Mathematical Background in Aircraft Structural Mechanics

CHAPTER 2.
Basic Equations

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2.1 Constitutive laws for isotropic materials

❖ **3 types of relationships for the sol. of elasticity problems**

- Equilibrium eqns
- Strain-displacement relationships
- Constitutive laws ⋯ mechanical behavior of the material

i) Homogeneity and isotropy

➢ "homogenous material" ⋯ physical properties are identical at each point

➢ "isotropic material" ⋯ physical properties are identical in all direction

- Ex) mild steel, aluminum ⋯ both homogeneous and isotropic
- Composite material ⋯ neither homogeneous nor isotropic -> heterogeneous, anisotropic

➢ "scale dependent" ⋯

① At atomic level, Al is neither homogenous nor isotropic

-> assumption of homogeniety and isotropy only hold for a very large number of atoms

② High temperature turbine blade applications {poly-crystalline / single crystal} materials

- single crystal ⋯ regular lattice structures -> homogeneous, but anisotropic
- poly-crystalline ⋯ {crystals oriented in a specific dir -> "
  ex) forged metals
  crystals arranged at random orientations -> {homogeneous / isotropic}
  ex) common structural metals(steel, Al)}

2.1 Constitutive laws for isotropic materials

③ Composite material ··· clearly anisotropic, but samples containing a very large number of fibers -> reasonably assumed as homogeneous

ii) Material testing

- If deformation very small -> linear stress-strain relationship
- If large deformation -> material is ductile or brittle
- Tensile test ··· strain $\varepsilon_1 = Al / l$
  stress $\sigma_1 = N / A$

$$\left.\begin{array}{ll} \text{strain} & \varepsilon_1 = Al / l \\ \text{stress} & \sigma_1 = N / A \end{array}\right\} \text{Stress−strain diagram}$$

2.1 Constitutive laws for isotropic materials

❖ 2.1.1 Homogeneous, isotropic, linearly elastic materials

- Small deformations -> linear stress-strain behavior

$$\sigma_1 = E\varepsilon_1 \qquad \text{Hooke's law} \qquad (2.1)$$

Young's modulus or modulus of elasticity [pa]

- Elongation of a bar ··· lateral contraction

$$\varepsilon_1 = \frac{1}{E}\begin{cases}\sigma_1 \\ \sigma_2\end{cases} \quad \varepsilon_2 = -\frac{\nu}{E}\begin{cases}\sigma_1 \\ \sigma_2\end{cases} \quad \varepsilon_3 = -\frac{\nu}{E}\begin{cases}\sigma_1 \\ \sigma_2\end{cases} \qquad \begin{matrix}(2.2) \\ (2.3)\end{matrix}$$

ν : Poisson's ratio

2.1 Constitutive laws for isotropic materials

i) Generalized Hooke's law

- Deformation under 3 stress components ⋯ sum of those obtained for each stress component

-> generalized Hooke's law

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \qquad (2.4a)$$

⋯ extensional strains depend only on the direct stress and not on the shear stress
<- isotropic material

ii) Shear stress – shear strain relationships

- Pure shear state in a plane stress state

  - 2 principal stresses $\sigma_{p2} = -\sigma_{p1}, \sigma_{p3} = 0$

$$\varepsilon_1 = \frac{1+\nu}{E}\sigma_{p1}, \varepsilon_2 = -\frac{1+\nu}{E}\sigma_{p1}, \gamma_{12} = 0 \qquad (2.5)$$

  - on faces oriented at a 45° angle w.r.t. the principal stress directions

$$\tau^*_{s_{12}} = \sigma_{p2} = -\sigma_{p1}, \sigma^*_{s_1} = \sigma^*_{s_2} = 0 \qquad (2.6)$$

*, s: specially rotated axis with max shear stress

2.1 Constitutive laws for isotropic materials

$$\varepsilon_1^* = \frac{\varepsilon_1 + \varepsilon_2}{2} + \frac{\varepsilon_1 - \varepsilon_2}{2}\cos 2\theta + \frac{\gamma_{12}}{2}\sin 2\theta, \qquad (1.94a)$$

$$\varepsilon_2^* = \frac{\varepsilon_1 + \varepsilon_2}{2} - \frac{\varepsilon_1 - \varepsilon_2}{2}\cos 2\theta - \frac{\gamma_{12}}{2}\sin 2\theta, \qquad (1.94b)$$

$$\gamma_{12}^* = -(\varepsilon_1 - \varepsilon_2)\sin 2\theta + \gamma_{12}\cos 2\theta. \qquad (1.94c)$$

Eq.(1.94) -> $\theta_s = 45^\circ, \gamma_{s_{12}}^* = -(\varepsilon_1 - \varepsilon_2) = -\frac{2(1+\nu)}{E}\sigma_{p1}, \varepsilon_{s1}^* = \varepsilon_{s2}^* = 0$ (2.7)

Eq.(2.6),(2.7) -> $\gamma_{s_{12}}^* = -\frac{2(1+\nu)}{E}\sigma_{p1} = 2(1+\nu)\frac{\tau_{s_{12}}^*}{E} = G\gamma_{s_{12}}^*$

=> $G = \frac{E}{2(1+\nu)}$ “shear modulus” (2.8)

··· generalized Hooke’s law for shear strains

$$\gamma_{23} = \tau_{23}/G, \gamma_{13} = \tau_{13}/G, \gamma_{12} = \tau_{12}/G \qquad (2.9)$$

2.1 Constitutive laws for isotropic materials

iii) Matrix form of the constitutive laws

- Compact matrix form of the generalized Hooke's law

$$\underline{\underline{\varepsilon}} = \underline{\underline{S}}\,\underline{\underline{\sigma}} \tag{2.10}$$

$$\underline{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{23}, \gamma_{13}, \gamma_{12}\}^T \tag{2.11a}$$

$$\underline{\sigma} = \{\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}\}^T \tag{2.11b}$$

2.1 Constitutive laws for isotropic materials

Eq(2.4)

$$\underline{\underline{S}} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \qquad (2.12)$$

Absence of coupling between
- Axial stresses
- Shear strains

And vice versa

Eq(2.9)

- Stiffness form of the same laws

$$\underline{\underline{\sigma}} = \underline{\underline{C}}\,\underline{\underline{\varepsilon}} \qquad (2.13)$$

$$\underline{\underline{C}} = [\ddots] \qquad (2.14)$$

2.1 Constitutive laws for isotropic materials

iv) Plane stress state

$$\underline{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \gamma_{12}\}^T \qquad (2.15a)$$

$$\underline{\sigma} = \{\sigma_1, \sigma_2, \gamma_{12}\}^T \qquad (2.15b)$$

$$\underline{\underline{C}} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \qquad (2.16)$$

ε_3 does not vanish due to Poisson's ratio effect, $\varepsilon_3 = -\nu(\sigma_1 + \sigma_2)$

v) Plane strain state

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \qquad (2.17)$$

σ_3 does not vanish due to Poisson's ratio effect, $\sigma_3 = \nu E \dfrac{(\varepsilon_1 + \varepsilon_2)}{(1+\nu)(1-2\nu)}$

2.1 Constitutive laws for isotropic materials

vi) The bulk modulus

- Volumetric strain ··· Eq.(1.75)

$$e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1-2\nu}{E} I_1 \qquad (2.18)$$

- Hydrostatic pressure, $\sigma_1 = \sigma_2 = \sigma_3 = p$

-> $p = \kappa e,$ (2.19)

$\kappa = \dfrac{E}{3(1-2\nu)}$: "bulk modulus" (2.20)

When $\nu \to \dfrac{1}{2}, \kappa \to \infty$ ··· "incompressible material" (ex: rubber)

❖ 2.1.2 Thermal effects

- Under a change in temperature, homogeneous isotropic materials will expand in all directions -> "thermal strain"

$$\varepsilon^t = \alpha \Delta T \qquad (2.21)$$

① Thermal strains are purely extensional, do not induce shear strains

② Thermal strains do not generate internal stresses ··· Unconfined material sample simply expands subject to a temp. change but remains unstressed

