Aircraft Structures

CHAPER 9.
Virtual Work Principle

Prof. SangJoon Shin

9.1 Introduction

Mechanical work : Scalar product of the force by the displacement through which it acts → scalar quantity → simpler to manipulate → very attractive

Newton's equilibrium condition : The sum of all force (regardless of externally applied loads, internal forces, and reaction forces) must vanish

Analytical mechanics : powerful tools for complex problems

- Scalar quantities, simpler analysis procedure
- Reaction forces can often be eliminated if the work involved vanishes.
- Systematic development of procedure for approximate solutions (ex : finite element method)

Why still need Newton's formulation? : to determine both magnitude and direction of all forces acting within a structure, to estimate failure condition

Principle of virtual work (PVW) ⟷ Newton's law

equivalent

9.2 Equilibrium and work fundamentals

9.2.1 Static equilibrium conditions

Newton's 1st law : every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it

- A particle at rest tends to remain at rest unless the sum of the externally applied force does not vanish.
- A particle is at rest if and only if the sum of the externally applied forces vanishes.
- A particle is in static equilibrium if and only if the sum of the externally applied forces vanishes.
- A particle is in static equilibrium iff $\sum \underline{F} = 0$ (9.1)

(1) The vector sum of all forces acting on a particle must be zero.

(2) The vector polygon must be closed.

(3) The component of the vector sum resolved in any coord. system must vanish.

$$\sum \underline{F} = F_1 \bar{i}_1 + F_2 \bar{i}_2 + F_3 \bar{i}_3 \rightarrow F_1 = F_2 = F_3 = 0$$

9.2 Equilibrium and work fundamentals

9.2.1 Static equilibrium conditions

Newton's 3rd law : If particle A exerts a force on particle B, particle B simultaneously exerts on particle A a force of identical magnitude and opposite direction.

- Two interacting particles exert on each other forces of equal magnitude, opposite direction, and sharing a common line of action.

Euler's 1st law

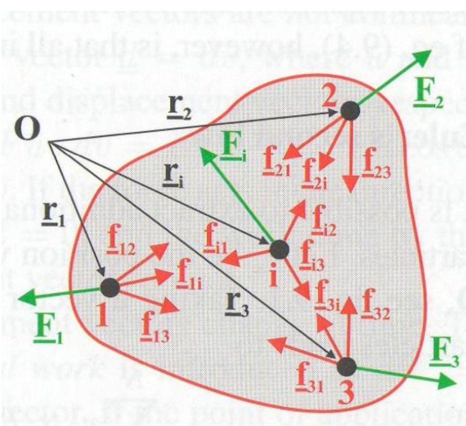


Fig. 9.1. A system of particles.

system consisting of N particles

Particle i subjected to an external force $\underline{F}_i$, $N-1$ interaction forces $\underline{f}_{ij}$, $j=1,2\ldots$, $N, j\neq i$

Newton's 1st law

$$\underline{F}_i + \sum_{j=1,\, j\neq i}^{N} \underline{f}_{ij} = 0 \qquad (9.2)$$

9.2 Equilibrium and work fundamentals

9.2.1 Static equilibrium conditions

Interaction forces : for rigid body, it will ensure the body shape remain unchanged
elastic body, stress resulting from deformation planetary
system, gravitational pull

Summation of *N* eqns. for *N* particles

$$\sum_{i=1}^{N} \underline{F}_i + \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \underline{f}_{ij} = 0$$

By Newton's 3rd law,

$$\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \underline{f}_{ij} = 0 \qquad (9.3)$$

Then,

$$\sum_{i=1}^{N} \underline{F}_i = 0 \qquad (9.4)$$

Euler's 1st law for a system of particles
necessary condition for a system of particles to be in static equilibrium but not a sufficient condition

9.2 Equilibrium and work fundamentals

9.2.1 Static equilibrium conditions

Euler's 2nd law

- Taking a vector product of

$$\sum_{i=1}^{N} \underline{F}_i + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} = 0$$

by $\underline{r}_i$, then summing over all particles

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{r}_i \times \underline{f}_{ij} = 0$$

then,

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \sum_{i=1}^{N} \underline{M}_i = 0 \qquad (9.6)$$

- Euler's 1st and 2nd law both necessary condition for the system of particles to be in static equilibrium, but not a sufficient condition.

9.2.2 Concept of mechanical work

- Definition
  - The work done by a force is the scalar product of the force by the displacement of its point of application.

    1. force, displacement, collinear : $\underline{F} = F\bar{u}$, $\underline{d} = d\bar{u}$, $W = Fd$
       + : if the same direction / - : if the opposite direction

    2. not collinear : $W = Fd\cos\theta$, θ angle between $\bar{u}$ and $\bar{v}$

    3. Perpendicular : $\cos\theta = \cos\frac{\pi}{2} = 0$, $W = 0$

  - "incremental work" : $dW = \underline{F} \cdot d\underline{r}$, total work : $W = \int_{\underline{r}_i}^{\underline{r}_f} dW = \int_{\underline{r}_i}^{\underline{r}_f} F \cdot d\underline{r}$ (9.7)
  - $\underline{F} = F_1\bar{e}_1 + F_2\bar{e}_2 + F_3\bar{e}_3$, $d\underline{r} = dr_1\bar{e}_1 + dr_2\bar{e}_2 + dr_3\bar{e}_3$, $dW = \underline{F} \cdot d\underline{r} = F_1dr_1 + F_2dr_2 + F_3dr_3$
  - $d\underline{r} = dr\bar{u}$, $\underline{F} = F_{\parallel}\bar{u} + F_{\perp}\bar{v} \rightarrow dW = \left(F_{\parallel}\bar{u} + F_{\perp}\bar{v}\right) \cdot dr\bar{u} = F_{\parallel}dr$

9.2.2 Concept of mechanical work

- superposition : $\underline{F} = \underline{F}_1 + \underline{F}_2$, $dW = \underline{F} \cdot d\underline{r} = \left(\underline{F}_1 + \underline{F}_2\right) \cdot d\underline{r} = \underline{F}_1 \cdot d\underline{r} + \underline{F}_2 \cdot d\underline{r} = dW_1 + dW_2$

- Why is work a quantity of interest for the static analysis?
  → Concept of "virtual work" that would be done by a force if it were to displace its point of application by a fictitious amount.

9.3 Principle of virtual work

- PVW

  - “arbitrary virtual displacement”, “arbitrary test virtual displacement” “arbitrary fictitious virtual displacement”

  - arbitrary : Displacement can be chosen arbitrarily without any restrictions imposed on their magnitude or orientations.

  - virtual, test, fictitious : Do not affect the forces acting on the particle.

9.3.1 PVW for a single particle

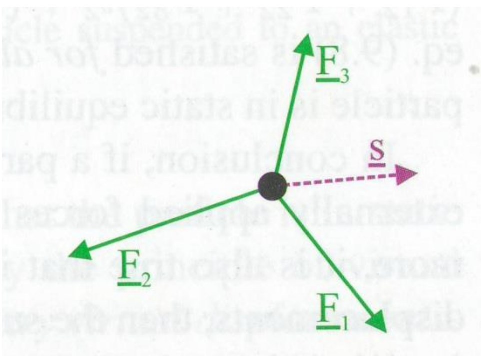


Fig. 9.2. A particle with applied forces subjected to a fictitious test displacement.

particle in static equilibrium under a set of externally applied loads, fictitious displacement of $\underline{s}$

virtual work done

$$W = \left[\sum \underline{F}\right] \cdot \underline{s} = 0 \qquad (9.8)$$

Assume that one of the externally applied forces, $\underline{F}_1$, is an elastic spring force. If for a real, arbitrary displacement, $\underline{d}$, the spring force will change to become $\underline{F}_1$', the sum of eventually applied forces,

$$\sum \underline{F}_1' \rightarrow \sum \underline{F}' \neq 0$$

For a virtual or fictitious displacement, do not affect the loads applied to the particle, it remains in static equilibrium, $W = \left[\sum \underline{F}\right] \cdot \underline{s} = 0$ holds.

If $W = \left[\sum \underline{F}\right] \cdot \underline{s} = 0$ is satisfied for all arbitrary virtual displacement, then $\sum \underline{F} = 0$, and the particle is in static equilibrium.

9.3 Principle of virtual work

9.3.1 PVW for a single particle

Principle 3 (PVW for a particle) : A particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary virtual displacement.

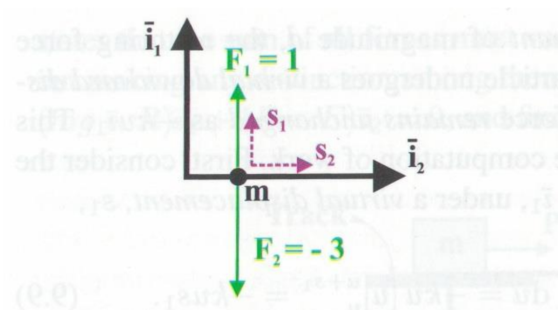


Fig. 9.3. A particle under the action of two forces.

Example 9.1 Equilibrium of a particle

$$\underline{F}_1 = 1\bar{i}_1 \qquad \underline{F}_2 = -3\bar{i}_1 \qquad \underline{s} = s_1\bar{i}_1 + s_2\bar{i}_2$$

Virtual work is

$$W = \left(1\bar{i}_1 - 3\bar{i}_2\right)\cdot\left(s_1\bar{i}_1 + s_2\bar{i}_2\right) = -2\bar{i}_1\cdot\left(s_1\bar{i}_1 + s_2\bar{i}_2\right) = -2s_1 \neq 0$$

Because the virtual work done by the externally applied forces does not vanish for all virtual displacement, the principle of virtual work, Principle 3, implied that the particle is not in static equilibrium.

9.3 Principle of virtual work

9.3.1 PVW for a single particle

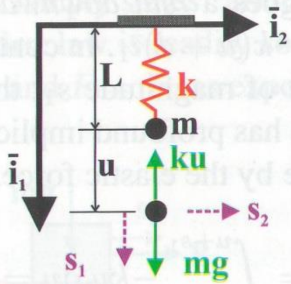


Fig. 9.4. A particle suspended to an elastic spring.

Example 9.2 Equilibrium of a particle connected to an elastic spring

$$W = \left(mg\bar{i}_1 - ku\bar{i}_1\right)\cdot\left(s_1\bar{i}_1 + s_2\bar{i}_2\right) = \left[mg - ku\right]s_1$$

$$\left[mg - ku\right]s_1 = 0$$

But, $s_1 = 0$ is not valid because, as implied by the principle of virtual work, s_1 is arbitrary.
In conclusion, the vanishing of the virtual work for all arbitrary virtual displacement implies that $mg - ku = 0$, and the equilibrium configuration of the system is found as $u = mg/k$.

9.3 Principle of virtual work

9.3.1 PVW for a single particle

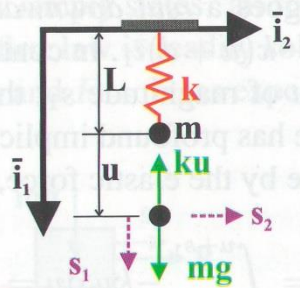


Fig. 9.4. A particle suspended to an elastic spring.

Consider the work done by the elastic force, $-ku\bar{i}_1 \cdot du\bar{i}_1$, under a virtual displacement, s_1,

$$W = \int_u^{u+s_1} -kudu = -ku\int_u^{u+s_1} du = -ku\left[u\right]_u^{u+s_1} = -kus_1 \quad (9.9)$$

It is possible to remove the elastic force, $-ku$, from the integral because this force remains unchanged by the virtual displacement, and hence, it can be treated as a constant.
In contrast, the work done by the same elastic force under a real displacement, d, is

$$W = \int_u^{u+d} -kudu = \left[-\frac{1}{2}ku^2\right]_u^{u+d} = -kud - \left[-\frac{1}{2}kd^2\right] \quad (9.10)$$

In this case, the real work includes an additional term that is quadratic in d and represents the work done by the change in force that develops due to the stretching of the spring.

