# Chapter 8
## The Complete Response of RL and RC Circuits

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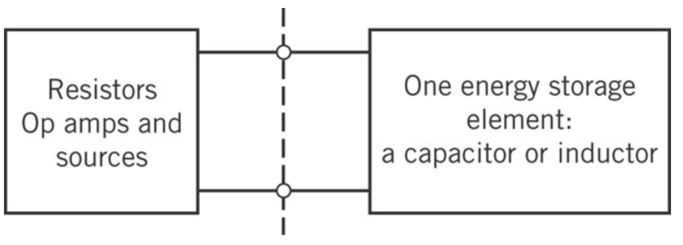
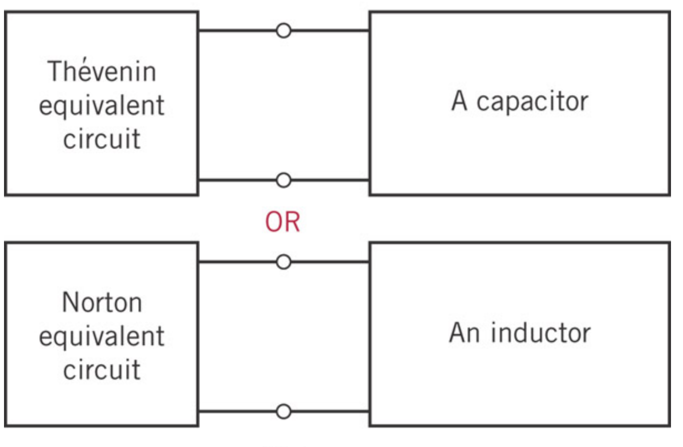
# What is First Order Circuits?

- Circuits that contain only one inductor or only one capacitor can be represented by a first-order differential equation. These circuits are called **first-order circuits**




*(a)*

(a) First, separate the energy storage element from the rest of the circuit.



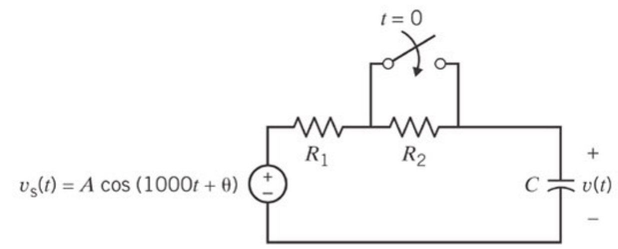

*(b)*

(b) Next, replace the circuit connected to a capacitor by its Thevenin equivalent circuit, or replace the circuit connected to an inductor by its Norton equivalent circuit. (because the voltage in capacitor circuit or the current in inductor circuit are to be continuous.)

# Response of the First Order Circuits

- Consider the first-order circuit shown in Figure 8.2-2
  - The circuit is at steady state before the switch is closed

$$v(t) = B\cos(1000t + \phi), \quad t < 0$$



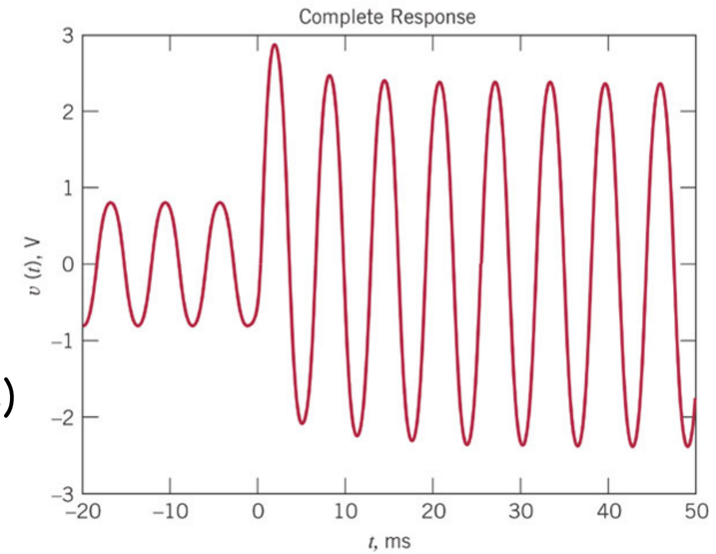
(a)

  - The switch closes at time t=0.

$$v(0) = B\cos(\phi), \quad t = 0$$

  - After the switch is closed, the capacitor voltage is

$$v(t) = \underbrace{Ke^{-t/\tau}}_{\text{Transient response}} + \underbrace{M\cos(1000t + \delta)}_{\text{Steady-state response}} \qquad (8.2\text{-}3)$$



(b)

# The responses are called:

- "transient part of the response" → **transient response**
- "steady-state part of the response" → **steady state response**
- The response, v(t), given by Eq. 8.2-3, is called the **complete response**

**complete response= transient response+ steady state response**

- In general, the complete response of a first-order circuit can be represented as the sum of two part, the **natural response (** which is the **transient response)** and the **forced response (**which is the **steady state response):**

**complete response= natural response+ forced response**

  - **Natural response:** the general solution of the (homogeneous) differential equation representing the first-order circuit, when **the input is zero**.
  - **Forced response:** a particular solution of the differential equation representing the circuit **when there is non-zero input**.

# The names

- The natural response of a first-order circuit will be of the form

$$\text{natural response} = Ke^{-(t-t0)/\tau}$$

When $t_0$=0, then

$$\text{natural response} = Ke^{-t/\tau}$$

The constant K in the natural response depends on the initial condition. For example, the capacitor voltage at time $t_0$

# Special Inputs to the First Order Circuits

- In this chapter, we will consider three cases.

In theses cases the input to the circuit after the disturbance will be

(1) a constant

$$v_s(t) = V_o$$

(2) an exponential

$$v_s(t) = V_0 e^{-t/\tau}$$

(3) a sinusoid

$$v_s(t) = V_0 \cos(\omega t + \theta)$$

These three cases are special because the forced response will have the same form as the input.

# Plans to find complete response

- Here is our plan for finding the complete response of first-order circuits:

Step 1: Find the steady state (forced) response before the disturbance. Evaluate this response at time $t=t_0$ to obtain the initial condition of the energy storage element. ($X(0)$: where it comes from.)

Step 2: Find the steady state (forced) response after the disturbance. ($X(\infty)$: where it goes ultimately.)

Step 3: Add the transient (natural) response$=Ke^{-t/\tau}$ to the steady state (forced) response to get the complete response. Use the initial condition to evaluate the constant K.

# The Response of a First-Order Circuit to a Constant Input

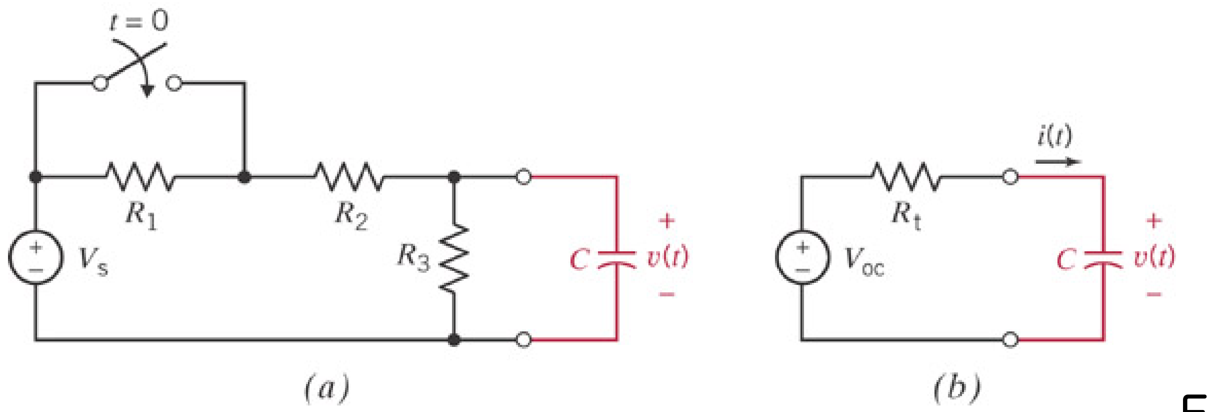



FIGURE 8.3−1

- In Figure 8.3-1a we find a first-order circuit.
- X(0)= (R3/(R1+R2+R3)*Vs is the initial Steady State response
- X(∞)= (R3/(R2+R3)*Vs is the final Steady State response.
- The transient response can be obtained using the Thevenin circuit shown in Fig.8.3-1b.

$$V_{oc} = \frac{R_3}{R_2 + R_3} V_s \quad \text{and} \quad R_t = \frac{R_2 R_3}{R_2 + R_3}{}_s$$

# The Response of a First-Order Circuit to a Constant Input

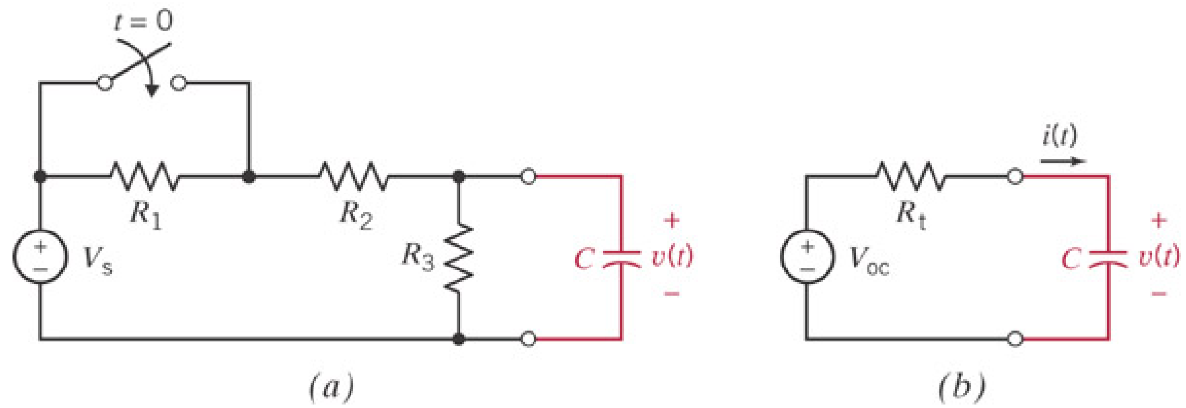



FIGURE 8.3-1

- The capacitor current is given by

$$i(t) = C\frac{d}{dt}v(t)$$

- Apply KVL to Figure 8.3-1b to get

$$V_{oc} = R_t i(t) + v(t) = R_t\left(C\frac{d}{dt}v(t)\right) + v(t)$$

- Therefore,

$$\frac{d}{dt}v(t) + \frac{v(t)}{R_t C} = \frac{V_{oc}}{R_t C} \qquad (8.3\text{-}1)$$