- Total strains ··· mechanical strains + thermal strains

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] + \alpha \Delta T \qquad (2.22a)$$

But shear stress-shear strain relationships unchanged

- Constrained material ··· a bar constrained at its two ends by rigid walls

2.1 Constitutive laws for isotropic materials

- Constrained material ··· a bar constrained at its two ends by rigid walls

$$\varepsilon_1 = \frac{1}{E}[\sigma_1] + \alpha\Delta T = 0 \rightarrow \sigma_1 = -E\alpha T$$

··· temp. change -> compressive stress("thermal stress")

❖ 2.1.4 Ductile materials

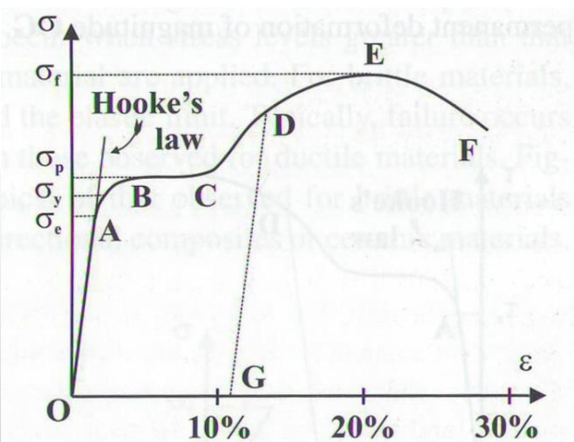


Fig. 2.5. Stress-strain diagram for a ductile material such as mild steel.

- O -> A ··· Hooke's law, slope=Young's modulus
- A ··· limit of proportionality, $\sigma_e \cong \sigma_y$("yield stress")
- B->C ··· "plastic flow" ($\varepsilon_1 = 5 \sim 10\%$)
- C->E ··· increasing stress, $\sigma_f = \max$
- "necking" ··· x-s area decrease
- E ··· "failure stress", σ_f

2.1 Constitutive laws for isotropic materials

- Large deformations before failure ··· B->E

- When unloading, will follow DG//AO, with a permanent deformation OG
- When reloading, will follow GD(higher yield stress at D <- "strain hardening"), and further DEF

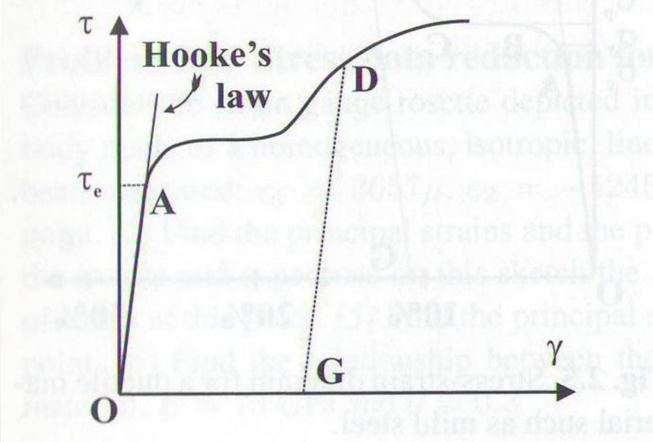


Fig. 2.6. Shear stress-shear strain diagram for a ductile material.

- Shear behavior ··· similar

2.1 Constitutive laws for isotropic materials

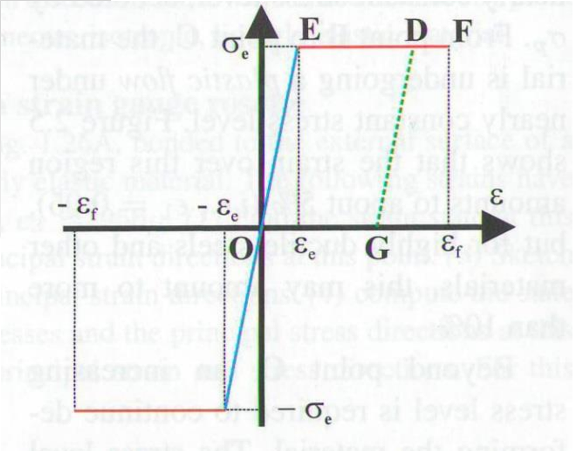


Fig. 2.7. Stress-strain diagram for an elastic-perfectly plastic material.

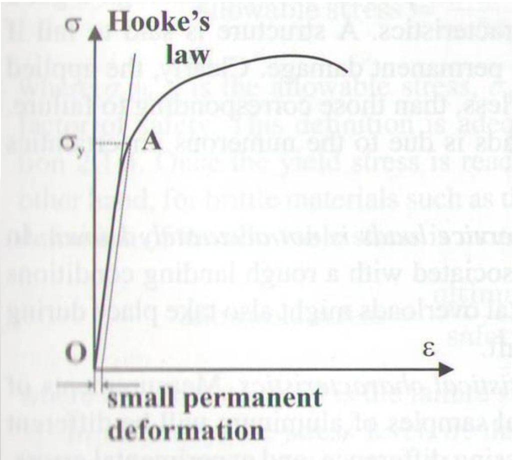


Fig. 2.8. Stress-strain diagram for a ductile material such as aluminum.

- Idealization ⋯ "elastic-perfectly plastic", mild steel, annealed Al

- Al, Cu, no plastic flow regime, specific permanent deformation defined for σ_y

- $ex)\varepsilon = 0.2\%$ for Al

2.1.5 Brittle materials

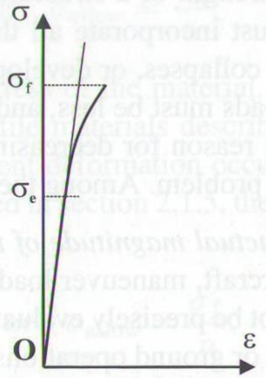


Fig. 2.9. Stress-strain diagram for a brittle material.

- Very little deformation beyond the elastic limit

Ex) glass, concrete, stone, wood, composites or ceramic

2.2 Allowable stress

- Factors influencing the design

① Strength of the structure <- focus of the present section
② Elastic deformation of the structure
③ Dynamic characteristics of the structure ··· natural frequencies and resonance
④ Stability characteristics of the structure ··· buckling
⑤ Time dependent deformations associated with creep ··· turbine engine design

- Numerous uncertainties which decrease service loads

① Actual magnitude of the applied service loads
② Strength of materials ··· statistical
③ Manufacturing variability
④ Corrosion, wear, chemically aggressive environment
⑤ Predicted stresses might be very different from their actual values

2.2 Allowable stress

- Load factor = failure load/ service load >1, as large as 10
- Factor of safety -> allowable stress = yield stress/safety factor, or

$$\sigma_{allow} = \frac{\sigma_y}{\eta} \qquad (2.26)$$

- ··· adequate for ductile materials, for brittle materials, allowable stress= ultimate stress/safety factor, or

$$\sigma_{allow} = \frac{\sigma_y}{\eta} \qquad (2.27)$$

2.3 Yielding under combined loading

- Proper yield criterion under multiple stress components acting
- Isotropic material ··· no directional dependency of the yield criterion state of stress

6 stress components defining the stress tensor

3 principal stresses, $\sigma_{p1}, \sigma_{p2}, \sigma_{p3}$ and the corresponding 3 orientations

No direction dependency -> only the magnitudes of the principal stress should appear

❖ 2.3.1 Tresca's criterion

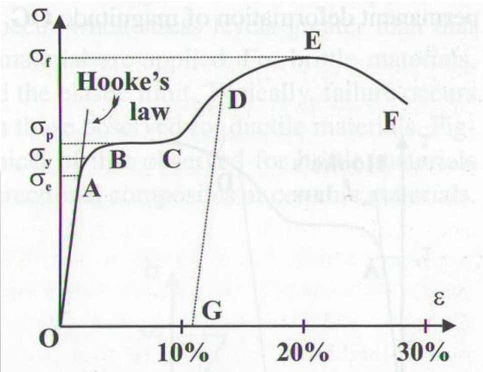


Fig. 2.5. Stress-strain diagram for a ductile material such as mild steel.

