# Chapter 11
## AC Steady-State Power

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# AC Steady-State Power

- Goal
  - Power in AC circuit
    - Represent the circuit in the frequency domain.
    - Average power, real and reactive power, complex power, power factor, rms values
    - Maximum power transfer using matching network.
  - Coupled inductor and/or ideal transformer
    - Represent the magnetically coupled coils in the frequency domain.

# Electric Power

- Controlling and distributing the energy is important.
- Why AC in transmission line?
  - Easy to convert magnitude of voltage by coupled inductor (e.g. transformer)
- Why high voltage AC in transmission line?
  - Reducing loss in the transmission line.

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# Instantaneous Power and Average Power

- **Instantaneous power**

$$p(t) = v(t)\, i(t)$$

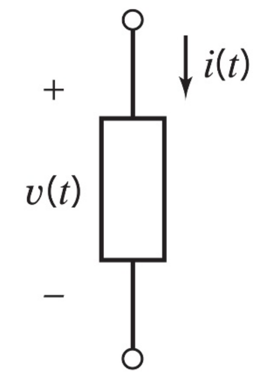


**Figure 11.3-1**

- Average power
  - Average power delivered to circuit.

- Instantaneous and average power delivered to the circuit 11.3-1 at specific time t

$$p(t) = v(t)\, i(t) \quad \Rightarrow \quad P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt$$

Integral of the time function over a complete period, divided by the period.

# Example 11.3-1 *Average Power*

- Find the average power delivered to a resistor R when the current through the resistor is i(t), as shown in Figure 11.3-2.

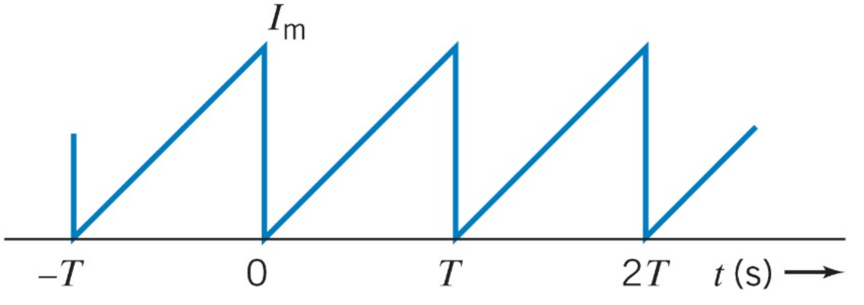



**Figure 11.3-2**

# Solution

- The current can be describe as

$$i = \frac{I_M}{T} t \quad 0 \leq t < T$$

- Instantaneous power is

$$p = i^2 R = \frac{{I_M}^2 t^2}{T^2} R \quad 0 \leq t < T$$

- Average power is

$$P = \frac{1}{T} \int_0^T p \, dt = \frac{{I_M}^2 R}{3} \; [W]$$

# Instantaneous Power and Average Power

Suppose $v(t) = V_m \cos(\omega t + \theta_V)$,

Then, $\mathrm{i}(t) = I_m \cos(\omega t + \theta_I)$ ($\because$ linear and steady state)

$$p(t) = v(t)\, i(t)$$

$$= V_m I_m \cos(\omega t + \theta_V) \cos(\omega t + \theta_I) = \frac{V_m I_m}{2}[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_I + \theta_V)]$$

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt$$

Then, average power will be,

$$P = \frac{V_\mathrm{m} I_\mathrm{m}}{2} \cos\left(\theta_\mathrm{V} - \theta_\mathrm{I}\right)$$

# Example 11.3-2 *Average Power*

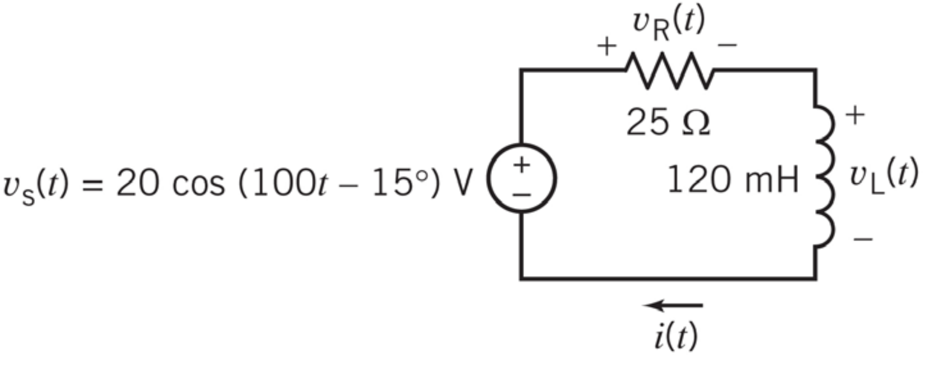
- The circuit shown in figure below is at steady state. The mesh current is

$$i(t) = 721\cos(100t - 41º)\ [\text{mA}]$$

The element voltages are

$$v_s(t) = 20\cos(100t - 15º)\ [\text{V}]$$
$$v_R(t) = 18\cos(100t - 41º)\ [\text{V}]$$
$$v_L(t) = 8.66\cos(100t + 49º)\ [\text{V}]$$

Find the average power delivered to each device in this circuit



# Solution

- Average power delivered is $$P = \frac{V_m I_m}{2} \cos(\theta_V - \theta_I)$$
- Average power delivered by the voltage source is

$$P_s = \frac{(20)(0.721)}{2} \cos\left(-15^\circ - \left(-41^\circ\right)\right) = 6.5\ W$$

- Similarly, Average power delivered to the resistor is 6.5W.

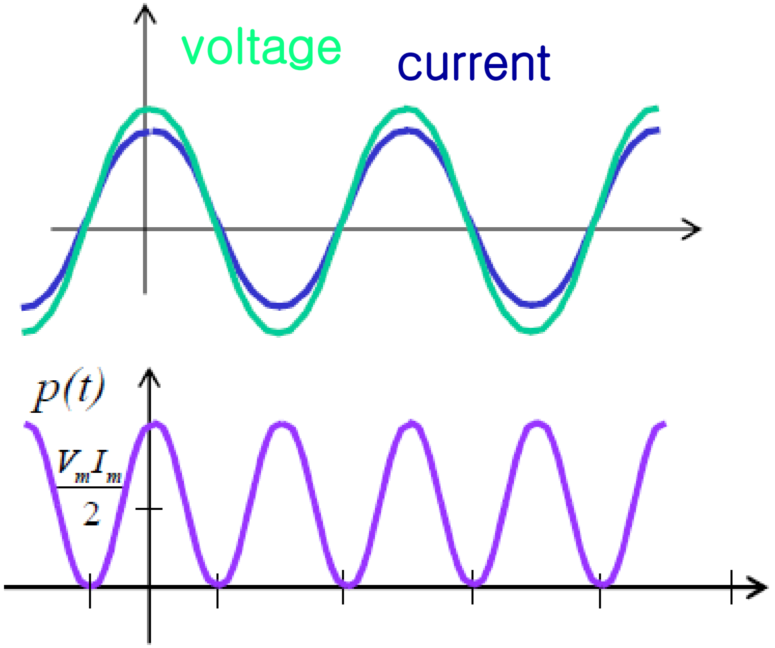
$$P_R = \frac{(18)(0.721)}{2} \cos\left(-41^\circ - \left(-41^\circ\right)\right) = 6.5\ W$$

  - The average power delivered to any inductor is **zero.**

$$P_L = \frac{(8.66)(0.721)}{2} \cos\left(49^\circ - \left(-41^\circ\right)\right) = 0\ W$$

# Power for purely resistive circuits

- Phase of voltage and current of the resistor is same.




- Instantaneous real power p(t)

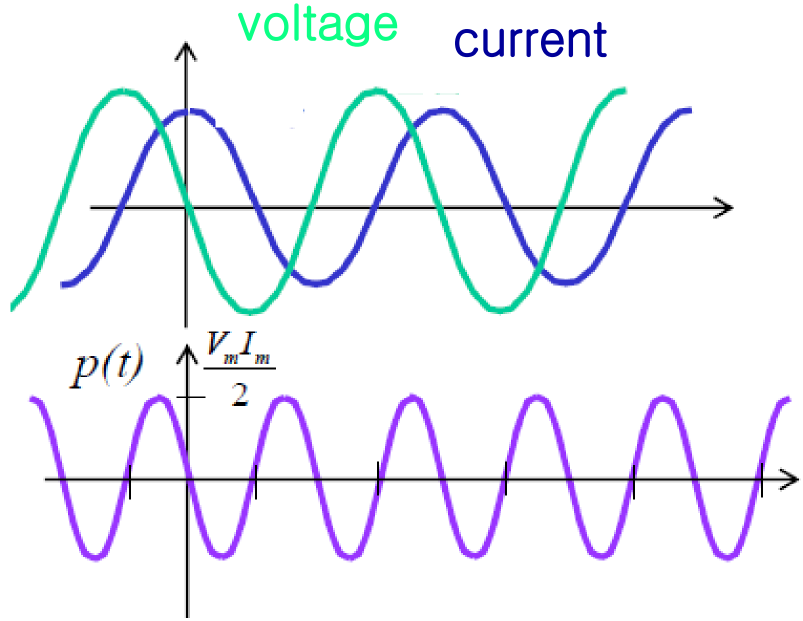
$$p(t) = \frac{V_m I_m}{2}[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_I + \theta_V)]$$

$$= \frac{V_m I_m}{2}[1 + \cos(2\omega t + 2\theta_I)]$$

- Average power

$$P = \frac{V_m I_m}{2}$$

# Power for purely inductive circuits




- Phase of current lags by 90º compared to phase of voltage.

