{\alpha_1} \int_{\alpha_2} (N_{11} A_2)_{,\alpha_1} \delta u_1 d\alpha_1 d\alpha_2 \tag{5.7.17}$$

Proceeding with all terms of Eq. (5.7.15) in this fashion we get

$$\int_{t_0}^{t_1} \delta U dt =$$

$$\int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} [(-(N_{11}A_2)_{,\alpha_1} - (N_{21}A_1)_{,\alpha_2} - N_{12}A_{1,\alpha_2} + N_{22}A_{2,\alpha_1} - Q_{13}\frac{A_1A_2}{R_1})\delta u_1$$

$$+(-(N_{12}A_2)_{,\alpha_1} - (N_{22}A_1)_{,\alpha_2} + N_{11}A_{1,\alpha_2} - N_{21}A_{2,\alpha_1} - Q_{23}\frac{A_1A_2}{R_2})\delta u_2$$

$$+(N_{11}\frac{A_1A_2}{R_1} + N_{22}\frac{A_1A_2}{R_2} - (Q_{13}A_2)_{,\alpha_1} - (Q_{22}A_1)_{,\alpha_2})\delta u_3$$

$$+(-(M_{21}A_1)_{,\alpha_2} - M_{12}A_{1,\alpha_2} + M_{22}A_{2,\alpha_1} - (M_{11}A_2)_{,\alpha_1} - Q_{13}A_1A_2)\delta\beta_1$$

$$+(-(M_{12}A_2)_{,\alpha_1} - (M_{22}A_1)_{,\alpha_2} - M_{21}A_{2,\alpha_1} - M_{11}A_{2,\alpha_2} - Q_{23}A_1A_2)\delta\beta_2]d\alpha_1 d\alpha_2 dt$$

$$+\int_{t_o}^{t_1} \int_{\alpha_2} [(N_{11}\delta u_1 + M_{11}\delta\beta_1 + N_{12}\delta u_2 + M_{22}\delta\beta_2 + Q_{13}\delta u_3)A_2 d\alpha_2 dt + \int_{t_o}^{t_1} \int_{\alpha_1} (...)A_2 d\alpha_2 dt$$

(5.7.18)

We are now ready to substitute Eqs. (5.7.18), (5.7.10), (5.7.9) and (5.7.8) in EQ. (5.7.3). This gives

$$\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\Big\{\ [Eq_1+(q_1+\rho h\ddot{u}_1)A_1A_2]\delta u_1+(Eq_2+(q_2+\rho h\ddot{u}_2)A_1A_2]\delta u_2$$

$$+(Eq_3+(q_3+\rho h\ddot{u}_3)A_1A_2)\delta u_3+(...)\delta\beta_1+(...)\delta\beta_2\ \Big\}d\alpha_1 d\alpha_2 dt$$

$$+\int_{t_o}^{t_1}\int_{\alpha_1}(..BC1..)A_1 d\alpha_1 dt+\int_{t_o}^{t_1}\int_{\alpha_2}(..BC2.)A_2 d\alpha_2 dt=0$$

(5.7.19)

The equation can be satisfied only if each of the triple and double intergral parts is zero individually. Moreover, since the variational displacements are arbitrary, each integral equation can be satisfied only if the coefficients of the variational displacements are zero. Thus the coefficients of the triple integral set to zero give the following five equations:

Thus, we can obtain (5.7.20)~(5.7.24)

5.8 Boundary Conditions

Examining Love's equations, the stress- strain and strain- displacement relations, we see that the equations are eighth- order partial differential equations in space. This means that we can accommodate at most four boundary conditions on each edge.

However, when set to zero, the two line integrals of Eq. (5.7.19) are satisfied only if the five coefficients in each are zero or if the virtual displacements are zero. This would define as necessary five boundary conditions. A similar problem was encountered by Kirchhoff [2.3] in the nine-teenth century, when he investigated the boundary conditions of a plate.

It appeared as if three conditions at each edge were needed, but the fourth-order equation would allow only two. Kirchhoff combined the three

conditions into two by noting that the twisting moment and shear boundary conditions were related.