9.3.1 PVW for a single particle

Example 9.3 Equilibrium of a particle sliding on a track

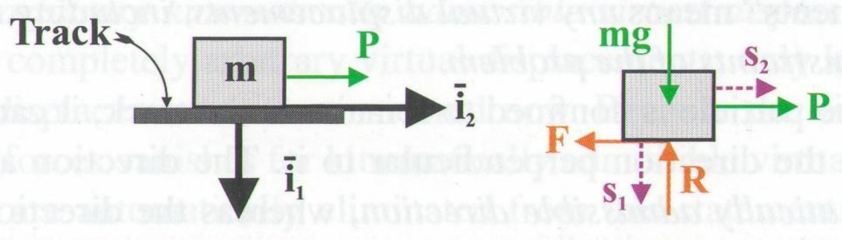


Fig. 9.5. A particle sliding on a track.

$$mg\bar{i}_1 - R\bar{i}_1 + P\bar{i}_2 - F\bar{i}_2 = 0$$

$$(mg - R)\bar{i}_1 + (P - F)\bar{i}_2 = 0$$

Finally, $R = mg$ $\quad F = P$

Next, by PVW,

$$W = (mg\bar{i}_1 - R\bar{i}_1 + P\bar{i}_2 - F\bar{i}_2)\cdot(s_1\bar{i}_1 + s_2\bar{i}_2) = [mg - R]s_1 + [P - F]s_2 = 0 \qquad (9.11)$$

9.3.2 Kinematically admissible virtual displacement

- "arbitrary virtual displacements" : including those that violate the kinematic constraints of the problem
  - "kinematically inadmissible direction", "infeasible direction" : s_1 in the track example → $\underline{s} = s_2 \bar{\imath}_2$ kinematically admissible
  - Reaction forces acts along the kinematically inadmissible direction

- Modified version of PVW : "a particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary kinematically admissible virtual displacements"
  - Constraint (reaction) forces are automatically eliminated.
  - Fewer number of equations

9.3 Principle of virtual work

9.3.3 Use of infinitesimal displacements as virtual displacements

- Special notation commonly used to denote virtual displacements

$$\underline{s} = \delta \underline{u}$$

  Virtual work done by a force undergoing virtual displacement → δW
- Convenient to use virtual displacements of infinitesimal magnitude
  → Often simplifies algebraic developments

1. Displacement dependent force → automatically remain unaltered

Ex 9.6 Consider a particle connected to an elastic spring. This is the same problem treated in Ex 9.2

$$\delta W = \left(mg\bar{i}_1 - ku\bar{i}_1\right)\cdot\left(\delta u\bar{i}_1 + \delta v\bar{i}_2\right) = \left[mg - ku\right]\delta u = 0$$

$$\delta u = du$$

$$\int_u^{u+du} -kudu = \left[-\frac{1}{2}ku^2\right]_u^{u+du} = -kudu - \frac{1}{2}k(du)^2 = -kudu$$

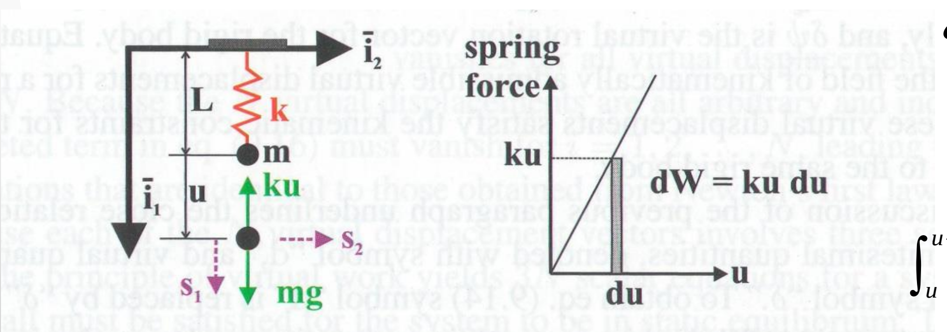


Fig. 9.7. Use of a differential displacement as a virtual displacement.

9.3 Principle of virtual work

9.3.3 Use of infinitesimal displacements as virtual displacements

2. Rigid bodies

- 2 point P, Q of a rigid body → must satisfy the rigid body dynamics

$$\underline{v}_P = \underline{v}_Q + \underline{\omega} \times \underline{r}_{QP}$$

$$\frac{d\underline{u}_P}{dt} = \frac{d\underline{u}_Q}{dt} + \frac{d\underline{\psi}}{dt} \times \underline{r}_{QP}$$

$$d\underline{u}_P = d\underline{u}_Q + \underline{d\psi} \times \underline{r}_{QP} \qquad (9.14)$$

- It is possible to write

$$\delta\underline{u}_P = \delta\underline{u}_Q + \underline{\delta\psi} \times \underline{r}_{QP}$$

field of kinematically admissible virtual displacements for a rigid body

9.3 Principle of virtual work

9.3.3 Use of infinitesimal displacements as virtual displacements

δ : virtual fictitious displacement, leave the forces unchanged, allowed to violate the kinematic constraints

d : real, infinitesimal displacement, no requirement for forces, cannot violate the kinematic constraints.

- $\underline{\delta\psi}$: vector quantity, but finite rotations are scalar quantity.
- Virtual displacements of infinitesimal magnitude greatly simplifies the treatment.

9.3 Principle of virtual work

9.3.4 PVW for a system of particles

- For a particle i,

$$\delta W_i = \left(\underline{F}_i + \sum_{j=1,\, j \neq i}^{N} f_{ij} \right) \cdot \delta \underline{u}_i \qquad (9.15)$$

- Sum of virtual work : All particles must also vanish.
  A system of particles is in static equilibrium if and only if

$$\delta W_i = \sum_{i=1}^{N} \left[\left(\underline{F}_i + \sum_{j=1,\, j \neq i}^{N} f_{ij} \right) \cdot \delta \underline{u}_i \right] = 0 \qquad (9.16)$$

for all virtual displacements, $\delta \underline{u}_i$, $i = 1, 2, 3, \cdots, N$

- 3N scalar eqn.s for a system of N particles → 3N D.O.F.'s

9.3 Principle of virtual work

9.3.4 PVW for a system of particles

- Internal and external virtual work
  - Internal forces : act and reacted within the system
  - External forces : act on the system but reacted outside the system

$$\delta W_E = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{u}_{ij}$$

$$\delta W_I = \sum_{i=1}^{N} \left(\sum_{j=1,\, j\neq i}^{N} f_{ij} \right) \cdot \delta \underline{u}_{ij} \qquad (9.17)$$

Eq. (9.16) becomes

$$\delta W = \delta W_E + \delta W_I = 0 \qquad (9.18)$$

9.3.4 PVW for a system of particles

- Principle 4 (Principle of virtual work)

  A system of particles is in static equilibrium if the sum of the virtual work done by the internal and external forces vanishes for all arbitrary virtual displacements.

  Actual displacements : $W = W_E + W_I = 0$ (9.19)

- Euler's law
  virtual displacement of a particle i

$$\delta \underline{u}_i = \delta \underline{u}_o + \underline{\delta \psi} \times \underline{r}_i \qquad (9.20)$$

$\delta \underline{u}_o$: virtual translation of a rigid body

$\underline{\delta \psi}$: virtual rotation → 6 independent virtual quantities, far few than 3N

9.3.4 PVW for a system of particles

$$\delta W = \sum_{i=1}^{N}\left[\left(\underline{F}_i + \sum_{j=1,\, j\neq i}^{N} \underline{f}_{ij}\right)\cdot\left(\delta\underline{u}_o + \underline{\delta\psi}\times\underline{r}_i\right)\right]$$

$$= \left(\sum_i \underline{F}_i\right)\cdot\delta\underline{u}_o + \left(\sum_i\sum_j \underline{f}_{ij}\right)\cdot\delta\underline{u}_o + \sum_i \underline{F}_i\cdot\left(\underline{\delta\psi}\times\underline{r}_i\right) + \sum_i\sum_j \underline{f}_{ij}\cdot\left(\underline{\delta\psi}\times\underline{r}_i\right)$$

$$= \delta\underline{u}_o\cdot\left(\sum_i \underline{F}_i\right) + \delta\underline{u}_o\cdot\left(\cancelto{0}{\sum_i\sum_j \underline{f}_{ij}}\right) + \underline{\delta\psi}\cdot\left(\sum_i \underline{r}_i\times\underline{F}_i\right) + \underline{\delta\psi}\cdot\left(\cancelto{0}{\sum_i\sum_j \underline{r}_i\times\underline{f}_{ij}}\right)$$

$$= \delta\underline{u}_o\cdot\left(\cancelto{0}{\sum_i \underline{F}_i}\right) + \underline{\delta\psi}\cdot\left(\cancelto{0}{\sum_i \underline{r}_i\times\underline{F}_i}\right)$$

Necessary but not sufficient condition for static equilibrium.

9.4 Principle of virtual work applied to mechanical systems

- Rigid body

$$\delta \underline{u}_i = \delta \underline{u}_O + \underline{\delta \psi} \times \underline{r}_i$$

  - Kinematically admissible virtual displacement field (3-dimensional)
  - 2 vector eqn.s

$$\sum_{i=1}^{N} \underline{F}_i = 0 \qquad \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \sum_{i=1}^{N} \underline{M}_i = 0$$

  or 6 scalar eqn.s

  - 2-dimensional or planar mechanism, $\delta \underline{u}_i = \delta \underline{u}_O + \underline{\delta \psi} \times \underline{r}_i$ becomes

$$\delta \underline{u}_i = \delta \underline{u}_O + \delta \phi \bar{i}_3 \times \underline{r}_i \qquad (9.21)$$

$$\underline{\delta \psi} = \delta \phi \bar{i}_3$$

9.4 Principle of virtual work applied to mechanical systems

Example 9.7
Consider the simple lever subjected to two vertical end forces, F_a and F_b acting at distance a and b, respectively, from the fulcrum.

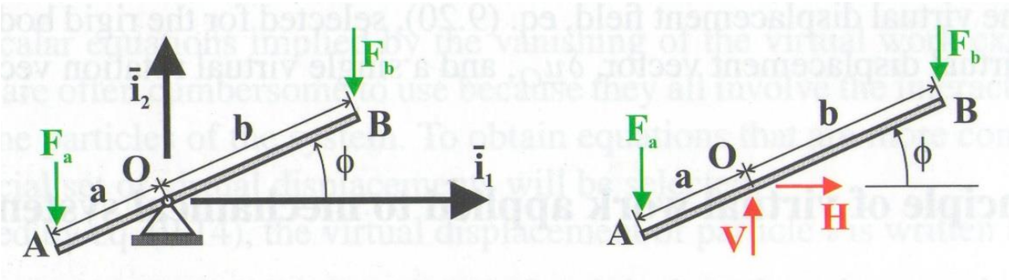


Fig. 9.9. Simple lever acted upon by two vertical end forces.