# Repeat the same for the inductor circuit using Norton eq. circuit.

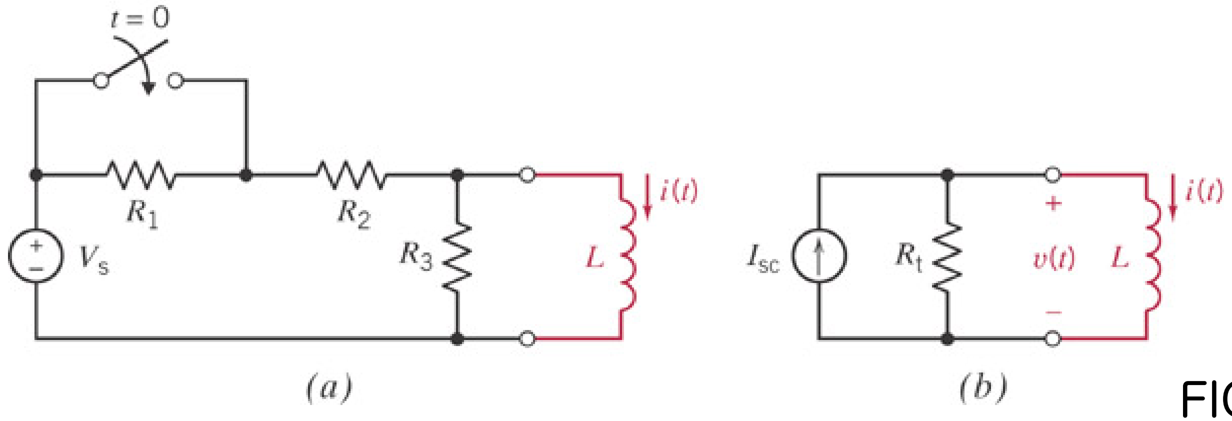



FIGURE 8.3−2

$$I_{sc} = \frac{V_s}{R_2} \qquad R_t = \frac{R_2R_3}{R_2 + R_{3\ s}}$$

# Solving the inductor circuit

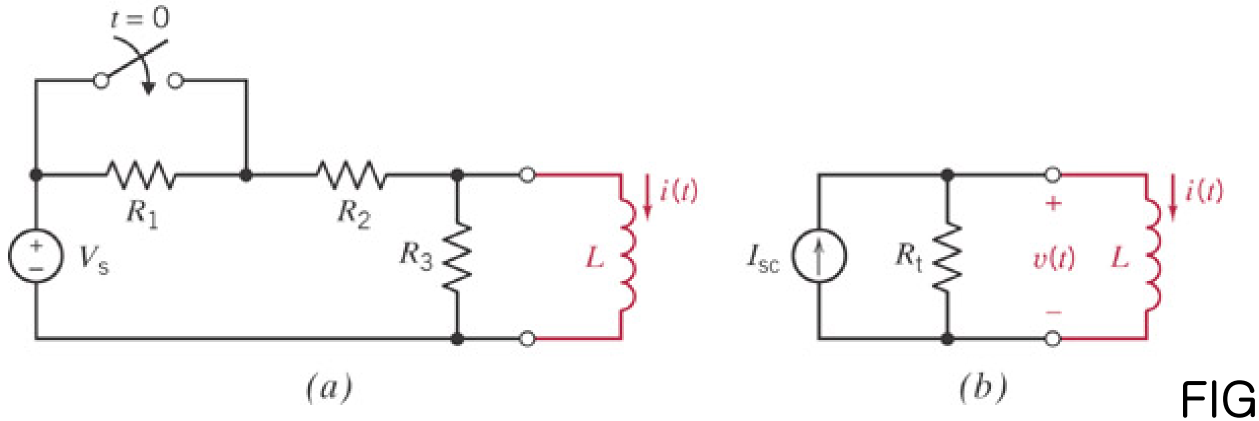



FIGURE 8.3−2

- The inductor voltage is given by

$$v(t) = L\frac{d}{dt}i(t)$$

- Apply KCL to Figure 8.3-1b to get

$$I_{sc} = \frac{v(t)}{R_t} + i(t) = \frac{L\frac{d}{dt}i(t)}{R_t} + i(t)$$

- Therefore,

$$\frac{d}{dt}i(t) + \frac{R_t}{L}i(t) = \frac{R_t}{L}I_{SC} \qquad (8.3\text{−}2)$$

# Now coming to a general form of solution

- Equation 8.3-1 and 8.3-2 have the same form. That is

$$\frac{d}{dt}x(t)+\frac{x(t)}{\tau}=K \qquad (8.3\text{–}3)$$

- The parameter τ is called the time constant. We will solve this differential equation by separating the variables and integrating. Then we will use the solution of Eq. 8.3-3 to obtain solutions of Eqs. 8.3-1 and 8.3- 2

- We may rewrite Eq. (3) as

$$\frac{dx}{dt}=\frac{K\tau-x}{\tau}$$

- Or, separating the variables,

$$\frac{dx}{x-k\tau}=-\frac{dt}{\tau}$$

# Second page--

- Forming the indefinite integral, we have

$$\int \frac{dx}{x-k\tau} = -\frac{1}{\tau}\int dt + D$$

where D is a constant of integration. Performing the integration, we have

$$\ln(x-K\tau) = -\frac{t}{\tau} + D$$

- Solving for x gives

$$x(t) = K\tau + Ae^{-t/\tau}$$

where $A=e^D$, which is determined from the initial condition, x(0).

## Here we go. Note the solution at the bottom of this page.

- To find A, let t=0. Then

$$x(0) = K\tau + Ae^{-0/\tau} = K\tau + A$$

or

$$A = x(0) - K\tau$$

- Therefore, we obtain

$$x(t) = K\tau + [x(0) - K\tau]e^{-t/\tau}$$

- Since

$$x(\infty) = \lim_{t\to\infty} x(t) = K\tau$$

- This can be written as

$$\boxed{x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}}$$

# The plot

- Figure 8.3-3 shows a plot of x(t) versus t.
- We can determine the values of
  (1) the slope of the plot at time t=0
  (2) the initial value of x(t)
  (3) the final value of x(t) from this plot.

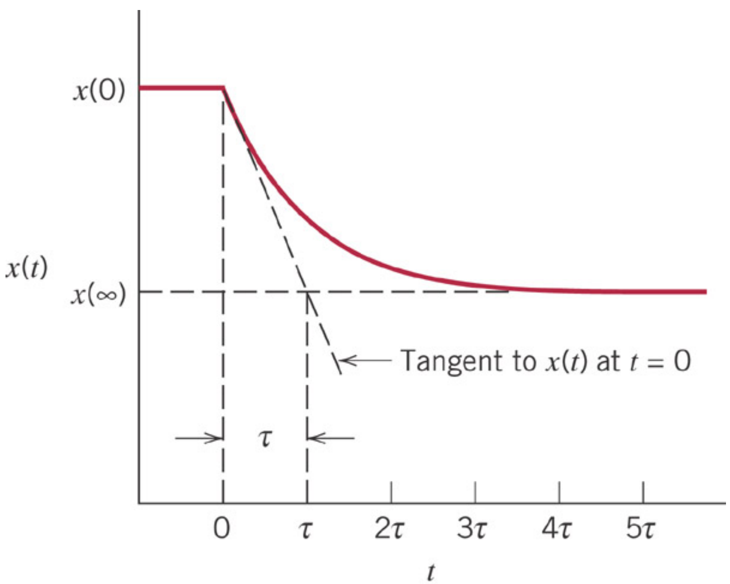



FIGURE 8.3-3

A graphical technique for measuring the time constant of a first-order circuit

# Apply this to the capacitor circuit

- Next, we apply these results to the RC circuit in Figure 8.3-1. Comparing Eqs. 8.3-1 and 8.3-3, we see that

$$x(t) = v(t), \ \ \tau = R_t C, \quad and \quad k = \frac{V_{oc}}{R_t C}$$

- Making these substitutions in Eq. 8.3-4 gives

$$v(t) = V_{oc} + (v(0) - V_{oc})e^{-t/(R_t C)}$$

- This is the steady-state or forced response. The sum of the natural and forced responses is the complete response;

$$\text{complete reponse} = v(t), \qquad \text{forced response} = V_{OC}$$

$$\text{natural reponse} = (v(0) - V_{OC})e^{-t/(R_t C)}$$

# Apply this to the inductor circuit

- Next, compare Eqs. 8.3-2 and 8.3-3 to find the solution of the RL circuit in Figure 8.3-2. We see that

$$x(t) = i(t), \quad \tau = \frac{L}{R_t}, \quad and \quad K = \frac{L}{R_t} I_{SC}$$

- Making these substitutions in Eq. 8.3-4 gives

$$i(t) = I_{SC} + (i(0) - I_{SC})e^{-(R_t/L)t}$$

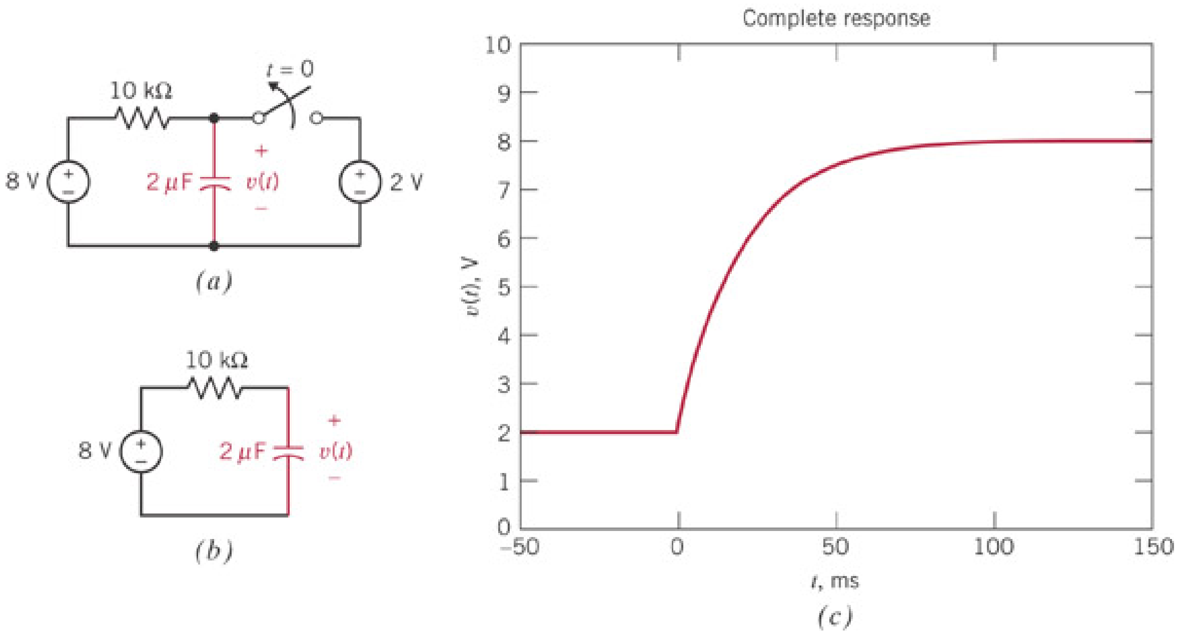
- Again, the complete response is the sum of the forced(steady-state) response and the transient(natural) response:

$$\text{complete reponse} = i(t), \quad \text{forced response} = I_{SC}$$

$$\text{natural reponse} = (i(0) - I_{SC})e^{-(R_t/L)t}$$

# Example 8.3-1 First-Order Circuit with a Capacitor

- Find the capacitor voltage after the switch opens in the circuit shown in Figure 8.3-4a. What is the value of the capacitor voltage 50ms after the switch opens?



# Solution

Initial condition

$$v(0) = 2\text{V}$$

Figure 8.3-4b shows the circuit after the switch opens.

$$R_t = 10k\Omega \qquad and \qquad V_{OC} = 8\text{V}$$

Time constant

$$\tau = R_t C = (10\times10^3)(2\times10^{-6}) = 20\times10^{-3} = 20ms$$

Capacitor voltage

$$v(t) = 8 - 6e^{-t/20}\text{V}$$

where t has units of ms.

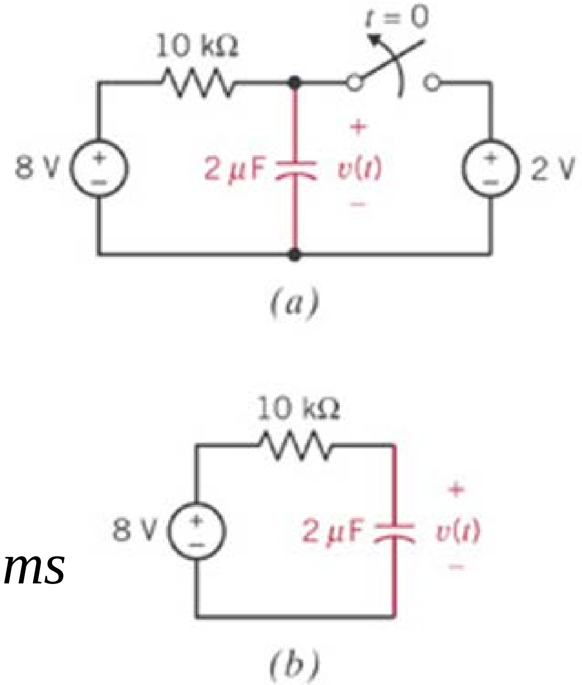



FIGURE 8.3–4

# Solution

To find the voltage 50ms after the switch opens, let t=50. Then

$$v(50) = 8 - 6e^{-50/20} = 7.51\text{V}$$

Figure 8.3-4c shows a plot of the capacitor voltage as a function of time

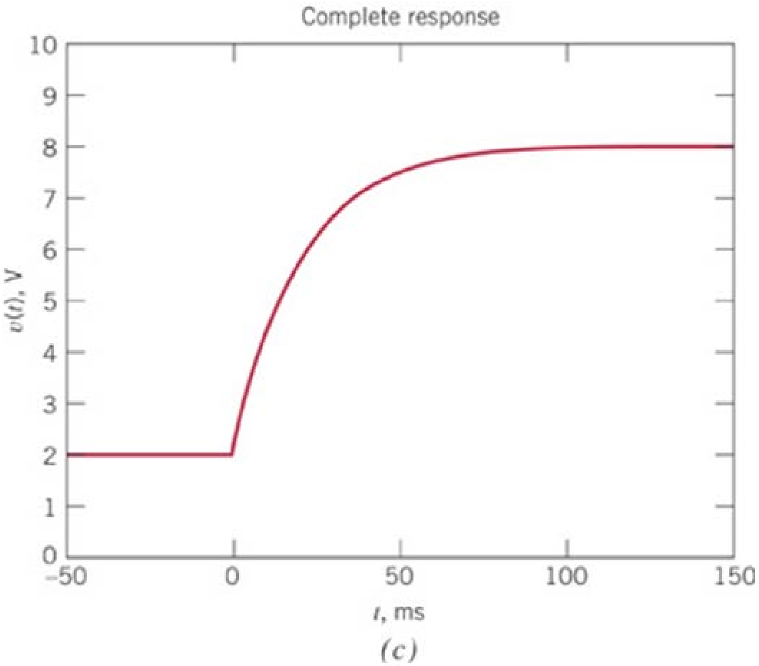



FIGURE 8.3−4

# Example 8.3-2 First-Order Circuit with an Inductor

- Find the inductor current after the switch closes in the circuit shown in Figure 8.3-5a. How long will it take for the inductor current to reach 2mA?