- Instantaneous real power p(t)

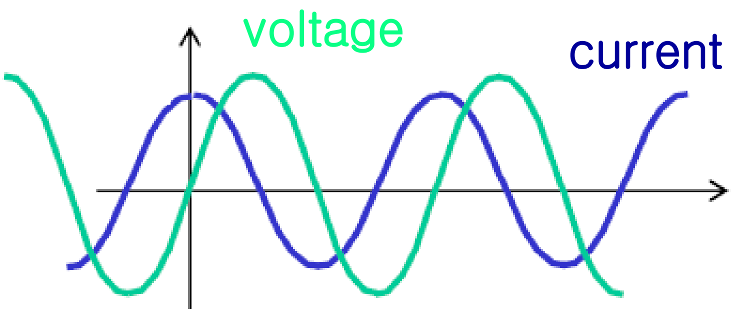
$$p(t) = \frac{V_m I_m}{2}[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_I + \theta_V)]$$

$$= \frac{V_m I_m}{2}[\cos(2\omega t + 2\theta_I + 90º)] = -\frac{V_m I_m}{2}[sin(2\omega t + 2\theta_I)]$$

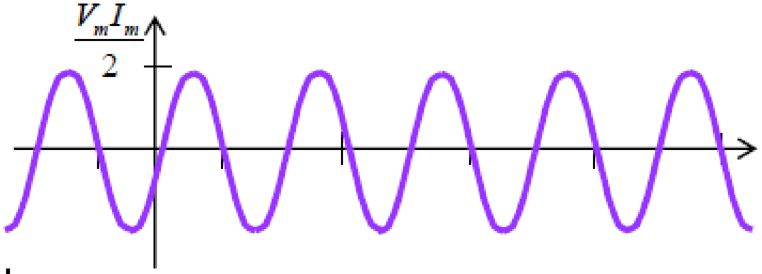
- Average power

$$P = 0$$

# Power for purely capacitive circuits




- Phase of current leads by 90º compared to phase of voltage.




- Instantaneous real power p(t)

$$p(t) = \frac{V_m I_m}{2}[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_I + \theta_V)]$$

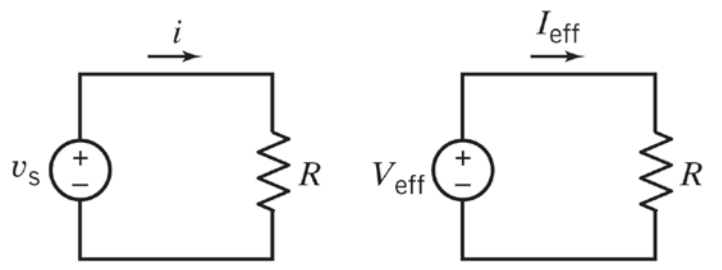
$$= \frac{V_m I_m}{2}[\cos(2\omega t + 2\theta_I - 90º)] = \frac{V_m I_m}{2}[sin(2\omega t + 2\theta_I)]$$

- Average power

$$P = 0$$

# Effective Value of Periodic Waveform

- We want to change AC voltage (or current) to **effective** DC voltage (or current) while average power remains still.




$$P = \frac{1}{T}\int_0^T i^2 R\, dt \quad = \quad P = I_{\text{eff}}^2 R$$

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\, dt}$$

- The effective value is commonly called as root-mean-square (rms) value.
- This is equivalent to an effective DC value in terms of power computation.

# Effective Value of Periodic Waveform

- The effective value of voltage in circuit is

$$V_{eff}{}^2 = Vrms^2 = \frac{1}{T}\int_0^T v^2\,dt$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2\,dt}$$

- $\mathbf{I_{rms}}$ of sinusoidally varying current $\mathrm{i}(t) = I_m cos\omega t,$

$$I_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T I_{\mathrm{m}}^2 \cos^2\omega t\,dt} = \sqrt{\frac{I_{\mathrm{m}}^2}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)\,dt} = \frac{I_{\mathrm{m}}}{\sqrt{2}}$$

# Example 11.3-2 *Effective Value*

- Find the effective value of current for the sawtooth waveform shown in below figure.

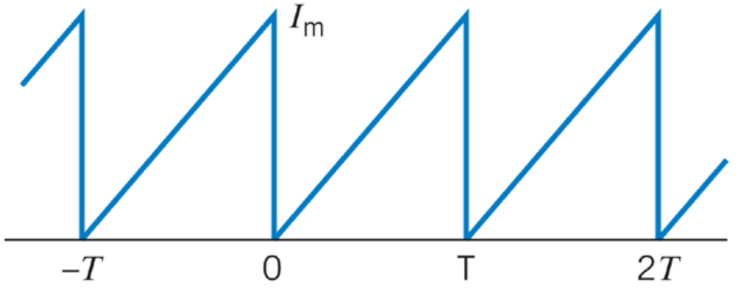



Fig_11-4-2

- The current can be describe as $i = \frac{I_m}{T} t \quad 0 \le t < T$

- The effective value is $I_{eff}{}^2 = \frac{1}{T}\int_0^T i^2\,dt = \frac{1}{T}\int_0^T (\frac{I_M}{T} t)2\,dt = \frac{I_M{}^2}{3}, \quad I_{eff} = \frac{I_M}{\sqrt{3}}$

# Instantaneous Power, Average Power, and Complex Power

Suppose $v(t) = V_m \cos(\omega t + \theta_V - \theta_I)$, $i(t) = I_m \cos(\omega t)$

- Instant power

$$p(t) = v(t)\, i(t)$$
$$= V_m I_m \cos(\omega t + \theta_V - \theta_I)\cos(\omega t) = \frac{V_m I_m}{2}[\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V - \theta_I)]$$
$$= P + P\cos(2\omega t) - Q\sin(2\omega t)$$

- Average powers

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$
$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

Real (Average) power

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

Watt

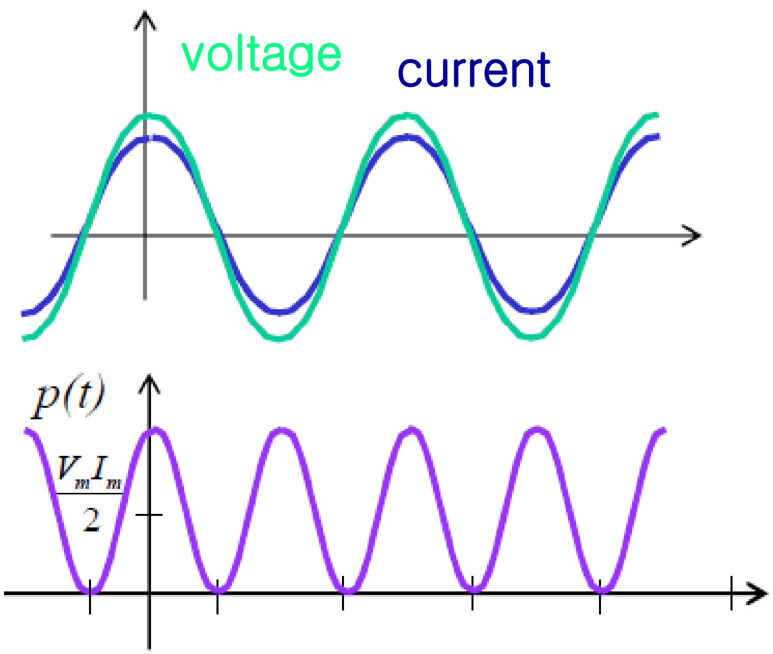
Reactive power

$$Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

VAR (Volt-Amp Reactive)

# Instantaneous Power, Average Power, and Complex Power for Circuit Elements

- Resistive circuits




***Current is in the same phase of voltage***

$$p(t) = P + P\cos 2\omega t - Q\sin 2\omega t$$

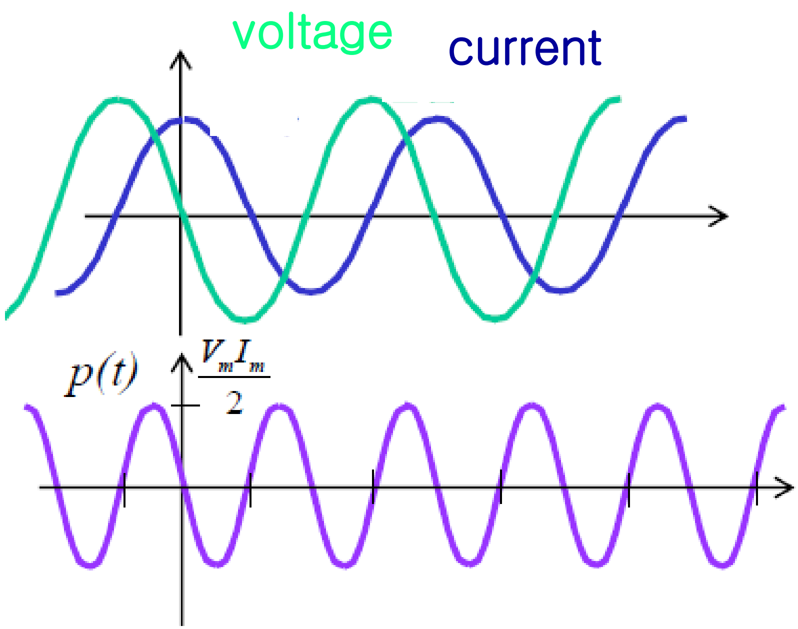
$$\theta_i = \theta_v$$

$$\cos(\theta_v - \theta_i) = 1, \ \sin(\theta_v - \theta_i) = 0$$

$$p(t) = P + P\cos 2\omega t$$

# Instantaneous Power, Average Power, and Complex power for Circuit Elements

- Inductive circuits




$$p(t) = P + P\cos 2\omega t - Q\sin 2\omega t$$

***Current lags voltage by 90°***

$$\theta_v = \theta_i + 90°$$

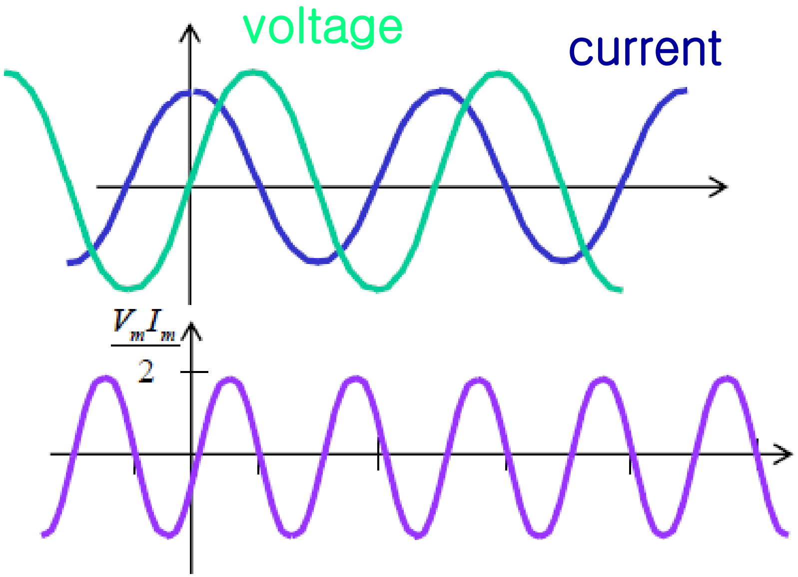
$$\cos(\theta_v - \theta_i) = 0,\ \sin(\theta_v - \theta_i) = 1$$

$$p(t) = -Q\sin 2\omega t$$

→ Average power is zero. Therefore, there is no energy conversion.