- Classical eqn. of statics by free body diagram

$$H = 0$$

$$V = F_a + F_b$$

$$aV\cos\phi = (a+b)F_b\cos\phi$$

$$aF_a = bF_b$$

Example 9.7

- Principle of virtual work (kinematically admissible virtual displacement)

  kinematically admissible virtual displacement field at A

$$\delta \underline{u}_A = \delta\phi \bar{i}_3 \times \underline{r}_{OA} = a\left(\sin\phi \bar{i}_1 - \cos\phi \bar{i}_2\right)\delta\phi$$

  kinematically admissible virtual displacement field at B

$$\delta \underline{u}_B = \delta\phi \bar{i}_3 \times \underline{r}_{OB} = b\left(-\sin\phi \bar{i}_1 + \cos\phi \bar{i}_2\right)\delta\phi$$

  virtual work

$$\delta W_E = (-F_a \bar{i}_2)\cdot\delta\underline{u}_A + (-F_b \bar{i}_2)\cdot\delta\underline{u}_B = \delta\phi[aF_a\cos\phi - bF_b\cos\phi]$$

- Principle of virtual work (kinematically violating virtual displacement)

  kinematically violating virtual displacement field at A

$$\delta \underline{u}_A = \delta u_1 \bar{i}_1 + \delta u_2 \bar{i}_2 = \delta \underline{u}_O + a\left(\sin\phi \bar{i}_1 - \cos\phi \bar{i}_2\right)\delta\phi$$

  kinematically violating virtual displacement field at B

$$\delta \underline{u}_B = \delta u_1 \bar{i}_1 + \delta u_2 \bar{i}_2 = \delta \underline{u}_O + b\left(-\sin\phi \bar{i}_1 + \cos\phi \bar{i}_2\right)\delta\phi$$

9.4 Principle of virtual work applied to mechanical systems

Example 9.7

- Principle of virtual work (kinematically violating virtual displacement)

virtual work

$$\delta W_E = (-F_a \bar{i}_2) \cdot \delta \underline{u}_A + (-F_b \bar{i}_2) \cdot \delta \underline{u}_B + (H \bar{i}_1 + V \bar{i}_2) \cdot \delta \underline{u}_O$$
$$= \delta u_1 [H] + \delta u_2 [V - F_a - F_b] + \delta \phi [a F_a \cos\phi - b F_b \cos\phi]$$

The virtual work done by the reaction forces at the fulcrum does not vanish. Thus they must be included in the formulation.
Three bracketed terms must vanish, leading to the three equilibrium eqns identical to those obtained by Newtonian approach

- Equivalence of PVW and Newton's first law

- Kinematically admissible virtual displacement field automatically eliminates the reaction forces when using PVW.

9.4 Principle of virtual work applied to mechanical systems

9.4.1 Generalized coordinates and forces

- Not convenient to work with Cartesian coord. in many cases
  - Will be represented in terms of N "generalized coord."

$$\underline{u} = \underline{u}(q_1, q_2, q_3, \cdots, q_N)$$

  - Virtual displacement

$$\delta \underline{u} = \frac{\partial \underline{u}}{\partial q_1}\delta q_1 + \frac{\partial \underline{u}}{\partial q_2}\delta q_2 + \frac{\partial \underline{u}}{\partial q_3}\delta q_3 + \cdots + \frac{\partial \underline{u}}{\partial q_N}\delta q_N$$

  - Virtual work done by a force $\underline{F}$

$$\delta W = \underline{F} \cdot \delta \underline{u} = \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_1}\right)\delta q_1 + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_2}\right)\delta q_2 + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_3}\right)\delta q_3 + \cdots + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_N}\right)\delta q_N$$

  - Generalized force

$$Q_i = \underline{F} \cdot \frac{\partial \underline{u}}{\partial q_i} \qquad (9.22)$$

9.4.1 Generalized coordinates and forces

- Then,

$$\delta W = Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3 + \cdots + Q_N\delta q_N = \sum_{i=1}^{N} Q_i \delta q_i \qquad (9.23)$$

virtual work = generalized forces X generalized virtual displacements

- Externally applied load or internal force

$$\delta W_I = \sum_{i=1}^{N} Q_i^I \delta q_i \qquad \delta W_E = \sum_{i=1}^{N} Q_i^E \delta q_i \qquad (9.24)$$

- PVW eqn.

$$\delta W_I + \delta W_E = \sum_{i=1}^{N} Q_i^I \delta q_i + \sum_{i=1}^{N} Q_i^E \delta q_i \sum_{i=1}^{N} \left[Q_i^I + Q_i^E \right] \delta q_i = 0$$

$$Q_i^I + Q_i^E = 0 \qquad i = 1, 2, 3, \cdots, N \qquad (9.25)$$

- If arbitrary virtual displacements, reaction forces must be included in Q_i^E .
- If kinematically admissible displacements, reaction forces are eliminated.

9.5 Principle of virtual work applied to truss structures

- Truss : like simple rectilinear spring of stiffness constant $k = EA/L$

  bar slenderness = 100

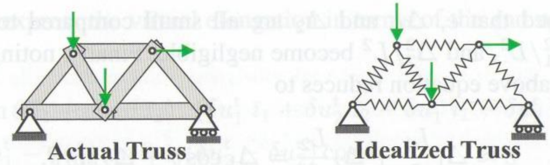


Fig. 9.28. Planar truss and its idealization as an assembly of rectilinear springs.

9.5.1 Truss structures

- Elongation : displacement equations

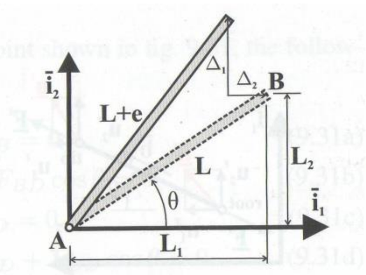


Fig. 9.29. Single bar of a planar truss.

displacement $\underline{\Delta} = \Delta_1 \bar{i}_1 + \Delta_2 \bar{i}_2$

e : elongation $(L+e)^2 = (L_1 + \Delta_1)^2 + (L_2 + \Delta_2)^2$

Δ_1, and Δ_2 small compared to the bar's length → can be linearized.

$$e \approx \Delta_1 \frac{L_1}{L} + \Delta_2 \frac{L_2}{L} = \Delta_1 \cos\theta + \Delta_2 \sin\theta \qquad (9.27)$$

Elongation is the projection of the relative displacement along the bar's direction

9.5.1 Truss structures

- Internal virtual work for a bar : general planar truss member

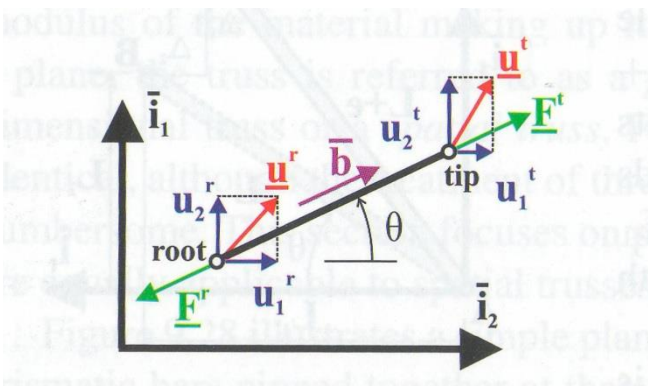


Fig. 9.30. Bar displacements and forces.

Virtual work done by the root and tip forces

$$\delta W = \underline{F}^r \cdot \delta \underline{u}^r + \underline{F}^t \cdot \delta \underline{u}^t = F\bar{b} \cdot \left(\delta \underline{u}^t - \delta \underline{u}^r\right)$$

Virtual work by the internal forces

$$\delta W_I = -\underline{F}^r \cdot \delta \underline{u}^r - \underline{F}^t \cdot \delta \underline{u}^t = -F\bar{b} \cdot \left(\delta \underline{u}^t - \delta \underline{u}^r\right) \quad (9.28)$$

Virtual elongation

$$\delta e = \bar{b} \cdot \left(\delta \underline{u}^t - \delta \underline{u}^r\right)$$

Then, $\delta W_I = -F\delta e$ (9.29)

$$\begin{aligned} \delta e &= \left(\sin\theta \bar{i}_1 + \cos\theta \bar{i}_2\right) \cdot \left(\delta u_1^t \bar{i}_1 + \delta u_2^t \bar{i}_2 - \delta u_1^r \bar{i}_1 - \delta u_2^r \bar{i}_2\right) \\ &= \left(\delta u_1^t - \delta u_1^r\right)\sin\theta + \left(\delta u_2^t - \delta u_2^r\right)\cos\theta \end{aligned} \quad (9.30)$$

9.5.2 Solution using Newton's law

- Internal virtual work for a bar : general planar truss member

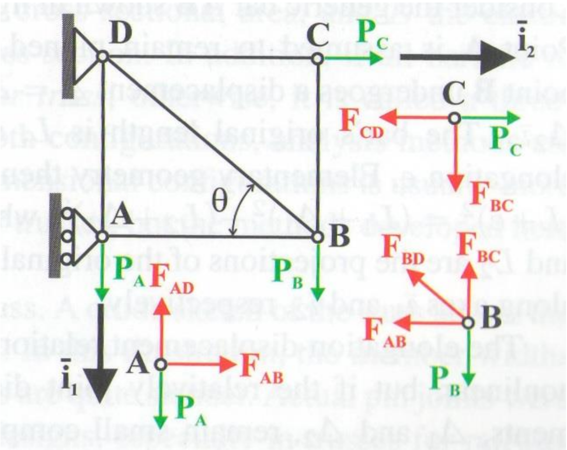


Fig. 9.31. Configuration of the 5-bar truss.

5-bars planar truss
Newton's law → equilibrium conditions at 4 joints A,B,C,D
Total 8 scalar eqn.s (method of joints)

$$P_A - F_{AD} = 0 \qquad H_A + F_{AB} = 0$$

$$P_B - F_{BC} - F_{BD}\sin\theta = 0 \qquad -F_{AB} - F_{BD}\cos\theta = 0$$

$$F_{BC} = 0 \qquad P_C - F_{CD} = 0 \tag{9.31}$$

$$V_D - F_{AD} - F_{BD}\sin\theta = 0 \qquad H_D + F_{CD} + F_{BD}\cos\theta = 0$$

9.5.3 Solution using kinematically admissible virtual displacement

- **Eq. (9.31)**
  - 5 corresponding to equilibrium in an unconstrained direction, multiplied by virtual displacements (kinematically admissible)

$$\left[P_A - F_{AD}\right]\delta u_1^A + \left[P_B - F_{BC} - F_{BD}\sin\theta\right]\delta u_1^B$$

$$+\left[-F_{BC} - F_{BD}\cos\theta\right]\delta u_2^B + \left[F_{BC}\right]\delta u_1^C + \left[P_C - F_{CD}\right]\delta u_2^C = 0 \quad (9.32)$$

  - Regrouping

$$\overbrace{P_A\delta u_1^A + P_B\delta u_1^B + P_C\delta u_2^C}^{\delta W_E}$$

$$\underbrace{-F_{AB}\delta u_2^B - F_{AD}\delta u_1^A - F_{BC}\left(\delta u_1^B - \delta u_1^C\right) - F_{BD}\left(\delta u_1^B\sin\theta + \delta u_2^B\cos\theta\right) - F_{CD}\delta u_2^C} = 0 \quad (9.33)$$

$$\delta W_I = -F_{AB}\delta e_{AB} - F_{AD}\delta e_{AD} - F_{BC}\delta e_{BC} - F_{BD}\delta e_{BD} - F_{CD}\delta e_{CD} \quad (9.35)$$

$$\rightarrow \delta W = \delta W_E + \delta W_I = 0 \quad (9.36)$$

- **Principle 5 (PVW)**

A structure is in static equilibrium if the sum of the internal and external virtual work vanishes for all kinematically admissible displacements.