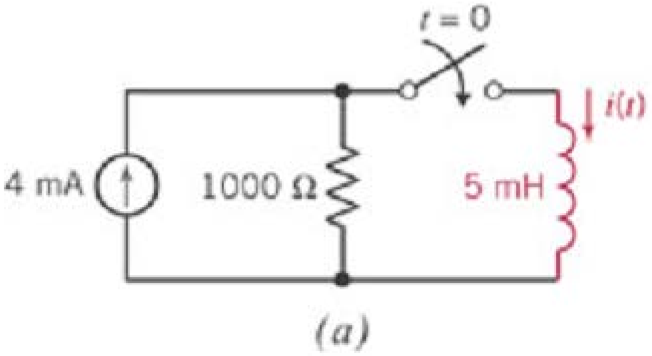



FIGURE 8.3−4

# Solution

Figure 8.3-5b shows the circuit after the switch closes.

$$R_t = 1000\Omega \quad and \quad I_{SC} = 4mA$$

Time constant

$$\tau = \frac{L}{R_t} = \frac{5\times10^{-3}}{1000} = 5\times10^{-6} = 5\mu s$$

Inductor current

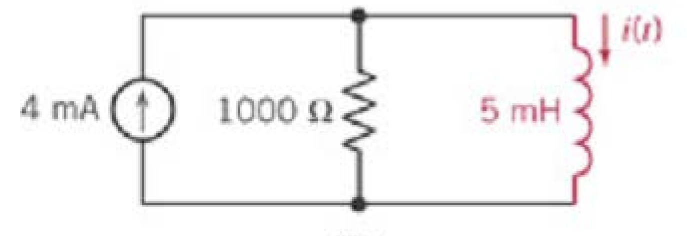
$$i(t) = 4 - 4e^{-t/5} mA$$

Find the time when the current reaches 2mA.

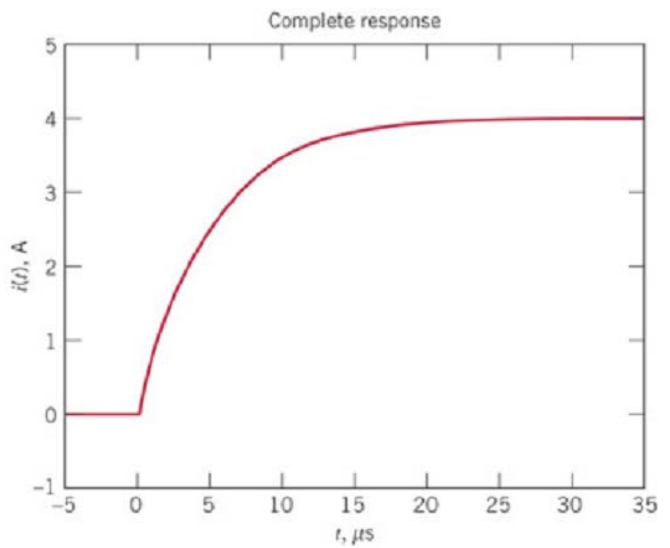
$$2 = 4 - 4e^{-t/5} mA$$

$$t = -5\times \ln(\frac{2-4}{-4}) = 3.47\mu s$$

Figure8.3-5c shows a plot of the inductor current as a function of time



(b)



(c)

# Example 8.3-3 First-Order Circuit

- The switch in Figure 8.3-6a has been open for a long time, and the circuit has reached steady state before the switch closes at time $t=0$. Find the capacitor voltage for $t \geq 0$.

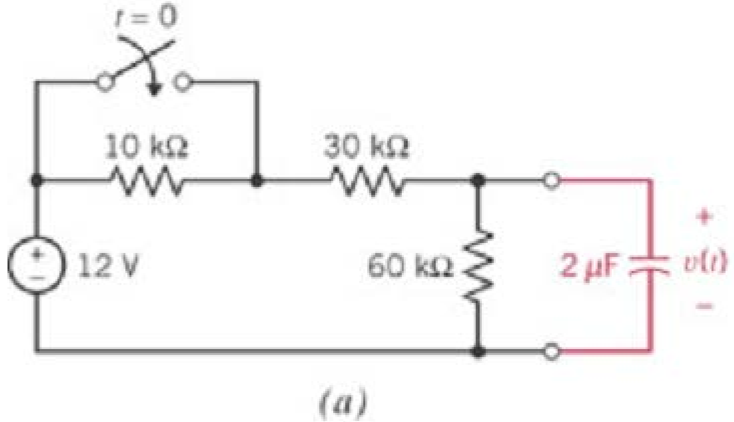
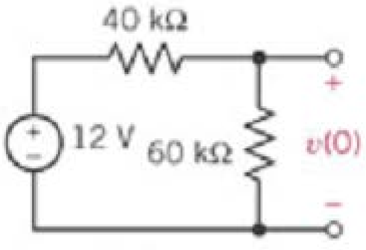



FIGURE 8.3−6

# Solution

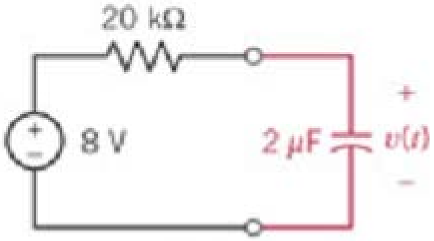
- Figure 8.3-6b shows the appropriate equivalent circuit while the switch is open.
  Analyzing the circuit in Figure 8.3-6b using voltage division gives




(b)

$$v(0) = \frac{60\times10^3}{40\times10^3 + 60\times10^3}12 = 7.2\text{V}$$

- Figure 8.3-6c shows the appropriate equivalent circuit after the switch closes.
  After the switch is closed




(c)

$$v_{OC} = \frac{60\times10^3}{30\times10^3 + 60\times10^3}12 = 8\text{V}$$

$$R_t = \frac{30\times10^3 \times 60\times10^3}{30\times10^3 + 60\times10^3} = 20\times10^3 = 20\text{k}\Omega$$

# Solution

- The time constant is

$$\tau = R_t \times C = (20\times10^{3})\times(2\times10^{-6}) = 40\times10^{-3} = 40\text{ms}$$

- Consequently

$$v(t) = 8 - 0.8e^{-t/40}\text{V}$$

where $t$ has units of ms.

# Example 8.3-4 First-Order Circuit

- The switch in Figure 8.3-7a has been open for a long time, and the circuit has reached steady state before the switch closes at time $t=0$. Find the inductor current for $t \geq 0$.

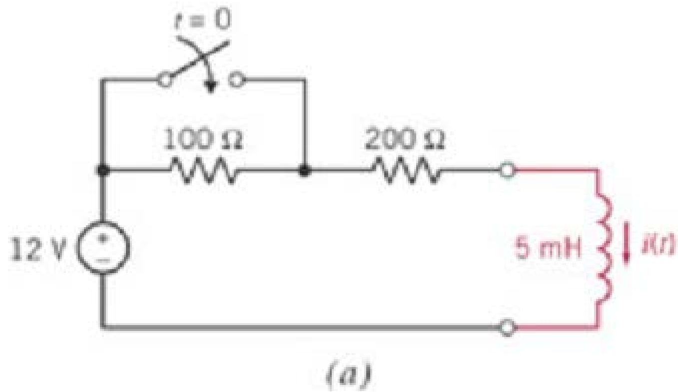
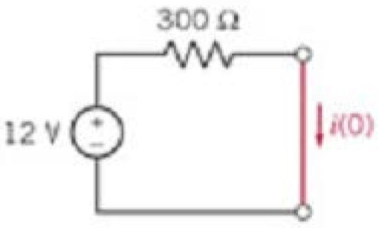



FIGURE 8.3−7

# Solution

1. Figure 8.3-7b shows the appropriate equivalent circuit while the switch is open.
   The initial inductor current can be calculated using Ohm's law:

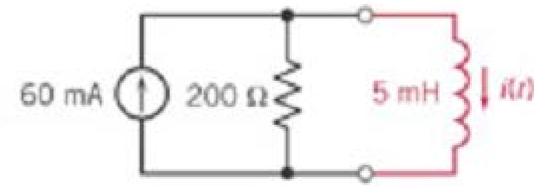
$$i(0) = \frac{12}{300} = 40\text{mA}$$



(b)

2. Figure 8.3-7c shows the appropriate equivalent circuit after the switch closes.
   After the switch is closed

$$I_{SC} = \frac{12}{200} = 60\text{mA} \quad \text{and} \quad R_t = 200\Omega$$



(c)

3. The time constant is

$$\tau = \frac{L}{R_t} = \frac{5\times 10^{-3}}{200} = 25\times 10^{-6} = 25\mu s$$

4. Consequently,

$$i(t) = 60 - 20e^{-t/25}\text{mA}$$

where $t$ has of microseconds.

# Example 8.3-5 First-Order Circuit

- The circuit in Figure 8.3-8a is at steady state before the switch opens. Find the current *i(t)* for $t>0$.

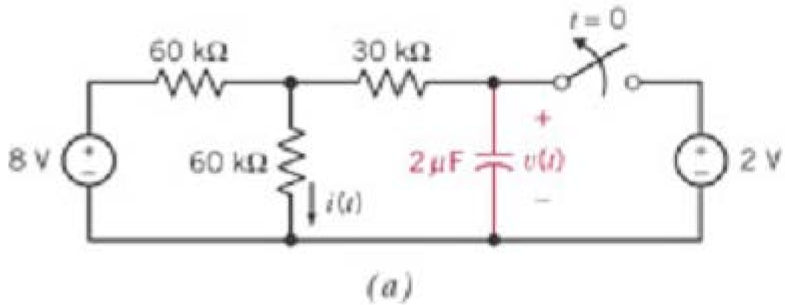


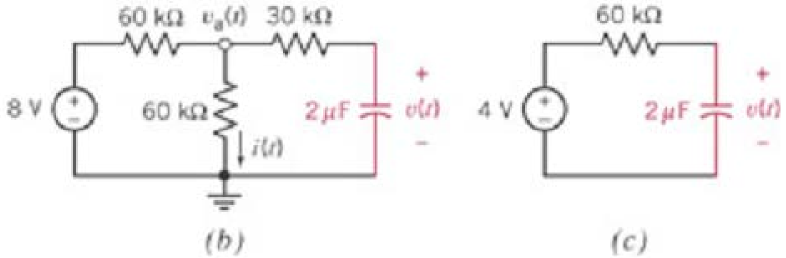

FIGURE 8.3−8

# Solution

1. We find the capacitor voltage. Before the switch opens, the capacitor voltage is equal to the voltage of the 2-volt source. The initial condition is

$$v(0) = 2\text{V}$$

2. Figure 8.3-8b shows the circuit as it will be after the switch is opened. The part of the circuit connected to the capacitor has been replaced by its Thévenin equivalent circuit in Figure 8.3-8c.




The parameters of the Thévenin equivalent circuit are

$$v_{OC} = \frac{60\times10^3}{60\times10^3 + 60\times10^3}8 = 4\text{V}$$

$$R_t = 30\times10^3 + \frac{60\times10^3\times60\times10^3}{60\times10^3+60\times10^3} = 60\times10^3 = 60\text{k}\Omega$$

# Solution

3. The time constant is

$$\tau = R_t \times C = (60\times10^3)\times(2\times10^{-6}) = 120\times10^{-3} = 120\text{ms}$$

therefore,

$$v(t) = 4 - 2e^{-t/120}\text{V}$$

where *t* has units of ms

4. The node voltage, $v_a(t)$ in Figure 8.3-8b

$$\frac{v_a(t)-8}{60\times10^3} + \frac{v_a(t)}{60\times10^3} + \frac{v_a(t)-v(t)}{30\times10^3} = 0$$

$$\frac{v_a(t)-8}{60\times10^3} + \frac{v_a(t)}{60\times10^3} + \frac{v_a(t)-(4-2e^{-t/120})}{30\times10^3} = 0$$

Solving for va(t), we get

$$v_a(t) = \frac{8+2(4-2e^{-t/120})}{4} = 4 - e^{-t/120}\text{V}$$

5. Finally, we calculate I(t) using Ohm's law:

$$i(t) = \frac{v_a(t)}{60\times10^3} = \frac{4-e^{-t/120}}{60\times10^3} = 66.7 - 16.7e^{-t/120}\,\mu A$$

# Example 8.3-6 First-Order Circuit with $t_0 \neq 0$

- Find the capacitor voltage after the switch opens in the circuit shown in Figure 8.3-9a. What is the value of the capacitor voltage 50ms after the switch opens?

