Mathematical Background in Aircraft Structural Mechanics

CHAPTER 1.
Linear Elasticity

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Basic equation of Linear Elasticity

- **Structural analysis … evaluation of deformations and stresses arising within a solid object under the action of applied loads**

  - if time is mot explicitly considered as an independent variable

    → the analysis is said to be static

    → otherwise, structural dynamic analysis or structural dynamics

- **Under the assumption of** { **Small deformation**, **Linearly elastic material behavior** }

  - Three dimensional formulation → a set of 15 linear 1st order PDE involving

    - displacement field (3 components)
    - stress field (6 components)
    - strain field (6 components)

  → simpler, 2-D formulations { plane stress problem, plane strain problem }

- **For most situations, not possible to develop analytical solutions**

  → analysis of structural components … bars, beams, plates, shells

1.1 The concept of Stress

1.1.1 The state of stress at a point

- **State of stress in a solid body… measure of intensity of forces acting within the solid**
  - distribution of forces and moments appearing on the surface of the cut …
    equipollent force $\underline{F}$, and couple $\underline{M}$
  - Newton's 3rd law → a force and couple of equal magnitudes and opposite directions
    acting on the two forces created by the cut

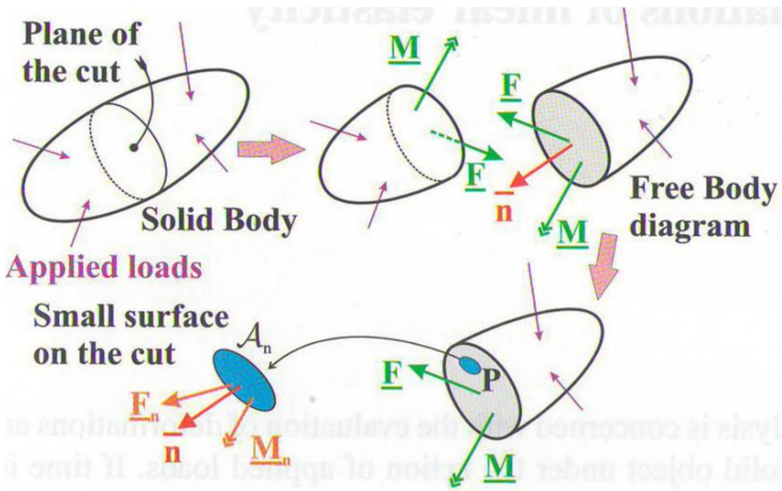

Fig. 1.1 A solid body cut by a plane to isolate a free body

1.1 The concept of Stress

- A small surface of area A_n located at point *P* on the surface generated by the cut
  → equipollent force $\underline{F}_n$, couple $\underline{M}_n$

  - limiting process of area → concept of "stress vector"

$$\underline{\tau}_n = \lim_{dA_n \to 0} \left(\frac{\underline{F}_n}{dA_n} \right) \tag{1.1}$$

  existence of limit : "fundamental assumption of continuum mechanics"

  - Couple $\underline{M}_n \to 0$ as $dA_n \to 0$
    … couple is the product of a differential element of force by a differential element of moment arm
    → negligible, second order differential quantity

  - Total force acting on a differential element of area, dA_n

$$\underline{F}_n = dA_n \underline{\tau}_n \tag{1.2}$$

  Unit : force per unit area, N/m^2 or Pa

1.1 The concept of Stress

- Surface orientation, as defined by the normal to the surface, is kept constant during the limiting process

  - Fig. 1.2 … Three different cut and the resulting stress vector
    first… solid is cut at point P by a plane normal to axis $\bar{i}_1$:
    differential element of surface with an area dA_1 , stress vector $\underline{\tau}_1$
    → No reason that those three stress vectors should be identical.

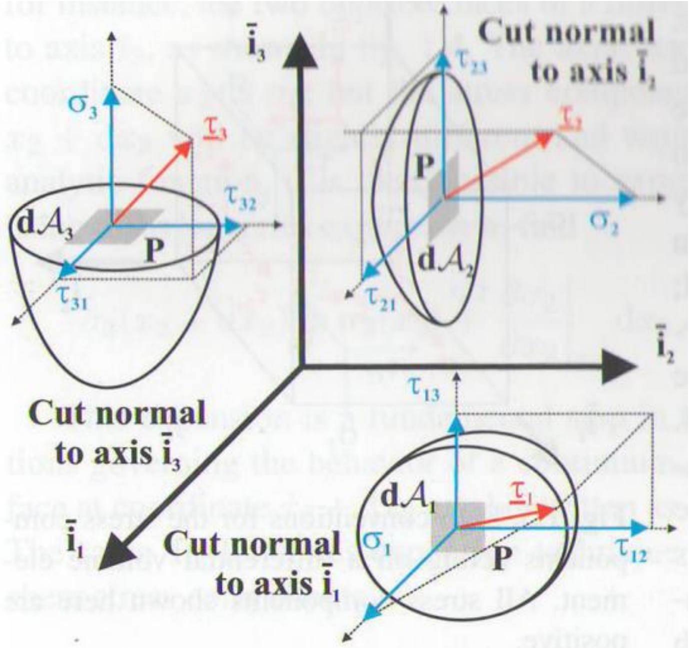

Fig. 1.2 A rigid body cut at point P by three planes orthogonal to the Cartesian axes.

1.1 The concept of Stress

- Component of each stress vectors acting on the three forces

$$\underline{\tau}_1 = \sigma_1 \bar{i}_1 + \tau_{12} \bar{i}_2 + \tau_{13} \bar{i}_3 \qquad (1.3a)$$

{ direct / normal } stress	{ shearing / shear } Stress

→ both act on the force normal to axis $\bar{i}_1$ in the direction of $\bar{i}_2$ and $\bar{i}_3$

→ "engineering stress components"
unit : force/area, Pa

"positive" force … the outward normal to the face,
i.e., the normal pointing away from the body,
is in the same direction as the axis
→ sign convention (Fig.1.3)

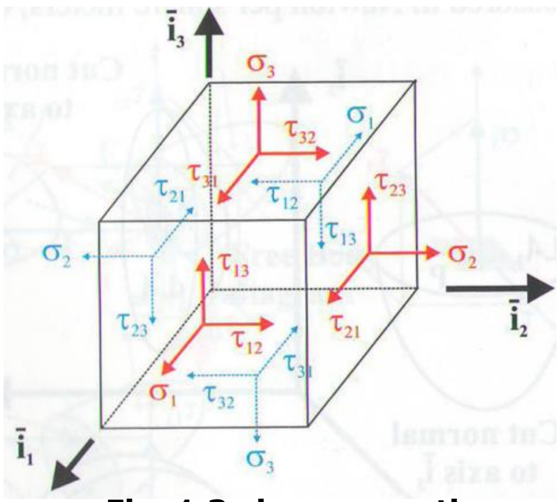

Fig. 1.3 sign conventions

- 9 components of stress components
→ fully characterize the state of stress at P

Force … vector quantity,
3 components of the force vector (1st order tensor)
Stress … 9 quantities (2nd order tensor)
→ Strain tensor
Bending stiffness of a beam
Mass moments of inertia

1.1.2 Volume equilibrium eqn.

❖ **Stress varies throughout a solid body**

- Fig. 1.4 axial stress component at the negative face : σ_2

  axial stress component at the positive face at coordinate $x_2 + dx_2 : \sigma_2(x_2 + dx_2)$

  if $\sigma_2(x_2)$ is an analytic function, using a Taylor series expansion

$$\sigma_2(x_2 + dx_2) = \sigma_2(x_2) + \left.\frac{\partial \sigma_2}{\partial x_2}\right|_{x_2} dx_2 + \ldots h.o.terms\ in\ dx_2$$

$$\sigma_2(x_2 + dx_2) = \sigma_2 + \frac{\partial \sigma_2}{\partial x_2} dx_2$$