- $$\left|\sigma_{p1}-\sigma_{p2}\right| \le \sigma_y, \left|\sigma_{p2}-\sigma_{p3}\right| \le \sigma_y, \left|\sigma_{p3}-\sigma_{p1}\right| \le \sigma_y$$

(2.29)

σ_y : yield stress observed in a uniaxial test

2.3 Yielding under combined loading

- Whenever any one of Eq.(2.29) is violated, yielding develops
- Interpretation -> $\tau_{23\max} \le \dfrac{\sigma_y}{2}, \tau_{13\max} \le \dfrac{\sigma_y}{2}, \tau_{12\max} \le \dfrac{\sigma_y}{2}, \text{or } \tau_{\max} \le \dfrac{\sigma_y}{2}$

…the material reaches the yield condition when the max, shear stress=half the yield stress under a uniaxial stress state.

- "max, shear stress criterion"
- Uniaxial state … $\sigma_{p1} \le \sigma_y$
- Plane state of stress … Eq.(2.31)
- Pure shear state … $\tau \le \sigma_y / 2$

$$2\sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau_{12}^2} \le \sigma_y, \left|\frac{\sigma_1+\sigma_2}{2} \pm \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau_{12}^2}\right| \le \sigma_y \qquad (2.31)$$

2.3.2 Von Mises' criterion

- $\sigma_{eq} = \dfrac{1}{\sqrt{2}}\sqrt{\left[(\sigma_{p1}-\sigma_{p2})^2+(\sigma_{p2}-\sigma_{p3})^2+(\sigma_{p3}-\sigma_{p1})^2\right]} \le \sigma_y$ (2.32)

\- Octahedral face -> shear stress acting on octahedral face

$$3\tau_{oc}^2 = \frac{2}{3}\sigma_{eq}^2 \qquad \sigma_{eq} = \frac{3}{\sqrt{2}}\tau_{oc} \qquad (2.33)$$

… "the yield coord. is reached when the octahedral shear stress $= \dfrac{3}{\sqrt{2}}$ of the yield stress for a uniaxial stress state, σ_y

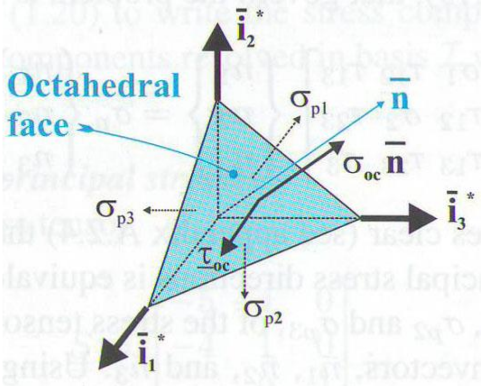


Fig. 1.8. The octahedral face.

2.3 Yielding under combined loading

- σ_{eq} can be expressed in terms of the stress invariants

$$\sigma_{eq}^2 = I_1^2 - 3I_2 \quad (2.34)$$

$$\rightarrow \sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 - \sigma_1\sigma_2 + 3(\tau_{23}^2 + \tau_{13}^2 + \tau_{12}^2)} \le \sigma_y \quad (2.35)$$

① Uniaxial stress state ⋯ $\sigma_{p1} \le \sigma_y$
② Plane state of stress ⋯ $\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 + 3\tau_{12}^2} \le \sigma_y$ (2.36)
③ Pure shear state ⋯ $\tau \le \frac{1}{\sqrt{3}}\sigma_y \cong 0.577(60\%)$ more accurate than that of Tresca's

❖ 2.3.3 Comparing Tresca's and Von Mises' criteria

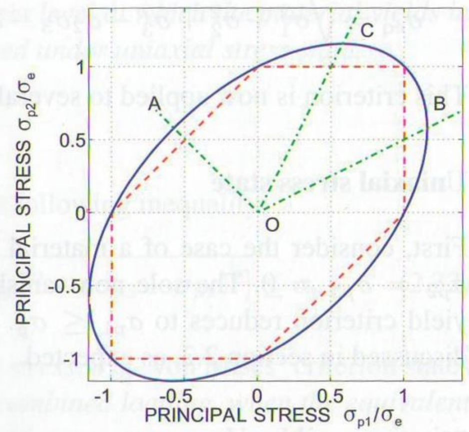


Fig. 2.10. Comparison of Tresca's and von Mises' criteria for a plane stress case.

- Plane stress problem, $\sigma_{p3} = 0$
- Tresca's criterion ⋯ 3 inequalities

$$\left|\frac{\sigma_{p1}}{\sigma_y}\right| < 1, \quad \left|\frac{\sigma_{p2}}{\sigma_y}\right| < 1, \quad \left|\frac{\sigma_{p2}}{\sigma_y} - \frac{\sigma_{p1}}{\sigma_y}\right| < 1$$

⋯ slightly more conservative

-> irregular hexagon enclosed by 6 dashed line segments

2.3 Yielding under combined loading

- Von Mises' criterion ··· oblique ellipse

$$\left(\frac{\sigma_{p1}}{\sigma_y}\right)^2 + \left(\frac{\sigma_{p2}}{\sigma_y}\right)^2 - \left(\frac{\sigma_{p1}}{\sigma_y}\right)\left(\frac{\sigma_{p2}}{\sigma_y}\right) = 1$$

··· often preferred since a single analytic expression

Table 2.1. Comparison of the Tresca and von Mises yield criteria.

Stress state	Radial line in fig. 2.10	Tresca's yield stress	von Mises' yield stress	Percent difference
$\sigma_{p1} = -\sigma_{p2} = \sigma$	**OA**	$\sigma_y/2$	$\sigma_y/\sqrt{3}$	15.5%
$\sigma_{p1} = 2\sigma_{p2} = \sigma$	**OB**	σ_y	$2\sigma_y/\sqrt{3}$	15.5%
$\sigma_{p2} = 2\sigma_{p1} = \sigma$	**OC**	σ_y	$2\sigma_y/\sqrt{3}$	15.5%

- 3 radial lines OA, OB, OC in Fig. 2.10
  -> max discrepancy between 2 criteria ··· 15.5 %

2.4 Material selection for structural performance

2.4 Material selection for structural performance

Table 2.2. Physical properties of a few metals.

	Ultimate stress [MPa]	Modulus of elasticity [GPa]	Density [kg/m^3]
Aluminum	620	73	2700
Titanium	1900	115	4700
Steel	4100	210	7700

… ultimate stress
Modulus of elasticity
Density
(Steel for superior, but heavier)

Table 2.3. Physical properties of a few fibers.

	Ultimate stress [MPa]	Modulus of elasticity [GPa]	Density [kg/m^3]
E-Glass	3400	72	2550
S-Glass	4800	86	2500
Carbon	1700	190	1410
Boron	3400	400	2570
Graphite	1700	250	1410

… fibers

- 3 categories of structural design
  - strength design
  - stiffness design
  - buckling design

2.4 Material selection for structural performance

❖ 2.4.1 Strength design

➢ For a given mass and geometry, the max. load it can carry

$$P_{\max} \propto \frac{\sigma_{ult}}{\rho} \qquad (2.38)$$

… material performance index

❖ 2.4.2 Stiffness design

➢ Cantilevered, thin-walled beam of length L, natural freq.

$$\omega \propto \frac{h}{L^2}\left[\frac{E}{\rho}\right]^{1/2} \qquad (2.40)$$

… material performance index

❖ 2.4.3 Buckling design

➢ Critical load that will cause the plate to buckle

$$P_{cr} \propto \frac{M^2}{b^4 L^3}\frac{E}{\rho^3}$$

… material performance index

2.5 Composite materials

Table 2.4. Structural design performance indices for a few metals.