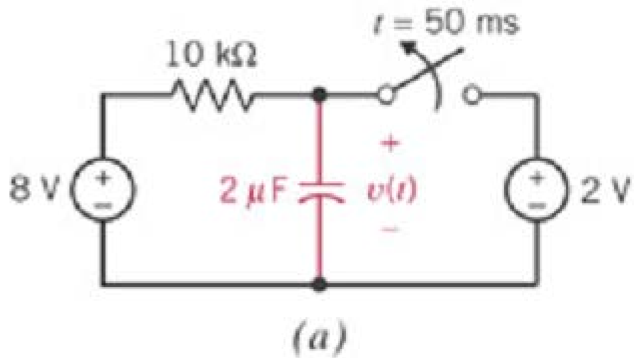



FIGURE 8.3-9

# Solution

1. The 2-volt voltage source forces the capacitor voltage to be 2 volts until the switch opens. Consequently,

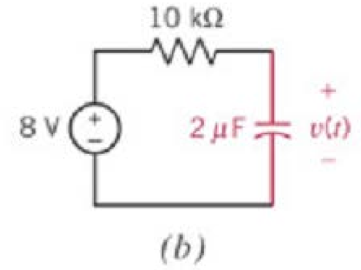
$$v(t) = 2\text{V} \quad for \quad t \le 0.05\text{s}$$

2. In particular, the in initial condition is

$$v(0.05) = 2\text{V}$$

3. Figure 8.3-8b shows the circuit after the switch opens. We see that

$$R_t = 10\text{k}\Omega \quad \text{and} \quad V_{oc} = 8\text{V}$$




4. The time constant for this first-order circuit containing a capacitor is

$$\tau = R_t C = 0.020\text{s}$$

5. Consequently, the voltage of the capacitor is given by

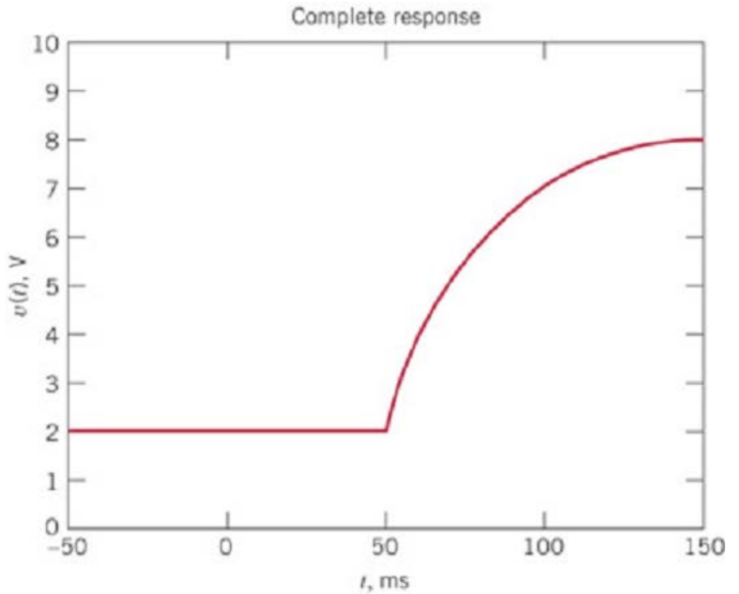
$$v(t) = 8 - 6e^{-(t-50)/20}\text{V}$$

# Solution

6. To find the voltage 50ms after the switch opens, let t=100ms. Then

$$v(100) = 8 - 6e^{-(100-50)/20} = 7.51\text{V}$$

7. Figure 8.3-9c shows a plot of the capacitor voltage as a function of time.




(c)

# Example 8.3-7 First-Order Circuit with $t_0 \neq 0$

- Find the inductor current after the switch closes in the circuit shown in Figure 8.3-10a. How long will it take for the inductor current to reach 2mA?

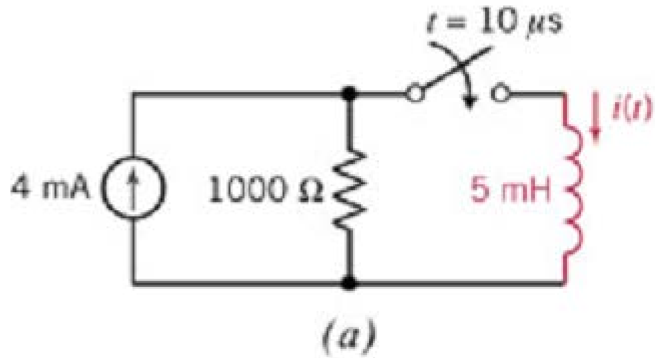
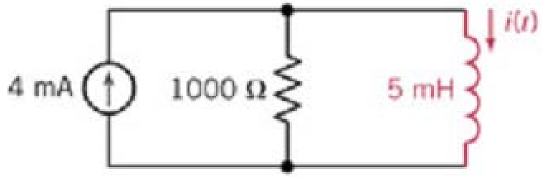



FIGURE 8.3−10

# Solution

1. The inductor current will be 0A until the switch closes. There, the initial condition is
$$i(10\mu s) = 0\text{A}$$
2. Figure 8.3-10b shows the circuit after the switch closes. We see that
$$R_t = 1000\Omega \quad \text{and} \quad I_{SC} = 4\text{mA}$$



(b)

3. The time constant for this first-order circuit containing an inductor is
$$\tau = \frac{L}{R_t} = \frac{5\times10^{-3}}{1000} = 5\times10^{-6} = 5\mu\text{s}$$
4. Consequently, the current of the inductor is given by
$$i(t) = 4 - 4e^{-(t-10)/5}\text{mA}$$

# Solution

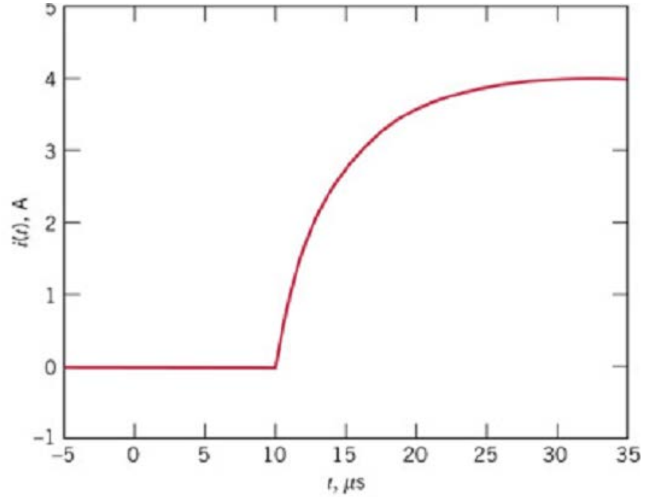
6. To find the time when the current reaches 2mA, substitute *i(t)*=2mA. Then

$$2 = 4 - 4e^{-(t-10)/5}\text{mA}$$

Solving for *t* gives

$$t = -5 \times \ln\left(\frac{2-4}{-4}\right) + 10 = 13.47\mu\text{s}$$

7. Figure 8.3-10c shows a plot of the inductor current as a function of time.



(c)

# Example 8.3-8 Exponential Response of a First-Order Circuit

- Figure 8.3-11a shows a plot of the voltage across the inductor in Figure 8.3-11b.
  a. Determine the equation that represents the inductor voltage as a function of time.
  b. Determine the value of the resistance R.
  c. Determine the equation that represents the inductor current as a function of time.

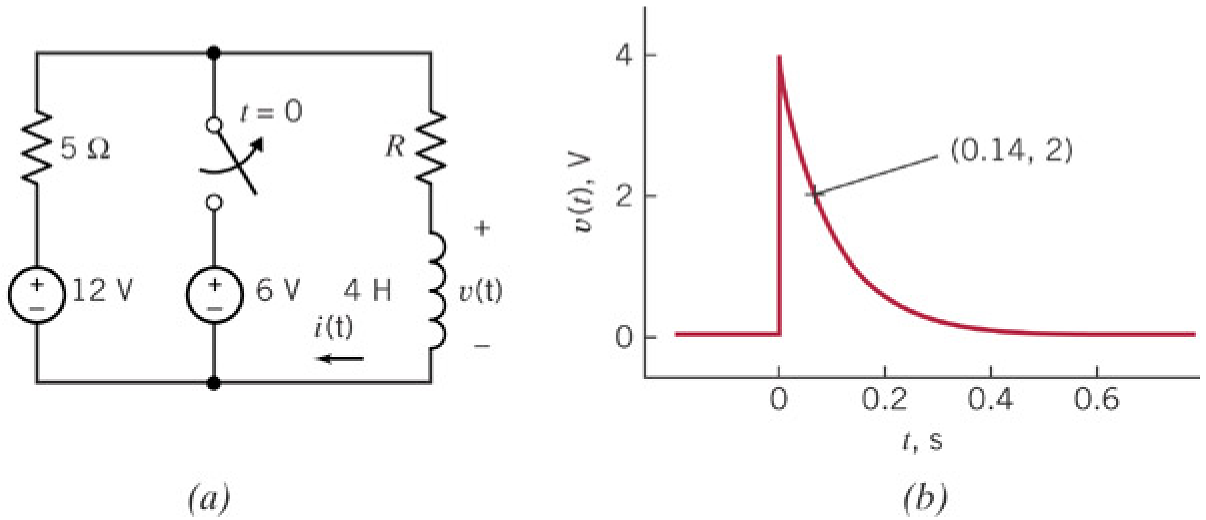



FIGURE 8.3-11

# Solution(a)

1. The inductor voltage is represented by an equation of the form

$$v(t)=\begin{cases} D & for \;\; t<0 \\ E+Fe^{-at} & for \;\; t\geq 0 \end{cases}$$

The constants D, E, and F are described by

$$D=v(t) \quad when \;\; t<0, \quad E=\lim_{t\to\infty} v(t), \quad and \quad E+F=\lim_{t\to 0+} v(t)$$

From the plot, we see that

$$D=0,\; E=0, \quad and \quad E+F=4\text{V}$$

Consequently,

$$v(t)=\begin{cases} 0 & for \;\; t<0 \\ 4e^{-at} & for \;\; t\geq 0 \end{cases}$$

2. One such point is labeled on the plot in Figure 8.3-11b. We see v(0.14)=2V;

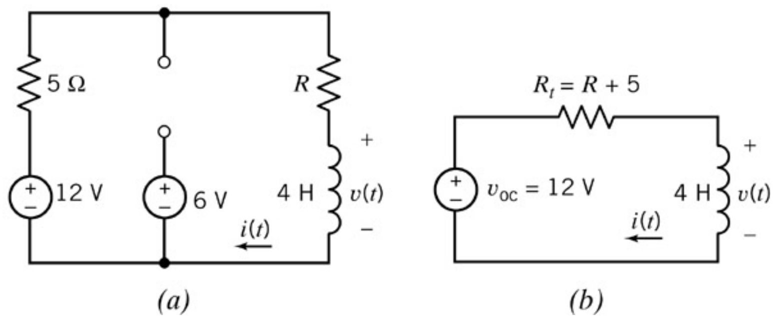
$$2=4e^{-a(0.14)} \Rightarrow a=\frac{\ln(0.5)}{-0.14}=5$$

Consequently,

$$v(t)=\begin{cases} 0 & for \;\; t<0 \\ 4e^{-5t} & for \;\; t\geq 0 \end{cases}$$

# Solution(b)

1. Figure 8.3-12a shows the circuit immediately after the switch opens.in Figure 8.3-12b, the part of the circuit connected to the inductor has been replaced by its Thévenin equivalent circuit.