# Instantaneous Power, Average Power, and Complex Power for Circuit Elements

- Capacitive circuits




$$p(t) = P + P\cos 2\omega t - Q\sin 2\omega t$$

**Current leads voltage by 90°**

$$\theta_i = \theta_v + 90^\circ$$

$$\cos(\theta_v - \theta_i) = 0, \ \sin(\theta_v - \theta_i) = -1$$

$$p(t) = Q\sin 2\omega t$$

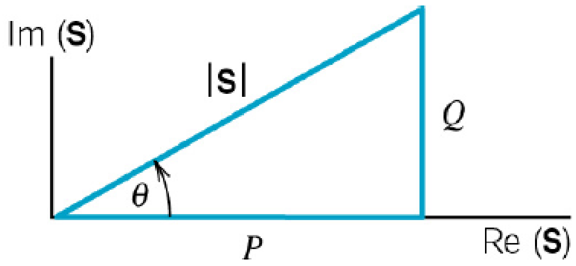
→ Average power is zero. Therefore, there is no energy conversion.

# Complex Power Calculation

- Complex power in terms of average power

$$\mathbf{S} = P + jQ \qquad P[\mathrm{W}], Q[\mathrm{VAR}], \mathbf{S}[\mathrm{VA}]$$

$$|\mathbf{S}| : apparent\ \ power, \quad \mathrm{VA}(volt - amps)$$




- Solving S (complex power),

$$\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)}$$

$$= V_{eff} I_{eff} e^{j(\theta_v - \theta_i)} = V_{eff} e^{j\theta_v} I_{eff} e^{-j\theta_i} = \mathbf{V}_{eff} \mathbf{I}_{eff}{}^{*}$$

$$\mathbf{S} = \mathbf{V}_{eff} \mathbf{I}_{eff}{}^{*} = \frac{1}{2}\mathbf{V}\mathbf{I}^{*}$$

# Complex Power

- Power calculated in the frequency domain.
  - Assume linear circuit with sinusoidal input is at steady state, all the element voltages and currents will be sinusoidal with same frequency as the input.

$$\mathbf{I}(\omega) = I_{\mathrm{m}} \angle \theta_{\mathrm{I}} \quad \text{and} \quad \mathbf{V}(\omega) = V_{\mathrm{m}} \angle \theta_{\mathrm{V}}$$

- **Complex power delivered to the element is defined to be**

$$\mathbf{S} = \frac{\mathbf{V}\mathbf{I}^*}{2} = \frac{(V_{\mathrm{m}} \angle \theta_{\mathrm{V}})\left(I_{\mathrm{m}} \angle -\theta_{\mathrm{I}}\right)}{2} = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} \angle \theta_{\mathrm{V}} - \theta_{\mathrm{I}}$$

$$|\mathbf{S}| = \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2}$$

- The magnitude of **S** is called apparent-power.

# Complex Power

- Complex power **S** can be represented as,

$$\mathbf{S} = \frac{V_m I_m}{2} \cos(\theta_V - \theta_I) + j \frac{V_m I_m}{2} \sin(\theta_V - \theta_I)$$

$$\mathbf{S} = P + jQ$$

- **P (real part of S): average power**

$$P = \frac{V_m I_m}{2} \cos(\theta_V - \theta_I) \quad = \boldsymbol{V_{rms} I_{rms} \cos(\theta_V - \theta_I)}$$

- **Q (imaginary part of S): reactive power**

$$Q = \frac{V_m I_m}{2} \sin(\theta_V - \theta_I) \quad = \boldsymbol{V_{rms} I_{rms} \sin(\theta_V - \theta_I)}$$

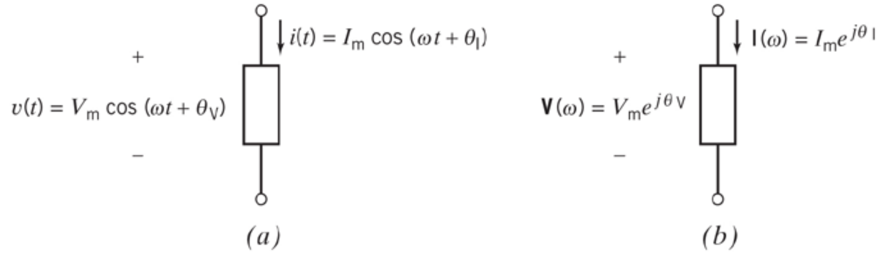
# Complex Power

- Units of the average power are watts, while units of complex power are volt-amps(VA), and the units of reactive power are volt-amp reactive (VAR)

| QUANTITY | RELATIONSHIP USING PEAK VALUES | RELATIONSHIP USING rms VALUES | UNITS |
|---|---|---|---|
| Element voltage, $v(t)$ | $v(t) = V_m \cos(\omega t + \theta_v)$ | $v(t) = V_{rms}\sqrt{2}\cos(\omega t + \theta_V)$ | V |
| Element current, $i(t)$ | $i(t) = I_m \cos(\omega t + \theta_I)$ | $i(t) = I_{rms}\sqrt{2}\cos(\omega t + \theta_I)$ | A |
| Complex power, $\mathbf{S}$ | $\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + j\frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$ | $\mathbf{S} = V_{rms} I_{rms}\cos(\theta_V - \theta_I) + jV_{rms} I_{rms}\sin(\theta_V - \theta_I)$ | VA |
| Apparent power, $\lvert\mathbf{S}\rvert$ | $\lvert\mathbf{S}\rvert = \frac{V_m I_m}{2}$ | $\lvert\mathbf{S}\rvert = V_{rms} I_{rms}$ | VA |
| Average power, $P$ | $P = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I)$ | $P = V_{rms} I_{rms}\cos(\theta_V - \theta_I)$ | W |
| Reactive power, $Q$ | $Q = \frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$ | $Q = V_{rms} I_{rms}\sin(\theta_V - \theta_I)$ | VAR |

# Complex Power in terms of impedance (alternate forms)

- Circuit in time and frequency domain.




$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m \angle \theta_V}{I_m \angle \theta_I} = \frac{V_m}{I_m} \angle \theta_V - \theta_I$$

$$\mathbf{Z}(\omega) = \frac{V_m}{I_m} \cos(\theta_V - \theta_I) + j\frac{V_m}{I_m} \sin(\theta_V - \theta_I)$$

- Think about the power for each electric components (resistor, capacitor, inductor) by their impedance.

# Complex Power in terms of impedance (alternate forms)

- Complex power can be expressed in terms of impedance

$$\mathbf{S} = \frac{V_m I_m}{2}\cos(\theta_V - \theta_I) + j\frac{V_m I_m}{2}\sin(\theta_V - \theta_I)$$

$$= \left(\frac{{I_m}^2}{2}\right)\frac{V_m}{I_m}\cos(\theta_V - \theta_I) + j\left(\frac{{I_m}^2}{2}\right)\frac{V_m}{I_m}\sin(\theta_V - \theta_I)$$

$$= \left(\frac{{I_m}^2}{2}\right)Re(\boldsymbol{Z}) + j\left(\frac{{I_m}^2}{2}\right)Im(\boldsymbol{Z})$$

$$= ({I_{rms}}^2)Re(\boldsymbol{Z}) + j({I_{rms}}^2)Im(\boldsymbol{Z})$$

$$\mathbf{Z}(\omega) = \frac{V_m}{I_m}\cos(\theta_V - \theta_I) + j\frac{V_m}{I_m}\sin(\theta_V - \theta_I)$$

- Average power delivered to the element is

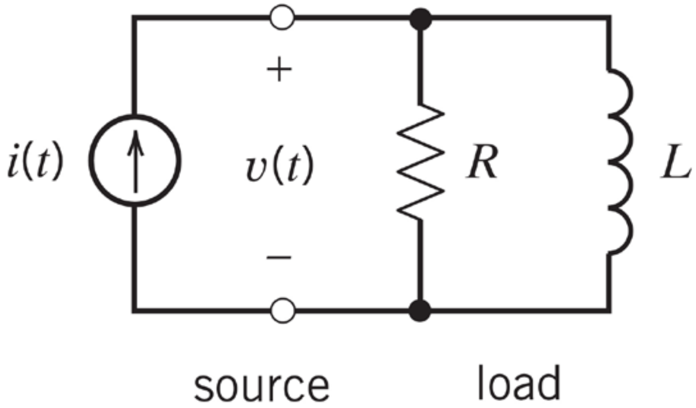
$$P = \left(\frac{I_m^2}{2}\right)\mathrm{Re}(\mathbf{Z})$$

# Example 11.5-1 *Complex Power*

- The circuit shown in below consists of a source driving a load. The current source is i(t) = 1.25cos(5t-15º) [A]

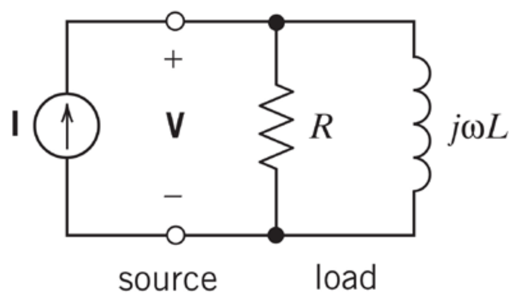
a. What is the value of the complex power delivered by the source to the load when R=20ohm and L = 3H?

b. What are the values of the resistance R and inductance L, when the source delivers 11.72+j11.72VA to the load?