Following Kirchhoff's lead, the two line integrals are rewritten utilizing the definitions of Eqs. (5.4.7) and (5.4.8). For instance, for the first line integral equation, we get

$$\int_{t_o}^{t_1} \int_{\alpha_1} \{(N_{22}^* - N_{22})\delta u_1 + (N_{21}^* - N_{21})\delta u_1 + (Q_{23}^* - Q_{23})\delta u_3$$
$$+(M_{22}^* - M_{22})\delta\beta_2 + (M_{21}^* - M_{21})[\frac{\delta u_1}{R_1} - \frac{1}{A_1}\frac{\partial}{\partial \alpha_1}(\delta u_3)]\ \}A_1 d\alpha_1 dt = 0$$

(5.8.1)

Before collecting coefficients of δu_3, we have to perform an integration by parts:

$$\int_{t_o}^{t_1} \int_{\alpha_1} \left\{ (M_{21}^* - M_{21}) \frac{\partial}{\partial \alpha_1} (\delta u_3) d\alpha_1 dt \right.$$

$$= \left| \int_{t_o}^{t_1} (M_{21}^* - M_{21}) \delta u_3 dt \right|_{\alpha_1} - \int_{t_o}^{t_1} \int_{\alpha_1} \frac{\partial}{\partial \alpha_1} (M_{21}^* - M_{21}) d\alpha_1 \delta u_3 dt = 0$$

(5.8.2)

Since $M_{21} = M^*_{21}$ along the entire edge, the term in parentheses is zero. Thus substituting Eq. (5.8.2) in Eq. (5.8.1) and collecting coefficients of virtual displacements yields

$$\int_{t_o}^{t_1}\int_{\alpha_1}\{(N_{22}^*-N_{22})\delta u_2+[(N_{21}^*+\frac{M_{21}^*}{R_1})-(N_{21}+\frac{M_{21}}{R_1})]\delta u_1$$

$$+(M_{22}^*-M_{22})\delta\beta_2+[(Q_{23}^*+\frac{1}{A_1}\frac{\partial M_{21}^*}{\partial\alpha_1})-(Q_{23}+\frac{1}{A_1}\frac{\partial M_{21}}{\partial\alpha_1})]\delta u_3\}$$

$$-A_1 d\alpha_1 dt=0 \tag{5.8.3}$$

Similarly, for the second line integral, we get

$$\int_{t_o}^{t_1}\int_{\alpha_2}\left\{(N_{11}^*-N_{11})\delta u_1+[(N_{12}^*+\frac{M_{12}^*}{R_2})-(N_{12}+\frac{M_{12}}{R_2})]\delta u_2\right.$$

$$+(M_{11}^*-M_{11})\delta\beta_1+[(Q_{13}^*+\frac{1}{A_2}\frac{\partial M_{12}^*}{\partial\alpha_2})-(Q_{13}+\frac{1}{A_2}\frac{\partial M_{12}}{\partial\alpha_2})]\delta u_3\}$$

$$-A_2 d\alpha_2 dt=0$$

(5.8.4)

These equations are satisfied if either the virtual displacements vanish or the coefficients of the virtual displacements vanish. Defining, in memory of Kirchhoff, the Kirchhoff effective shear stress resultants of the first kind

$$V_{13}=Q_{13}+\frac{1}{A_2}\frac{\partial M_{12}}{\partial\alpha_2} \quad \text{and} \quad V_{23}=Q_{23}+\frac{1}{A_1}\frac{\partial M_{21}}{\partial\alpha_1}$$

(5.8.5, 5.8.6)

and as the kirchhoff effective shear stress resultants of the second kind

$$T_{12}=N_{12}+\frac{M_{12}}{R_2} \qquad \text{and} \qquad T_{21}=N_{21}+\frac{M_{21}}{R_1} \qquad (5.8.7,\ 5.8.8)$$

We may write the integrals as

$$\int_{t_o}^{t_1}\int_{\alpha_1}\Big[\ (N_{22}^{*}-N_{22})\delta u_2+(T_{21}^{*}-T_{21})\delta u_2$$
$$+(M_{22}^{*}-M_{22})\delta\beta_2+(V_{23}^{*}-V_{23})\delta u_3\Big]A_1 d\alpha_1 dt=0 \qquad (5.8.9)$$

and

$$\int_{t_o}^{t_1} \int_{\alpha_2} \Big[\; (N_{11}^* - N_{11})\delta u_1 + (T_{12}^* - T_{12})\delta u_2$$

$$+ (M_{11}^* - M_{11})\delta\beta_1 + (V_{13}^* - V_{13})\delta u_3 \Big] A_2 \, d\alpha_2 \, dt = 0 \qquad (5.8.10)$$