2. The time constant of the circuit is given by $\tau = \dfrac{L}{R_t} = \dfrac{4}{R+5}$

$$\tau = \frac{L}{R_t} = \frac{4}{R+5}$$

Also, the time constant is related to the exponent in v(t) by $-5t = -\dfrac{t}{\tau}$. Consequently

$$5 = \frac{1}{\tau} = \frac{R+5}{4} \quad => \quad R = 15\Omega$$

# Solution(c)

1. The inductor current is related to the inductor voltage by

$$i(t)=\frac{1}{L}\int_0^t v(\tau)\,d\tau+i(0)$$

2. Figure 8.3-13 show the circuit before the switch opens.

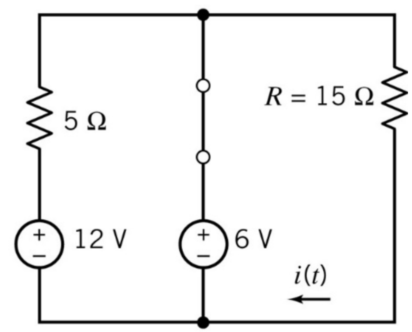



FIGURE 8.3−13

The inductor current is given by

$$i(t)=\frac{6}{15}=0.4\text{A}$$

In particular, *i(0-)=0.4*A. The current in an inductor is continuous, so *i(0+)=i(0-)*. Consequently,

$$i(0)=0.4\text{A}$$

3. Returning to the equation for the inductor current, after the switch opens we have

$$i(t)=\frac{1}{4}\int_0^t 4e^{-5\tau}\,d\tau+0.4=\frac{1}{-5}(e^{-5t}-1)+0.4=0.6-0.2e^{-5t}$$

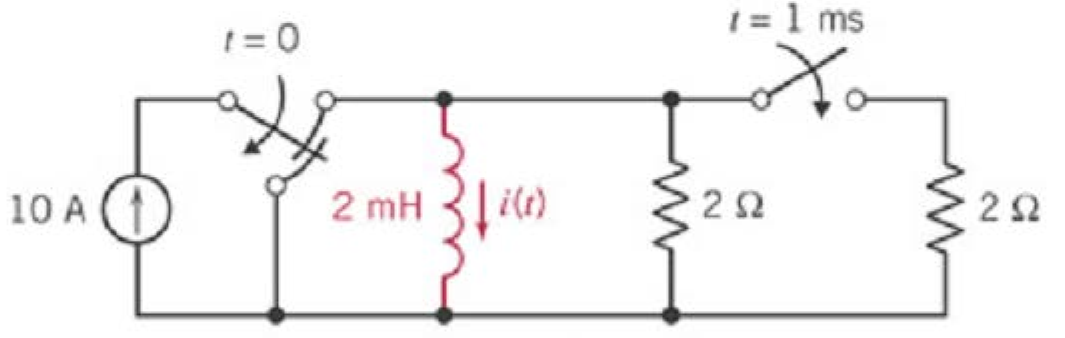
4. In summary,

$$i(t)=\begin{cases} 0.4 & \text{for } t<0 \\ 0.6-0.2e^{-5t} & \text{for } t\geq 0 \end{cases}$$

# Sequential Switching

- **Sequential switching** occurs when a circuit contains two or more switches that change state at different instants.

- Figure 8.4-1a is an example of sequential switching.




(a)

FIGURE 8.4-1

# Sequential Switching (cont'd)

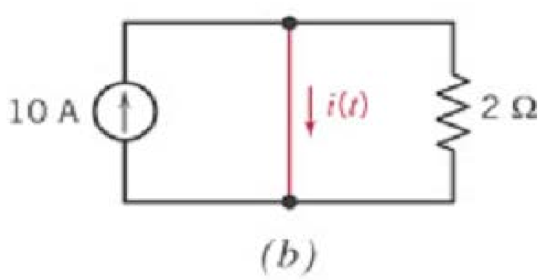



FIGURE 8.4-1

- Figure 8.4-1b shows the equivalent circuit that is appropriate for t<0.

$$i(t) = 10[A] \qquad t < 0$$

Before the switch changes state at time t=0.

$$i(0^-) = 10[A]$$

After the switch changes state at time t=0.

$$i(0^+) = 10[A]$$

This is the initial condition at t=0.

# Sequential Switching (cont'd)

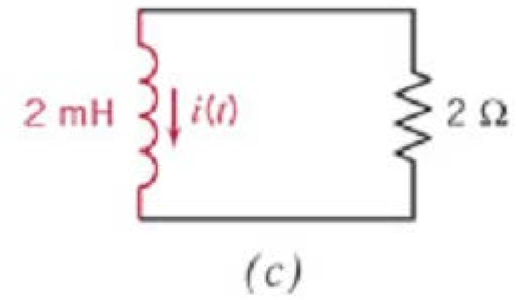



FIGURE 8.4–1

- Figure 8.4-1c shows the equivalent circuit at 0< t <1 ms

  $$I_{sc} = 0[A] \quad and \qquad R_t = 2[\Omega]$$

  Time constant

  $$\tau = \frac{L}{R_t} = \frac{2\times10^{-3}}{2} = 1\times10^{-3} = 1ms$$

  Inductor current

  $$i(t) = i(0)e^{-t/\tau} = 10e^{-t}[A] \quad for\ 0 < t < 1ms$$

  Immediately before t=1ms

  $$i(1^-) = 10e^{-1} = 3.68[A]$$

  Immediately after t=1ms

  $$i(1^+) = 3.68[A]$$

  This is the initial condition at time t=1ms.

# Sequential Switching (cont'd)

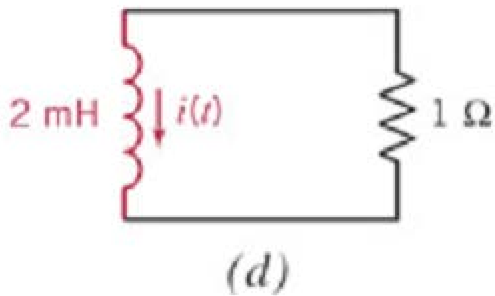



FIGURE 8.4−1

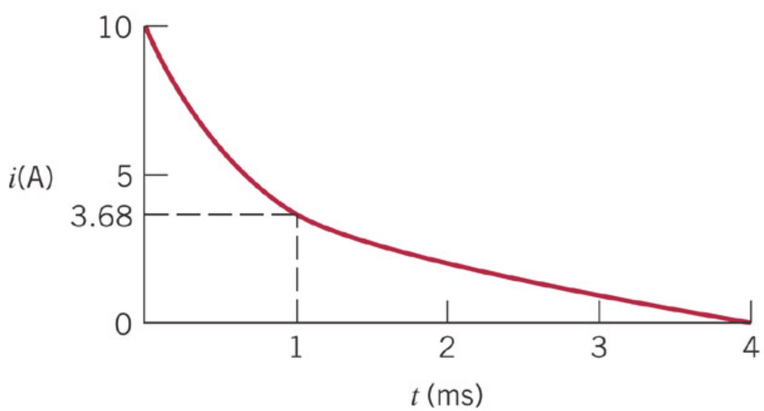



FIGURE 8.4−2

- Figure 8.4-1d shows the appropriate equivalent circuit.

  $$I_{sc} = 0[A] \quad and \qquad R_t = 1[\Omega]$$

  Time constant

  $$\tau = \frac{L}{R_t} = \frac{2\times10^{-3}}{1} = 2\times10^{-3} = 2ms$$

  Inductor current

  $$i(t) = i(t_0)e^{-(t-t_0)/\tau} = 3.68e^{-(t-1)/2}[A] \qquad for\ t > 1ms$$

  $t_0$ denotes the time when the switch changes state – 1ms in this example.

- Figure 8.4-2 shows a plot of the inductor current.

# Sequential Switching (cont'd)

- In some applications, switching occurs at prescribed voltage values rather than at prescribed times. Figure 8.4-3 a device, called comparator, that can be used to accomplish this kind of switching.

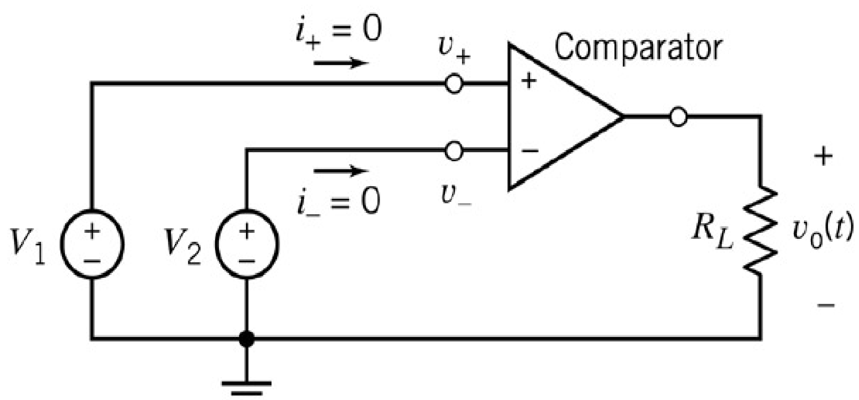



FIGURE 8.4−3

$$v_o(t) = \begin{cases} V_H & \text{if } v_+ > v_- \\ V_L & \text{if } v_+ < v_- \end{cases}$$

# Sequential Switching (cont'd)

- In figure 8.4-4, a comparator is used to compare the capacitor voltage to a threshold voltage $V_T$, Suppose

  $$V_A > V_T > v_c(0)$$

  The input voltages of the comparator are

  $$v_+ = v_c(t) \quad and \quad v_- = v_T$$

  so the output voltage of the comparator is

$$v_o(t) = \begin{cases} V_H & if \;\; v_c(t) > v_T \\ V_L & if \;\; v_c(t) < v_T \end{cases}$$

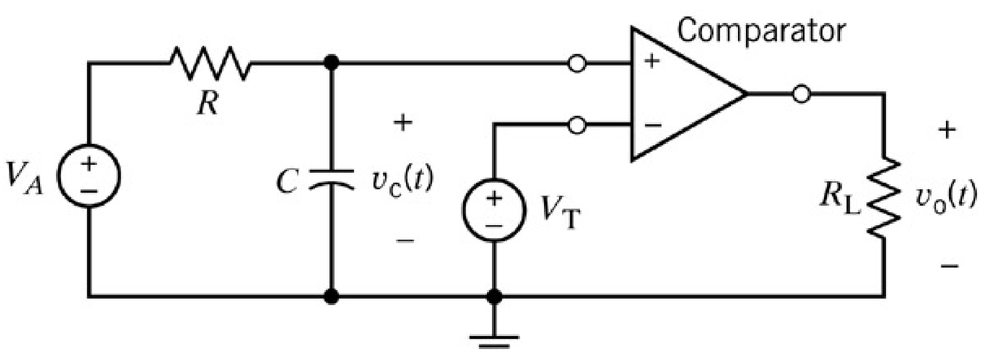



FIGURE 8.4−4

# Sequential Switching (cont'd)

- We know that the capacitor voltage of this first-order circuit will be

$$v_c(t) = V_A + (v_c(0) - V_A)e^{-t/(RC)}$$

Let $t_1$ denote the time when the comparator output voltage switches from $V_L$ to $V_H$. Then $v_c(t_1)=V_T$, so

$$V_T = V_A + [v_c(0) - V_A]e^{-t_1/(RC)}$$

Solving for $t_1$ gives

$$t_1 = RC\ln\left(\frac{v_c(0) - V_A}{V_T - V_A}\right)$$

# Example) Comparator Circuit

- Consider the circuit shown in Figure 8.4-5. The initial value of the capacitor voltage is $v_c(0)=1.667$ volts. What value of resistance, R, is required if the comparator is to switch from $V_L=0$ to $V_H=5$ volts at time $t_1=1$ ms

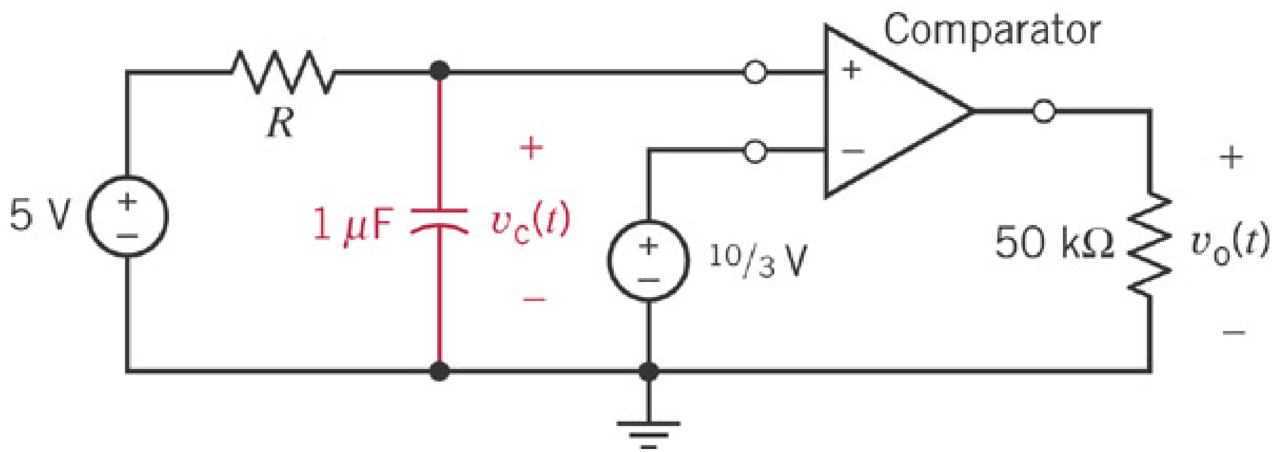



FIGURE 8.4−5

# Solution

- Figure 8.4-5 shows a specific example of the circuit in Figure 8.4-4.