Performance index	Strength design σ_{ult}/ρ [10^3 m^2/sec^2]	Stiffness design $\sqrt{E/\rho}$ [10^3 m/sec]	Buckling design E/ρ^3 [m^8/(kg^{2}sec^2)]
Aluminum	230	5.2	3.7
Titanium	405	4.9	1.1
Steel	530	5.2	0.46

Table 2.5. Structural design performance indices for a few fibers.

Performance Index	Strength design σ_{ult}/ρ [10^3 m^2/sec^2]	Stiffness design $\sqrt{E/\rho}$ [10^3 m/sec]	Buckling design E/ρ^3 [m^8/(kg^2 sec^2)]
E-Glass	1330	5.3	4.3
S-Glass	1920	5.9	5.5
Carbon	1200	11.6	68
Boron	1320	12.5	23
Graphite	1200	13.3	89

- ··· performance indices for metals and fibers

strength design ··· steel is the best
Stiffness design ··· 3 equally well
Strength and buckling ··· Al >> steel and Ti

Remarkably high performance indices of fibers -> potential use in structural applications

2.5 Composite materials

❖ 2.5.1 Basic characteristics

- Embedding fiber aligned in a single direction, in a matrix material
- Matrix material ··· thermostat polymeric material, ex) epoxy
- "rule of mixture" ··· strength

$$S_c = V_f S_f + V_m S_m$$

S: strength, V: volume fraction, $V_f + V_m = 1$
Ex) graphite fiber ($V_f = 0.6$) embedded in an epoxy matrix ($V_m = 0.4$)

$$S_c = 1,700 \times 0.6 + 50 \times 0.4 = 1,040 (MPa)$$

- Stiffness ··· assuming that perfectly bonded together

$$\varepsilon_m = \varepsilon_f = \varepsilon_c \qquad (2.47)$$

- Average stress

$$P = A_c \sigma_c = A_f \sigma_f + A_m \sigma_m \qquad (2.48)$$

Dividing by

$$\sigma_c = \frac{A_f}{A_c}\sigma_f + \frac{A_m}{A_c}\sigma_m = V_f \sigma_f + V_m \sigma_m \qquad (2.49)$$

2.5 Composite materials

- Fiber, matrix → linearly elastic, isotropic

$$\sigma_f = E_f \varepsilon_f \qquad \sigma_m = E_m \varepsilon_m \qquad (2.50)$$

- Modulus of elasticity for the composite, E_c

$$\sigma_c = E_c \varepsilon_c \qquad (2.51)$$

Eq. (2.50), (2.51) → (2.49) : $E_c = V_f E_f + V_m E_m$ (2.52)

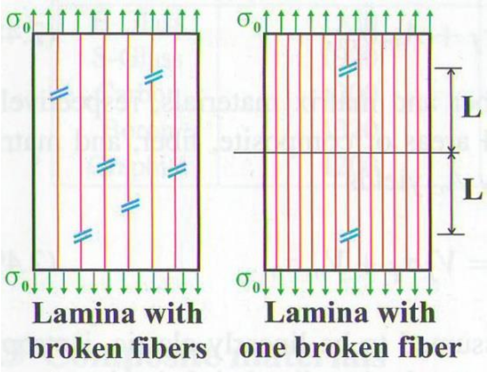
Ex) graphite-epoxy: $E_c = 250 \times 0.6 + \underbrace{3.5 \times 0.4}_{\text{contributes little}} = 150\,GPa$

- What is the role of the matrix material?
  ① Keep all the fibers together
  ② Diffuse the stresses among the otherwise isolated fibers

2.5 Composite materials

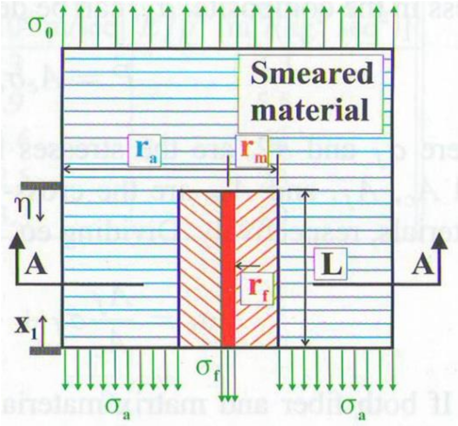
2.5.2 stress diffusion in composites

❖ **Fig. 2.12: single broken fiber of length 2L**



→ Matrix material adjacent to the broken fiber will transfer stress from the surrounding material to the broken fiber
"stress diffusion process"

❖ **Fig. 2.13: simplified model**



- Assumptions
  ① Matrix carries shear stress only
  ② Axial stress in the fiber is uniformly distributed
  ③ Existence of individual fibers ignored in the remaining composite
  ④ Perfectly bonded together

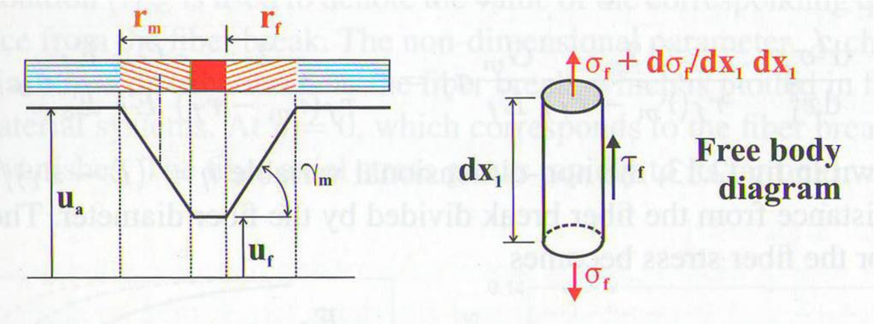
2.5 Composite materials

- Strain-displacement relationship

$$\varepsilon_f = \frac{du_f}{dx_1}\ ,\qquad \varepsilon_a = \frac{du_a}{dx_1}\ ,\qquad \gamma_m = \frac{u_a - u_f}{r_m - r_f} \qquad (2.54)$$

- Axial force equilibrium of a differential element of fiber

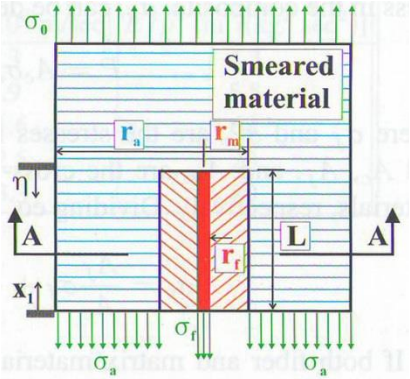
$$\frac{d\sigma_f}{dx_1} + \frac{2}{r_f}\tau_m = 0 \qquad (2.55)$$



- Overall equilibrium of an entire model

$$\sigma_a = \frac{\sigma_0}{1-\frac{r_m^2}{r_a^2}} - \frac{r_f^2}{r_a^2}\frac{\sigma_f}{1-\frac{r_m^2}{r_a^2}} \approx \sigma_0 \qquad (2.56)$$

$\frac{r_f}{r_a} << 1$ $\rightarrow$2nd term negligible, $\frac{r_m}{r_a} << 1$



2.5 Composite materials

- Constitutive laws for fiber, composite, and matrix

$$\sigma_f = E_f \varepsilon_f \ , \quad \sigma_a = E_a \varepsilon_a \ , \quad \tau_m = G_m \gamma_m \qquad (2.57)$$

- Eq.(2.57c), (2.54c) → Eq.(2.55)

$$\frac{d^2 \sigma_f}{dx_1} + \frac{2G_m}{r_f (r_m - r_f)} \left(u_a - u_f \right) = 0$$

- Differentiate w.r.t. x_1 and substituting Eqs. (2.54a), (2.54b), (2.57a), (2.57b)

$$\frac{d^2 \sigma_f}{dx_1} + \frac{2G_m}{r_f (r_m - r_f)} \left(\frac{\sigma_a}{E_a} - \frac{\sigma_f}{E_f} \right) = 0$$