# Solution (1/2)

- Represent the circuit in the frequency domain where $I = 1.25\angle -15º$ [A]




$$Z = \frac{j\omega LR}{R + j\omega R}$$

- (a) Find complex power.

$$Z = \frac{\text{j300}}{\text{20+j15}} = 12\angle 53 \text{ ohm}$$

$$\text{V=IZ=}(1.25\angle -15º)(12 \angle -53º) = 15\angle 38º \text{ [V]}$$

$$S = \frac{\text{VI}*}{2} = \frac{(15\angle 38º)(1.25\angle 15º)^*}{2} = 9.375 \angle 53º\text{[VA]}$$

# Solution (2/2)

- (b) Find R, L at given power.

$$S = \frac{VI^*}{2} \rightarrow V = \frac{2S}{I^*} = \frac{2 * 16.57\angle 45º}{1.25\angle 15º} = 26.52\ \angle 30º\ [V]$$

- Equivalent impedance

$$Z = \frac{V}{I} = \frac{26.52\ \angle 30º}{1.25\ \angle -15º} = 21.21\ \angle 45º\ [\mathrm{ohm}] = \frac{j\omega LR}{R + j\omega R}$$

$$R = 30\ [\mathrm{ohm}], L = 6\ [H]$$

# Complex Power

- Complex power is conserved.

$$\sum_{\substack{all \\ elements}} \frac{\mathbf{V}_k \mathbf{I}_k^*}{2} = 0$$

$$\sum_{all\ elements} Re\left(\frac{\boldsymbol{V_K I_K}^*}{2}\right) = \sum_{all\ elements} P_K = 0$$

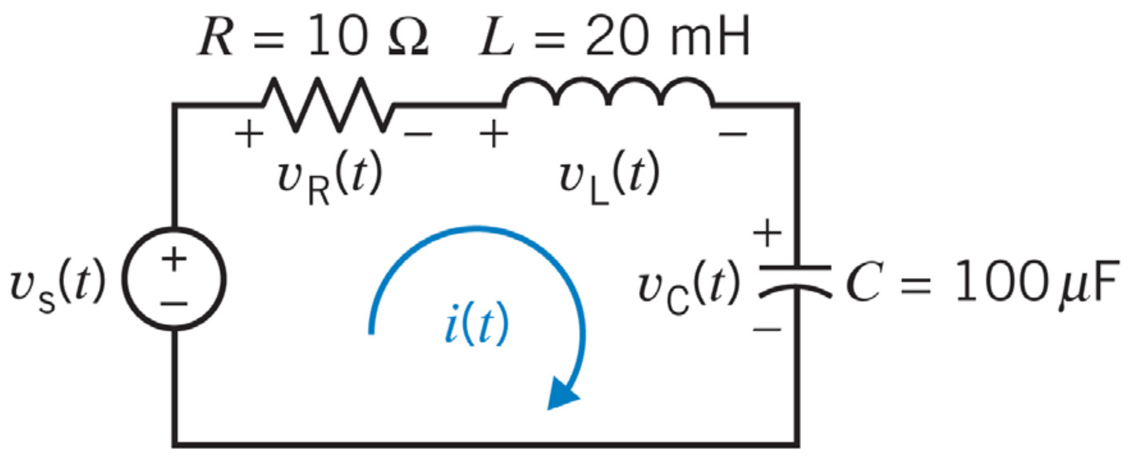
$$\sum_{all\ elements} Im\left(\frac{\boldsymbol{V_K I_K}^*}{2}\right) = \sum_{all\ elements} Q_K = 0$$

  - **The total complex power supplied by the source is equal to the total complex power received by the other elements**

$$\sum_{sources} \frac{\mathbf{V}_k \mathbf{I}_k^*}{2} = \sum_{\substack{other \\ elements}} \frac{\mathbf{V}_k \mathbf{I}_k^*}{2}$$

# Example 11.5-3 ***Complex Power***

- Verify that complex power is conserved in the circuit below when $v_s$=100cos1000t [V].



# Solution (1/2)

- Verify total complex power supplied by the source is equal to the total complex power received by the other elements

$$\sum_{sources} \frac{\mathbf{V}_k \mathbf{I}_k^*}{2} = \sum_{\substack{other \\ elements}} \frac{\mathbf{V}_k \mathbf{I}_k^*}{2}$$

- Find V and I for each element

$$Vs(\omega) = 100\angle 0[V]$$

$$I(\omega) = \frac{Vs(\omega)}{R + j\omega R - j\dfrac{1}{\omega C}} = 7.07\angle -45\ [A]$$

Then,

$$V_R(\omega) = RI(\omega) = 70.7\angle -45[V]$$
$$V_L(\omega) = j\omega LI(\omega) = 141.4\angle 45[V]$$
$$V_C(\omega) = -j\frac{1}{\omega C}I(\omega)70.7\angle -135[V]$$

# Solution (2/2)

- Equate the complex power from the voltage and current.
  - Source

$$S_V = \frac{V_s I^*}{2} = 353.5\angle 45º \text{ [VA]}$$

  - Other elements

$$S_R = \frac{V_R I^*}{2} = 250\angle 0º \text{ [VA]}$$
$$S_L = \frac{V_L I^*}{2} = 500\angle 90º \text{ [VA]}$$
$$S_C = \frac{V_C I^*}{2} = 250\angle -90º \text{ [VA]}$$

- $S_V = 353.5\angle 45º$ , $S_R + S_L + S_C = 353.5\angle 45º$

# Power Factor (pf)

- Average power absorbed by the element can be represented as pf.

$$P = \frac{V_m I_m}{2} \cos(\theta_V - \theta_I)$$

$$\frac{V_m I_m}{2} : apparent\ power$$

- The ratio of the average power to the apparent power is called power factor (pf).

  pf = $cos(\theta_V - \theta_I)$ , $(\theta_V - \theta_I)$: power factor angle

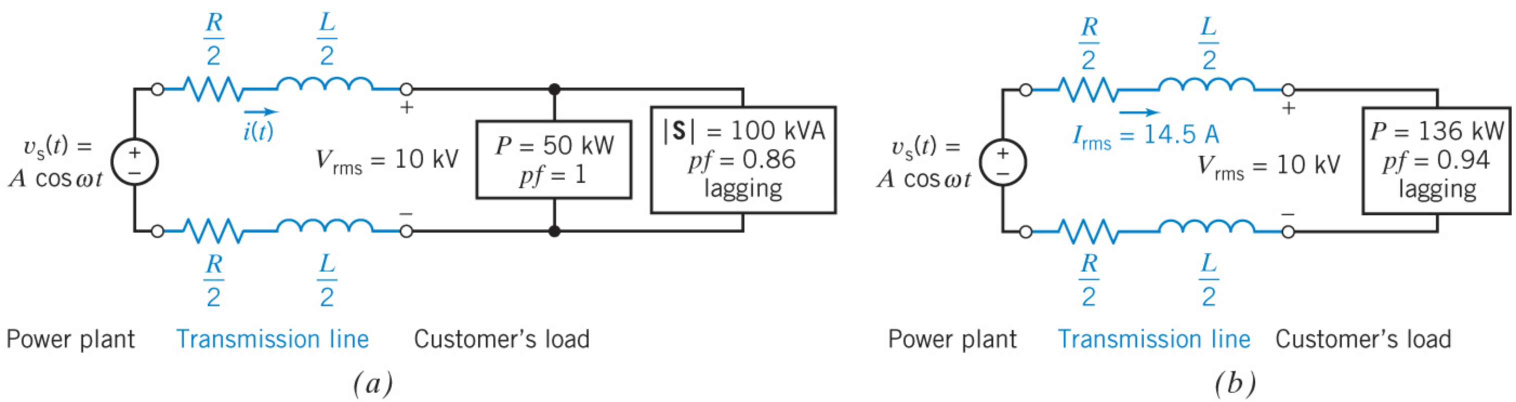
$$P = \frac{V_m I_m}{2} pf$$

- lagging or leading
  - Leading: $\theta_V - \theta_I < 0$
  - Lagging: $\theta_V - \theta_I > 0$

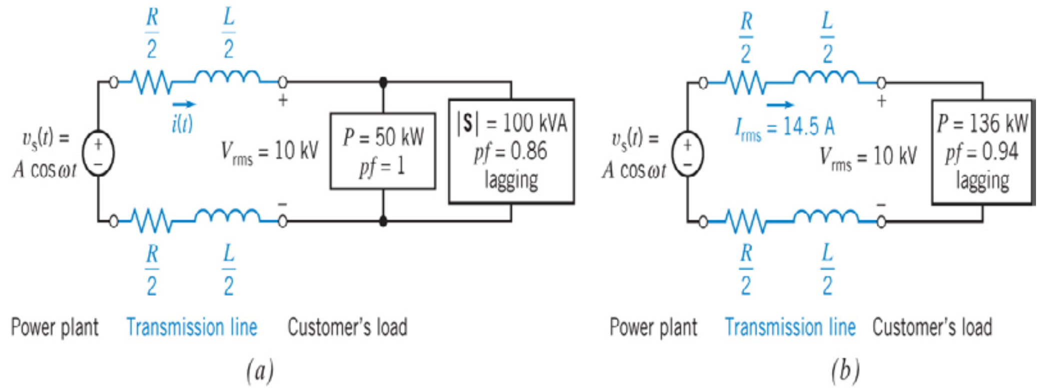
- Power factor refers transmission loss.