- body forces $\underline{b}$ … gravity, inertial, electric, magnetic origin

$$\underline{b} = b_1 \bar{i}_1 + b_2 \bar{i}_2 + b_3 \bar{i}_3$$ → unit : force/volume, N / m^3

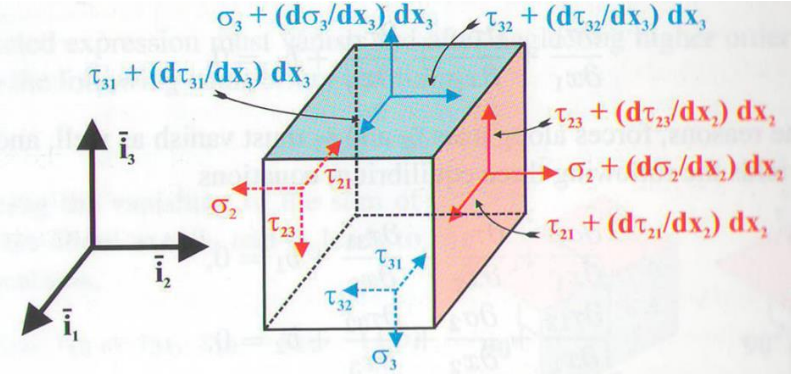

Fig. 1.4 Stress components acting on a differential element of volume.

1.1 The concept of Stress

i) Force equilibrium

direction of axis $\bar{i}_1$

$$\begin{cases} \dfrac{\partial \sigma_1}{\partial x_1} + \dfrac{\partial \tau_{21}}{\partial x_2} + \dfrac{\partial \tau_{31}}{\partial x_3} + b_1 = 0 \\ \cdots \\ \cdots \end{cases} \qquad (1.4a)$$

must be satisfied at all points inside the body

- equilibrium should be enforced on the DEFORMED configuration (strictly)

Unknown, unfortunately

"linear theory of elasticity"… assumption that the displacements of the body under the applied loads are very small, and hence the difference between deformed and undeformed is very small.

ii) Moment equilibrium

about axis $\bar{i}_1$ $\rightarrow \tau_{23} - \tau_{32} = 0$

→ "principle of reciprocity of shear stress" (Fig. 1.5)

- only 6 independent components in 9 stress components

→ symmetry of the stress tensor (1.6)

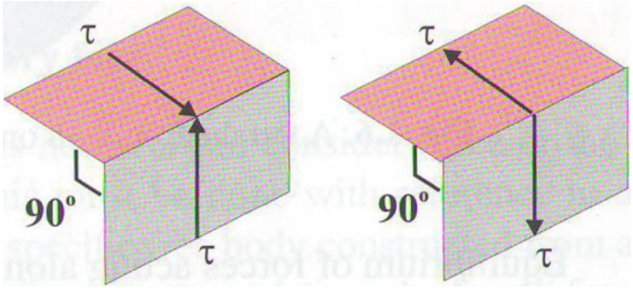

Fig. 1.5 Reciprocity of the shearing stresses acting on two orthogonal faces

1.1.3 Surface equilibrium eqn.

❖ **At the outer face of the body.**

$$\left\{\begin{array}{l}\text{stress acting}\\ \text{inside the body}\end{array}\right\} \xleftrightarrow{\text{equilibrium}} \left\{\begin{array}{l}\text{Externally applied}\\ \text{Surface tractions}\end{array}\right\} \underline{t}$$

$$\underline{t} = t_1\bar{i}_1 + t_2\bar{i} + t_3\bar{i}_3$$

➢ Fig. 1.6 … free body in the form of a differential tetrahedron bounded by

$$\left\{\begin{array}{l}\text{3 negative faces cut through the body in directions normal to axes } \bar{i}_1, \bar{i}_2, \bar{i}_3\\ \text{A fourth face, ABC, of area } dA_n\end{array}\right.$$

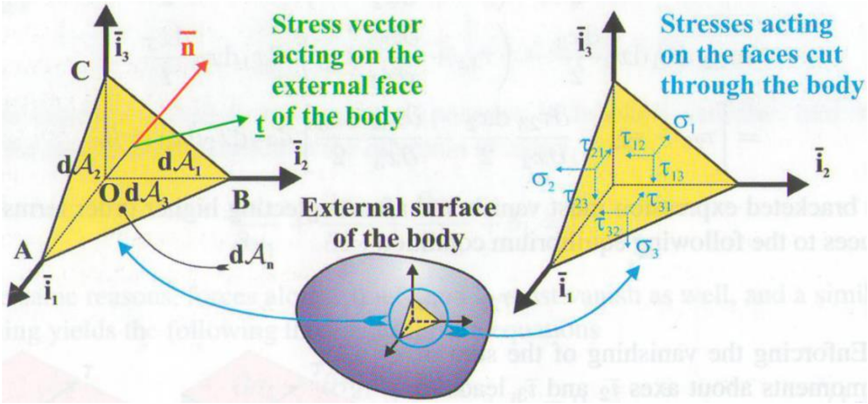

Fig. 1.6 A tetrahedron with one face along the outer surface of the body.

1.1 The concept of Stress

- Force equilibrium along $\bar{i}_1$, and dividing by dA_n

$$t_1 = \sigma_1 n_1 + \tau_{21} n_2 + \tau_{31} n_3 \qquad (1.9a)$$

body force term vanishes since it is a h.o. differential term.

A body is said to be in equilibrium if eqn (1.4) is satisfied at all points inside the body and eqn (1.9) is satisfied at all points of its external surface.

1.2 Analysis of the state of stress at a point

➢ It is fully defined once the stress components acting on three mutually orthogonal faces at a point are known.

1.2.1 Stress components acting on an arbitrary face

➢ Fig 1.7… "Cauchy's tetrahedron" with a fourth face normal to unit vector $\bar{n}$ of arbitrary orientation

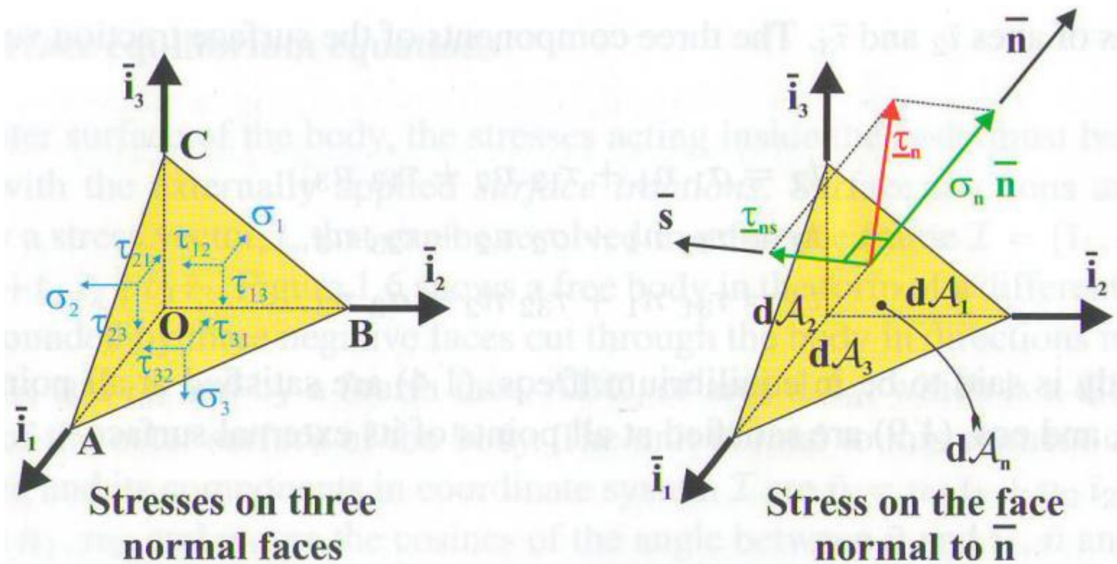

Fig. 1.6 Differential tetrahedron element with one face, ABC, normal to unit vector $\bar{n}$ and the other three faces normal to axes $\bar{i}_1, \bar{i}_2$ and $\bar{i}_3$, respectively.