1. We get

$$1\times10^{-3} = R(1\times10^{-6})\ln\left(\frac{\frac{5}{3}-5}{\frac{10}{3}-5}\right) = R(1\times10^{-6})\ln(2)$$

2. Then, solving for R:

$$R = \frac{1\times10^{-3}}{\ln(2)\times10^{-6}} = 1.44k\Omega$$

# Example) Comparator Circuit

- In Figure 8.4-6, a comparator is used to compare the resistor voltage, $v_R(t)$, to a threshold voltage, $V_T$. Suppose

$$V_A > V_T > Ri_L(0)$$

Determine the time $t_1$ when the comparator output voltage switches from $V_L$ to $V_H$

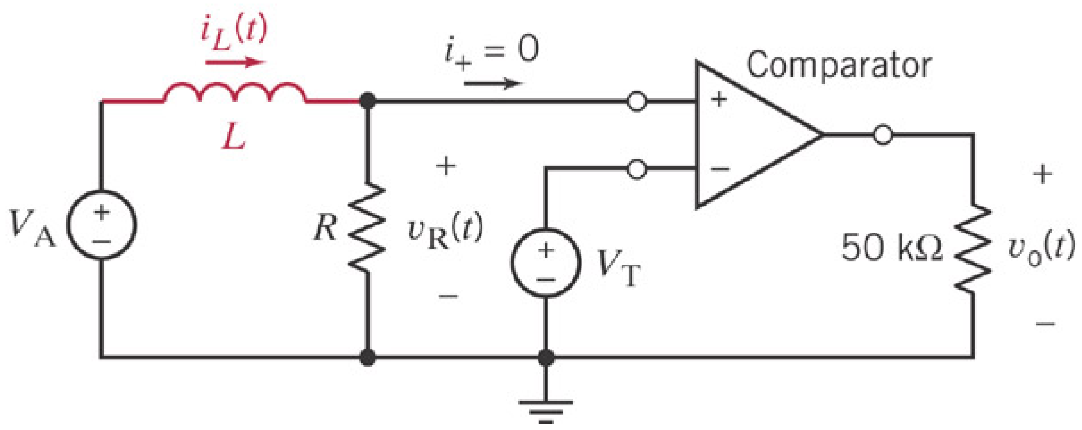



FIGURE 8.4−6

# Solution

1. The resistor current is equal to the inductor current, so

$$v_R(t) = Ri_L(t)$$

2. The comparator does not disturb the first-order circuit consisting of the voltage source, resistor, and inductor. The inductor current is

$$i_L(t) = \frac{V_A}{R} + (i_L(0) - \frac{V_A}{R})e^{-(R/L)/t}$$

3. Next, $t_1$ is the time when $Ri_L(t_1)=V_T$, so

$$V_T = V_A + (Ri_L(0) - V_A)e^{-(R/L)/t_1}$$

4. Solving for $t_1$ gives

$$t_1 = \frac{L}{R}\ln(\frac{Ri_L(0) - V_A}{V_T - V_A})$$

# Stability of First-Order Circuits

- Complete response

$$x(t) = x_n(t) + x_f(t)$$

$$x_n(t) = Ke^{-t/\tau} \quad (\text{natural response})$$

$$x_f(t) \quad (\text{forced response})$$

- The circuit is *stable*
  - When τ>0 , the natural response vanishes as t->0.
- The circuit is *unstable*
  - When τ<0, the natural response grows without bound as t->0.
  - In most applications, the behavior of unstable circuits is undesirable and is to be avoided.
- How can we design first-order circuits to be stable?
  - $R_t$>0 is required to make a first-order circuit be stable. $(\tau = R_t C \;\; or \;\; \tau = L / R_t)$

# Example 8.5-1 Response of an Unstable First-Order Circuit

- The First-order circuit shown in Figure 8.5-1a is at steady state before the switch closes at t=0. This circuit contains a dependent source and so may be unstable. Find the capacitor voltage, v(t), for t>0.

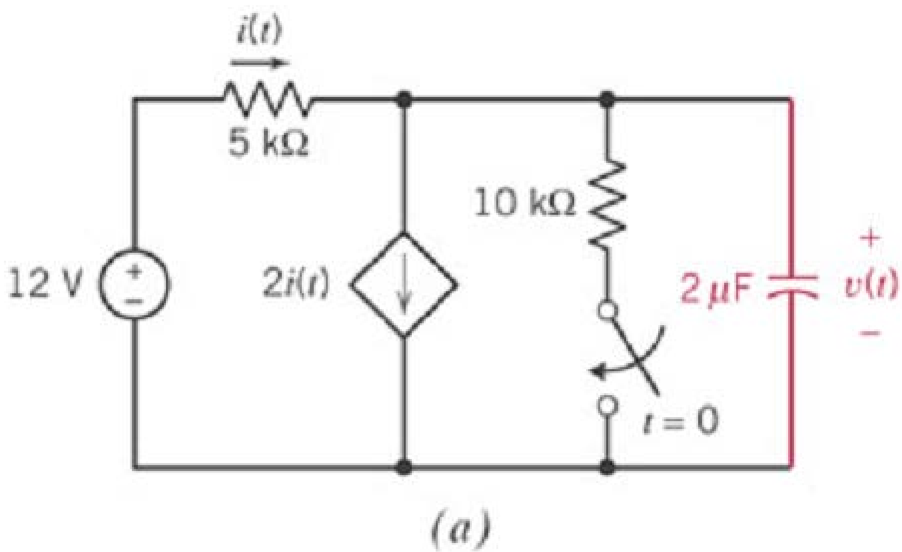
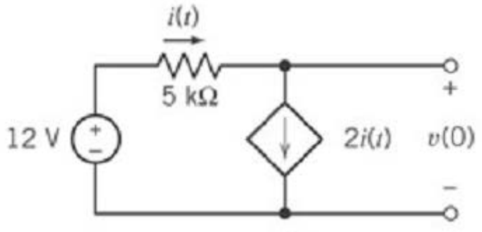



FIGURE 8.5−1

# Solution

1. We calculate the initial condition from the circuit in Figure 8.5-2b.




(b)

   1. Apply KCL to the top node of the dependent current source

$$-i + 2i = 0$$

$$i = 0$$

   2. Consequently, There is no voltage drop across the resistor and

$$v(0) = 12\text{V}$$

2. Calculate the open-circuit voltage using the circuit in Figure 8.5-1c.
   1. Writing a KVL equation, we get

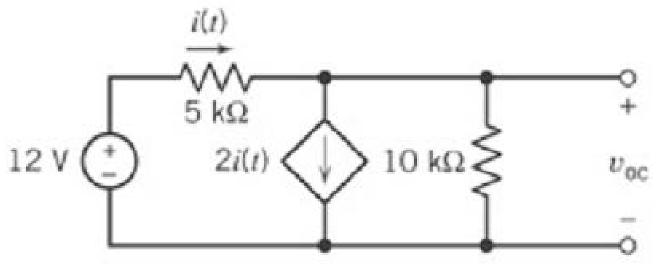
$$12 = (5\times10^3)\times i + (10\times10^3)\times(i - 2i)$$

   2. We find

$$i = -2.4\text{mA}$$

   3. Applying Ohm's law to the 10-kΩ resistor, we get

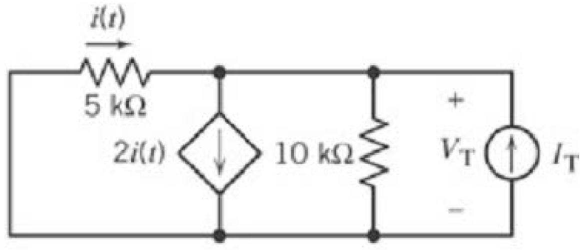
$$V_{OC} = (10\times10^3)\times(i - 2i) = 24\text{V}$$




(c)

# Solution

3. Calculate the Thévenin resistance using the circuit shown in Figure 8.5-1d.
   1. Apply KVL to the loop consisting of the two resistors to get
   $$0=(5\times10^3)\times i+\left(10\times10^3\right)\times\left(I_T+i-2i\right)$$
   2. Solving for the current,
   $$i=2I_T$$
   3. Applying Ohm's law to the 10-kΩ resistor, we get
   $$V_T=10\times10^3\times\left(I_T+i-2i\right)=-10\times10^3\times I_T$$
   4. The Thévenin resistance is given by
   $$R_t=\frac{V_T}{I_T}=-10\text{k}\Omega$$
   5. The time constant is
   $$\tau=R_tC=-20\text{ms}$$



(d)

4. The complete response is
$$v(t)=24-12e^{t/20}$$

# Example 8.5-2 Designing First-Order Circuits to Be Stable

- The circuit considered in Example 8.5-1 has been redrawn in Figure 8.5-2a, with the gain of the dependent source represented by the variable B. What restrictions must be placed on the gain of the dependent source to ensure that it is stable? Design this circuit to have a time constant of +20ms.

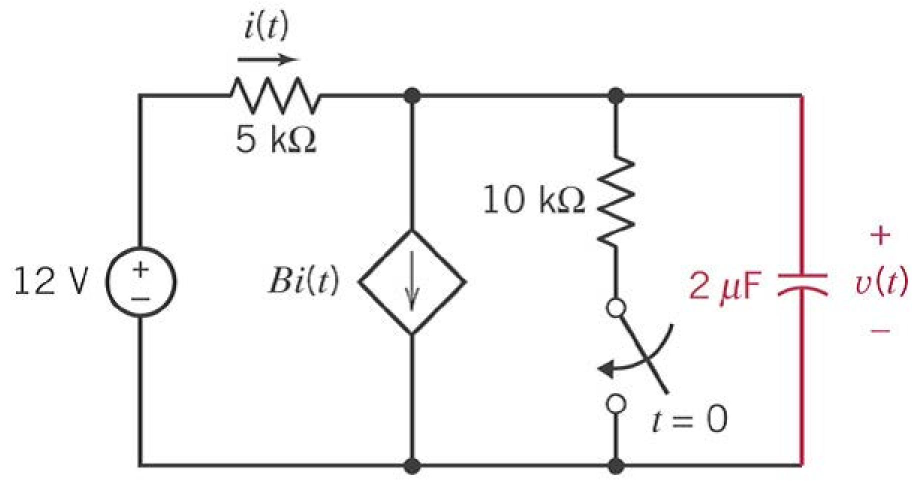



FIGURE 8.5−2

# Solution

- Figure 8.5-2b the circuit used to calculate $R_t$

1. Applying KVL to the loop consisting of the two resistors.

$$5\times10^3\times i+V_T=0$$

2. Solving for the current gives

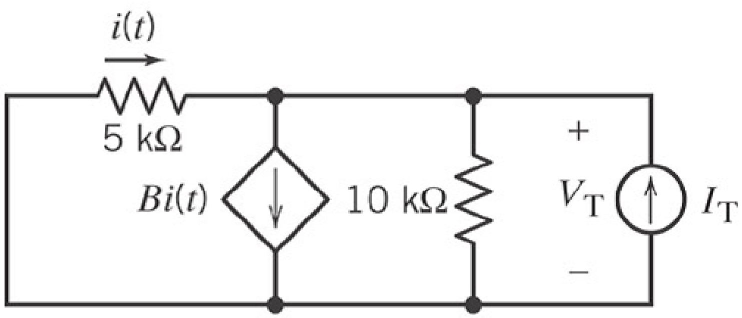
$$i=\frac{V_T}{5\times10^3}$$

3. Applying KCL to the top node of the dependent source, we get

$$-i+Bi+\frac{V_T}{10\times10^3}-I_T=0$$

4. Combining these equations, we get

$$\left(\frac{1-B}{5\times10^3}+\frac{1}{10\times10^3}\right)V_T-I_T=0$$



(b)

# Solution

5. The Thevenin resistance is given by

$$R_t = \frac{V_T}{I_T} = -\frac{10 \times 10^3}{2B - 3}$$

   - B<3/2 is required to ensure that $R_t$ is positive and the circuit is stable.

6. To obtain a time constant of +20ms requires

$$R_t = \frac{\tau}{C} = \frac{20 \times 10^{-3}}{2 \times 10^{-6}} = 10 \times 10^3 = 10\ [k\Omega]$$

7. which in turn requires

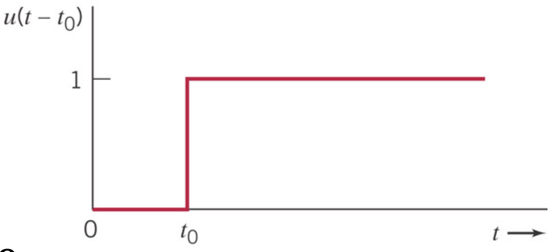
$$10 \times 10^3 = -\frac{10 \times 10^3}{2B - 3}$$

   - Therefore B=1. This suggests that we can fix the unstable circuit by decreasing the gain of the dependent source from 2A/A to 1 A/A

# The Unit Step Source

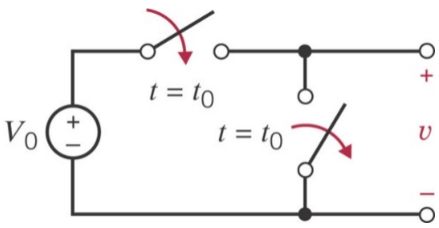
- The Unit step forcing function as a function of time that is zero for $t<t_0$, and unity for $t>t_0$.

$$u(t-t_0)=\begin{cases}0 & t<t_0\\ 1 & t>t_0\end{cases}$$

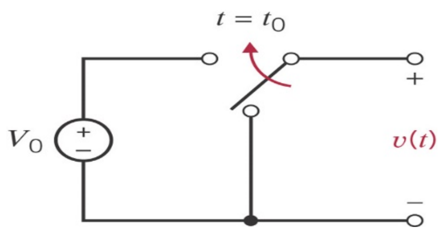



- Application of a constant-voltage source at $t=t_0$ using two switches both acting at $t=t_0$.