# Example 11.6-1 *Parallel Loads*

- A customer's plant has two parallel loads connected to the power utility's distribution lines. The first load consist of 50 kW of heating and is resistive. The second load is a set of motors that operate at 0.86 lagging power factor. The motors' load is 100 KVA. Power is supplied to the plant at 10,000 volts rms. Determine the total current flowing form the utility's lines into the plant and the plant's overall power factor.



# Solution




- Determine total complex power
  - Power at resistive load ($S_1$)

$$S_1 = P_1 = 50\ [kW]$$

  - Power at 0.86 lagging load ($S_2$)

$$S_2 = |S_2|\angle\theta_2 = 100\cos^{-1}(0.86)\ [V] = 100\angle + 30.7°\ [kVA]$$

  - Sum of $S_1$ and $S_2$

$$S = S_1 + S_2 = 145.2\angle 20.6°\ [kVA]$$

  - Calculate pf

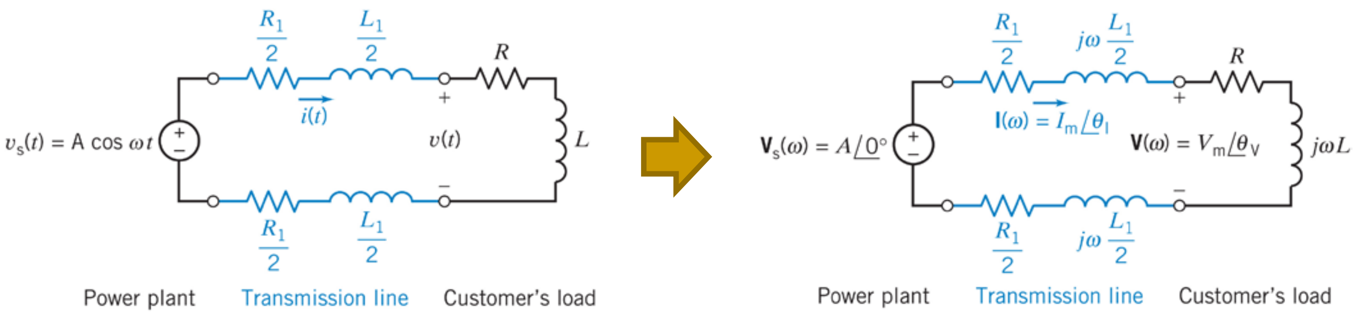
$$pf = \cos(20.6°) = 0.94\ lagging$$

- Find current from the calculated power

$$|S| = \frac{V_m I_m}{2} = Vrms\, I_{rms} \qquad I_{rms} = \frac{145.2k}{10k} = 14.52\ [A]$$

# Corrected Power Factor

- Power plant supply electric power to the customer through transmission line. It is essential to minimize the loss at the transmission line.
- Power loss at transmission line as a function of pf.




- Impedance of the transmission line

$$Z_{line}(\omega) = R_1 + jwL_1$$

- Average power absorbed by the line

$$P_{line} = \frac{I_m^{\ 2}}{2}Re(Z_{line}) = \frac{I_m^{\ 2}}{2}R_1 \leftarrow I_m = \frac{2P}{V_m pf}$$

$$\boldsymbol{P_{line} = 2\left(\frac{P}{V_m pf}\right)^2 R_1}$$

# Corrected Power Factor

- Power loss at transmission line as a function of pf.

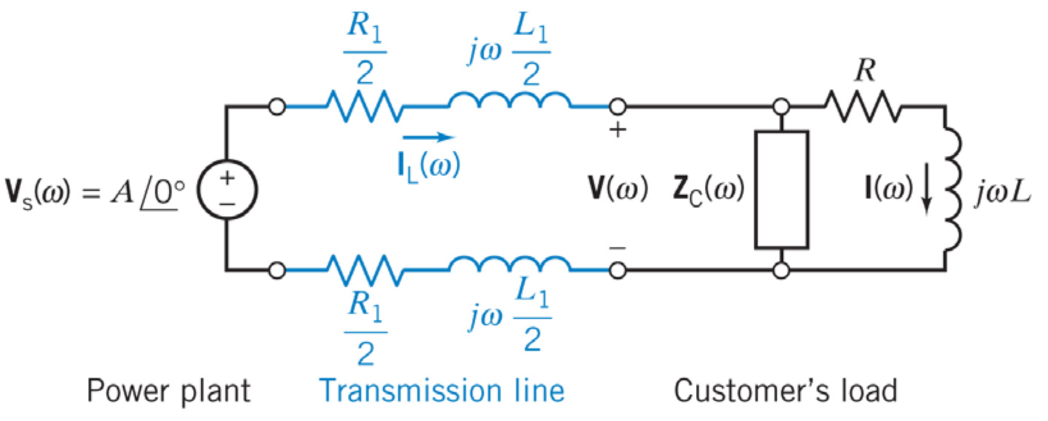
$$P_{line} = 2\left(\frac{P}{V_m pf}\right)^2 R_1$$

- Increasing pf will reduce the power absorbed in the transmission line.
  - If, pf is 1, then load should appear resistive. ($\theta_V = \theta_I$)
  - If pf becomes small, power loss at the line becomes large. $I_m = \frac{2P}{V_m pf}$

- Add a compensating impedance (**Zc(w)**) to make pf close to ‘1’.

# Corrected power Factor

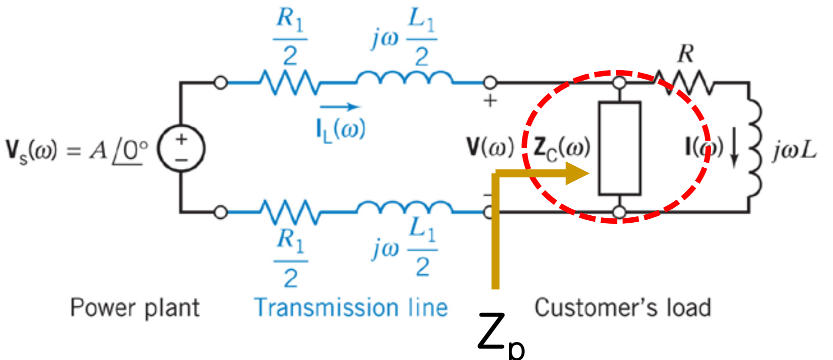
- Add compensating impendance Zc(w) parallel to the Z(w)




- Add Zc so the phase of current $I_L$(w) (= $I_C$(w) + I(w)) is equal to phase of voltage $V_S$(w).

# Corrected power Factor (pfc)

- Compensating impedance:




  - $Z_C$(w): reactive element, since we want $Z_C$ to absorb no average power.

$$Z_P = (Zp||Z) = \frac{ZZ_C}{Z+Zc} = Rp + jXp = Z\angle\theta_P$$

  - Angle of power factor is equal to angle of Zp.

$$\mathrm{p}f = \cos(\theta_V - \theta_\mathrm{I}) = \cos\big(0 - (-\theta_P)\big) = \boldsymbol{\cos(\theta_P)}$$

# Corrected power Factor (pfc)

- Find corrected power factor using Zp.

$$pfc = \cos\theta_P = \cos\left(\tan^{-1}\frac{X_P}{R_P}\right)$$

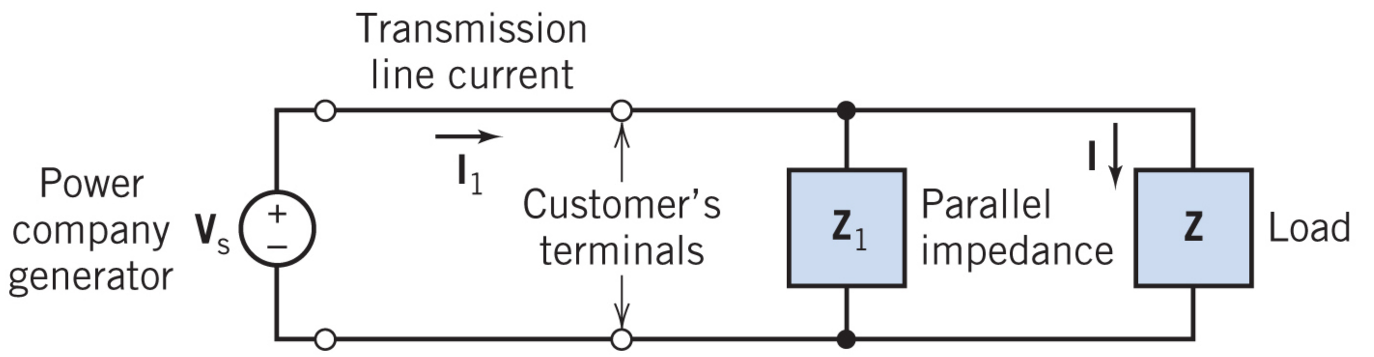
- Find the compensated impedance at given corrected power factor.

$$X_C = \frac{R^2 + X^2}{R\tan\left(\cos^{-1} pfc\right) - X}$$

→ Textbook page 498, 499.

# Example 11.6-2 *Power factor correction*

- A load as shown figure below has an impedance of Z = 100 + j100 ohm. Find the parallel capacitance required to correct the power factor to (a) 0.95 lagging and (b)1.0. Assume that the source is operating at w = 377 rad/s.