- Since $\sigma_a \approx \sigma_0$ (Eq.2.56),

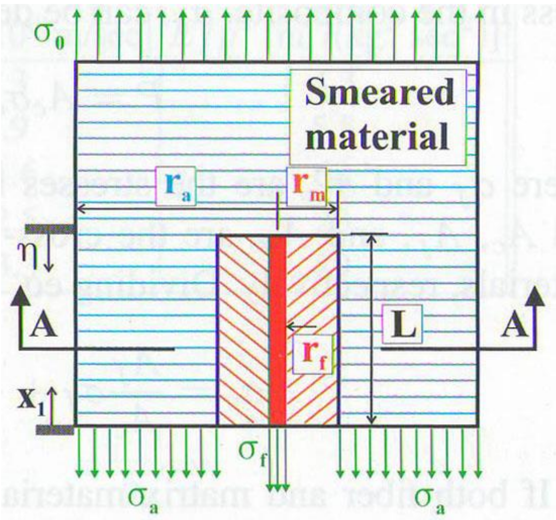
$$\frac{d^2 \sigma_f}{dx_1^2} - \frac{2}{r_f (r_m - r_f)} \frac{G_m}{E_f} \sigma_f = - \frac{2}{r_f (r_m - r_f)} \frac{G_m}{E_f} \frac{E_f}{E_a} \sigma_0$$

2.5 Composite materials

- Non-dimensional variable $\eta = (L - x_1)/(2r_f)$
- Then, the governing eqn.

$$\sigma_f'' - \lambda^2 \sigma_f = -\lambda^2 \frac{E_f}{E_a} \sigma_0$$ ( )′: derivative w.r.t η

$$\lambda^2 = 8 \frac{G_m}{E_f} \frac{r_f}{r_m} \frac{1}{1 - r_f / r_m}$$



- $\dfrac{E_f}{E_a} = \dfrac{E_f}{V_f E_f + V_m E_m} \approx \dfrac{E_f}{V_f E_f} = \dfrac{1}{V_f}$ since $E_m << E_f$

- Governing eqn.

$$\sigma_f'' - \lambda^2 \sigma_f = -\lambda^2 \frac{\sigma_0}{V_f} \quad (2.58)$$

where $\lambda^2 = 8 \dfrac{G_m}{E_f} \dfrac{\sqrt{V_f}}{1 - \sqrt{V_f}}$ (2.59)

B.C.: $\sigma_f = 0$ at $\eta = 0$ (broken fiber)

$\sigma_f' = 0$ at $\eta = L/2r_f$ (symmetry)

2.5 Composite materials

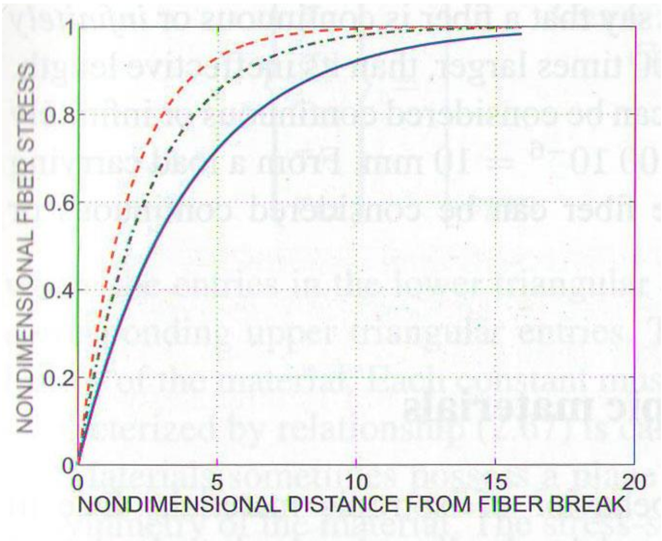
- Solution

$$\frac{\sigma_f}{\sigma_0}=\frac{1}{V_f}\left(1-\frac{\cosh\lambda\left(L/2r_f-\eta\right)}{\cosh\left(\lambda L/2r_f\right)}\right)\approx\frac{1}{V_f}\left(1-e^{-\lambda\eta}\right)\quad(2.60)$$

Since $\sigma_0=V_f\sigma_{f\infty}+(1-V_f)\sigma_{m\infty}\approx V_f\sigma_{f\infty}$,

Eq. (2.60) → $\frac{\sigma_f}{\sigma_{f\infty}}=1-e^{-\lambda\eta}$ (2.61)

:fiber axial stress distribution near the fiber break



2.5 Composite materials

- Ineffective length δ : the distance where the fiber stress reaches 95% of its for field value

$$0.95 = 1 - \exp\left(-\lambda\delta / d_f\right) \rightarrow \frac{\delta}{d_f} \approx \left[\frac{E_f}{G_m}\frac{1-\sqrt{V_f}}{\sqrt{V_f}}\right]^{1/2} \quad (2.62)$$

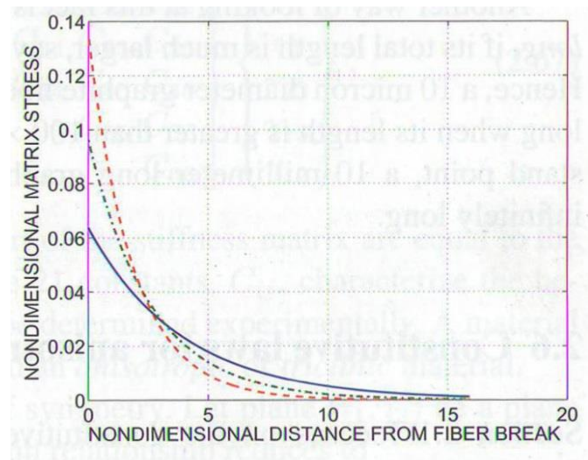
:length of fiber, near a fiber break, that does not carry axial stress at fully capacity

→ Matrix material transfers the load from the surrounding material to the broken fiber very rapidly ("shear lag")
Shear stress in the matrix is effectively transferring the load to the fiber

$$\frac{\tau_m}{\sigma_{f\infty}} = \frac{\lambda}{4}e^{-\lambda\eta} \quad (2.63)$$

- Zone affected by a fiber break → about 2δ in length

Ex) graphite of dia.10micron
→ Zone of only 200 microns in length



2.6 Constitutive laws for anisotropic material

- **Unidirectional composite materials** → fiber dir., dominated by that of fiber
  - transverse to fiber, dominated by that of matrix

- **Linear relationship between the stress and strain**

$$\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon} \;:\; \underline{\varepsilon} = \underline{\underline{S}}\,\underline{\sigma} \quad (2.64) \qquad \underline{\underline{S}} = \underline{\underline{C}}^{-1} \quad (2.65)$$

↑ 6 x 6 stiffness ↑ 6 x 6 compliance

  - Strain energy: $A = \frac{1}{2}\underline{\varepsilon}^T\underline{\sigma} = \frac{1}{2}\underline{\varepsilon}^T\underline{\underline{C}}\,\underline{\varepsilon} = \frac{1}{2}\underline{\sigma}^T\underline{\underline{S}}\,\underline{\sigma}$

  → both $\underline{\underline{C}}$ and $\underline{\underline{S}}$ are symm. and positive definite

2.6 Constitutive laws for anisotropic material

- **Due to symmetry, 6x6=36 independent consts → 21** (2.67)

  "anisotropic" or "triclinic" material

  - Plain of symmetry: (i_1 , i_2) plane of symmetry

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \quad (2.68)$$

  - If $C_{14} \neq 0$, ε_1 would give rise to τ_{23} → violate the symmetry of response

    → 21-8=13 independent consts "monoclinic" material

2.6 Constitutive laws for anisotropic material

- 2 mutually orthogonal planes of symmetry: (i_1, i_2), (i_2, i_3)