9.5.4 Solution using arbitrary virtual displacements

- **Eq. (9.31)**
  - 8 equilibrium multiplied by a virtual displacement

$$\left[P_A - F_{AD}\right]\delta u_1^A + \left[H_A + F_{AB}\right]\delta u_2^A + \left[P_B - F_{BC} - F_{BD}\sin\theta\right]\delta u_1^B$$
$$+\left[-F_{BC} - F_{BD}\cos\theta\right]\delta u_2^B + \left[F_{BC}\right]\delta u_1^C + \left[P_C - F_{CD}\right]\delta u_2^C$$
$$+\left[V_D + F_{AD} + F_{BD}\sin\theta\right]\delta u_1^D + \left[H_D + F_{CD} + F_{BD}\cos\theta\right]\delta u_2^D = 0 \qquad (9.37)$$

  - Regrouping

$$\overbrace{P_A\delta u_1^A + P_B\delta u_1^B + P_C\delta u_2^C + H_A\delta u_2^A + V_D\delta u_1^D + H_D\delta u_2^D}^{\delta W_E}$$
$$\underbrace{\begin{aligned}&-F_{AB}\left(\delta u_2^B - \delta u_2^A\right) - F_{AD}\left(\delta u_1^A - \delta u_1^D\right) - F_{BC}\left(\delta u_1^B - \delta u_1^C\right)\\&-F_{BD}\left[\left(\delta u_1^B - \delta u_1^D\right)\sin\theta + \left(\delta u_2^B - \delta u_2^D\right)\cos\theta\right] - F_{CD}\left(\delta u_2^C - \delta u_2^D\right)\end{aligned}}_{\delta W_I} = 0 \qquad (9.38)$$

- **Principle 6 (PVW)**

  A structure is in static equilibrium if the sum of the internal and external work vanishes for all virtual displacements.

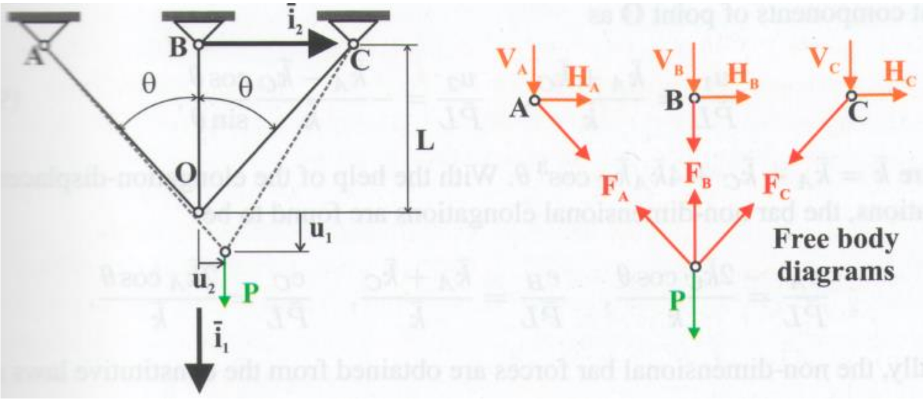
Example 9.13 Three-bar truss using PVW

- Simple hyperstatic truss with a single free joint
- Subjected to a vertical load *P* at joint **O**, where the three bars are pinned together
- Cross sectional area of the bars **A**, **B**, and **C**: A_A, A_B, A_C
- Young's moduli: E_A, E_B, E_C
- Axial stiffness of the three bars: $k_A = (EA)_A / L_A = (EA)_A \cos\theta / L,$

$$k_B = (EA)_B / L,$$

$$k_C = (EA)_C \cos\theta / L$$

- Hyperstatic system of order 1, can be solved using either the displacement or force method (Example 4.4, 4.6)



Three-bar truss configuration with free-body diagram

Example 9.13 Three-bar truss using PVW

- Virtual displacement vector for point **O**

$$\delta u = \delta u_1 \bar{i}_1 + \delta u_2 \bar{i}_2$$

- Bar virtual elongation for **A**, **B**, and **C**, by Eq. (9.30)

$$\delta e_A = \delta u_1 \cos\theta + \delta u_2 \sin\theta,$$
$$\delta e_B = \delta u_1,$$
$$\delta e_C = \delta u_1 \cos\theta - \delta u_2 \sin\theta$$

- PVW: for kinematically admissible virtual displacements

$$\delta W = \delta W_E + \delta W_I$$
$$= P\delta u_1 - F_A\left(\delta u_1 \cos\theta + \delta u_2 \sin\theta\right) - F_B \delta u_1 - F_C\left(\delta u_1 \cos\theta - \delta u_2 \sin\theta\right)$$
$$= -\left[F_A \cos\theta + F_B + F_C \cos\theta - P\right]\delta u_1 - \sin\theta\left[F_A - F_C\right]\delta u_2 = 0$$

- Two bracketed terms must vanish, leading to two equilibrium eqns.

$$F_A \cos\theta + F_B + F_C \cos\theta = P, F_A = F_C$$

9.5 PVW applied to truss structures

Example 9.13 Three-bar truss using PVW

- PVW: for arbitrary virtual displacements

$$\delta W_E = V_A \delta u_1^A + H_A \delta u_2^A + V_B \delta u_1^B + H_B \delta u_2^B + V_C \delta u_1^C + H_C \delta u_2^C + P \delta u_1^O$$

$$\begin{aligned}
\delta W_I = & -F_A \left(\cos\theta \bar{i}_1 + \sin\theta \bar{i}_2\right) \bullet \left[\left(\delta u_1^O - \delta u_1^A\right)\bar{i}_1 + \left(\delta u_2^O - \delta u_2^A\right)\bar{i}_2\right] \\
& -F_B \bar{i}_1 \bullet \left[\left(\delta u_1^O - \delta u_1^B\right)\bar{i}_1 + \left(\delta u_2^O - \delta u_2^B\right)\bar{i}_2\right] \\
& -F_C \left(\cos\theta \bar{i}_1 - \sin\theta \bar{i}_2\right) \bullet \left[\left(\delta u_1^O - \delta u_1^C\right)\bar{i}_1 + \left(\delta u_2^O - \delta u_2^C\right)\bar{i}_2\right]
\end{aligned}$$

- Invoking PVW Principle 6

$$\begin{aligned}
& \left[V_A + F_A \cos\theta\right]\delta u_1^A + \left[H_A + F_A \sin\theta\right]\delta u_2^A + \left[V_B + F_B\right]\delta u_1^B + \left[H_B\right]\delta u_2^B \\
& + \left[V_C + F_C \cos\theta\right]\delta u_1^C + \left[H_C - F_C \sin\theta\right]\delta u_2^C \\
& + \left[P - F_A \cos\theta - F_B - F_C \cos\theta\right]\delta u_1^O + \left[F_A \sin\theta - F_C \sin\theta\right]\delta u_2^O = 0
\end{aligned}$$

- All the bracketed terms must vanish.

9.6 Principle of Complementary Virtual Work

❖ **Fig. 9.33**

… Basic equations of linear elasticity (Chap.1)

➢ 3 Groups { equilibrium equations
strain-displacement relations
constitutive laws

➢ Strain compatibility equations:
do not form an independent set of equations and are not required to solve elasticity problems

➢ However, it is a over-determined problem since 6 strain components are expressed in terms of 3 displacement components only

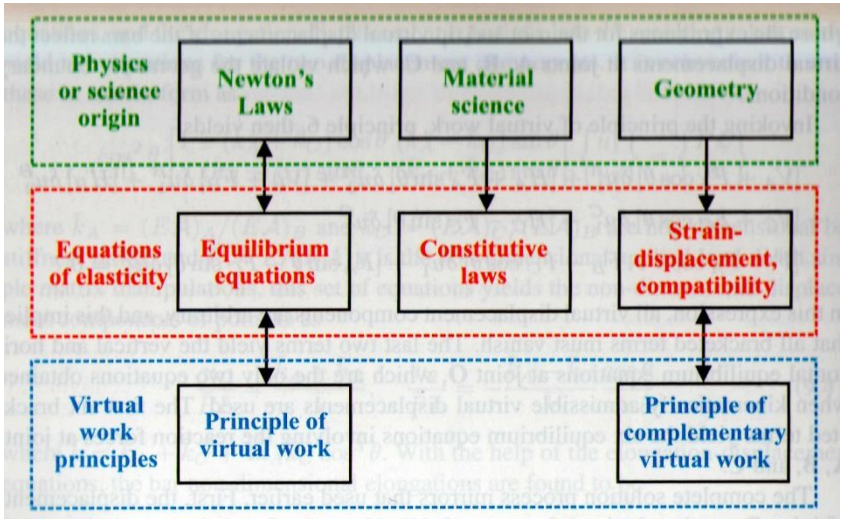


Fig. 9.33. Relationship between the equations of elasticity and virtual work principles.

9.6 Principle of Complementary Virtual Work

❖ **Solution of any elasticity problem requires 3 groups of basic eqn.s (Fig. 9.33)**

- ➢ PVW alone does not provide enough information to solve the problems

$\Longrightarrow$ PCVW will augment equilibrium equations and constitutive laws to derive complete solutions, entirely equivalent to the compatibility equations

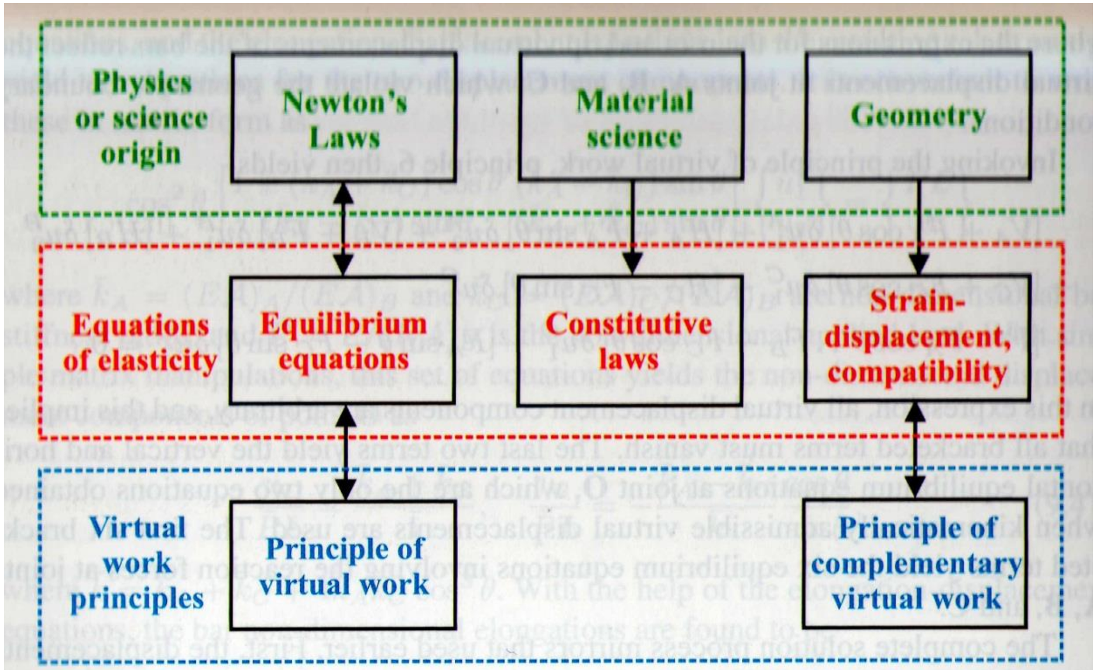


Fig. 9.33. Relationship between the equations of elasticity and virtual work principles.

9.6 Principle of Complementary Virtual Work

9.6.1 Compatibility equations for a planar truss

❖ Compatibility conditions

- Fig. 9.34 ⋯ 2-bar truss, arbitrary elongations e_A, e_C configuration of the truss compatible with these elongations is easily found
  $\longrightarrow$ intersection of 2 circles (of radii $L_A + e_A$, $L_C + e_C$) $\longrightarrow$ O′

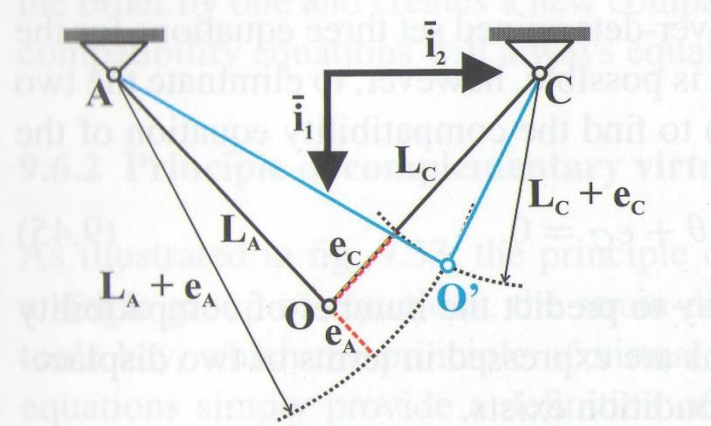


Fig. 9.34. Two-bar truss in the original and deformed configurations.