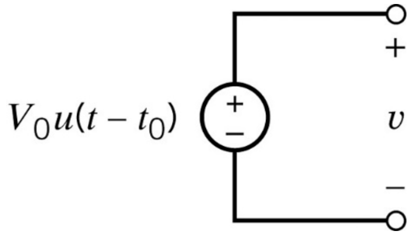
$$v(t)=V_0u(t-t_0)$$




- Single-switch equivalent circuit for the step voltage source




- Symbol for the step voltage source

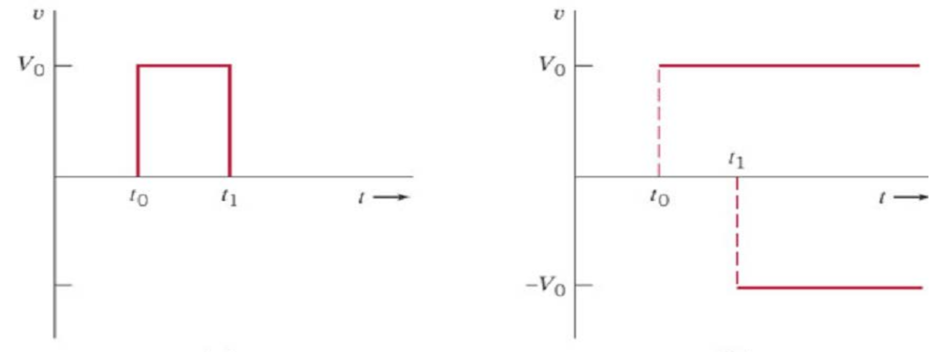


# The Unit Step Source

- A pulse signal has a constant nonzero value for a time duration of $\Delta t = t_1 - t_0$

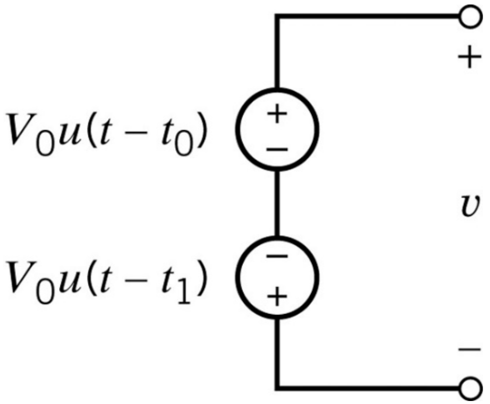
  - *Pulse source*

$$v(t) = V_0 u(t - t_0) - V_0 u(t - t_1)$$
$$= \begin{cases} 0 & t < t_0 \\ V_0 & t_0 < t < t_1 \\ 0 & t_1 < t \end{cases}$$



(a) (b)

  - Two-step voltage sources



# The Unit Step Source

- Let us consider the application of a pulse to an RL circuit as shown in Figure 8.6-7. Here we let $t_0$=0. The pulse is applied to the RL circuit when i(0)=0.

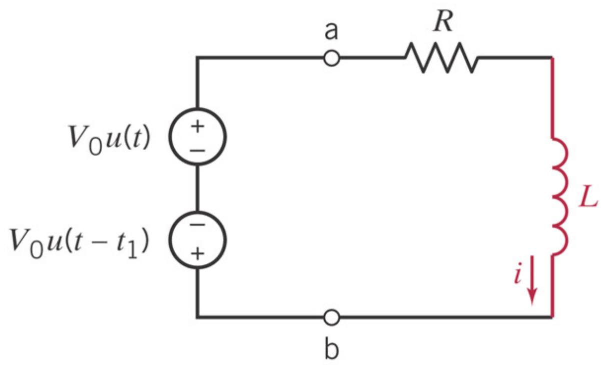



FIGURE 8.6−7

- Since the circuit is linear, we may use the principle of superposition, so that

$$i = i_1 + i_2$$

where $i_1$ is the response to $V_0$u(t) and $i_2$ is the response to $V_0$u(t-$t_1$)

- The response of an RL circuit to a constant forcing function applied at t=$t_n$ is

$$i = \frac{V_0}{R}(1 - e^{-(t-t_n)/\tau}) \quad \text{when } t > t_n$$

where τ=L/R.

# The Unit Step Source

- The two solutions to the two-step sources are

$$i_1 = \frac{V_0}{R}(1 - e^{-t/\tau}) \quad when \;\; t \geq 0$$

$$i_2 = \frac{-V_0}{R}(1 - e^{-(t-t_1)/\tau}) \quad when \;\; t > t_1$$

- Adding the responses, we have

$$i = \begin{cases} \dfrac{V_0}{R}(1 - e^{-t/\tau}) & 0 < t \leq t_1 \\ \dfrac{-V_0}{R} e^{-t/\tau}(e^{t/\tau} - 1) & t > t_1 \end{cases}$$

- The response at t=t1 is

$$i(t_1) = \frac{V_0}{R}(1 - e^{-t_1/\tau})$$

- If $t_1$ is greater than τ, the response will approach $V_0$/R before starting its decline, as shown in Figure 8.6-8. The response at t=2$t_1$ is

$$i(2t_1) = \frac{V_0}{R} e^{-2(t_1/\tau)}(e^{-t_1/\tau} - 1) = \frac{V_0}{R}(e^{-t_1/\tau} - e^{2(-t_1/\tau)})$$

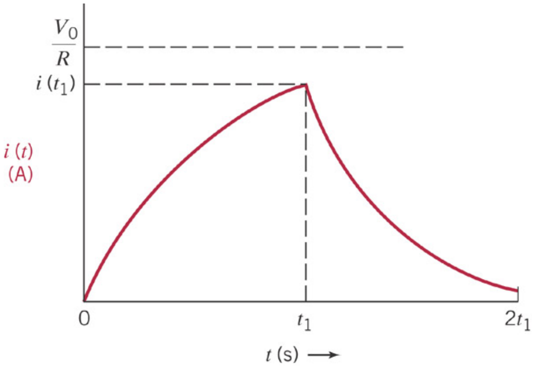



FIGURE 8.6−8

# Example 8.6-1 First-Order Circuit

- Figure 8.6-9 shows a first-order circuit. The input to the circuit is the voltage of the voltage source, $v_s(t)$. The output is the current of the inductor, $i_0(t)$. Determine the output of this circuit when the input is $v_s(t)=4-8u(t)$ [V].

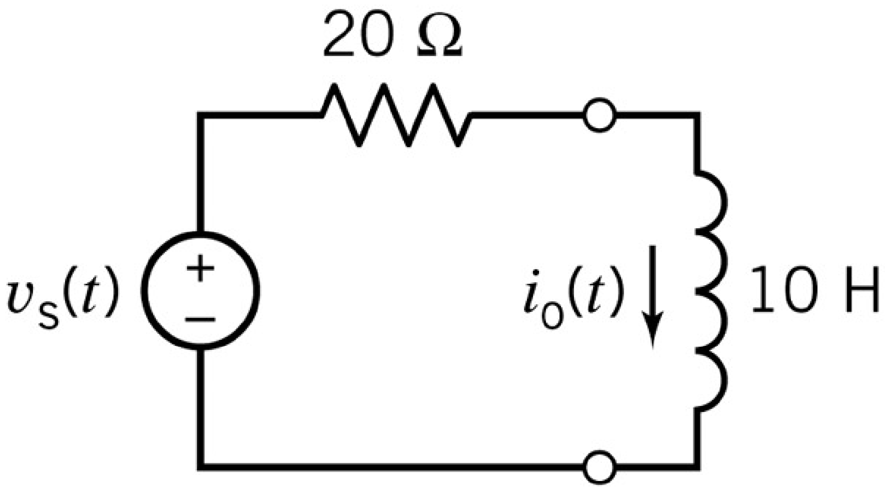



FIGURE 8.6−8

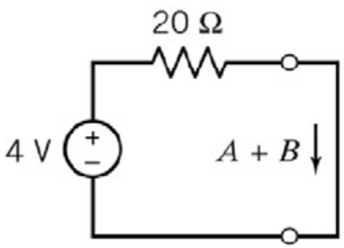
# Solution

1. The response of the first-order circuit will be

$$i_o(t) = A + Be^{-at} \quad for \quad t > 0$$

2. Circuits used to calculate the steady-state reponse

(a)before t=0

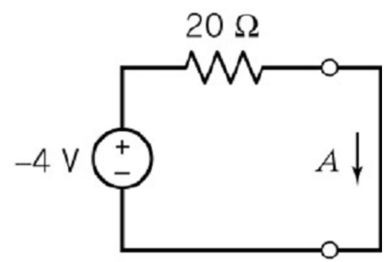



(a)

$$i_0(0) = A + Be^{-a(0)} = A + B$$

$$A + B = 0.2A$$

(b) after t=0




(b)

$$i_0(\infty) = A + Be^{-a(\infty)} = A$$

$$A = -0.2\text{A}$$

$$\rightarrow \quad B = 0.4\text{A}$$

3. The value of the constant a is determined from the time constant τ.

$$\frac{1}{a} = \tau = \frac{L}{R_t}$$

# Solution

4. Figure 8.6-11 shows the circuit used to calculate $R_t$.

$$R_t = 20\Omega$$

Therefore,

$$a = \frac{20}{10} = 2\frac{1}{s}$$

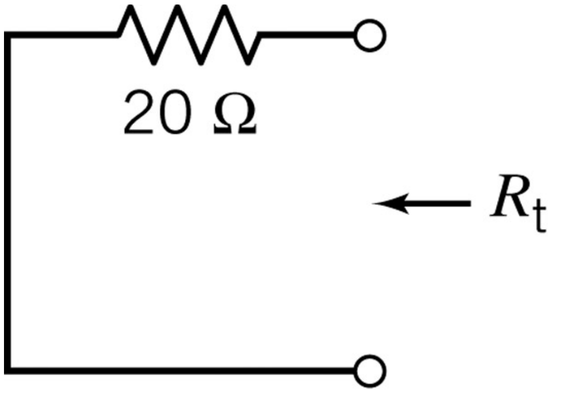



FIGURE 8.6-11

5. Substituting the values of A, B and a gives

$$i_0(t) = \begin{cases} 0.2[A] & for \quad t \le 0 \\ -0.2 + 0.4e^{-2}[A] & for \quad t \ge 0 \end{cases}$$

# Example 8.6-2 First-Order Circuit

- Figure 8.6-12 shows a first-order circuit. The input to the circuit is the voltage of the voltage source, $v_s(t)$. The output is the voltage across the capacitor, $v_o(t)$. Determine the output of this circuit when the input is $v_s(t)=7-14u(t)$V.

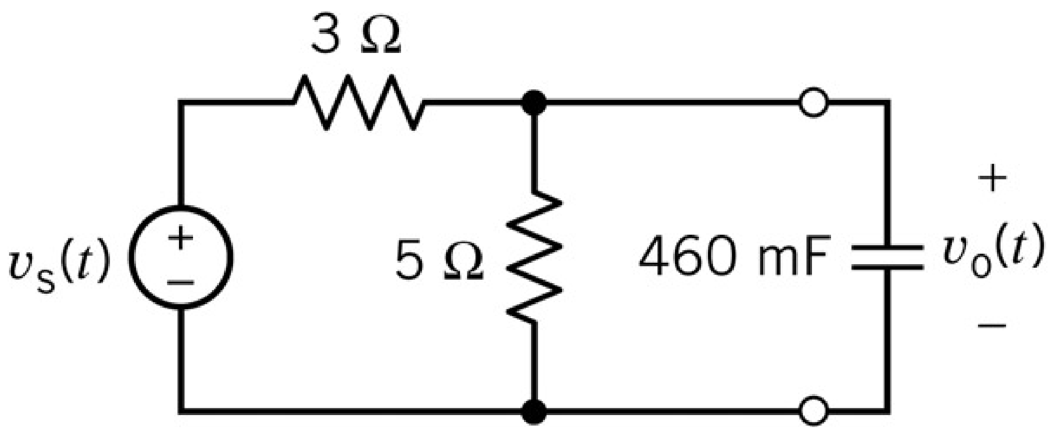



FIGURE 8.6−12

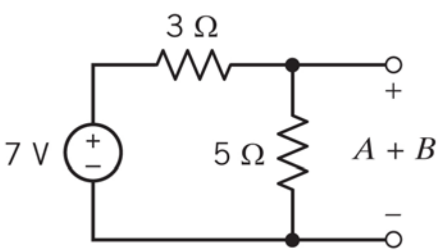
# Solution

1. The response of the first-order circuit will be

$$v_o(t) = A + Be^{-at} \qquad for \quad t > 0$$

2. Circuits used to calculate the steady-state reponse

(a)before t=0

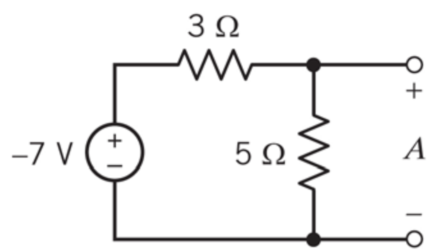



(a)