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \qquad (2.69)$$

→ 21-12=9 independent consts, "orthotropic" material

- Laminated composite material → $\begin{cases} \text{2 orthogonal plane of symmetry: } (i_1, i_2), (i_2, i_3) \\ \text{1 plane of isotropy: } (i_2, i_3) \end{cases}$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix} \qquad (2.70)$$

→ 5 constants, "transversely isotropic"

2.6 Constitutive laws for anisotropic material

- Isotropic

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ & & & & \frac{C_{11}-C_{12}}{2} & 0 \\ & & & & & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \quad (2.71)$$

→ 2 constants

- $\begin{cases} \text{Not clear about } C_{11}, C_{12} \\ \text{“Engineering consts”: Young`s modulus, Poisson`s ratio} \end{cases}$

→ experimental determination and physical interpretation

2.6 Constitutive laws for anisotropic material

2.6.1 Constitutive laws for a lamina in the fiber aligned triad

❖ **Thin sheet of composite material made of unidirectional fibers**

$\bar{i}_1^*$: fiber direction $\quad \bar{i}_2^*$: transverse direction

$\bar{i}_3^*$: perpendicular to the plane of thin sheet

→ "fiber aligned triad"

→ can be assumed as a homogeneous, transversely isotropic material

❖ **Plane stress state: constitutive laws in compliance form**

$$\begin{Bmatrix} \varepsilon_1^* \\ \varepsilon_1^* \\ \gamma_{12}^* \end{Bmatrix} = \begin{bmatrix} 1/E_1^* & -\nu_{21}^*/E_2^* & 0 \\ -\nu_{12}^*/E_1^* & 1/E_2^* & 0 \\ 0 & 0 & 1/G_{12}^* \end{bmatrix} \begin{Bmatrix} \sigma_1^* \\ \sigma_1^* \\ \tau_{12}^* \end{Bmatrix} \qquad (2.72)$$

- $E_1^*,\ E_2^*,\ \nu_{12}^*,\ G_{12}^*$: engineering consts
- Symm. → $\nu_{12}^*/E_1^* = \nu_{21}^*/E_2^*$ → one of 5 consts is not an independent quantity

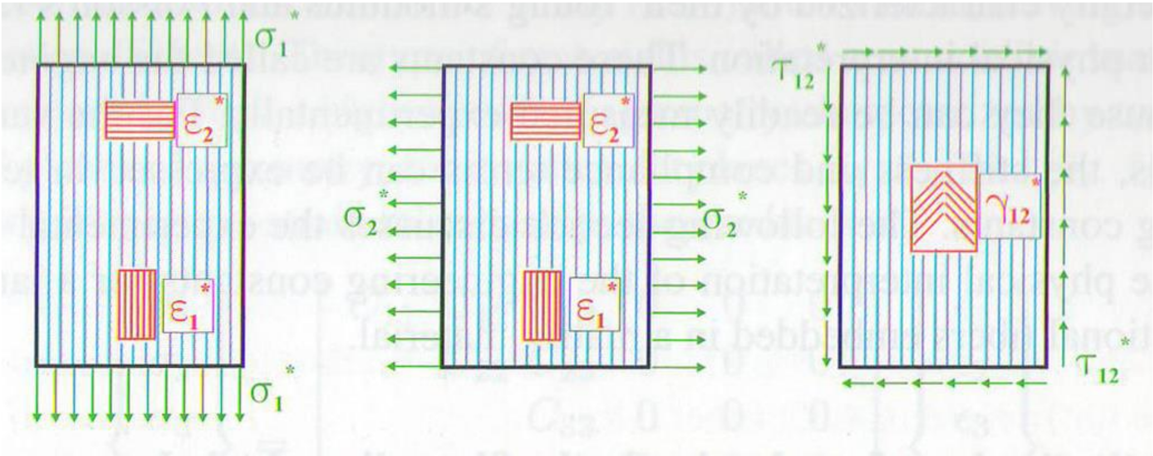
2.6 Constitutive laws for anisotropic material

- Simple test of a known stress σ_1^* , then $\sigma_2^* = \tau_{12}^* = 0$
  - ① of Eq.(2.72) → $\varepsilon_1^* = \sigma_1^* / E_1^*$, E_1^* can be determined
  - ② of Eq.(2.72) → $\varepsilon_2^* = -\nu_{12}^* \sigma_1^* / E_1^*$, ν_{12}^* can be determined

- 2nd test of a known stress σ_2^* , then $\sigma_1^* = \tau_{12}^* = 0$

  $\varepsilon_2^* = \sigma_2^* / E_2^*$, E_2^* can be obtained

- Last test of a known τ_{12}^*, then $\sigma_1^* = \sigma_2^* = 0$
  - ③ of Eq.(2.72) → $\gamma_{12}^* = \tau_{12}^* / G_{12}^*$, G_{12}^* can be obtained



2.6 Constitutive laws for anisotropic material

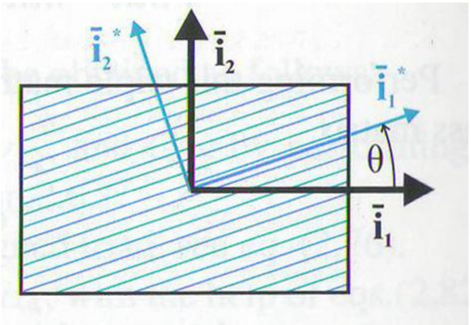
➢ Stiffness matrix: by inverting Eq. (2.72)

$$\begin{Bmatrix} \sigma_1^* \\ \sigma_1^* \\ \tau_{12}^* \end{Bmatrix} = \begin{bmatrix} \dfrac{E_1^*}{1-\nu_{12}^{*2}E_2^*/E_1^*} & \dfrac{\nu_{12}^*E_2^*}{1-\nu_{12}^{*2}E_2^*/E_1^*} & 0 \\ \dfrac{\nu_{12}^*E_2^*}{1-\nu_{12}^{*2}E_2^*/E_1^*} & \dfrac{E_2^*}{1-\nu_{12}^{*2}E_2^*/E_1^*} & 0 \\ 0 & 0 & G_{12}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_1^* \\ \varepsilon_1^* \\ \gamma_{12}^* \end{Bmatrix} \quad (2.73)$$

2.6 Constitutive laws for anisotropic material

2.6.2 Constitutive laws for a lamina in an arbitraty triad

❖ **Fig. 2.18**



Laminar of a direction that might not coincide with that of fiber
counterclockwise θ orientation of fiber w.r.t. ref. direction
← formulae for stresses and strains in a rotated axis system

1) Rotations of the stiffness matrix

- Constitutive laws for a lamina in the fiber aligned triad

$$\underline{\sigma}^* = \underline{\underline{C}}^* \underline{\varepsilon}^*$$

- Introducing the rotation formulae, Eqs. (1.47), (1.91)

$$\begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \underline{\underline{C}}^* \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2-n^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

where $m = \cos\theta$, $n = \sin\theta$

- Multiplying from the left by the inverse of the rotation matrix for the stress

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \underbrace{\begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix} \underline{\underline{C}}^* \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2-n^2 \end{bmatrix}}_{\underline{\underline{C}}} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (2.79)$$