1.2 Analysis of the state of stress at a point

- force equilibrium

$$\tau_1 dA_1 + \tau_2 dA_2 + \tau_3 dA_3 = \tau_n dA_n + \underline{b} dV$$

dividing by dA_n and neglecting the body force term (since it is multiplied by a h.o. term)

$$\underline{\tau}_n = \underline{\tau}_1 n_1 + \underline{\tau}_2 n_2 + \underline{\tau}_3 n_3$$

- Expanding the 3 stress vectors,

$$\underline{\tau}_n = (\sigma_1 \bar{i}_1 + \tau_{12} \bar{i}_2 + \tau_{13} \bar{i}_3) n_1 + (\tau_{21} \bar{i}_1 + \sigma_2 \bar{i}_2 + \tau_{23} \bar{i}_3) n_2 + (\tau_{31} \bar{i}_1 + \tau_{32} \bar{i}_2 + \sigma_3 \bar{i}_3) n_3 \tag{1.10}$$

- To determine the direct stress σ_n, project this vector eqn in the direction of $\bar{n}$

$$\bar{n} \cdot \underline{\tau}_n = (\sigma_1 n_1 + \tau_{12} n_2 + \tau_{13} n_3) n_1 + (\tau_{21} n_1 + \sigma_2 n_2 + \tau_{23} n_3) n_2 + (\tau_{31} n_1 + \tau_{32} n_2 + \sigma_3 n_3) n_3$$

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 + 2\tau_{23} n_2 n_3 + 2\tau_{13} n_1 n_3 + 2\tau_{12} n_1 n_2 \tag{1.11}$$

- stress component acting in the plane of face *ABC* : τ_{ns}

… by projecting Eq. (1.10) along vector $\bar{s}$

$$\tau_{ns} = \sigma_1 n_1 s_1 + \sigma_2 n_2 s_2 + \sigma_3 n_3 s_3 + \tau_{12}(n_2 s_1 + n_1 s_2) + \tau_{13}(n_1 s_3 + n_3 s_1) + \tau_{23}(n_2 s_3 + n_3 s_2) \tag{1.12}$$

1.2 Analysis of the state of stress at a point

Eqns (1.11), (1.12)… Once the stress components acting on 3 mutually orthogonal faces are known, the stress components on a face of arbitrary orientation can be readily computed.

- How much information is required to fully determine the state of stress at a point *P* of a solid?

Compute definition of the state of stress at a point only requires knowledge of the stress vectors, or equivalently of the stress tensor components, acting on three mutually orthogonal faces.

1.2.2 Principal stresses

- Is there a face orientation for which the stress vector is exactly normal to the face ? Does a particular orientation, $\bar{n}$ exist for which the stress vector acting on this face consists solely of $\underline{\tau}_n = \sigma_p \bar{n}$, where σ_p is the yet unknown?
  … projecting Eq.(1.10) along axes $\bar{i}_1, \bar{i}_2, \bar{i}_3$ → 3 scalar eqns

$$\begin{bmatrix} \sigma_1 - \sigma_p & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_2 - \sigma_p & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_3 - \sigma_p \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0 \qquad (1.13)$$

1.2 Analysis of the state of stress at a point

- Determinant of the system = 0, non-trivial sol. exists.

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0 \qquad (1.14)$$

"stress invariants" (1.15)

- Solution of Eq (1.14) … "principal stress"

  3 solutions $\sigma_{p_1}, \sigma_{p_2}, \sigma_{p_3}$ → non-trivial sol. for the direction cosines

  "principal stress direction"

  Homogeneous eqns → arbitrary constant → enforcing the normality condition

$$\bar{n}_1^2 + \bar{n}_2^2 + \bar{n}_3^2 = I$$

1.2.3 Rotation of stresses

➢ Arbitrary basis : $I^* = (\bar{i}_1^*, \bar{i}_2^*\ \bar{i}_3^*) \longrightarrow \begin{cases} \sigma_1^*, \sigma_2^*, \sigma_3^* \\ \tau_{23}^*, \tau_{13}^*, \tau_{12}^* \end{cases}$

- orientation of basis $\underline{J}^*$ relative to $\underline{J}$

  → matrix of direction cosine, or rotation matrix $\underline{\underline{R}}$ (A.36)

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\bar{i}_1^*, \bar{i}_1) & \cos(\bar{i}_2^*, \bar{i}_1) & \cos(\bar{i}_3^*, \bar{i}_1) \\ \cos(\bar{i}_1^*, \bar{i}_2) & \cos(\bar{i}_2^*, \bar{i}_2) & \cos(\bar{i}_3^*, \bar{i}_2) \\ \cos(\bar{i}_1^*, \bar{i}_3) & \cos(\bar{i}_2^*, \bar{i}_3) & \cos(\bar{i}_3^*, \bar{i}_3) \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & n_2 & m_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \qquad (A.36)$$

1.2 Analysis of the state of stress at a point

- Eq(1.11) $\rightarrow \sigma_1^*$ in terms of those resolved in axis $\underline{J}$

$$\bar{n} \cdot \underline{\tau}_n = (\sigma_1 n_1 + \tau_{12} n_2 + \tau_{13} n_3) n_1 + (\tau_{21} n_1 + \sigma_2 n_2 + \tau_{23} n_3) n_2 + (\tau_{31} n_1 + \tau_{32} n_2 + \sigma_3 n_3) n_3$$

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 + 2\tau_{23} n_2 n_3 + 2\tau_{13} n_1 n_3 + 2\tau_{12} n_1 n_2 \quad (1.11)$$

$$\sigma_1^* = \sigma_1 l_1^2 + \sigma_2 l_2^2 + \sigma_3 l_3^2 + 2\tau_{23} l_2 l_3 + 2\tau_{13} l_1 l_3 + 2\tau_{12} l_1 l_2 \quad (1.18)$$

l_1, l_2, l_3 : direction cosines of unit vector $\bar{i}_1^*$

Similar eqns for $\sigma_2^* \cdots m_1, m_2, m_3$

$\sigma_3^* \cdots n_1, n_2, n_3$

Shear component : Eq. (1.12) $\rightarrow$

$$\tau_{12}^* = \sigma_1 l_1 m_1 + \sigma_2 l_2 m_2 + \sigma_3 l_3 m_3 + \tau_{12}(l_2 m_1 + l_1 m_2) + \tau_{13}(l_3 m_1 + l_1 m_3) + \tau_{23}(l_2 m_3 + l_3 m_2) \quad (1.19)$$

Compact matrix eqn.

$$\begin{bmatrix} \sigma_1^* & \tau_{12}^* & \tau_{13}^* \\ \tau_{21}^* & \sigma_2^* & \tau_{23}^* \\ \tau_{31}^* & \tau_{32}^* & \sigma_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_2 & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_3 \end{bmatrix} \underline{\underline{R}} \quad (1.20)$$