9.6 Principle of Complementary Virtual Work

- Fig. 9.35 ··· 3-bar truss, again arbitrary elongations e_A, e_C but configurations of bar **B** is now uniquely defined, since it must join B and O'

  $e_B = L'_B - L_B$ 3 elongations are no longer independent

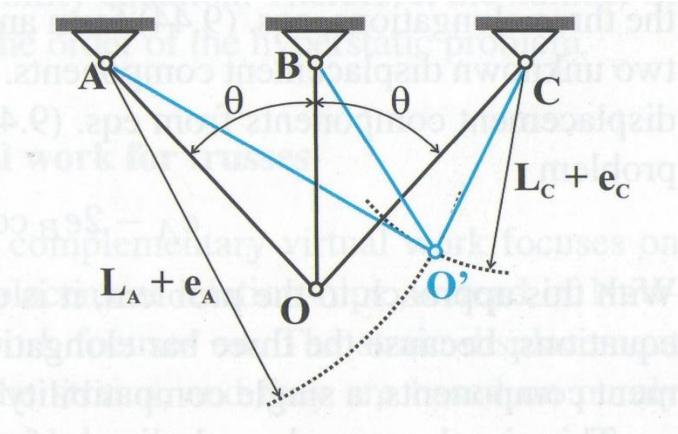


Fig. 9.35. Three-bar truss in the original and deformed configurations.

- Some conclusion can be reached by the elongation-displacement relationship instead of the geometric reasoning

elongation ··· projection of displacement vector along bar`s direction. Eq. (9.27)

$$e_A = u_1 \cos\theta + u_2 \sin\theta,\ e_C = u_1 \cos\theta - u_2 \sin\theta$$

··· for a 2-bar truss, final configuration is uniquely determined if the 2 displacement components, u_1 and u_2, are given

9.6 Principle of Complementary Virtual Work

- 3-bar truss (Fig. 9.35)

  $$e_A = u_1 \cos\theta + u_2 \sin\theta, \; e_B = u_1, \; e_C = u_1 \cos\theta - u_2 \sin\theta \tag{9.44}$$

  It is not possible to express the 2 displacement components in terms of 3 elongations. Because 3 elongations form an over-determined set for 2 unknown to eliminate 2 displacement components

- However, it is possible to express to eliminate 2 displacement components to obtain the compatibility equation

  $$e_A - 2e_B \cos\theta + e_C = 0 \tag{9.45}$$

  ⋯ 3 elongation in terms of 2 displacement components
  $\rightarrow$1 compatibility equation

- 2-bar truss ⋯ isostatic, order of redundancy, number of equation = 0
  3-bar truss ⋯ hyperstatic, number of compatibility equation
  = order of redundancy of the hyperstatic problem

- 3-bar truss ⋯ 3 force components, 2 equilibrium equations $\rightarrow$ hyperstatic of degree 1
  3 elongation, 2 displacement components $\rightarrow$ 1 compatibility equation

9.6 Principle of Complementary Virtual Work

9.6.2 PCVW for truss

❖ **3-bar truss under applied load**

➢ Fig. 9.36 ⋯ assumed to undergo compatible deformations so that the 3-bar elongations satisfy the elongation-displacement relationship, Eq.(9.44)

$$\delta W' = -\left[e_A - u_1\cos\theta - u_2\sin\theta\right]\delta F_A - \left[e_B - u_1\right]\delta F_B - \left[e_C - u_1\cos\theta + u_2\sin\theta\right]\delta F_C \qquad (9.46)$$

"Complementary VW"

"virtual forces"

$$\begin{aligned}\delta W' &= -e_A\delta F_A - e_B\delta F_B - e_C\delta F_C \\ &\quad + u_1\left(\delta F_A\cos\theta + \delta F_B + \delta F_C\cos\theta\right) + u_2\sin\theta\left(\delta F_A - \delta F_C\right) = 0\end{aligned} \qquad (9.47)$$

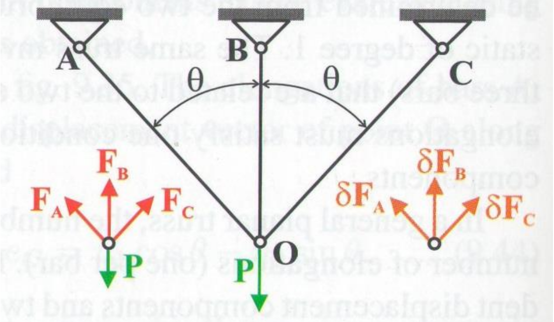


Fig. 9.36. Three-bar truss with applied load.

9.6 Principle of Complementary Virtual Work

- Free body diagram ⟶ equilibrium equations
  $F_A \cos\theta + F_B + F_C \cos\theta = P,\ F_A - F_C = 0$
  ⋯ A set of forces that satisfies these equilibrium equations is said to be "statically admissible"

- "statically admissible virtual forces"
  $$\begin{cases} \delta F_A \cos\theta + \delta F_B + \delta F_C \cos\theta = 0 \\ \delta F_A - \delta F_C = 0 \end{cases} \qquad (9.48)$$
  ⋯ do not include the externally applied loads since $\delta P = 0$,
  geometry of the system is given ⟶ $\delta\theta = 0$

- Eq. (9.47) becomes much simpler due to Eq.(9.48)
  $$\delta W' = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = 0 \qquad (9.49)$$
  for all statically admissible virtual forces

- Eq. (9.49) ⋯ "internal complementary VW"
  $$\delta W_I' = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = -\sum_{i=1}^{N_b} e_i \delta F_i \qquad (9.50)$$
  Eq. (9.49) ⟶ $\delta W' = \delta W_I' = 0$ (9.51)
  for all statically admissible virtual forces

9.6 Principle of Complementary Virtual Work

❖ **3-bar truss under prescribed displacement**

➢ Fig. 9.37 ··· instead of a concentrated load, downward vertical displacement is prescribed of magnitude Δ

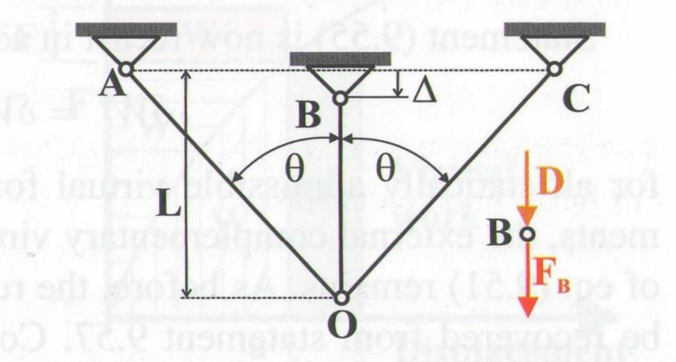


Fig. 9.37. Three-bar truss with prescribed displacement.

➢ "driving force" D required to obtain the specified displacement, as yet unknown

Eq. (9.46) $\longrightarrow$

$$\delta W' = -\left[e_A - u_1\cos\theta - u_2\sin\theta\right]\delta F_A - \left[e_B - u_1 + \Delta\right]\delta F_B - \left[e_C - u_1\cos\theta + u_2\sin\theta\right]\delta F_C = 0 \quad (9.52)$$

$$= -e_A\delta F_A - e_B\delta F_B - e_C\delta F_C - \Delta\delta F_B + u_1\left(\cos\theta\delta F_A + \delta F_B + \cos\theta\delta F_C\right) + u_2\sin\theta\left(\delta F_A - \delta F_C\right) = 0 \quad (9.53)$$

9.6 Principle of Complementary Virtual Work

- Set of statically admissible virtual forces that satisfy the following equilibrium eqns

$$\underbrace{\delta F_A \cos\theta + \delta F_B + \delta F_C \cos\theta = 0,\ \delta F_A - \delta F_C = 0}_{\text{equilibrium at joint O}},\ \underbrace{\delta F_B + \delta D = 0}_{\text{equilibrium at joint B}} \quad (9.54)$$

- If the virtual forces are required to be statically admissible,
  Eq. (9.53) will be simpler

$$\delta W' = \Delta\delta D \underbrace{- e_A \delta F_A - e_B \delta F_B - e_C \delta F_C}_{} = 0 \quad (9.55)$$

external complementary VW $\longrightarrow \delta W_E' = \Delta\delta D$ (true displacement X virtual) (9.56)

Eq. (9.55) $\longrightarrow \delta W' = \delta W_E{}' + \delta W_I{}' = 0$ (9.57)

for all statically admissible virtual forces

- **Principle 7 (PCVW)**

  A truss undergoes compatible deformations if the sum of the internal and external complementary VW vanishes for all statically admissible virtual forces

9.6 Principle of Complementary Virtual Work

- If the CVW is required to vanish for all arbitrary virtual forces, i.e., for all independently chosen arbitrary $\delta F_A,\ \delta F_B,\ \delta F_C,\ \delta D$
  $\longrightarrow$ Eq. (9.55) $e_A = e_B = e_C = \Delta = 0 \longrightarrow$ truss can not deform
  $\longrightarrow$ NOT correct

- For a statically admissible virtual forces, must satisfy Eq. (9.54),
  3 equations for 4 statically admissible virtual forces
  $\longrightarrow$ possible to express 3 of the virtual forces in terms of the 4th:
  $\delta F_B = -2\delta F_A \cos\theta,\ \delta F_C = \delta F_A,\ \delta D = 2\delta F_A \cos\theta$
  $\longrightarrow$ PCVW: $\delta W' = \Delta\left(2\delta F_A \cos\theta\right) - e_A \delta F_A - e_B\left(-2\delta F_A \cos\theta\right) - e_C \delta F_A$

$$= \left[2\Delta\cos\theta - e_A + 2e_B\cos\theta - e_C\right]\delta F_A = 0$$

$= 0$ $\cdots$ compatibility equation

9.6 Principle of Complementary Virtual Work

9.6.3 CVW

$\begin{cases} \text{CVW: work done by virtual forces acting through real displacement} \\ \text{VW: work done by real forces acting through virtual displacement} \end{cases}$

↳ real quantities remain fixed

- **Fig. 9.38 ⋯ not necessarily linear elastic material**

$$W = \int_0^u ku\,du = \frac{1}{2}ku^2 = \frac{1}{2}Fu$$

linearly elastic material

$$W' = \int_0^F \frac{F}{k}dF = \frac{1}{2K}F^2 = \frac{1}{2}Fu = W$$

only when linearly elastic material

- **Fig. 9.38 ⋯ shaded areas for "VW" and "CVW"**

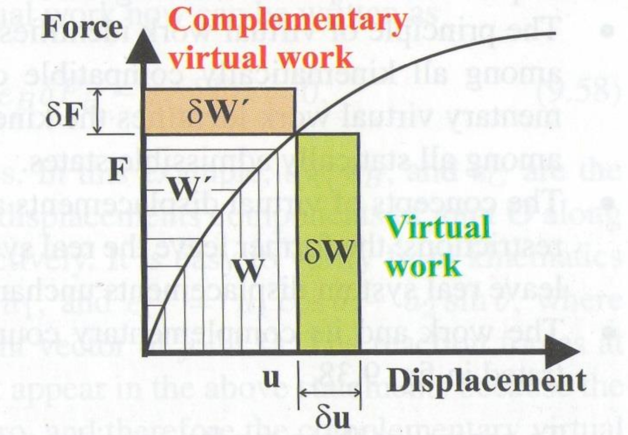


Fig. 9.38. Work and complementary work and their virtual counterparts.