# Solution

- 공식 이용
  - Determine compensating impedance
    $$X_C = \frac{R^2 + X^2}{R\tan\left(\cos^{-1} pfc\right) - X} \qquad \mathbf{Z}_C = \frac{-j}{\omega C} = jX_C$$

a) pfc = 0.95 $\quad X_C = -297.9\ [ohm] \rightarrow C = 8.9\ [uF]$

b) pfc = 1 $\quad X_C = -200\ [ohm] \rightarrow C = 13.3\ [uF]$

- pf 의 정의를 이용하여 풀기
  - 전류의 phase를 구하기

$$\frac{Vs}{jXc} + \frac{Vs}{100 + j100} = Vs\left(\frac{1}{200} + j\left(\frac{-Xc}{Xc^2} + \frac{1}{200}\right)\right) \rightarrow \theta_I = \tan^{-1}\left[\frac{\frac{-Xc}{Xc^2} + \frac{1}{200}}{\frac{1}{200}}\right]$$

a) pfc = 0.95

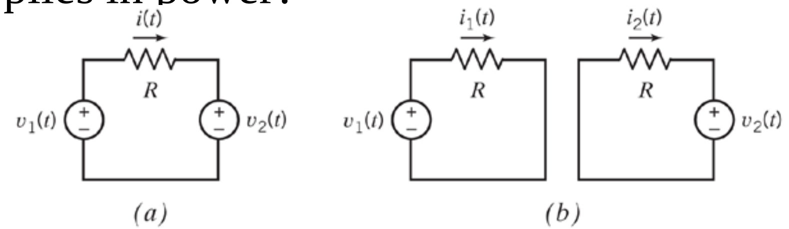
$$cos(\theta_V - \theta_I) = 0.95 \rightarrow \theta_V - \theta_I = 0 - \theta_I = \cos^{-1}(0.95) \rightarrow Xc = -297.9\ ohm$$

b) pfc = 1

$$cos(\theta_V - \theta_I) = 1 \rightarrow \theta_V - \theta_I = 0 - \theta_I = -\theta_I = \cos^{-1}(1) \rightarrow Xc = -200\ ohm$$

# The Power Superposition Principle

- Circuit with two or more sources.
  - Superposition principle applies to current and voltage.
  - Does it applies in power?




$$i = i_1 + i_2$$

$$\begin{aligned} P &= \frac{1}{T}\int_0^T p\,dt = \frac{R}{T}\int_0^T \left(i_1^2 + i_2^2 + 2i_1 i_2\right) dt \\ &= \frac{R}{T}\int_0^T i_1^2\,dt + \frac{R}{T}\int_0^T i_2^2\,dt + \frac{2R}{T}\int_0^T i_1 i_2\,dt = P_1 + P_2 + \frac{2R}{T}\int_0^T i_1 i_2\,dt \end{aligned}$$

**Zero only if sources have different frequencies!**

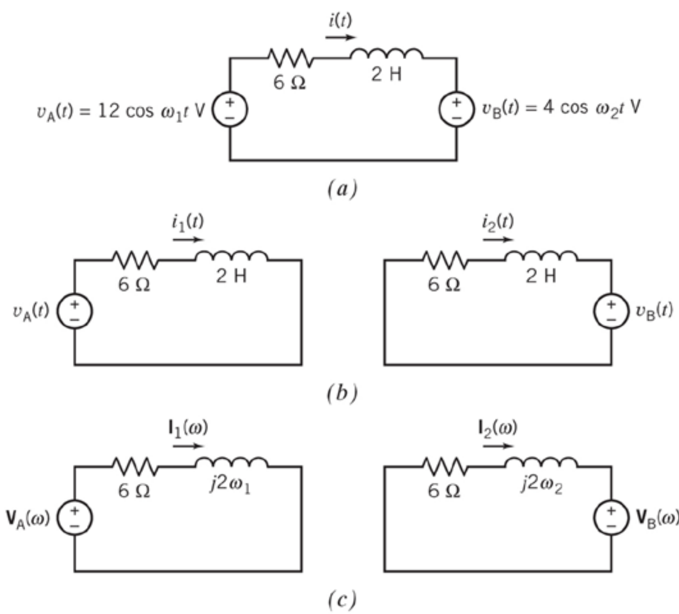
$$P = P_1 + P_2$$

- Power superposition applies only if the sources have different frequencies.

# Example 11.7-1 *Power Superposition*

- The circuit in figure below contains two sinusoidal sources. To illustrate power superposition, consider two cases:
  - (1) $V_A(t) = 12\cos 3t$ [V] and $V_B(t) = \cos 4t$ [V]
  - (2) $V_A(t) = 12\cos 4t$ [V] and $V_B(t) = \cos 4t$ [V]

  Find the average power absorbed by the 6 ohm resistor.



# Solution

- (a) Find the average power using power superposition.

$$P_1 = \frac{(12/\sqrt{2})^2}{|6+j6|^2}6 = 6\ \text{W} \quad , \quad P_2 = \frac{(4/\sqrt{2})^2}{|6+j8|^2}6 = 0.48\ \text{W}$$

$$P = P_1 + P_2 = 6.48\ \text{W}$$

$$P = P_1 + P_2$$

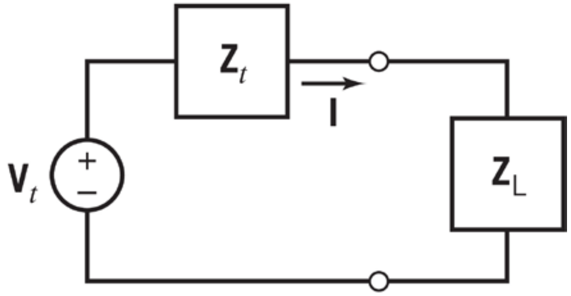
- (b) Find total current and calculate the average power.

$$v(t) = v_A(t) + v_B(t) = 8\cos 4t$$

$$P = \frac{(8/\sqrt{2})^2}{|6+j8|^2}6 = 1.92\ \text{W}$$

# The Maximum Power Transfer Theorem

- Maximum power transfer in a sinusoidal steady state circuit containing reactive impedance




$$Z_t = Rt + jXt$$
$$Z_L = R_L + jXL$$

- Average power delivered to the load is

$$I = \frac{V_t}{Z_t + ZL} = \frac{V_t}{(Rt + jXt) + (RL + jX_L)} \qquad P = \frac{{I_m}^2}{2} R_L = \frac{|V_t|^2 R_L / 2}{(R_t + R_L)^2 + (X_t + X_L)^2}$$

# The Maximum Power Transfer Theorem

- To maximize P
  - $X_t + X_L$ can be eliminated by setting $X_L = -X_t$

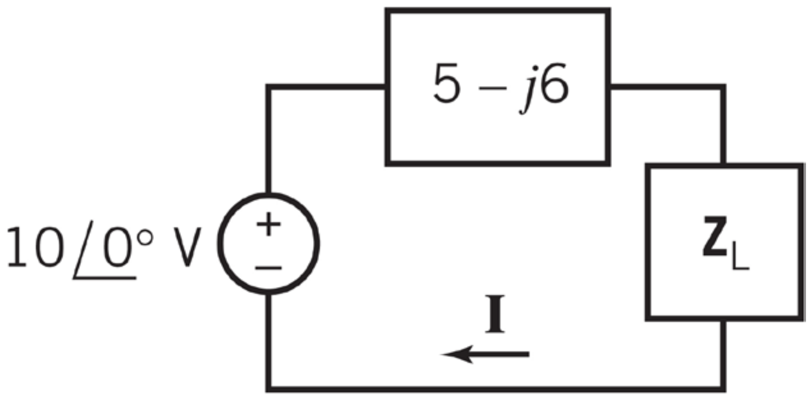
$$P = \frac{|V_t|^2 R_L/2}{(R_t + RL)^2}$$

  - The value of $R_L$ that maximize P is determined by taking the derivative $dP/dR_L$ and setting it to zero. P maximize at $R_L = R_t$

- **P maximizes when $Z_L = Z_t^*$**

# Example 11.8-1 *Maximum Power Transfer*

- Find the load impedance that transfers maximum power to the load and determine the maximum power delivered to the load for the circuit shown below.



# Solution

- Select $Z_L$ to have complex conjugate of $Z_t$

$$Z_L = Z_t{}^{*} = 5 + j6\ [ohm]$$

- Maximum power delivered to the load

$$I = \frac{10\angle 0}{5+5}\ [\mathrm{A}]$$

$$P = \frac{I_m{}^2}{2} R_L = 2.5\ [W]$$

# Coupled Inductors

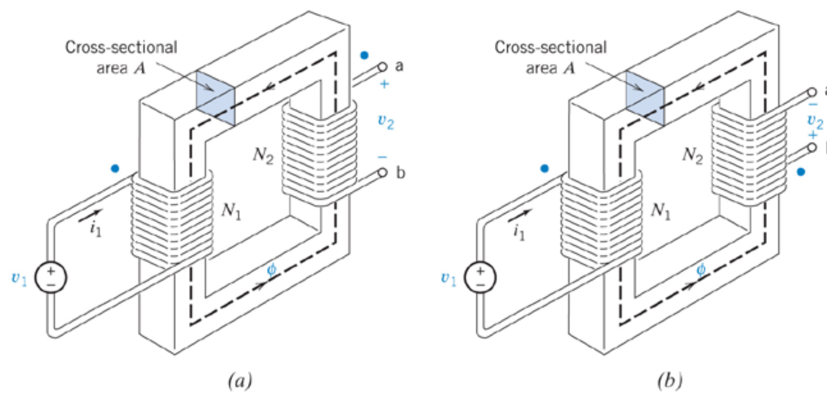
- **Coupled inductors**, or coupled coil, are magnetic devices that consist of two or more multiturn coils wound on a common core.
- Current and voltage relation in coupled coil.