1.2 Analysis of the state of stress at a point

- "Stress invariant". … invariant w.r.t. a change of coordinate system. (1.21)

$$I_1 = \sigma_1^* + \sigma_2^* + \sigma_3^* = \sigma_1 + \sigma_2 + \sigma_3, \qquad (1.21a)$$

$$I_2 = \sigma_1^*\sigma_2^* + \sigma_2^*\sigma_3^* + \sigma_3^*\sigma_1^* - \tau_{12}^{*2} - \tau_{13}^{*2} - \tau_{23}^{*2}$$

$$= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 - \tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2, \qquad (1.21b)$$

$$I_3 = \sigma_1^*\sigma_2^*\sigma_3^* - \sigma_1^*\tau_{23}^{*2} - \sigma_2^*\tau_{13}^{*2} - \sigma_3^*\tau_{12}^{*2} + 2\tau_{12}^{*2}\tau_{13}^{*2}\tau_{23}^{*2}$$

$$= \sigma_1\sigma_2\sigma_3 - \sigma_1\tau_{23}^2 - \sigma_2\tau_{13}^2 - \sigma_3\tau_{12}^2 + 2\tau_{12}\tau_{13}\tau_{23} \qquad (1.21c)$$

1.3 The state of plane stress

- All stress components acting along the direction of axis $\bar{i}_3$ are assumed to vanish or to be negligible. Only non-vanishing components : $\sigma_1,\ \sigma_2,\ \tau_{12}$

Independent of x_3

Vary thin plate or sheet subject to loads applied in its own plane (Fig. 1.11)

1.3 The state of plane stress

1.3.1 Equilibrium eqns

- Considerably simplified from the general, 3-D case → 2 remaining eqns

$$\frac{\partial \sigma_1}{\partial x_1}+\frac{\partial \tau_{21}}{\partial x_2}+b_1=0,\quad \frac{\partial \tau_{12}}{\partial x_1}+\frac{\partial \sigma_2}{\partial x_2}+b_2=0 \qquad (1.26)$$

- Surface tractions

$$t_1=n_1\sigma_1+n_2\tau_{21},\quad t_2=n_1\tau_{121}+n_2\sigma_2 \qquad (1.27)$$

Very thin plate or sheet subject to loads applied in its own plane Fig. 1.11

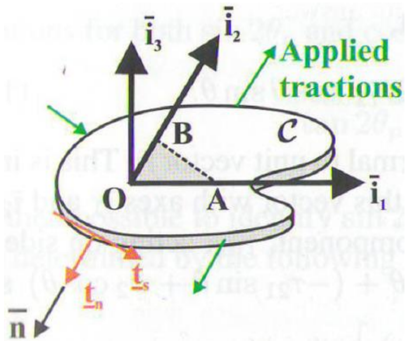

Fig. 1.11 Plane stress problem in thin sheet with in-plane tractions

1.3 The state of plane stress

- Fig. 1.11 … outer normal unit vector $\bar{n} = n_1 \bar{i}_1 + n_2 \bar{i}_2,$

  $n_1 = \cos\theta,\ n_2 = \sin\theta,\ n_3 = 0$

  tangent unit vector $\bar{s} = s_1 \bar{i}_1 + s_2 \bar{i}_2$

  $s_1 = -\sin\theta,\ s_2 = \cos\theta,\ s_3 = 0$

Eq.(1.11) → $$t_n = \cos^2\theta\sigma_1 + \sin^2\theta\sigma_2 + 2\sin\theta\cos\theta\tau_{12} \tag{1.28}$$

Eq.(1.12) → $$t_s = \sin\theta\cos\theta(\sigma_2 - \sigma_1) + (\cos^2\theta - \sin^2\theta)\tau_{12} \tag{1.29}$$

1.3.2 Stress acting on an arbitrary face within the sheet

- Fig.1.12 : 2-D version of Cauchy's tetrahedron (Fig. 1.7)

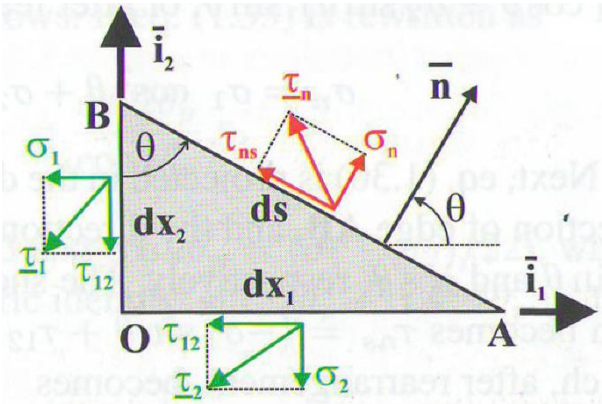

Fig. 1.12 Differential element with a face at an angle θ

1.3 The state of plane stress

- Equilibrium of forces

$$\underline{\tau}_2 dx_1 + \underline{\tau}_1 dx_2 = \underline{\tau}_n ds + \underline{b} dx_1 dx_2 \frac{1}{2}$$

dividing by ds

$$\underline{\tau}_n = \underline{\tau}_1 n_1 + \underline{\tau}_2 n_2 - \underline{b} dx_1 dx_2 \frac{1}{2ds}$$

↑ Neglected since multiplied by h.o. term

$$\underline{\tau}_n = \left(\sigma_1 \bar{i}_1 + \tau_{12} \bar{i}_2\right)\cos\theta + \left(\tau_{21} \bar{i}_1 + \sigma_2 \bar{i}_2\right)\sin\theta \qquad (1.30)$$

- Projecting in the dir. of unit vector $\bar{n} \rightarrow \sigma_n$

$$\sigma_n = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau_{12}\cos\theta\sin\theta \qquad (1.31)$$

- Projecting in the dir. of normal to $\bar{n} \rightarrow \tau_{ns}$

$$\tau_{ns} = -\sigma_1 \cos\theta\sin\theta + \sigma_2 \cos\theta\sin\theta + \tau_{12}(\cos^2\theta - \sin^2\theta) \qquad (1.32)$$

→ Knowledge of σ_1, σ_2, τ_{12} or 2 orthogonal faces allows computation of the stress components acting on a face with an arbitrary orientation

1.3 The state of plane stress

1.3.3 Principal stress

- Simply write Eqn. (1.13)-(1.15) with $\sigma_3 = \tau_{13} = \tau_{23} = 0$
  or, using Eq. (1.31)… particular orientation, θ_p , that maximizes (or minimizes) σ_n

$$\rightarrow \frac{d\sigma_n}{d\theta} = 0 \rightarrow \tan 2\theta_p = \frac{2\tau_{12}}{\sigma_1 - \sigma_2} = \frac{\sin 2\theta_p}{\cos 2\theta_p} \qquad (1.33)$$

2 sol.s θ_p and $\theta_p + \pi/2$ corresponding to 2 mutually orthogonal principal stress directions.

$$\begin{cases} \sin 2\theta_p = \dfrac{\tau_{12}}{\Delta}, \ \cos 2\theta_p = \dfrac{(\sigma_1 - \sigma_2)}{2\Delta}, & (1.34) \\ \Delta = \left[\left(\dfrac{\sigma_1 - \sigma_2}{2} \right) + (\tau_{12})^2 \right]^{1/2} & (1.35) \end{cases}$$

where Δ is determined by $\sin^2 2\theta_p + \cos^2 2\theta_p = 1$

$\rightarrow$ Unique solution for θ_p

- Max./Min. axial stress : "principal stress" by introducing Eq.(1.34) into (1.31)

$$\sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + \Delta \ ; \ \sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} - \Delta \qquad (1.36)$$

Where the shear stress vanishes

1.3 The state of plane stress

- Max. shear stress $\rightarrow \theta_s \rightarrow \dfrac{d\tau_{ns}}{d\theta} = 0$ using Eq.(1.32) (1.37)

$$\rightarrow \tan 2\theta_s = -\frac{\sigma_1 - \sigma_2}{2\tau_{12}} = \frac{1}{\tan 2\theta_p} \quad (1.38)$$

2 sol.s θ_s and $\theta_s + \pi/2$ corresponding to 2 mutually orthogonal faces directions.