Now we may argue that each of these integrals can be satisfied only if the coefficients of the virtual displacements, the virtual displacements, or one of the two for each term are zero. Since virtual displacements are zero only when the boundary displacements are prescribed, this translates into following possible boundary conditions for an $\alpha_1 = \text{constant edge}$ [Eq.(5.8.10)]:

$$N_{11}=N_{11}^{*}..or..u_1=u_1^{*}$$
$$M_{11}=M_{11}^{*}..or..\beta_1=\beta_1^{*}$$
$$V_{13}=M_{13}^{*}..or..u_3=u_3^{*}$$
$$T_{12}=M_{12}^{*}..or..u_2=u_2^{*}$$

(5.8.11)~(5.8.14)

This states the intuitively obvious fact that we have to prescribe at a boundary either forces(moments) or displacements (angular displacements). However, four conditions have to be identified per edge. In a later chapter we see that under certain simplifying assumptions we may reduce this number even further. Similarly, examining Eq. (5.8.3), which describes an $\alpha_2 = \text{constant edge}$, the four boundary conditions have to be

$$N_{22}=N_{22}^{*}..or..u_2=u_2^{*}$$
$$M_{22}=M_{22}^{*}..or..\beta_2=\beta_2^{*}$$
$$V_{23}=V_{23}^{*}..or..u_3=u_3^{*}$$
$$T_{21}=V_{21}^{*}..or..u_1=u_1^{*}$$

(5.8.15)~(5.8.18)

We can therefore state in general that if n denotes the subscript that defines the normal direction to the edge and if t denotes the subscript that defines the tangential direction to the edge, the necessary boundary conditions are

$$N_{nn}=N^*_{nn}..or..u_n=u^*_n$$
$$M_{nn}=M^*_{nn}..or..\beta_n=\beta^*_n$$
$$V_{n3}=V^*_{n3}..or..u_3=u^*_3$$
$$T_{nt}=V^*_{nt}..or..u_t=u^*_t$$

(5.8.19)~(5.8.22)

Let us consider a few examples. First there is the case where the edge is completely free. This means that no forces of moments act on this edge

$$N_{nn}=0\ ,\quad M_{n\bar{n}}=0\ ,\quad V_{3n}=0\ ,\quad T_{n\bar{t}}=0$$

(5.8.23)~ (5.8.26)

The other extreme is the case where the edge is completely prevented from deflecting by being clamped

$$u_n=0,\quad u_t=0,\quad u_3=0,\quad \beta_n=0$$

(5.8.27)~(5.8.30)

If the edge is supported on knife edges such that it is free to rotate in normal direction but is prevented from having any transverse deflection,

clearly the two conditions

$$u_3 = 0, \quad M_{nn} = 0 \qquad (5.8.31)\sim(5.8.32)$$

apply. If the knife edges are such that the shell is free to slide between them, the other two conditions are

$$N_{nn} = 0, \quad T_{nt} = 0 \qquad (5.8.33)\sim(5.8.34)$$

If the shell is somehow prevented from sliding, the conditions

$$u_n = 0, \quad u_t = 0 \qquad (5.8.35)\sim(5.8.36)$$

should be used.

5.9 Hamilton's Principle

Hamilton's principle is a minimization principle that seems to apply to all of mechanics and most classical physics. It is the end of a development that started in the second century B.C with Hero of Alexandria, who stated that light always takes the shortest path. This indeed governs reflections, by the minimum principle that includes refraction was not found until Fermat in 1657 postulated that light travels from point to point in the shortest time. On theological grounds, Maupertius in 1747 asserted that dynamical motion takes place with minimum action, where action is defined as the product of distance and momentum, or energy and time. Lagrange formulated the mathematical foundation of this principle in 1760. In 1828, Gauss formulated the principle of least constraint, which was extended later by Hertz when formulating the principle of least curvature. Finally, in 1834, Hamilton announced his general principle, which included all the others. He postulated that while there are usually several possible paths along which a dynamic system may move from one point to

another in space and time, the path actually followed is the one that minimizes the time integral of the difference between the kinetic and potential energies. In terms of the calculus of variations, developed primarily by Euler and Bernoulli in the eighteenth century, it is usually stated as

$$\delta\int_{t_0}^{t_1} (T-U+W_{nc})dt=0 \qquad \delta\overline{r_i} = 0, \tag{5.9.1}$$

where $\delta\overline{r_i}$ are the variations of displacements(virtual displacements,) T the kinetic energy, U the strain energy, W_{nc} any additional energy input to the system, and δ the variation, operationally equivalent to a total differential. In general, Hamilton's principle can be viewed as an axiom, replacing the axiom of Newton's second law for dynamic problems. With other words, we either accept Newton's second law for dynamic problems. With other words, we either accept Newton's second law as an axiom and derive Hamilton's principle from it for dynamic problems, or we accept