2.6 Constitutive laws for anisotropic material

- More compact manner of the relationship

$$\underline{C}(\theta) = \underline{\underline{x}}(\theta)\underline{\alpha} \quad (2.86)$$

where

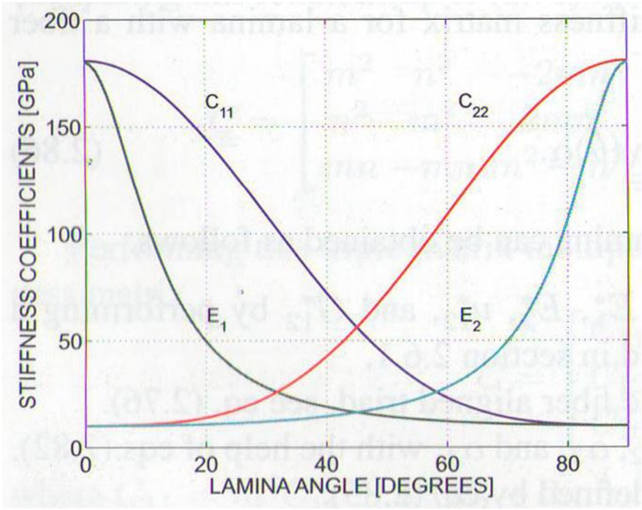
$$\underline{C} = \{C_{11} \quad C_{22} \quad C_{12} \quad C_{66} \quad C_{16} \quad C_{26}\}^T \quad (2.84)$$

$$\underline{\underline{x}}(\theta) = \begin{bmatrix} 1 & 1 & \cos 2\theta & \cos 4\theta \\ 1 & 1 & -\cos 2\theta & \cos 4\theta \\ 1 & -1 & 0 & -\cos 4\theta \\ 0 & 1 & 0 & -\cos 4\theta \\ 0 & 0 & \frac{1}{2}\sin 2\theta & \sin 4\theta \\ 0 & 0 & \frac{1}{2}\sin 2\theta & -\sin 4\theta \end{bmatrix} \quad (2.83)$$

2.6 Constitutive laws for anisotropic material

$$\underline{\alpha} = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\}^T$$ "material invariants" (2.85)

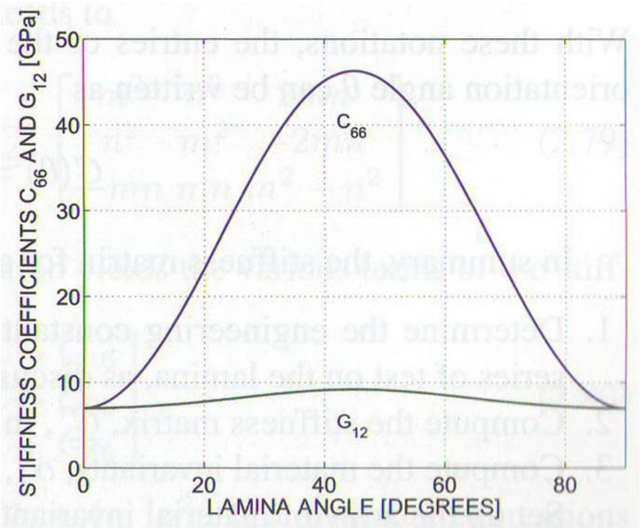
with Eq. (2.82)



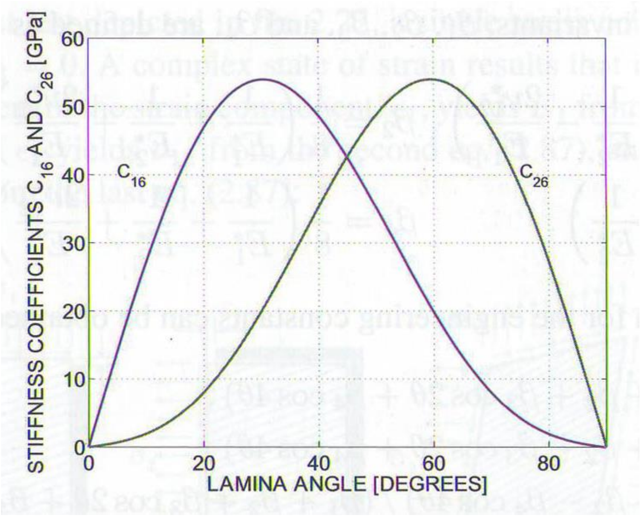
C_{11} , C_{12} in terms of θ , sharp decline

→ high directionality

2.6 Constitutive laws for anisotropic material



C_{66} very high near $\theta = 45°$



$C_{11}, C_{26} \neq 0$ in terms of θ, coupling between extension and shearing

$C_{11}, C_{26} = 0$ in $\underline{\underline{C}}^*$

$\leftarrow$ response of the systems must be symmetric precluding extension-shear couple

2.6 Constitutive laws for anisotropic material

2) Rotations of the compliance matrix

$$\underline{\underline{S}} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} \underline{\underline{S}}^* \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \quad (2.88)$$

$$= \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & \nu_{61}/G_{12} \\ -\nu_{12}/E_1 & 1/E_2 & \nu_{62}/G_{12} \\ \nu_{16}/E_1 & \nu_{26}/E_2 & 1/G_{12} \end{bmatrix}$$

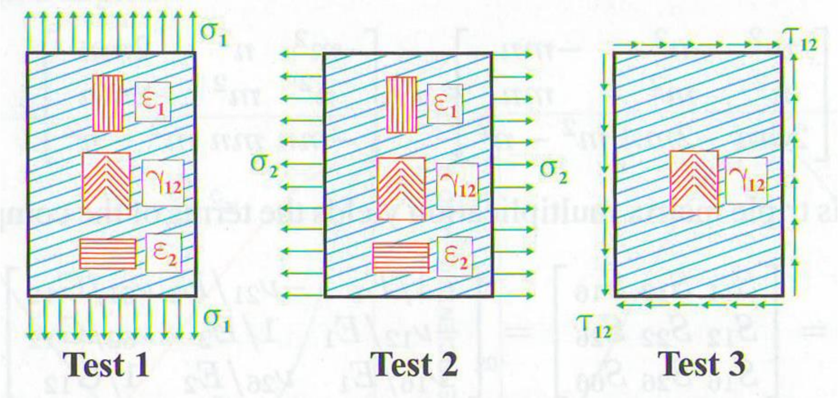
- $E_1, E_2, \nu_{12}, G_{12}, \nu_{16}, \nu_{26}$: engineering constants in the arbitrary triad
- $\underline{\underline{S}}$ must be symmetric

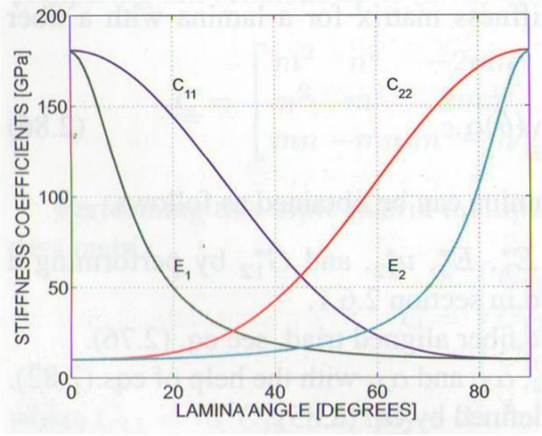
2.6 Constitutive laws for anisotropic material

- Alternative expression for engineering const. – Eq. (2.92)

  Various tests to determine the engineering const.s

  Similar to those in sec 2.6.1, but currently stress is applied at θ





E_1 shows precipitous drop w.r.t. θ

2.6. Constitutive laws for anisotropic material

❖ **Difference between C_{11} and E_1**

- $E_1 = 1/S_1,\ 1/S_{11} \neq C_{11}$ since the inverse of a matrix is not simply the inverse of its items

Fig. 2.22 – to measure E_1 , σ_1 is applied, $\sigma_2 = \tau_{12} = 0,\ \varepsilon_1 \rightarrow E_1,\ \varepsilon_2 \rightarrow \nu_{12},\ \gamma_{12} \rightarrow \nu_{16}$ in Eq (2.87)

Fig. 2.23 – to measure C_{11} , ε_1 is applied, $\varepsilon_2 = \gamma_{12} = 0$

but test is very difficult to perform since would have to be constrained to prevent any deformations except ε_1