9.6 Principle of Complementary Virtual Work

9.6.4 Application to truss

❖ **Planer truss with a number of bars connected of N nodes**

- PVW ⟶ 2N equilibrium equations
  PCVW ⟶ n equations produced for a hyperstatic truss of order n
  for an isostatic truss, no compatibility equations

❖ **PCVW** ··· enables the development of the force method,
in general, n≪N, only a few eqn generated ⟶ simpler solution procedure
But major drawback ··· must be statically admissible virtual forces,
self-equilibrating, requires much more extensive work for generation of
the equations ⟶ PVW is used much more widely used

9.6 Principle of Complementary Virtual Work

9.6.6 Unit load method for trusses

❖ **PCVW** → "unit load method" ··· determine deflections at specific points of structure

- Fig. 9.40 ··· 2-bar truss

  PCVW, imagine the displacement Δ prescribed at O,

  external complementary work $\delta W_E{}' = \Delta \delta D$

  δD : virtual driving force

- PCVW → $\delta W_E{}' + \delta W_I{}' = 0,\ \Delta \delta D = -\delta W_I{}'$ (9.62)

  for all statically admissible virtual forces

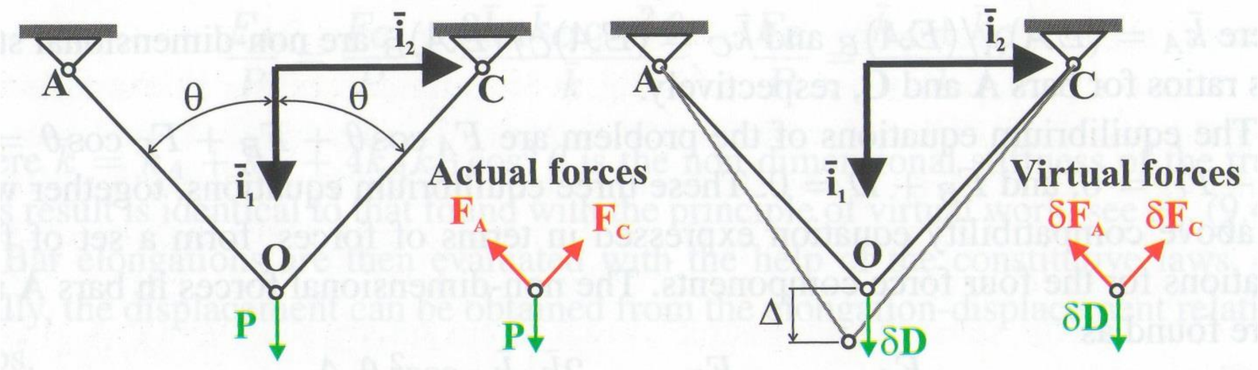


Fig. 9.40. The unit load method for a two-bar truss.

9.6 Principle of Complementary Virtual Work

- Internal CVW: $\delta W_I' = -e_A \delta F_A - e_C \delta F_C$

  then, Eq. (9.62) $\Delta \delta D = e_A \delta F_A + e_C \delta F_C$

  for a more general truss consisting of N_b bars, $\Delta \delta D = \sum_{i=1}^{N_b} e_i \delta F_i$ (9.63)

  for all statically admissible virtual forces

  $\delta D,\ \delta F_A,\ \delta F_C$ $\cdots$ a set of statically admissible virtual forces, free body diagrams

  $\rightarrow \delta F_A - \delta F_C = 0,\ \delta D - (\delta F_A + \delta F_C)\cos\theta = 0$

  $\cdots$ 2 equilibrium equations of the system linking 3 virtual forces

- Unit load method $\cdots$ the virtual driving force is selected to be a unit load, $\delta D = 1$

  $$\rightarrow \delta F_A = \delta F_C = \Delta \delta D / (2\cos\theta) = 1/(2\cos\theta)$$

- Simplified notation $\cdots$ when $\delta D = 1 \rightarrow \delta F_A = \delta \hat{F}_A,\ \delta F_C = \delta \hat{F}_C$

  Eq. (9.63) $\rightarrow \Delta = \sum_{i=1}^{N_b} \widehat{F}_i e_i$ (9.64)

  F_i: actual forces that develop due to the externally applied load must satisfy all equilibrium conditions, and the associated elongations must be compatible

  $\widehat{F}_i$: the unit forces $\cdots$ a set of statically admissible forces.
  must satisfy the equilibrium equations, but the associated elongations are NOT required to be compatible.

9.6 Principle of Complementary Virtual Work

- For a linearly elastic material, $e_i = \dfrac{F_i L_i}{E_i A_i}$

  Eq. (9.64) $\longrightarrow$ $\Delta = \sum_i \dfrac{\widehat{F}_i F_i L_i}{E_i A_i}$ (9.65)

- To determine rotation of the structure $\longrightarrow$: "unit moment method"

  $\Phi \delta M = -\delta W_I'$ (9.66)

9.6 Principle of Complementary Virtual Work

➢ Example. 9.16: Joint deflection in a simple 2-bar truss

- Step 1: determination of the bar forces and extensions due to externally applied loads

$$F_A = F_C = \frac{P}{(2\cos\theta)} \qquad e_A = \frac{F_A L_A}{(EA)_A}, e_C = \frac{F_C L_C}{(EA)_C}$$

$$e_A = \frac{PL}{(EA)_A}\frac{1}{2\cos^2\theta}, e_C = \frac{PL}{(EA)_C}\frac{1}{2\cos^2\theta}$$

- Step 2: unit load applied at the point and in the direction of the desired direction component.

$$\hat{F}_A = \hat{F}_C = \frac{1}{(2\cos\theta)}$$

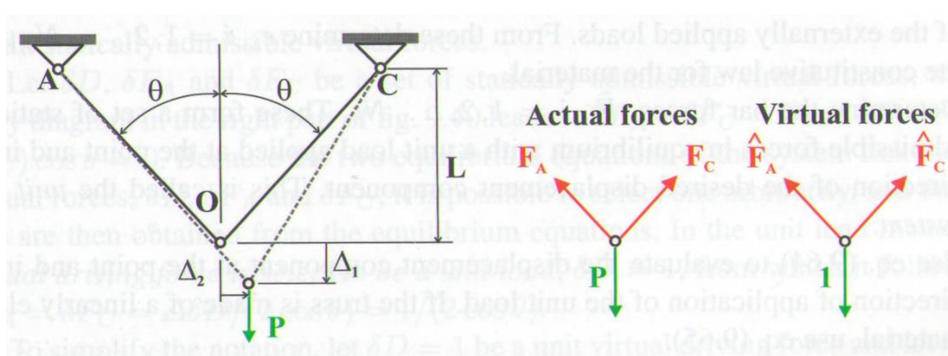


Fig. 9.41. Two-bar truss with unsymmetric properties and vertical load at joint.

- Example. 9.16: Joint deflection in a simple 2-bar truss
  - Step 3: find the vertical displacement of joint **O**

$$\Delta_1 = \sum_i \sum_{i=1}^{N_b} \widehat{F}_i e_i = \frac{1}{2\cos\theta}\frac{PL}{2\cos^2\theta (EA)_A} + \frac{1}{2\cos\theta}\frac{PL}{2\cos^2\theta (EA)_C}$$

$$= \frac{PL}{4\cos^3\theta}\frac{(EA)_A + (EA)_C}{(EA)_A (EA)_C}$$

  - Horizontal deflection component

$$\hat{F}_A = \frac{1}{(2\sin\theta)}, \hat{F}_C = -\frac{1}{(2\sin\theta)}$$

$$\Delta_2 = \sum_i \sum_{i=1}^{N_b} \widehat{F}_i e_i = \frac{1}{2\sin\theta}\frac{PL}{2\cos^2\theta (EA)_A} - \frac{1}{2\sin\theta}\frac{PL}{2\cos^2\theta (EA)_C}$$

$$= \frac{PL}{4\sin\theta\cos^2\theta}\frac{(EA)_A - (EA)_C}{(EA)_A (EA)_C}$$

9.7.1 Beam bending

❖ **Plane $\left(\bar{i}_1, \bar{i}_2\right)$… plane of symmetry**

$M_3\left(x_1\right)$ bending moment, $\Phi_3\left(x_1\right)$ rotation,

$\bar{u}_2\left(x_1\right)$ transverse displacement

➢ Infinitesimal slice of a beam (Fig. 9.48)
$\longrightarrow$ curvature of the differential element $\kappa_3 = \Phi_3' = \bar{u}_2''$

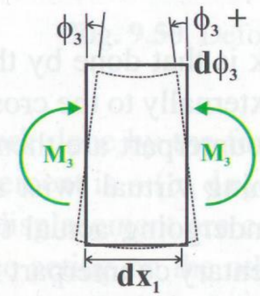


Fig. 9.48. Bending deformation of an infinitesimal segment of a beam.

➢ Work done by the moment acting on the left-hand side: $-M_3\Phi_3$
((-) since moment and rotation are counted (+) about opposite axes)
Work done by the moment acting on the other side: $M_3\left(\Phi_3 + d\Phi_3\right)$

net work done by the 2 moments: $dW = M_3 d\Phi_3 = M_3\left(\dfrac{d\Phi_3}{dx_1}\right)dx_1$

total internal work done by the moment distribution acting along the beam

$$W_I = -\int_0^L M_3 \frac{d\Phi_3}{dx_1}\,dx_1 = -\int_0^L M_3\kappa_3\,dx_1 \qquad (9.66)$$

(-) due to internal moment, which is opposite to externally applied moment

9.7 Internal virtual work in beams and solid

❖ **Internal VW**

$$\delta W_I = -\int_0^L M_3 \delta\kappa_3 dx_1 \qquad (9.66)$$

❖ **Internal CVW**

$$\delta W_I' = -\int_0^L \kappa_3 \delta M_3 dx_1$$

9.7 Internal virtual work in beams and solid

9.7.2 Beam twisting

❖ **Fig. 9.49**

… differential rotation of 2 cross section`s

→ twist rate of the differential element, $\kappa_1 = \Phi_1'$

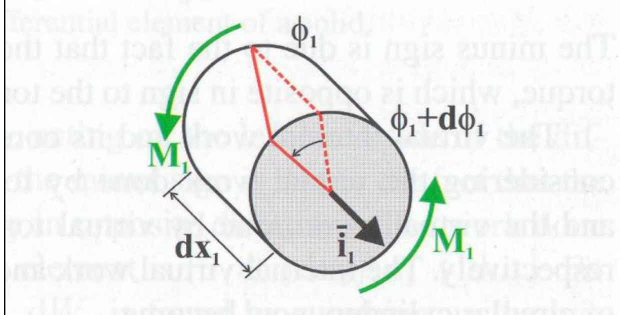


Fig. 9.49. Torsional deformation of an infinitesimal segment of a beam.