Hamilton's principle as an axiom and derive Newton's second law from it. In the following, let us derive Hamiltion's principle from the axiom of Newton's second law, utilizing D'Alembert's principle for the restricted case of interest here, namely, the motion of masses under constraints. Let the virtual displacements $\delta x_1, \delta y_1, \delta z_{1,} \cdots, \delta x_n, \delta y_n, \delta z_n$ be infinitesimal, arbitrary changes in the displacement coordinates of a system. They must be compatible with the constraints of the system. If each mass particle $i = 1, \ldots, n,$ is acted on by forces with the resultant $\bar{F}_i$, it must be that

$$\sum_{i=1}^{n} (\bar{F}_i - \dot{\bar{p}}_i) \bullet \delta \bar{r}_i = 0 \tag{5.9.2}$$

where $\dot{\bar{p}}_i$ is the rate of change of the linear momentum vector $\bar{p}_i$ and $\delta \bar{r}_i = \delta x_i \bar{i} + \delta y_i \bar{j} + \delta z_i \bar{k}$. Since

$$\dot{\bar{p}}_i = m_i \ddot{\bar{r}}_i \tag{5.9.3}$$

Eq. (5.9.2) may be written as

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \delta \bar{r}_i = \delta W \tag{5.9.4}$$

where

$$\delta W = \sum_{i=1}^{n} \bar{F}_i \delta \bar{r}_i \tag{5.9.5}$$

and represents the virtual work due to the applied forces alone. Using the mathematical identify

$$\ddot{\bar{r}}_i \bullet \delta \bar{r}_i = \frac{d}{dt}(\dot{\bar{r}}_i \bullet \delta \bar{r}_i) - \delta(\frac{1}{2}\dot{\bar{r}}_i \bullet \dot{\bar{r}}_i)$$

(5.9.6)

gives, after multiplying it by m_i and summing over all particles,

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \bullet \delta \bar{r}_i = \sum_{i=1}^{n} m_i \frac{d}{dt}(\dot{\bar{r}}_i \bullet \delta \bar{r}_i) - \delta \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\bar{r}}_i \bullet \dot{\bar{r}}_i$$

(5.9.7)

Recognizing that the kinetic energy is

$$K=\sum_{i=1}^{n}\frac{1}{2}m_i\dot{\bar{r}}_i\bullet\dot{\bar{r}}_i \tag{5.9.8}$$

and that work done by the applied forces is equal to the input work minus what is stored in terms of strain energy,

$$W=W_{in}-U \tag{5.9.9}$$

we obtain, utilizing Eq. (5.9.4),

$$\delta(K-U+W_{in})dt=\sum_{i=1}^{n}m_i\frac{d}{dt}(\dot{\bar{r}}_i\bullet\delta\bar{r}_i) \tag{5.9.10}$$

Integrating between two points in time, t_1 and t_2, where the virtual

displacements or variations are zero, we obtain

$$\delta \int_{t_1}^{t_2} (K - U + W_{in}) dt = \sum_{i=1}^{n} \int_{t_1}^{t_2} m_i \frac{d}{dt}(\dot{\bar{r}}_i \bullet \delta \bar{r}_i) dt = \sum_{i=1}^{n} m_i \dot{\bar{r}}_i \bullet \delta \bar{r}_i \Big|_{t_1}^{t_2} \tag{5.9.11}$$

If we select $\delta \bar{r}_i$ such that

$$\delta \bar{r}_i = 0 \quad t = t_1, t_2 \tag{5.9.12}$$

The final result is

$$\delta \int_{t_1}^{t_2} (K - U + W_{in}) dt = 0 \quad : \quad \text{Hamilton Priciple !} \tag{5.9.13}$$