(in terms of inductance and/or number of turns of coils)

$$\lambda_i = \sum_{j=1}^{n} L_{ij} i_j \qquad v_i = \frac{d\lambda_i}{dt} = \sum_{j=1}^{n} L_{ij} \frac{di_j}{dt}$$

코일이 두개라면,

$$\boxed{\begin{aligned} v_1 &= L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} \\ v_2 &= L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} \end{aligned}}$$




$L_{11}$, $L_{22}$: self inductance
$L_{12}$, $L_{21}$: mutual inductance

# Coupled Inductors

- **Self inductance and mutual inductance**
  - Sinusoidal current in coil 1 → Magnetic field induced by coil 1 (determined by self inductance $L_1$) → Induced magnetic field at coil 2 → Induced current as well as voltage in coil 2 (determined by Mutual inductance M)

$$L_{ij} \equiv \frac{\lambda_i}{i_j \text{ (이때 } i_j \text{외의 전류는 영)}}$$

i=j self inductance
i≠j mutual inductance

# Coupled Inductors: Self and mutual inductance

- Self inductance and mutual inductance
  - Self inductance

$$\phi = c_1 N_1 i_1$$

$$v_1 = N_1 \frac{d\phi}{dt} = N_1 \frac{d}{dt}(c_1 N_1 i_1) = c_1 N_1^2 \frac{di_1}{dt}$$

$$L_1 = c_1 N_1^2$$

  - Mutual inductance qualitatively shows how two coils interact.

$$v_2 = N_2 \frac{d\phi}{dt} = c_M N_1 N_2 \frac{di_1}{dt} = M \frac{di_1}{dt}$$

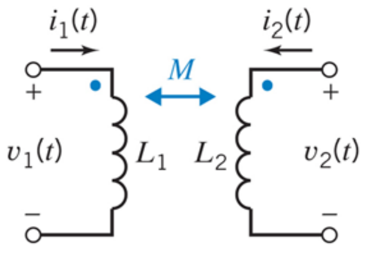
$$L_1 L_2 = (c_1 N_1^2)(c_2 N_2^2) = c_1 c_2 (N_1 N_2)^2 = \left(\frac{c_M N_1 N_2}{k}\right)^2 = \frac{M^2}{k^2}$$

$$k = \frac{M}{\sqrt{L_1 L_2}}$$ k: coupling coefficient $0 \le k \le 1$

# Coupled Inductors

- Dot convention
  - is used to indicate winding direction.
  - **affects the sign of induced voltage.**

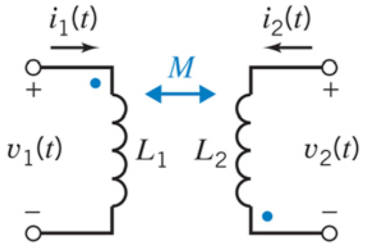
$$V = Vself + V_{induced}$$




(a)

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$



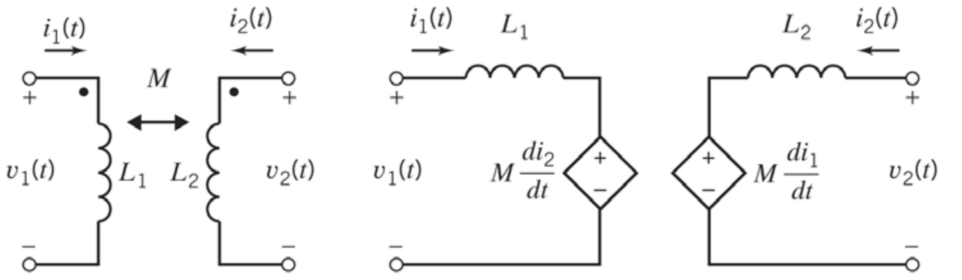

(b)

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

# Coupled Inductors

- Equivalent circuit of coupled inductors using dependent sources



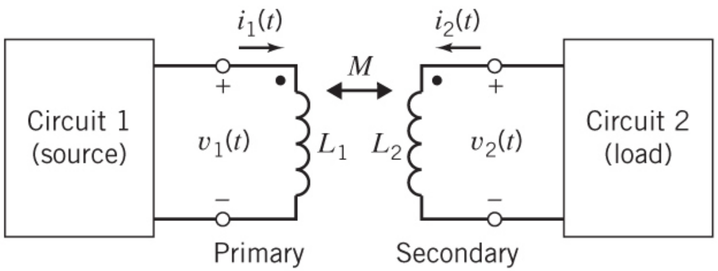

(a) (b)

$$\mathbf{V}_1 = j\omega L_1 \mathbf{I}_1 + j\omega M\,\mathbf{I}_2$$

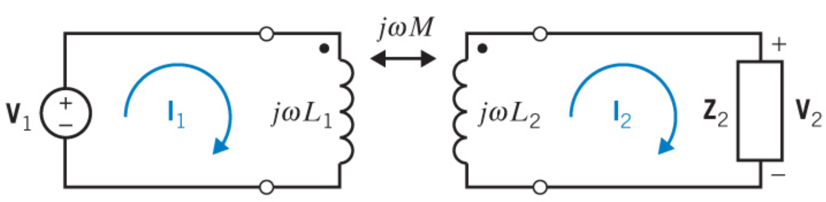
$$\mathbf{V}_2 = j\omega L_2 \mathbf{I}_2 + j\omega M\,\mathbf{I}_1$$

# Coupled Inductors

- Voltage across each coil




(a)




(b)

  - The ratio of voltage across each coil is same as the ratio of turns of each coil.

$$jwL_1I_1 - jwMI_2 = V_1$$
$$-jwMI_1 + (jwL_2 + Z_2)I_2 = 0$$

$$\mathbf{I}_2 = \left[\frac{j\omega\, M}{\left((j\omega)^2(L_1L_2 - M^2) + (j\omega\, L_1\mathbf{Z}_2)\right)}\right]\mathbf{V}_1$$

For ideal transformer, $M^2 = L_1L_2$

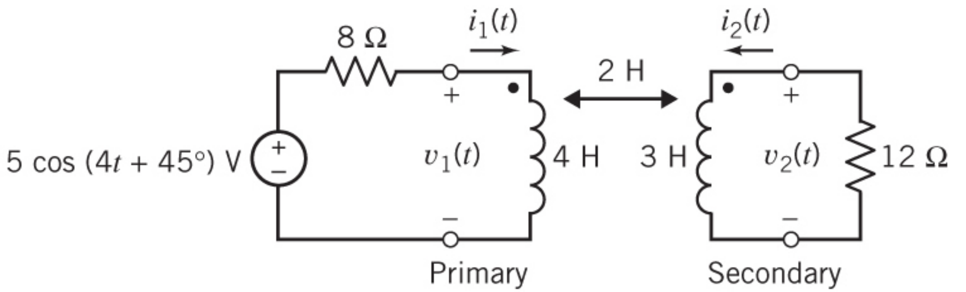
$$\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 = \sqrt{\frac{L_2}{L_1}}\mathbf{V}_1$$

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = n^2$$

$$\mathbf{V}_2 = n\mathbf{V}_1$$

# Example 11.9-1 *Coupled Inductors*

- Find the voltage $v_2(t)$ in the circuit shown below.

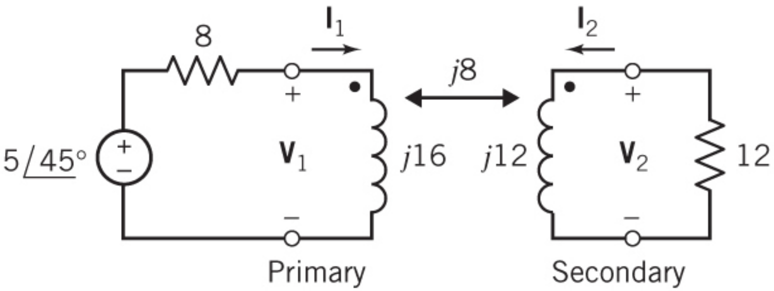


(a)

# Solution

- Represent the circuit in the frequency domain and express the coil voltages.

$$V_1 = j16I_1 + j8I_2$$
$$V_2 = j8I_1 + j12I_2$$




(b)

- Write mesh equations for each mesh

$$5\angle 45 = 8I_1 + V_1$$
$$V_2 = -12I_2$$

- Solving for $V_2$ gives

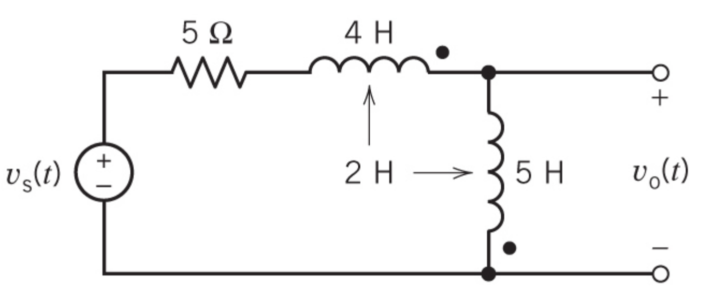
$$V_2 = 1.656\angle 39\ [V] \rightarrow v_2(t) = 1.656\cos(4t + 39)\ [V]$$

# Example 11.9-2 *Coupled Inductors*

- The input to the circuit shown in figure below is the voltage of the voltage source,

$$v_s(t) = 5.94\cos(3t + 140)\ [V]$$

The output is the voltage across the right-hand coil, $v_o(t)$. Determine the output voltage $v_o(t)$.