- Max. shear stress $\tau_{\max} = \Delta = \dfrac{\sigma_{p1} - \sigma_{p2}}{2}$ (1.40)

$$\theta_s = \theta_p - \frac{\pi}{4} \quad (1.41)$$

Max. shear stress occurs at a face inclined at a 45° angle w.r.t. the principal stress directions

$$\sigma_{1s} = \sigma_{2s} = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{p1} + \sigma_{p2}}{2} \quad (1.42)$$

1.3.4 Rotation of Stresses

- Eq (1.31) $\longrightarrow$ $\sigma_1^* = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau_{12}\sin\theta\cos\theta$ (1.45)

  (1.32) $\longrightarrow$ $\tau_{12}^* = -\sigma_1 \sin\theta\cos\theta + \sigma_2 \sin\theta\cos\theta + \tau_{12}\left(\cos^2\theta - \sin^2\theta\right)$ (1.46)

- Compact matrix form

$$\begin{bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{1.47}$$

  can be easily inverted by simply replacing θ by $-\theta$

- Knowledge of the stress components $\sigma_1, \sigma_2, \tau_{12}$ on 2 orthogonal faces allows computation of those acting on a face with an arbitrary orientation

1.3.5 Special state of stresses

i) Hydrostatic stress state …… $\sigma_{p_1} = \sigma_{p_2} = P$ "hydrostatic pressure"

$\tau_{12} = 0$ With any arbitrary orientation

ii) Pure shear state

$$\sigma_{p_2} = -\sigma_{p_1} \qquad \text{(Fig. 1.13)}$$

At the face inclined at a 45° angle w.r.t. the principal stress direction

$$\tau^*_{12} = -\sigma_{p_1} \quad \sigma^*_1 = \sigma^*_2 = 0 \qquad (1.51)$$

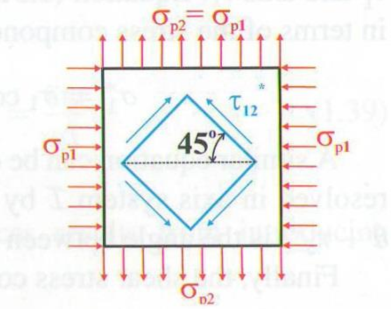

Fig. 1.13. A differential plane stress element in pure shear.

iii) Stress state in thin-walled pressure vessels

Fig.14 … cylindrical pressure vessel subjected to internal pressure P_i

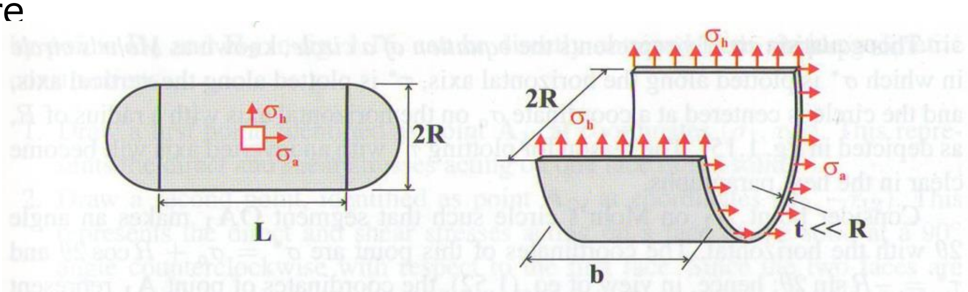

Fig. 1.14. Long, thin-walled cylindrical pressure vessel (left) and free body diagram (right) used to calculate in-plane stresses σ_h and σ_a.

1.3 The state of plane stress

2 in-plane stress components
- σ_a (axial direction)
- σ_h (circumferential or "hoop" direction)

Possibly a shear stress, τ_{ah}

- Axial force equilibrium $\sigma_a \pi R t = p_i \pi R^2 / 2$ Left/right

Area of cylinder cross-section in the direction of axial

$$\sigma_a = p_i R^2 / 2t$$

- tangential (hoop) direction $2\sigma_h bt = p_i 2Rb$

Internal area of cylinder projected to the tangential (hoop) direction

$$\sigma_h = p_i R / t$$

$$\tau_{ah} = 0$$

1.3.6 Mohr's circle for plane stress

◦ σ_{p_1}, σ_{p_2} … Principal stresses at a point

◦ Eq.(1.49) -> stresses acting on a face oriented at an angle θ w.r.t. the principal stress direction

$$\sigma^* = \sigma_a + R\cos 2\theta \quad \tau^* = -R\sin 2\theta \qquad (1.52)$$

1.3 The state of plane stress

◦ where

$$\sigma_a = \left(\sigma_{p_1} + \sigma_{p_2}\right)/2, \quad R = \left(\sigma_{p_1} - \sigma_{p_2}\right)/2$$

$$\Rightarrow \left(\sigma^* - \sigma_a\right)^2 + \left(\tau^*\right)^2 = R^2 \qquad (1.53)$$

◦ equation of a circle "Mohr's circle"

σ^* :horizontal axis, τ^* :vertical axis ("inverted")

center at a coordinate σ_a on the horizontal axis

R :radius,

……each point on Mohr's circle represents the state of stress acting at a face at a specific orientation

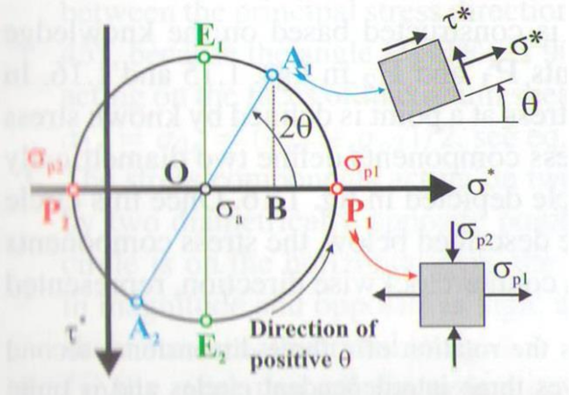

Fig. 1.15. Mohr's circle for visualizing plane stress state.

◦ Observations

1) At point, P_1, $\sigma^* = \sigma_{p_1}$, $\tau^* = 0$ …..principal stress direction
P_2, second principal stress direction

2) At point E_1, $\theta = \dfrac{\pi}{4}$, $\tau^*_{\max} = R = \left(\sigma_{p_1} - \sigma_{p_2}\right)/2$ -> Max. shear stress orientation
E_2

3) At point, A_1, A_2 two faces oriented 90° apart, the shear stresses are equal in magnitude and of opposite sign -> principle of reciprocity

1.3 The state of plane stress

◦ Construction procedure

1) First point A_1 at (σ_1, τ_{12})

2) Second point A_2, $(\sigma_2, -\tau_{12})$ at a 90° angle counterclockwise w.r.t. the first point

3) Straight line joining A_1 and A_2

4) Stress component at β an angle
…….a new diameter B_1OB_2 rotated 2β degree from the ref. diameter A_1OA_2

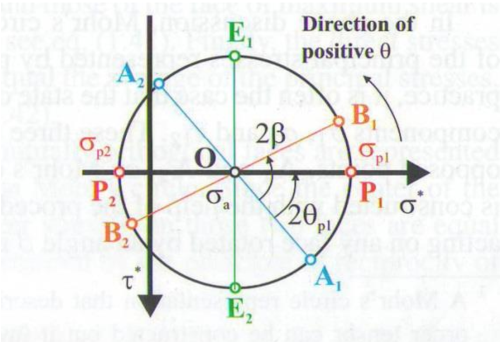

Fig. 1.16. Mohr's circle construction procedure.