➢ Work done by the torque acting on the left-hand side: $-M_1\Phi_1$

((-) due to the torque and rotation are (+) about opposite axes)

Work done by the torque acting on the other side: $M_1\left(\Phi_1 + d\Phi_1\right)$

net work by 2 torques: $dW = M_1 d\Phi_1 = M_1\left(\dfrac{d\Phi_1}{dx_1}\right)dx_1$

total internal work done by the torque distribution

$$W_I = -\int_0^L M_1 \frac{d\Phi_1}{dx_1}dx_1 = -\int_0^L M_1\kappa_1 dx_1 \qquad (9.71)$$

(-) due to internal torque

9.7 Internal virtual work in beams and solid

- **Internal VW**

$$\delta W_I = -\int_0^L M_1 \delta\kappa_1 dx_1 \qquad (9.72)$$

- **Internal CVW**

$$\delta W_I' = -\int_0^L \kappa_1 \delta M_1 dx_1$$

based on the kinematics of Saint-Venant's theory of uniform torsion

9.7 Internal virtual work in beams and solid

9.7.3 Three-dimensional solid

- Work done by each 6 stress components are computed separately and then are summed up

Axial Stresses

- Fig. 9.50 ··· infinitesimal differential element of a solid

  work done by the force, $\sigma_1 dx_2 dx_3$, acting on the left side: $-(\sigma_1 dx_2 dx_3) u_1$

  ((-) due to that force and displacement are counted (+) in opposite directions)

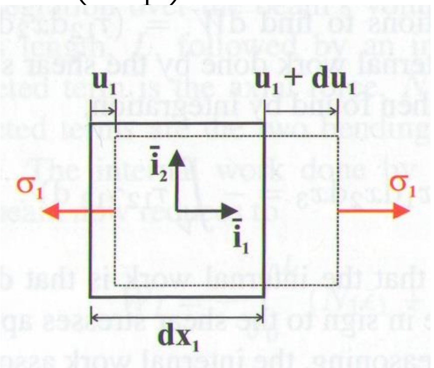
  work done by the force acting on the other side: $(\sigma_1 dx_2 dx_3)(u_1 + du_1)$

  net work by the 2 forces: $dW = (\sigma_1 dx_2 dx_3) du_1 = (\sigma_1 dx_2 dx_3)\left(\dfrac{\partial u_1}{\partial x_1}\right) dx_1$

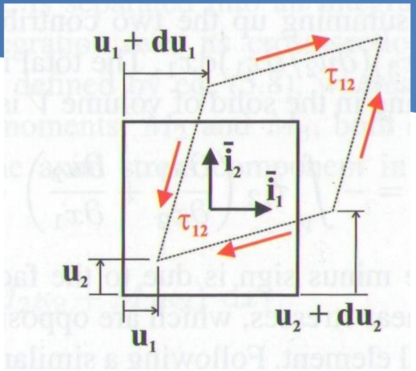
  total internal work done by the axial stress distribution

  $$W_I = -\int_V \sigma_1 \frac{\partial u_1}{\partial x_1} dx_1 dx_2 dx_3 = -\int_V \sigma_1 \varepsilon_1 dV \quad (9.73)$$

  (-) due to the internal axial stresses



9.7 Internal virtual work in beams and solid



❖ Shear stresses

- Due to the principle of reciprocity, shear stress components will act on right, left edges, also on the top, bottom edges

- Work done by the force, $\tau_{12}dx_1dx_3$, acting on the bottom edges: $-(\tau_{12}dx_1dx_3)u_1$
  ((-) due to that force and displacement are opposite)
  Work done by the force, $\tau_{12}dx_1dx_3$, acting on the top edges: $(\tau_{12}dx_1dx_3)(u_1 + du_1)$

- Net work done by these 2 forces: $dW = (\tau_{12}dx_1dx_3)du_1 = (\tau_{12}dx_1dx_3)\dfrac{\partial u_2}{\partial x_2}dx_2$

- Work done by $\tau_{12}dx_2dx_3$, acting on the left edge: $-(\tau_{12}dx_2dx_3)u_2$
  ((-) due to that force and displacement are counted (+) in opposite direction)
  Work done by $\tau_{12}dx_2dx_3$, acting on the right edge: $(\tau_{12}dx_2dx_3)(u_2 + du_2)$

- Net work done by these 2 forces: $dW = (\tau_{12}dx_2dx_3)du_2 = (\tau_{12}dx_2dx_3)\dfrac{\partial u_2}{\partial x_1}dx_1$

9.7 Internal virtual work in beams and solid

- Total internal work by the shear stress distribution

$$W_I = -\int_V \tau_{12} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) dx_1 dx_2 dx_3 = -\int_V \tau_{12} \gamma_{12} dV \qquad (9.72)$$

((-) due to the internal shear stresses)

9.7 Internal virtual work in beams and solid

- Total work done by all 6 stress components

$$W_I = -\int_V \left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3 + \tau_{23}\gamma_{23} + \tau_{13}\gamma_{13} + \tau_{12}\gamma_{12}\right) dV \quad (9.75)$$

$$= -\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV \quad (9.76)$$

- Internal virtual work $\delta W_I = -\int_V \underline{\sigma}^T \delta\underline{\varepsilon}\, dV$
- Internal CVW work $\delta W_I = -\int_V \underline{\varepsilon}^T \delta\underline{\sigma}\, dV$

(9.77)

9.7.4 Euler-Bernoulli beam

❖ **Viewed as a 3-dim. Solid**

… in Euler-Bernoulli beam, all strain components vanish, except for the axial strain

Eq. (9.75) $\longrightarrow W_I = -\int_V \sigma_1 \varepsilon_1 dV = -\int_0^L \int_A \sigma_1 \left(\bar{\varepsilon}_1 + x_3 \kappa_2 - x_2 \kappa_3 \right) dA dx_1$

$$= -\int_0^L \left\{ \left[\int_A \sigma_1 dA \right] \bar{\varepsilon}_1 + \left[\int_A \sigma_1 x_3 dA \right] \kappa_2 + \left[-\int_A \sigma_1 x_2 dA \right] \kappa_3 \right\} dx_1$$

$\downarrow$ N_1 by Eq. (5.8) $\quad\downarrow$ M_2 $\quad\downarrow$ M_3 by Eq. (5.10)

$$W_I = -\int_0^L \left(N_1 \bar{\varepsilon}_1 + M_2 x_2 + M_3 x_3 \right) dx_1 \quad (9.78)$$

- Internal VW $\quad \delta W_I = -\int_0^L \left(N_1 \delta \bar{\varepsilon}_1 + M_2 \delta \kappa_2 + M_3 \delta \kappa_3 \right) dx_1$
- Internal CVW $\quad \delta W_I' = -\int_0^L \left(\bar{\varepsilon}_1 \delta N_1 + \kappa_2 \delta M_2 + \kappa_3 \delta M_3 \right) dx_1$

(9.79)

9.7.6 Unit load method for beams

❖ **If Δ is prescribed of a point of the beam**

- PCVW, Eq. (9.57) $\longrightarrow \Delta\delta D + \delta W_I' = 0$
  for statically admissible virtual forces, (δD: virtual driving force)

- By Eq. (9.79b),

$$\Delta\delta D = \int_0^L \left(\bar{\varepsilon}_1 \delta N_1 + \kappa_2 \delta M_2 + \kappa_3 \delta M_3\right) dx_1 \quad (9.80)$$

- $\delta D = 1$ and $\delta N_1 = \widehat{N}_1$, $\delta M_2 = \widehat{M}_1$, $\delta M_3 = \widehat{M}_3$: resulting statically admissible axial forces and bending moments

$$\Delta = \int_0^L \left(\widehat{N}_1 \bar{\varepsilon}_1 + \widehat{M}_2 \kappa_2 + \widehat{M}_3 \kappa_3\right) dx_1 \quad (9.81)$$

9.7 Internal virtual work in beams and solid

- If $\begin{cases}\text{linearly elastic material}\\ \text{the origin of the axis system is at the centroid of the cross section}\end{cases}$

  $\longrightarrow$ sectional constitutive law, Eq. (6.13) can be applicable

$$\Delta = \int_0^L \left[\frac{\widehat{N}_1 N_1}{S} + \frac{\widehat{M}_2 \left(H_{33}^C M_2 + H_{23}^C M_3 \right)}{\Delta_H} + \frac{\widehat{M}_3 \left(H_{23}^C M_2 + H_{22}^C M_3 \right)}{\Delta_H} \right] dx_1 \qquad (9.82)$$

$$\Delta_H = H_{22}^C H_{33}^C - H_{23}^C H_{23}^C$$

- If the principle axes of bending

$$\Delta = \int_0^L \left[\frac{\widehat{N}_1 N_1}{S} + \frac{\widehat{M}_2 M_2}{H_{22}^C} + \frac{\widehat{M}_3 M_3}{H_{33}^C} \right] dx_1 \qquad (9.83)$$

- Example. 9.19: Deflection of a tip-loaded cantilevered beam
  - Cantilevered beam of length *L* subjected to a concentrated load *P* at beams tip, $\alpha = 1$

- evaluation of the bending moment distribution under the externally applied loads

$$M_3(x_1) = P(x_1 - L)$$

- vertical unit load applied at the tip

$$\hat{M}_3(x_1) = -1(x_1 - L)$$

- tip deflection by Eq. (9.83)

$$\Delta = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} dx_1 = \int_0^L \frac{\left[P(x_1 - L)\right]\left[-(x_1 - L)\right]}{H_{33}^c} dx_1 = -\frac{PL^3}{3H_{33}^c}$$

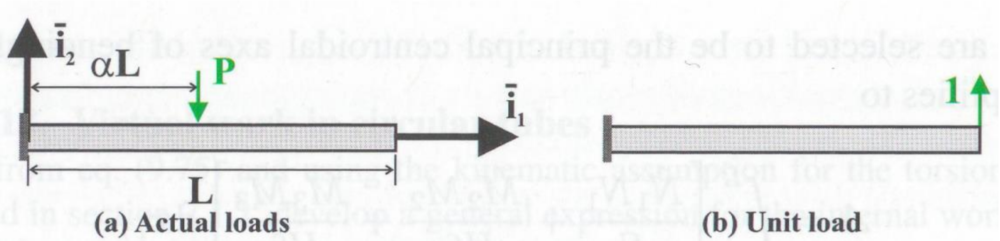


Fig. 9.51. Cantilevered beam under tip load.

9.8 Application of the unit method to hyperstatic problem

❖ Unit load method

⋯ determination of 2 sets of statically admissible forces corresponding to 2 distinct loading cases

$\begin{cases} \text{① associated with the externally applied loads} \\ \text{② associated with the unit load} \end{cases}$

⟶ applied equally to iso- and hyperstatic systems

❖ Hyperstatic systems

$\begin{cases} \text{displacement or stiffness method} \\ \text{force or flexibility method} \end{cases}$

⋯ focuses on the determination of internal forces / moments and reactions

key step: development of the compatibility equations

PCVW ⋯ equivalent to the compatibility equations

⟶ logical to combine the force method with PCVW

❖ Force method

⋯ intuitively described as "method of cuts"

for each cut, the order of the hyperstatic system is decreased by 1.

statically admissible forces are then solely obtained from the equilibrium equations

9.8 Application of the unit method to hyperstatic problem

- **2 crucial step**

  ① determine the relative displacements at the cuts under the externally applied load alone

  ② evaluate the internal forces applied at the cuts that are required to eliminate the relative displacements at the cuts

  $\longrightarrow$ PCVW is a powerful total to solve both problems

- **Fig. 9.65 ··· single bar of truss**
  - R: set of self-equilibrating forces applied at the cut
  - C external VW $\delta W_E' = d_1 \delta R - d_2 \delta R = \left(d_1 - d_2\right) \delta R$

    relative displacement at the cut: $\Delta = d_1 - d_2$
  - PCVW, Eq. (9.57) ··· $\delta W_E' + \delta W_I' = 0$

    $\longrightarrow \quad \Delta \delta R = -\delta W_I' \qquad (9.84)$

    ··· very similar to Eq. (9.62), but Δ: relative displacement at the cut,

    δR: set of self-equilibrating virtual forces applied at the cut

9.8 Application of the unit method to hyperstatic problem

- **Right of Fig. 9.65 ⋯ a cantilevered beam**
  - C external VW: $\delta W_E' = \theta_1 \delta M - \theta_2 \delta M = (\theta_1 - \theta_2)\delta M$
  - PCVW $\rightarrow \Phi \delta M = -\delta W_I'$ (9.85)

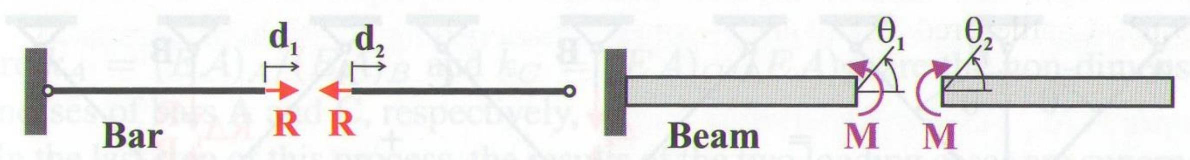


Fig. 9.65. Relative displacements and rotations.