$$v_0(0) = A + Be^{-a(0)} = A + B$$

$$A + B = \frac{5}{3+5} \times 7 = 4.38\text{V}$$

(b) after t=0




(b)

$$v_0(\infty) = A + Be^{-a(\infty)} = A$$

$$A = \frac{5}{3+5} \times (-7) = -4.38\text{V}$$

$$\rightarrow \quad B = 8.76\text{V}$$

3. The value of the constant a is determined from the time constant τ.

$$\frac{1}{a} = \tau = R_t C$$

# Solution

4. Figure 8.6-14 shows the circuit used to calculate $R_t$.

$$R_t = \frac{(5)(3)}{5+3} = 1.875\Omega$$

Therefore,

$$a = \frac{1}{(1.875)(460\times10^{-3})} = 1.16\frac{1}{s}$$

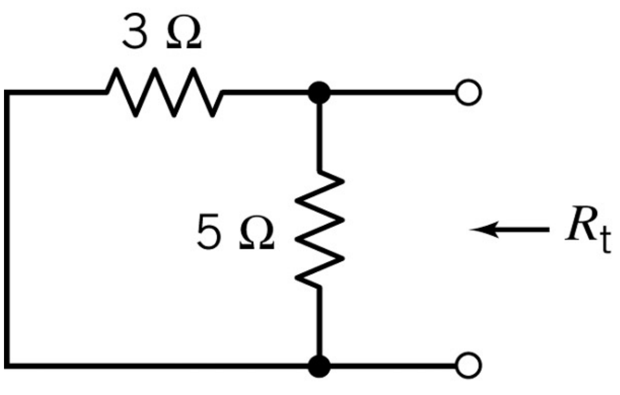



FIGURE 8.6−14

5. Substituting the values of A, B and a gives

$$v_0(t) = \begin{cases} -4.38\text{V} & for \quad t \le 0 \\ -4.38 + 8.76e^{-1.16t}\text{V} & for \quad t \ge 0 \end{cases}$$

# The Response of a First-Order Circuit to a Nonconstant Source

- The differential equation an RL or RC circuit is represented by the general form

$$\frac{dx(t)}{dt} + ax(t) = y(t) \qquad (8.7\text{-}1)$$

- Consider the derivative of a product of two terms such that

$$\frac{d}{dt}(xe^{at}) = \frac{dx}{dt}e^{at} + axe^{at} = (\frac{dx}{dt} + ax)e^{at} \qquad (8.7\text{-}2)$$

- The term within the parentheses on the right-hand side of Eq.8.7-2 is exactly the form on the left-hand side of Eq.8.7-1.
  Therefore,

$$(\frac{dx}{dt} + ax)e^{at} = ye^{at} \quad or \quad \frac{d}{dt}(xe^{at}) = ye^{at}$$

- Integrating both sides of the second equation, we have

$$xe^{at} = \int ye^{at}dt + K$$

# The Response of a First-Order Circuit to a Nonconstant Source

- Therefore,

$$x = e^{-at}\int ye^{at}dt + Ke^{-at} \qquad (8.7\text{–}1)$$

- For the case where the source is a constant so that y(t)=M, we have

$$x = e^{-at}M\int e^{at}dt + Ke^{-at} = \frac{M}{a} + Ke^{-at} = x_f + x_n \qquad (8.7\text{–}2)$$

$$\text{natural response}: \; x_n = ke^{-at}$$

$$\text{forced response}: \; x_f = M/a$$

- Consider the case where y(t), the forcing function, is not a constant.

$$\text{natural response}: \; x_n = ke^{-at}$$

$$\text{forced response}: \; x_f = e^{-at}\int e^{bt}e^{at}dt = e^{-at}\int e^{(a+b)t}dt = \frac{1}{a+b}e^{-at}e^{(a+b)} = \frac{e^{bt}}{a+b}$$

# Example 8.7-1 First-Order Circuit with Nonconstant Source

- Find the current i for the circuit of Figure 8.7-1a for t>0 when

$$v_s = 10e^{-2t}u(t)\text{V}$$

Assume the circuit is in steady state at $t=0^-$

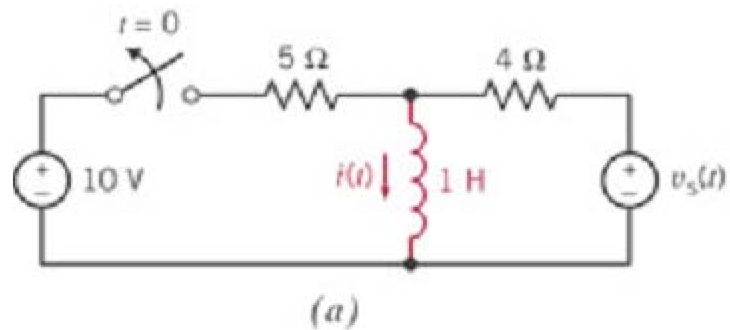



FIGURE 8.7-1

# Solution

1. We expect $i_f$ to be

$$i_f = Be^{-2t}$$

2. Writing KVL arround the right-hand mesh, we have

$$L\frac{di}{dt} + Ri = v_s \qquad or \qquad \frac{di}{dt} + 4i = 10e^{-2t}$$

3. Substituting $i_f = Be^{-2t}$ , we have

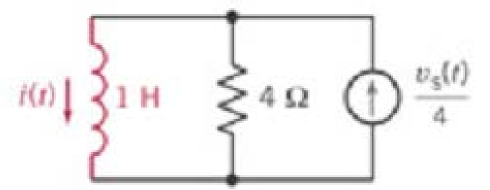
$$-2Be^{-2t} + 4Be^{-2t} = 10e^{-2t} \qquad or \qquad (-2B + 4B)e^{-2t} = 10e^{-2t}$$

Hence, B=5 and

$$i_f = 5e^{-2t}$$

4. The natural response can be obtained by considering the circuit shown in Figure 8.7-1b. This is the equivalent circuit after the switch opens. The natural response is

$$i_n = Ae^{-(R_t/L)t} = Ae^{-4t}$$



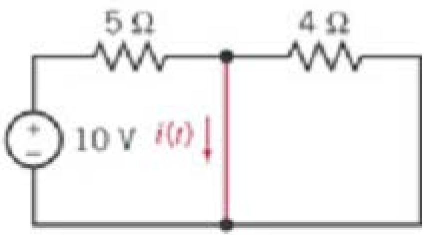
(b)

# Solution

5. The complete response is

$$i = i_n + i_f = Ae^{-4t} + 5e^{-2t}$$

6. The constant A can be determined from the value of the inductor current at time t=0. The initial inductor current, i(0), can be obtained by considering the circuit shown in Figure 8.7-1c. This is the equivalent circuit that is appropriate before the switch opens.



(c)

7. From Figure 8.7-1c

$$i(0) = \frac{10}{5} = 2A$$

8. Therefore, at t=0

$$i(0) = Ae^{-4\times 0} + 5e^{-2\times 0} | = A + 5$$

$$2 = A + 5$$

$$A = -3$$

9. Therefore,

$$i = (-3e^{-4t} + 5e^{-2t})\text{A} \quad t>0$$

# Differential Operators

- We can define a *differential operators* such tat

$$sx = \frac{dx}{dt} \quad and \quad s^2 x = \frac{d^2 x}{dt^2}$$

- Use of the *s* operator is particularly attractive when higher-order differential equations are involved. Then we use the *s* operator, so that

$$s^n x = \frac{d^n x}{dt^n} \qquad for\ \ n \geq 0$$

- We assume that $n$=0 represents no differentiation, so that

$$s^0 = 1$$

which implies $s^0 x = x$.

- Because integration is the inverse of differentiation, we define

$$\frac{1}{s} x = \int_{-\infty}^{t} x d\tau$$

- The operator *1/s* must be shown to satisfy the usual rules of algebraic manipulations. Of these rules, the commutative multiplication property presents the only difficulty. Thus, we require

$$s \cdot \frac{1}{s} = \frac{1}{s} \cdot s = 1$$

# Differential Operators

- First, we examine Eq. 8.8-1. Multiplying Eq. 8.8-1 by *s* yields

$$s \cdot \frac{1}{s} x = \frac{d}{dt} \int_{-\infty}^{t} x \, d\tau \qquad or \qquad x = x$$

- We try the reverse order by multiplying *sx* by the integration operator to obtain

$$\frac{1}{s} sx = \int_{-\infty}^{t} \frac{dx}{dt} d\tau = x(t) - x(-\infty)$$

Therefore

$$\frac{1}{s} sx = x \quad only \ when \ x(-\infty) = 0$$

- From a physical point of view, we require that all capacitor voltages and inductor currents be zero at $t = -\infty$. Then the operator *1/s* can be said to satisfy Eq. 8.8-2 and can be manipulated as an ordinary algebraic quantity.

- Differential operators can be used to find the natural solution of a differential equation.
- For example, consider the first-order differential equation

$$\frac{d}{dt} x(t) + ax(t) = by(t)$$

# Differential Operators

- The natural solution of this differential equation is

$$x_n(t) = Ke^{st}$$

- The homogeneous form of this equation is

$$\frac{d}{dt}x(t) + ax(t) = 0$$

- To see that $x_n$(t) is a solution of the homogeneous form of the differential equation,

$$\frac{d}{dt}(Ke^{st}) + a(Ke^{st}) = sKe^{st} + aKe^{st} = 0$$

- To obtain the parameters *s*,

$$sx + ax = (s+a)x = 0$$

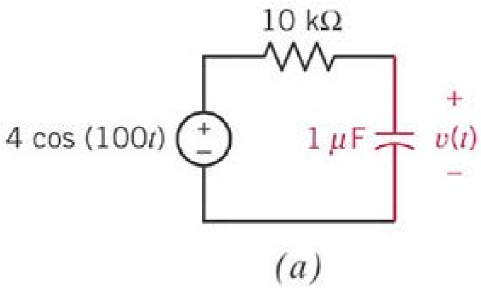
  we use the solution $s$=-$a$.

- Consequently,

$$x_n(t) = Ke^{-at}$$

# Differential Operators

- As a second application of differential operators, consider using the computer program MATLAB to find the complete response of a first-order circuit.
- Differential operators are used to describe differential equations to MATLAB. The natural solution of this differential equation is
- To represent this circuit by a differential equation, apply KVL to get




$$10\times 10^{3}\left(1\times 10^{-6}\frac{d}{dt}v(t)\right)+v(t)-4\cos(100t)=0$$

$$or \qquad 0.01\frac{d}{dt}v(t)+v(t)=4\cos(100t)$$

- In the syntax used by MATLAB, the differential operator is represented by *D* instead of *s*. Replace by the differential operator *D* to get

$$0.01Dv+v=4\cos(100t)$$

- Entering the MATLAB commands

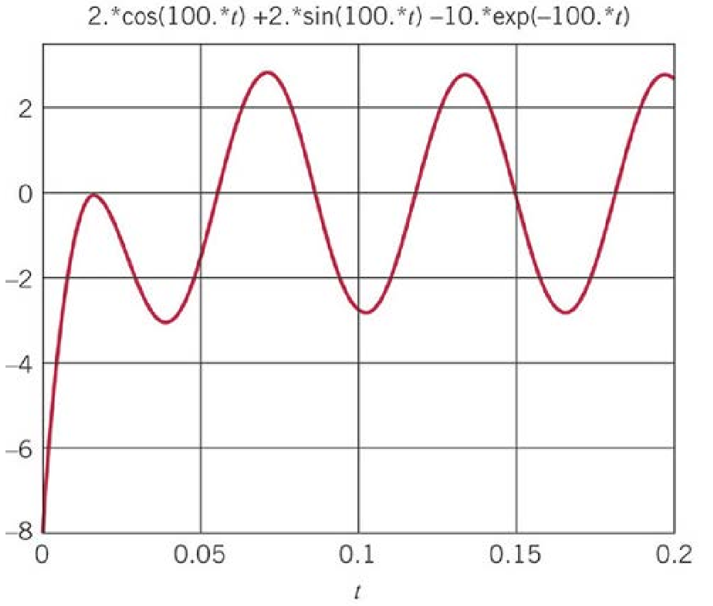
```
v = dsolve('0.01*Dv + v = 4*cos(100*t)', 'v(0)=-8')
ezplot(v, [0, 2])
```

# Differential Operators

- MATLAB responds by providing the complete solution of the differential equation

  v = 2.*cos(100*t)+2.*sin(100*t)-10.*exp(-100.*t)

- The plot of v(t) versus t shown in Figure 8.8-1b.




(b)