◦ Important features

1) Principal stress $\sigma_{p_1}, \sigma_{p_2}$ -> points P_1 and P_2, direct stress Max/Min. shear stress=0

2) Max. shear stress…….. Vertical line E_1=radius, $\tau_{\max} = \left(\sigma_{P_1} - \sigma_{P_2}\right)/2$

direction…..45° since $P_1OE_1 = 90°$

3) Stress components acting on 2 mutually orthogonal faces ……..2 diametrically opposite points on Mohr's circle

4) All the point on Mohr's circle represent the same state of stress at one point of the solid

1.3 The state of plane stress

1.3.7 Lame's ellipse

Eq. (1.30) → When selecting the principal stress direction

→ $\underline{\tau}_n = \sigma_{1p} \cos\theta \bar{i}_1^* + \sigma_{2p} \sin\theta \bar{i}_2^*$

(x_1, x_2) : Tip of the stress vector, $\underline{\tau}_n = x_1 \bar{i}_1^* + x_2 \bar{i}_2^*$

$$x_1 = \sigma_{1p} \cos\theta \qquad x_2 = \sigma_{2p} \sin\theta$$

Eliminating θ

$$\left(\frac{x_1}{\sigma_{1p}}\right)^2 + \left(\frac{x_2}{\sigma_{2p}}\right)^2 = 1 \qquad (1.54)$$

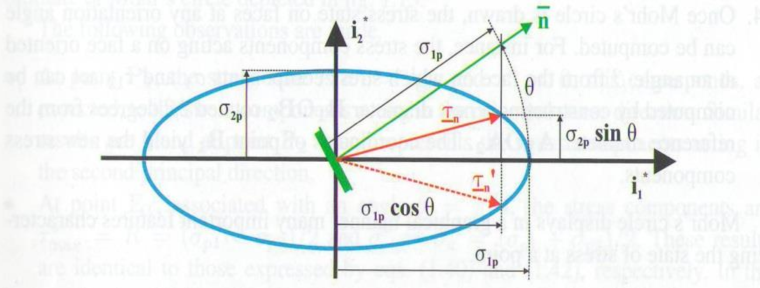

Fig. 1.17. Lamé's ellipse. Stress vector $\underline{\tau}_n$ corresponds to positive principal stresses whereas stress vector $\underline{\tau}_n$ corresponds to $\sigma_{p1} > 0$ and $\sigma_{p2} < 0$.

→ Equation of ellipse with semi-axis equal to $|\sigma_{p_1}|$ and $|\sigma_{p_2}|$ (Fig 1.17)

◦ Pure shear.....ellipse → circle (Fig.1.18)

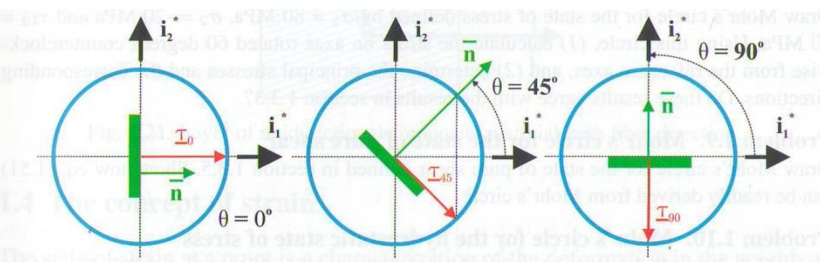

Fig. 1.18. Lamé's ellipse for the case of pure shear; the three figures illustrate the stress vectors acting on faces at 0, 45, and 90 degrees with respect to axis $\bar{i}_1^*$.

1.4 The concept of strain

◦ State of strain……..characterization of the deformation in the neighborhood of a material point in a solid

at a given point P, located by a position vector $\underline{r} = x_1 \bar{i}_1 + x_2 \bar{i}_2 + x_3 \bar{i}_3$ (Fig.1.22)

small rectangular parallelepiped PQRST of differential size
" reference configuration," undeformed state

→ "deformed configuration" PQRST

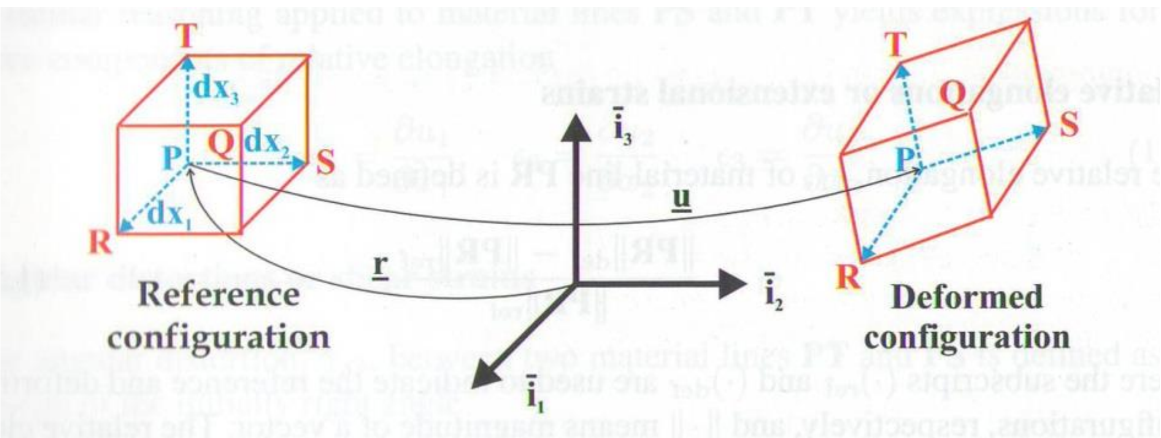

Fig. 1.22. The neighborhood of point **P** in the reference and deformed configurations.

◦ displacement vector $\underline{u}$ …..measure of how much a material point moves. (1.56)

two parts:
- Rigid body motion…..translation, rotation -> does not produce strain
- Deformation or straining -> strain-displacement relation

1.4 The concept of strain

1.4.1 The state of strain at a point

- Material line PR in the ref. conf. $\longrightarrow$ Material line PR in the ref. conf. in the deformed configure

assumed to be still straight, but a parallelogram

- 2 factors in the measure of state of strain
  - Stretching of a material line …… $\varepsilon_1, \varepsilon_2, \varepsilon_3$
  - Angular distortion between 2 material lines……$\gamma_{23}, \gamma_{13}, \gamma_{12}$

i) Relative elongations or extensional strain

$\|\ldots\|$: magnitude

$$\varepsilon_1 = \frac{\|PR\|_{def} - \|PR\|_{ref}}{\|PR\|_{def}} \tag{1.57}$$

…. Non-dimensional quantity

$$\|PR\|_{ref} = \|dx_1 \bar{i}_1\| = dx_1$$

$$\|PR\|_{def} = \|dx_1 \bar{i}_1 + \underline{u}(x_1 + dx_1) - \underline{u}(x_1)\|$$

$$\|PR\|_{def} = \left\| dx_1 \bar{i}_1 + +\underline{u}(x_1) + \frac{\partial \underline{u}}{\partial x_1} dx_1 - \underline{u}(x_1) \right\| = \left\| dx_1 \bar{i}_1 + \frac{\partial \underline{u}}{\partial x_1} dx_1 \right\|$$

$$= \left\| \bar{i}_1 dx_1 + \left(\frac{\partial u_1}{\partial x_1} \bar{i}_1 + \frac{\partial u_2}{\partial x_1} \bar{i}_2 + \frac{\partial u_3}{\partial x_1} \bar{i}_3 \right) dx_1 \right\| \tag{1.59}$$

1.4 The concept of strain

$$= \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}\, dx_1$$

Then

$$\varepsilon_1 = \sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}\, dx_1 - 1 \qquad (1.60)$$