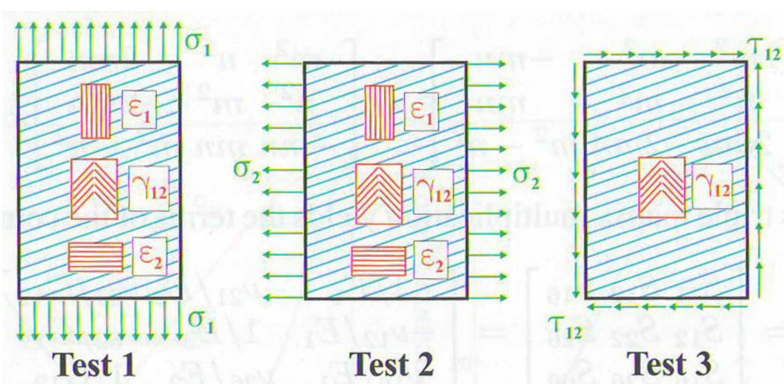


Fig. 2.22

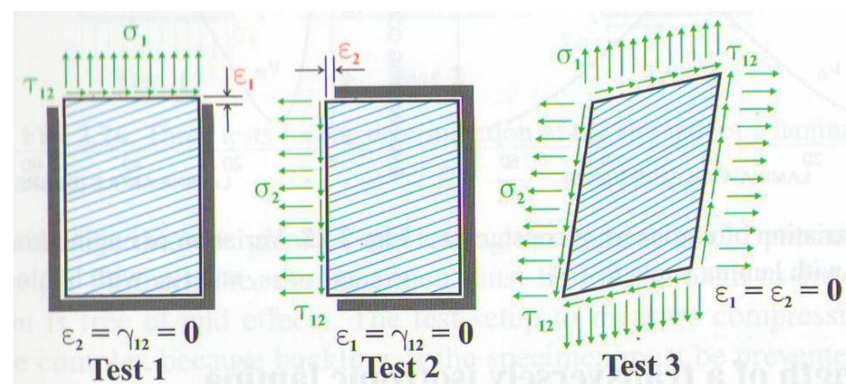


Fig. 2.23

2.6 Constitutive laws for anisotropic material

➢ Effect of these constraints → considerably stiffen the material

ex) $C_{11} >> E_1$ (Fig. 2.19)

$C_{66} >> G_{12}$ (Fig. 2.20)

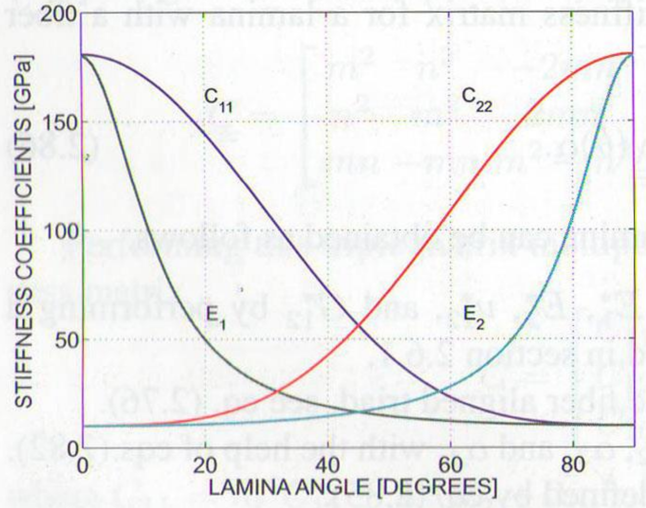


Fig. 2.19. Variation of the stiffness coefficients, C_{11} and C_{22}, and the engineering constants, E_1 and E_2, as a function of θ.

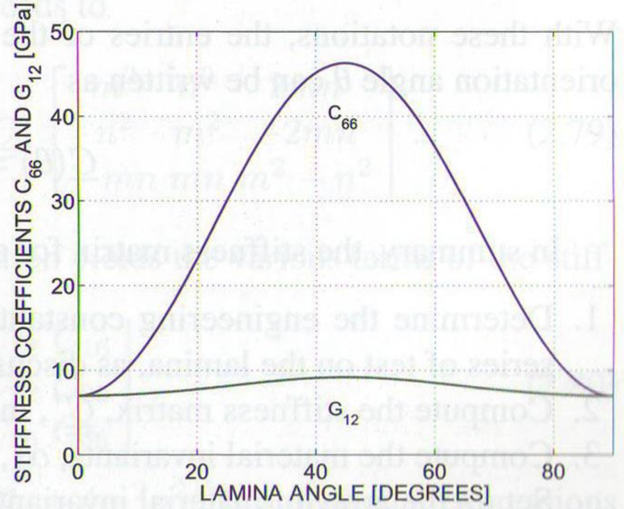
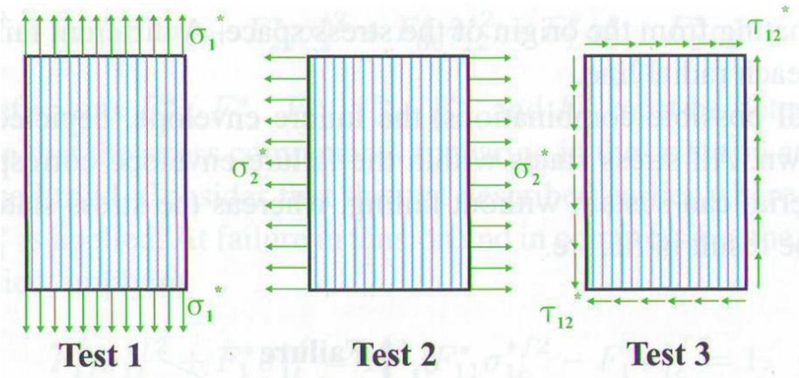


Fig. 2.20. Variation of the stiffness coefficient, C_{66}, and engineering constant, G_{12} as a function of θ.

2.7 Strength of a transversely isotropic lamina

2.7.1 Strength of a lamina under simple loading condition

❖ **Fig. 2.26**



① σ_1^* applied in the fiber direction, and $\sigma_2^* = \tau_{12}^* = 0$
will provide σ_{1t}^{*f} and σ_{1c}^{*f} (not equal, generally)

② σ_2^* applied in the transverse direction, and $\sigma_1^* = \tau_{12}^* = 0$
will provide σ_{2t}^{*f} and σ_{2c}^{*f}

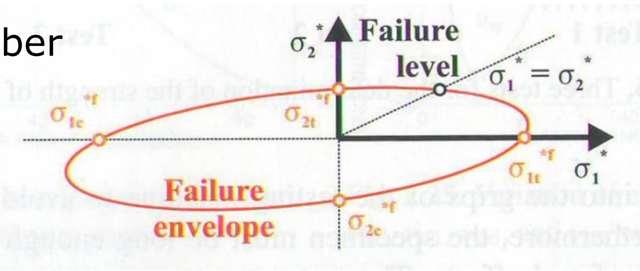
③ Shear stress τ_{12}^* applied and
$\rightarrow \tau_{12}^{*f}$, no dependence on sign

❖ **Tests can be very difficult to perform in practice**

2.7.2 Strength of a lamina under combined loading conditions

❖ **Fig. 2.27**

Failure envelope, rather than performing a large number of experiments, apply a failure criterion
→ many different failure criteria, widely used



❖ **Matrix failure** – not always a catastrophic event
Fiber failure – completely eliminates load carrying capability

2.7 Strength of a transversely isotropic lamina

2.7.3 The Tsai-Wu failure criterion

❖ **Combined stresses applied**

$$F_{11}^{*}\sigma_1^{*2} + 2F_{12}^{*}\sigma_1^{*}\sigma_2^{*} + F_{22}^{*}\sigma_2^{*} + F_{66}^{*}\tau_{12}^{*2} + F_{11}^{*}\sigma_1^{*} + F_2^{*}\sigma_2^{*} = 1 \qquad (2.93)$$

① Test with a single stress component σ_1^{*} applied

$$F_{11}^{*}\sigma_{1t}^{*2} + F_{11}^{*}\sigma_{1t}^{*f} = 1, \quad F_{11}^{*}\sigma_{1c}^{*2} - F_{11}^{*}\sigma_{1c}^{*f} = 1$$

② σ_2^{*} only

③ τ_{12}^{*} only

→ then, can find 5 coefficients in Eq.(2.93)