9.8 Application of the unit method to hyperstatic problem

9.8.1 Force method for trusses

- **Fig. 9.66**
  - 3-bar hyperstatic truss, hyperstatic system of order 1,
    a single cut is applied at the middle bar
    Then, the actual system is viewed as a superposition of 2 problem

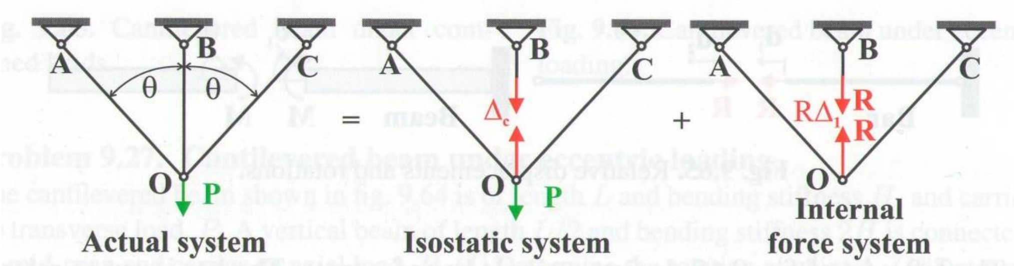


Fig. 9.66. Force method for the three-bar truss.

9.8 Application of the unit method to hyperstatic problem

① Isostatic system subjected to the externally applied loads

- Unit load method is directly applicable

$$\Delta_C = \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{(EA)_i} \qquad (9.86)$$

where F_i : bar forces subjected to the externally applied loads

$\hat{F}_i$: statically admissible virtual forces corresponding to the self-equilibrating unit load system applied at the cut

$$F_A = F_C = P/(2\cos\theta),\ F_B = 0$$

$$\hat{F}_A = \hat{F}_C = -1/(2\cos\theta),\ \hat{F}_B = 1$$

$$\rightarrow \quad \Delta_C = -\left(\frac{1}{(EA)_A} + \frac{1}{(EA)_C}\right)\frac{PL}{4\cos^3\theta}$$

② Internal force system

… Relative displacement at the cut, Δ_1, due to a unit internal force in bar B

- Eq. (9.84) $\rightarrow \Delta_1 = \sum_{i=1}^{N_b} \frac{\hat{F}_i^2 L_i}{(EA)_i}$ (9.87)

$$\Delta_1 = \frac{L}{(EA)_B 4\cos^3\theta} \frac{\bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta}{\bar{k}_A\bar{k}_C}$$

③ Superposition of 2 loading cases

- Compatibility condition at the cut

$$\Delta_C + R\Delta_1 = 0 \qquad (9.88)$$

$$\rightarrow \ R = -\frac{\Delta_C}{\Delta_1} = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C \cos^3\theta} \qquad (9.89)$$

- Bar forces $F_i + R\widehat{F}_i, \ i = 1, 2, \cdots, N_b$ (9.90)

9.8.2 Force method for beams

- Beam structures becomes hyperstatic due to the presence of multiple supports

❖ **Fig. 9.70**

… cantilevered beam with additional mid-span support

⟶ additional reaction R

- Eliminating or cutting the appropriate number of supports to render the beam isostatic

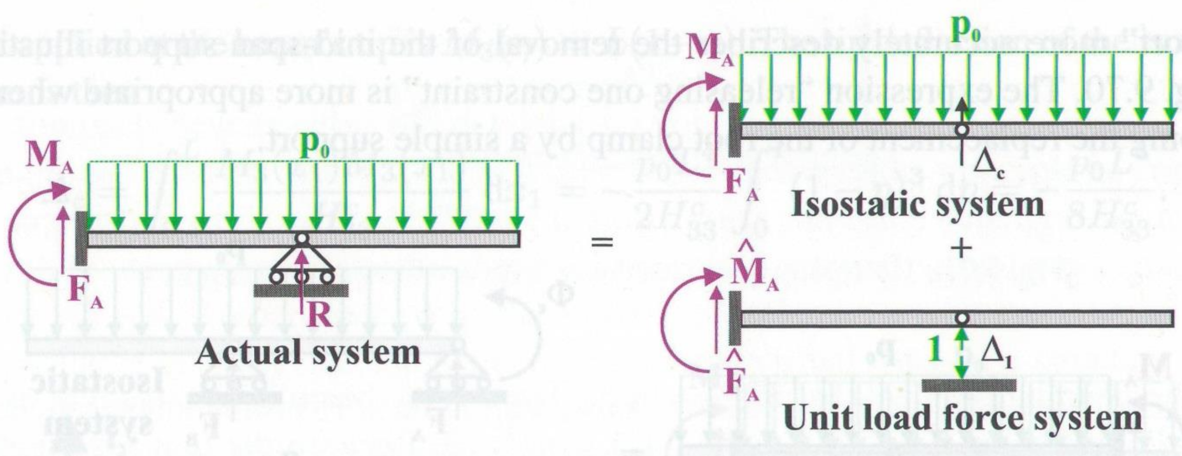


Fig. 9.70. Cantilever with a mid-span support. The isostatic system is obtained by eliminating the mid-span support.

9.8 Application of the unit method to hyperstatic problem

i) Δ_C is computed by unit load method, Eq. (9.83)

$$\Delta_C = \int_0^L \frac{M_3 \widehat{M}_3}{H_{33}^C} dx_1 \quad (9.91)$$

$M_3(x_1)$: bending moment distribution in the isostatic beam subjected to the externally applied loads

$\widehat{M}_3(x_1)$: statically admissible bending moment distribution in the isostatic beam subjected to a set of self-equilibrating unit forces applied at the support

ii) Δ_1 relative deflection at the support due to a set of self-equilibrating, unit load. Eq. (9.84)

$$\Delta_1 = \int_0^L \frac{\widehat{M}_3^2}{H_{33}^C} dx_1 \quad (9.92)$$

iii) Displacement compatibility equation at the support

$$\Delta_C + R\Delta_1 = 0 \quad (9.93) \qquad R = -\Delta_C / \Delta_1 \quad (9.94)$$

reaction forces : $F_A + R\widehat{F}_A$ at the root
bending moments : $M_A + R\widehat{M}_A$
bending moments distribution $M_3(x_1) + R\widehat{M}_3(x_1)$

9.8 Application of the unit method to hyperstatic problem

- Alternative way to eliminate the support (or "releasing one constraint")

  ··· Replacement of the root clamp by a simple support (Fig. 9.71)

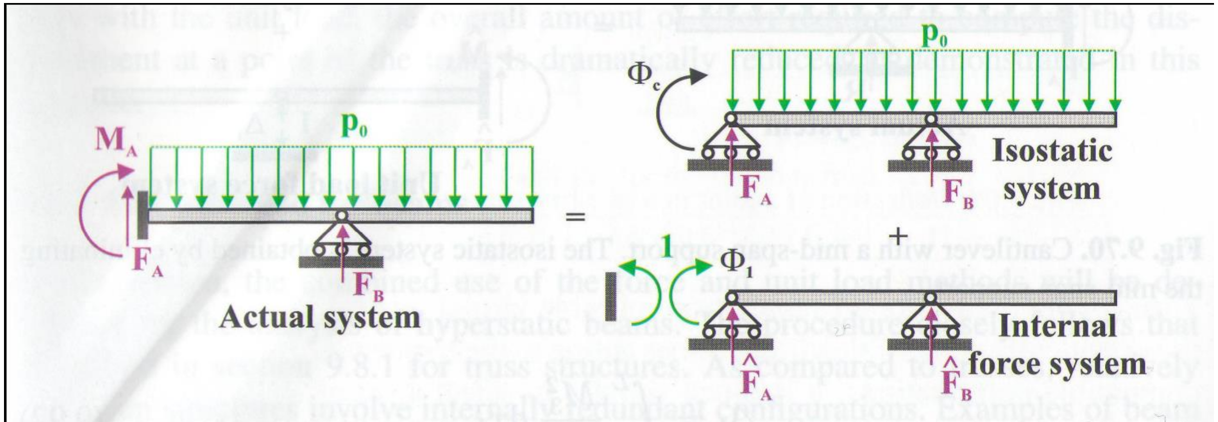


Fig. 9.71. Cantilever with a mid-span support. The isostatic system is obtained by eliminating the mid-span support.

i) Φ_C : relative root rotation in the isostatic structure, Eq (9.85)

ii) Φ_1 : associated root rotation

iii) root rotation compatibility eqn. : $\Phi_C + M_A \Phi_1 = 0, \quad M_A = -\Phi_C / \Phi_1$

9.8 Application of the unit method to hyperstatic problem

- Example. 9.29: Cantilevered beam with a tip support
  - Cantilevered beam of length L subjected to a uniform loading distribution p_o

  - Isostatic system: the tip support is eliminated (tip constraint is released)

  - Tip deflection of the beam by unit load method

    bending moment distribution in the isostatic beam

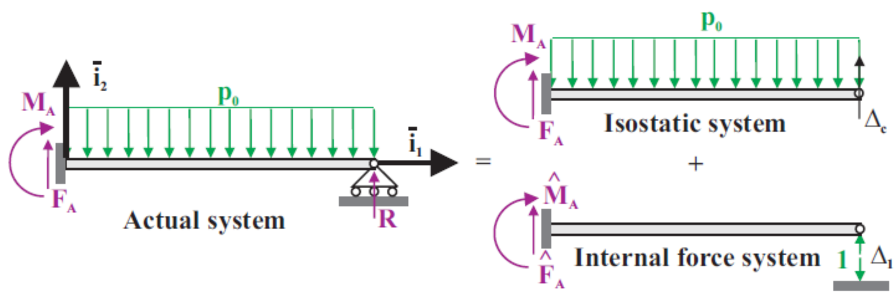
$$M_3(\eta) = -p_o L^2 (1-\eta) 2/2, \eta = x_1/L$$

  - statically admissible bending moment distribution associated with a unit load appied at the tip

$$\hat{M}_3(\eta) = L(1-\eta)$$

  - tip deflection of the isostatic beam

$$\Delta_c = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} dx_1 = -\frac{p_o L}{2H_{33}^c} \int_0^L (1-\eta)^3 d\eta = -\frac{p_o L^3}{8H_{33}^c}$$



9.8 Application of the unit method to hyperstatic problem

- tip deflection of the isostatic beam subjected to a set of self-equilibrating tip unit loads by unit load method

$$\Delta_1 = \int_0^L \frac{\hat{M}_3^2(x_1)}{H_{33}^c} dx_1 = -\frac{L^3}{H_{33}^c} \int_0^L (1-\eta)^2 \, d\eta = \frac{L^3}{3H_{33}^c}$$

- Compatibility condition, Eq. (9.93), allows determination of the reaction force at the tip support

$$R = -\frac{\Delta_c}{\Delta_1} = \frac{p_o L^4}{8H_{33}^c} \frac{3H_{33}^c}{L^3} = \frac{3p_o L}{8}$$

- Solution of the original hyperstatic problem: by superposition
  Bending moment distribution

$$M_3 + R\hat{M}_3 = -\frac{p_o L^2}{2}(1-\eta) + \frac{3p_o L^2}{8}(1-\eta) = \frac{p_o L^2}{8}\left[3(1-\eta) - 4(1-\eta)^2\right]$$

Q & A