◦ fundamental assumption of linear elasticity….all displacement components remain very small so that all 2nd order terms can be neglected.
And, using the binomial expansion

$$\left.\begin{aligned} &\varepsilon_1 \cong 1 + \frac{\partial u_1}{\partial x_1} - 1 = \frac{\partial u_1}{\partial x_1} && (1.62) \\ &\varepsilon_2 = \frac{\partial u_2}{\partial x_2},\ \varepsilon_3 = \frac{\partial u_3}{\partial x_3} && (1.63) \end{aligned}\right\} \text{“direct strains” or “axial strains”}$$

1.4 The concept of strain

ii) Angular distortions or shear strains

γ_{23} ....between two material lines PT and PS, defined as the change of the initially right angle

$$\gamma_{23} = \langle TPS \rangle_{ref} - \langle TPS \rangle_{def} = \frac{\pi}{2} - \langle TPS \rangle_{def} \tag{1.64}$$

Non-dimensional quantities

<......> : angle between segment

$$\sin\gamma_{23} = \sin\left(\frac{\pi}{2} - \langle TPS \rangle_{def}\right) = \cos\langle TPS \rangle_{def} \tag{1.65}$$

by law of cosine

$$\|TS\|_{def} = \|PT\|_{def}^2 + \|PS\|_{def}^2 - \underbrace{2\cos\langle TPS \rangle_{def}}_{2\sin\gamma_{23}} \|PT\|_{def} \|PS\|_{def} \tag{1.66}$$

$$\gamma_{23} = \arcsin\frac{\|PT\|_{def}^2 + \|PS\|_{def}^2 - \|TS\|_{def}^2}{2\|PT\|_{def} \|PS\|_{def}} \tag{1.67}$$

$$PT_{def} = \left(\bar{i}_3 + \frac{\partial \underline{u}}{\partial x_3}\right) dx_3 = A, \; PS_{def} = \left(\bar{i}_2 + \frac{\partial \underline{u}}{\partial x_2}\right) dx_2 = B$$

$$PS_{def} = PS_{def} - PT_{def} = B - A$$

1.4 The concept of strain

Numerator

$$N = 2\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}\right)dx_2 dx_3$$

Denominator

$$D = 2\sqrt{A \cdot A}\sqrt{B \cdot B}$$

-with the help of small displacement assumption

$$N \cong 2\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)dx_2 dx_3 \qquad D \cong 2\left(1 + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)dx_2 dx_3$$

$$\Rightarrow \quad \gamma_{23} \cong \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \qquad (1.70)$$

$$\gamma_{13} \cong \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \qquad \gamma_{12} \cong \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \qquad (1.71)$$

"shearing strain"

"shear strain"

-Strain-displacement relationship, Eqs. (1.63), (1,71) ….. Under the small displacement assumption

Large displacement → Eqns. (1.60),(1.67) should be used

1.4 The concept of strain

iii) Rigid body rotation

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \qquad (1.73a)$$

Rotation vector $\underline{\omega}^T = \{\omega_1, \omega_2, \omega_3\}$……the rotation of the solid about axes $\bar{i}_1, \bar{i}_2, \bar{i}_3$ respectively

1.4.2 The volumetric strain

◦ After deformation

$$v \approx (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)\,dx_1dx_2dx_3 \approx (1+\varepsilon_1+\varepsilon_2+\varepsilon_3)\,dx_1dx_2dx_3 \qquad (1.74)$$

where high order strain quantities are neglected

◦ relative change in volume

$$e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad \text{“volumetric strain”} \qquad (1.75)$$

1.5 Analysis of the state of strain at a point

- Arbitrary reference frame $\underline{J}^* = \left(\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*\right)$

  -> Strain-displacement relationship in $\underline{J}^*$(1.76), (1.77)

1.5.1 Rotation of strains

◦ chain rule

$$\varepsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial x_2}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial x_3}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1} l_1 + \frac{\partial u_1^*}{\partial x_2} l_2 + \frac{\partial u_1^*}{\partial x_3} l_3 \qquad (1.78)$$

Where Eq. (A.39) is used

- Next, u_1^* in terms of the components in $\underline{J}$

$$\varepsilon_1^* = l_1 \frac{\partial}{\partial x_1}\left(l_1 u_1 + l_2 u_2 + l_3 u_3\right) + l_2 \frac{\partial}{\partial x_2}\left(l_1 u_1 + l_2 u_2 + l_3 u_3\right) + l_3 \frac{\partial}{\partial x_3}\left(l_1 u_1 + l_2 u_2 + l_3 u_3\right) \qquad (1.79)$$

Using Eq. (1.63) and (1.71)

$$\varepsilon_1^* = \varepsilon_1 l_1^2 + \varepsilon_2 l_2^2 + \varepsilon_3 l_3^2 + \gamma_{12} l_1 l_2 + \gamma_{13} l_1 l_3 + \gamma_{23} l_2 l_3$$

Similar eqns. (1.80), (1.81)

$$\varepsilon_{23} = \frac{\gamma_{23}}{2} \qquad \varepsilon_{13} = \frac{\gamma_{13}}{2} \qquad \varepsilon_{12} = \frac{\gamma_{12}}{2} \qquad (1.82)$$

Engineering shear strain comp.

Tensor shear strain component

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \underline{\underline{R}} \begin{Bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{Bmatrix} \Rightarrow \begin{Bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{Bmatrix} = \underline{\underline{R}}^T \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} \qquad (A.39)$$

- Compact matrix form

$$\begin{bmatrix} \varepsilon_1^* & \varepsilon_{12}^* & \varepsilon_{13}^* \\ \varepsilon_{12}^* & \varepsilon_2^* & \varepsilon_{23}^* \\ \varepsilon_{13}^* & \varepsilon_{23}^* & \varepsilon_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_2 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_3 \end{bmatrix} \underline{\underline{R}} \qquad (1.83)$$

1.5.2 Principal strains

◦ Is there a coordinate system $\underline{J}^*$ for which the shear strains vanish?

$$\begin{bmatrix} \varepsilon_1^* & 0 & 0 \\ 0 & \varepsilon_2^* & 0 \\ 0 & 0 & \varepsilon_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_2 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_3 \end{bmatrix} \underline{\underline{R}}$$

1.5 Analysis of the state of strain at a point

- Pre-multiplying $\underline{\underline{R}}$ and reversing the equality

$$\begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_2 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_3 \end{bmatrix} \underline{\underline{R}} = \underline{\underline{R}} \begin{bmatrix} \varepsilon_{p_1} & 0 & 0 \\ 0 & \varepsilon_{p_2} & 0 \\ 0 & 0 & \varepsilon_{p_3} \end{bmatrix}$$

where the orthogonality of $\underline{\underline{R}}$, Eq. (A.37), is used

$$\underline{\underline{R}}^T \underline{\underline{R}} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{I}} \qquad \text{(A.37)}$$

- $\varepsilon_{p_1}, \varepsilon_{p_2}, \varepsilon_{p_3}$: sol. of 3 system of 3 eqns

$$\begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_2 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \varepsilon_p \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

1.5 Analysis of the state of strain at a point

Determinant of the system vanishes-> non-trivial solution.

Cubic eqn. $\varepsilon_p^3 - I_1\varepsilon_p^2 + I_2\varepsilon_p - I_3 = 0$ (1.85)

"Strain invariant" (1.86)

3 sol: $\varepsilon_{p_1}, \varepsilon_{p_2}, \varepsilon_{p_3}$ -> corresponding "principal strain direction"

-> homogeneous eqn.-> arbitrary constant ->normality condition

◦ displacement component along $\bar{i}_3$ is assumed to vanish, or to be negligible

Example: a very long buried pipe aligned with $\bar{i}_3$ dir.

1.6 The state of plane strain

1.6.1 Strain-displacement relations for plane strain

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_2 = \frac{\partial u_2}{\partial x_2} \qquad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{1.87}$$

1.6.2 Rotation of strains

- chain rule

$$\varepsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial x_2}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\cos\theta + \frac{\partial u_1^*}{\partial x_2}\sin\theta$$

Eq (A.43)

1.6 The state of plane strain

- u_1^* in terms of those in $\underline{J}$

$$\varepsilon_1^* = \cos\theta \frac{\partial}{\partial x_1}\left(u_1 \cos\theta + u_2 \sin\theta\right) + \sin\theta \frac{\partial}{\partial x_2}\left(u_1 \cos\theta + u_2 \sin\theta\right) \tag{1.88}$$

-Then,

$$\varepsilon_1^* = \cos^2\theta\varepsilon_1 + \sin^2\theta\varepsilon_2 + \sin\theta\cos\theta\gamma_{12} \tag{1.89}$$

-Matrix form

$$\begin{Bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \\ \varepsilon_{12}^* \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} \tag{1.91}$$

can be readily inverted by replacing θ by -θ

1.6 The state of plane strain

1.6.3 Principal strains

◦ θ_p , in which the max (or min) elongation occurs

$$\longrightarrow \quad \frac{d\varepsilon_1^*}{d\theta} = 0 = -\frac{\varepsilon_1 - \varepsilon_2}{2} 2\sin 2\theta_p + \frac{\gamma_{12}}{2} 2\cos 2\theta_p = 0 \qquad (1.95)$$

$$\tan 2\theta_p = \frac{\gamma_{12}/2}{(\varepsilon_1 - \varepsilon_2)/2} \qquad (1.96)$$

2 sols. --- $\theta_{p_1}, \; \theta_{p_2} = \theta_{p_1} + \frac{\pi}{2}$..... 2 mutually orthogonal principal strain directions

$$\varepsilon_{p_1} = \frac{\varepsilon_1 + \varepsilon_2}{2} + \Delta \qquad \varepsilon_{p_2} = \frac{\varepsilon_1 + \varepsilon_2}{2} - \Delta \qquad (1.99)$$

where shear strain vanishes

- The orientation of the { Principal stresses / Principal strains } are not necessarily identical

1.6.4 Mohr's circle for plane strain

- Strains along a direction defined by angle θ w.r.t. the principal strain direction

$$\varepsilon^* = \varepsilon_a + R\cos 2\theta \qquad \frac{\gamma^*}{2} = -R\sin 2\theta \tag{1.100}$$

Where,

$$\varepsilon_a = \frac{\left(\varepsilon_{p_1} + \varepsilon_{p_2}\right)}{2} \qquad R = \frac{\left(\varepsilon_{p_1} - \varepsilon_{p_2}\right)}{2}$$

=> $\left(\varepsilon^* - \varepsilon_a\right)^2 + \left(\frac{\gamma^*}{2}\right) = R^2$ Mohr's circle (1.101)

Fig.1.23, positive angle θ …. counterclockwise dir.
shear strain …….positive downward

Vertical axis ……strain tensor, $\left(\frac{\gamma_{12}}{2}\right)$

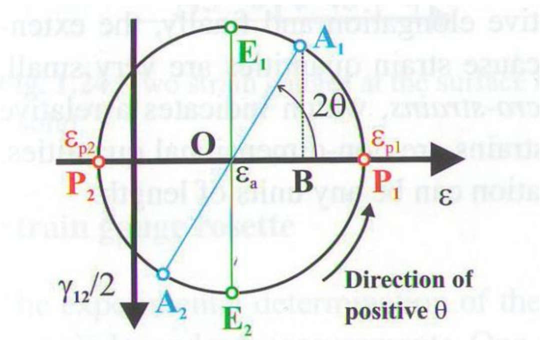

Fig. 1.23. Mohr's circle for visualizing plane strain state.

1.7 Measurement of strains

No practical experimental device for direct measurement of STRESS …….indirect measurement of strain first
→ constitutive laws

i) strain gauges

◦ measurement of extensional strains on the body's external surfaces

- Very thin electric wire, or an etched foil pattern
- extension …. wire's cross-section reduced by Poisson's effect. Slightly increasing its electrical resistance
  compression….increasing its electrical reduced resistance
- Wheatstone bridge …..accurate measurement

"micro-strains" …$\mu\, m/m = 10^{-6}\ m/m$

ii) Chevron strain gauges

Fig 1.24….. e_{+45} and e_{-45}, experimentally measured relative elongations

Using Eq (1.94a),

$$e_{+45} = \frac{\varepsilon_1 + \varepsilon_2}{2} + \frac{\gamma_{12}}{2} \qquad e_{-45} = \frac{\varepsilon_1 + \varepsilon_2}{2} - \frac{\gamma_{12}}{2}$$

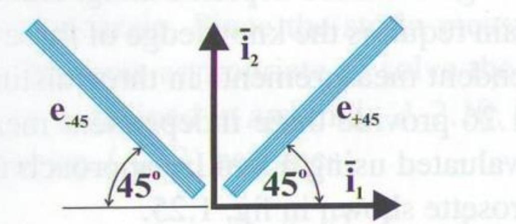

Fig. 1.24. Two strain gauges at the surface of a solid.

……Not sufficient to determine the strain state at the point

3 measurements would be required $\begin{cases} \varepsilon_1, \varepsilon_2, \gamma_{12} \\ \text{2 principal strains \& dir.} \end{cases}$

However, can uniquely determine $\gamma_{12} = e_{+45} - e_{-45}$ (1.102)

iii) Strain gauge rosette

Fig.1.25……3 independent measurements, “delta rosette”

Eq. (1.94a) → $\varepsilon_1 = e_1 \qquad \varepsilon_2 = \dfrac{2}{3}\left(e_2 + e_3 - \dfrac{e_1}{2}\right) \qquad \gamma_{12} = \dfrac{2}{\sqrt{3}}\left(e_2 - e_3\right)$ (1.103)

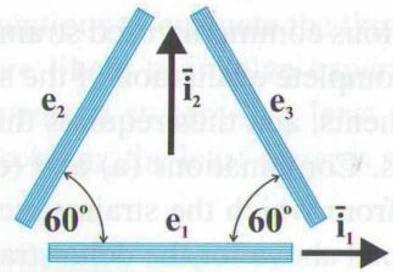

Fig. 1.25. Three strain gauges forming a rosette at the surface of a solid.

Fig.1.26 …..various arrangement of strain gauges

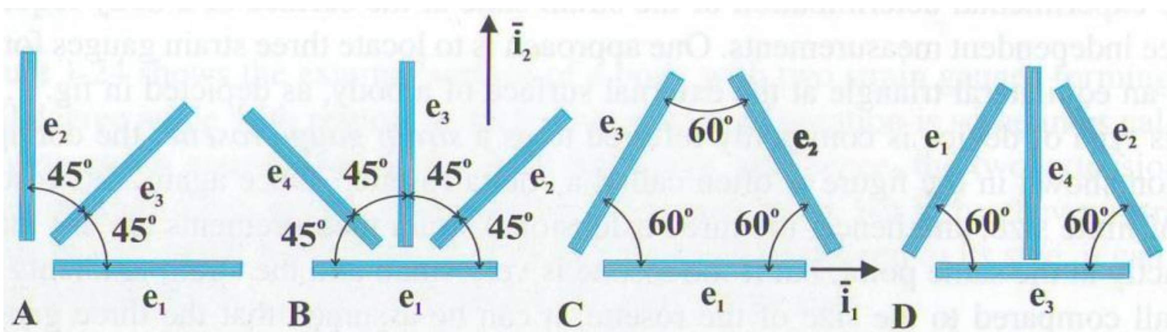

Fig. 1.26. Various commonly used strain gauge arrangements.