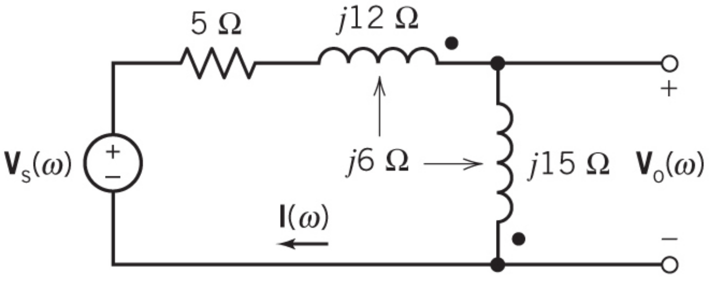



(a)

# Solution

- Represent the circuit in the frequency domain and express the coil voltages.

$$V_{coil1} = j12I + j6I$$
$$V_{coil2} = j15I + j6I$$




(b)

- Write mesh equations

$$5I + (j12 + j6)I + (j15 + j6)I - 5.94\angle 140 = 0$$

- Solving for I gives

$$I = 0.151\angle 57 \; [A]$$

- Solving for $V_O$

$$V_O = j15I + j6I = 3.17\angle 147 \; [V] \rightarrow vO(t) = 3.17(3t + 147) \; [V]$$

# The Ideal Transformer

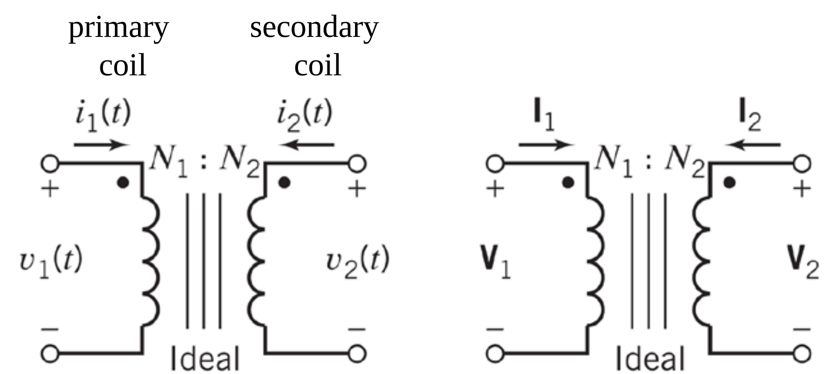
- Coupling coefficient is ‘1’.
- **Voltage and current**
  - Time domain

$n = N_2/N_1$
(Turn ratio)

$$v_2(t) = \frac{N_2}{N_1} v_1(t) \qquad i_1(t) = -\frac{N_2}{N_1} i_2(t)$$

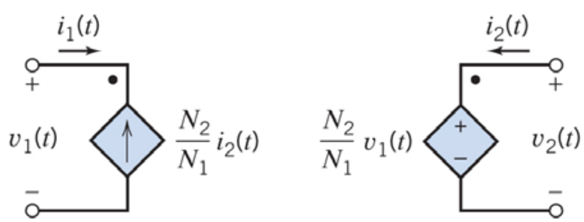
  - Frequency domain

$$\mathbf{V}_2 = \frac{N_2}{N_1}\mathbf{V}_1 \qquad \mathbf{I}_1 = -\frac{N_2}{N_1}\mathbf{I}_2$$



(a) (b)

- Ideal transformer modeled using dependent source.

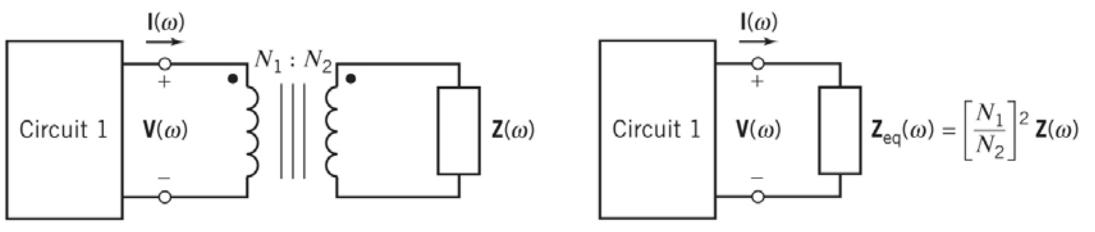


(b)

# The Ideal Transformer

- Power loss in ideal transformer
  - Lossless
  - Zero complex power, zero average power, zero reactive power.

- **Load impedance seen at primary coil**

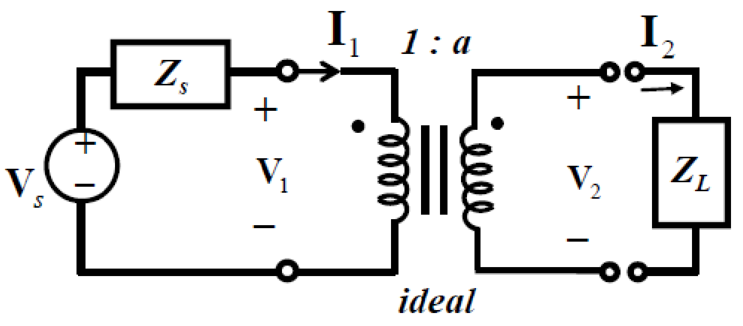



(a) (b)

  - Primary coil is a source, secondary coil has a load! → Source in the primary coil operate load in the secondary coil even though there is no connection.
  - Load impedance Z(w) can be seen as **$n^2$Z(w)** at primary coil. → Actual load impedance has scaled by a factor of $n^2$ when connecting ideal transformer

# Impedance matching

- Load impedance seen at primary coil




전원 쪽에서 본 임피던스

$$Z_{in} = \frac{\mathbf{V}_1}{\mathbf{I}_1}$$

전압, 전류의 관계

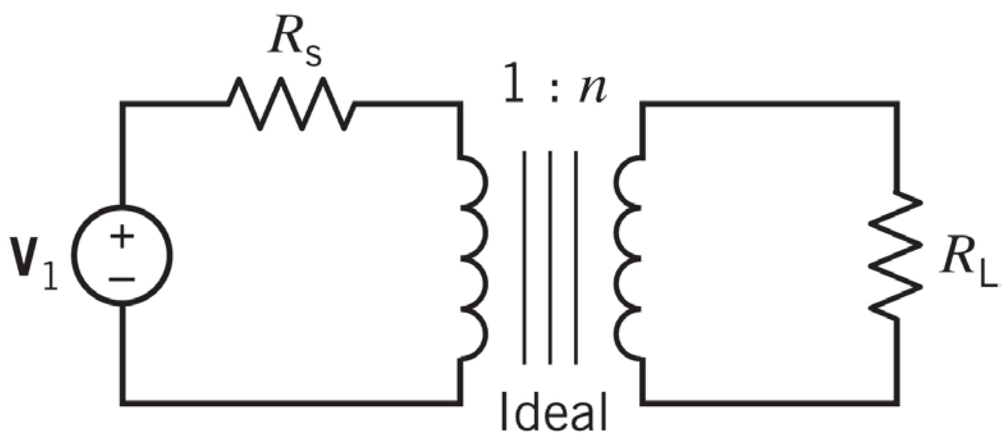
$$\mathbf{V}_2 = j\omega M\mathbf{I}_1 = j\omega M\frac{\mathbf{V}_1}{j\omega L_1}$$
$$= a\mathbf{V}_1$$

따라서, $$Z_{in} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\frac{1}{a}\mathbf{V}_2}{a\mathbf{I}_2} = \frac{1}{a^2}\frac{\mathbf{V}_2}{\mathbf{I}_2} = \frac{Z_L}{a^2}$$

→ If $Z_{in}$ is complex conjugate of $Z_s$ , then we could deliver maximum power.

# Example 11.10-1 ***Maximum Power Transfer***

- Often, we can use an ideal transformer to represent a transformer that connects the output of a stereo amplifier $V_1$ to a stereo speaker, as shown in figure below. Find the value of the turns ratio n that is required to cause maximum power to be transferred to the load when $R_L$=8 ohm and $R_S$=48 ohm.



# Solution

- Maximum power transfer is achieved when $R_S = R_L$. Find the $R_L$ seen at primary coil.

$$Z_1 = \frac{R_L}{n^2} = \frac{8}{n^2}$$

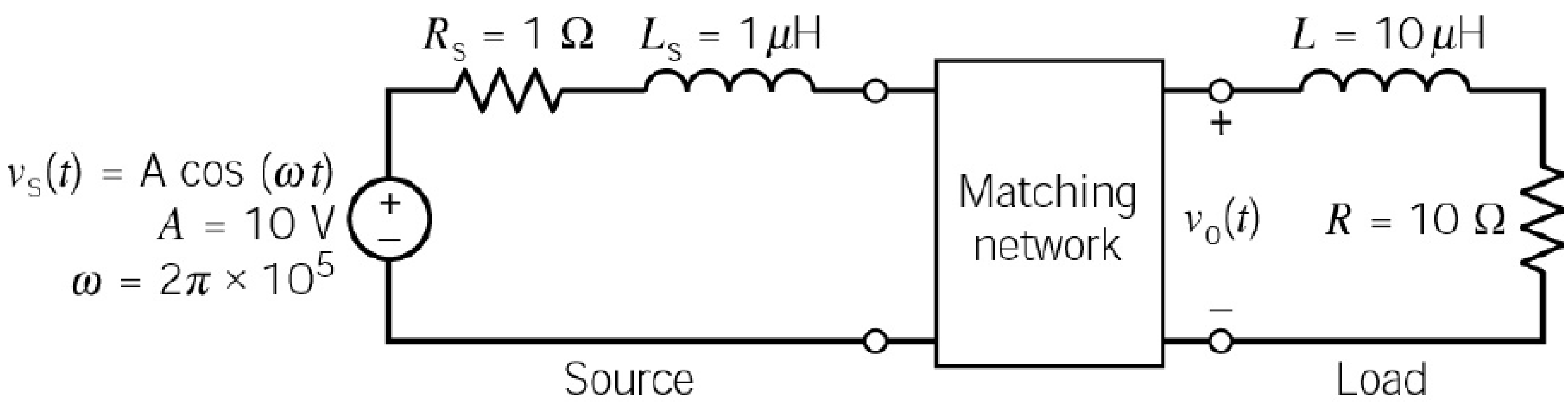
- Find n which satisfies $R_S = R_L$.

$$Z_1 = \frac{R_L}{n^2} = \frac{8}{n^2} = 48 \rightarrow n^2 = \frac{1}{6}, \qquad n = N_2/N_1$$

$$N_1 = \sqrt{6}N_2$$

# Maximum Power Transfer_예제(Pf.김용권)

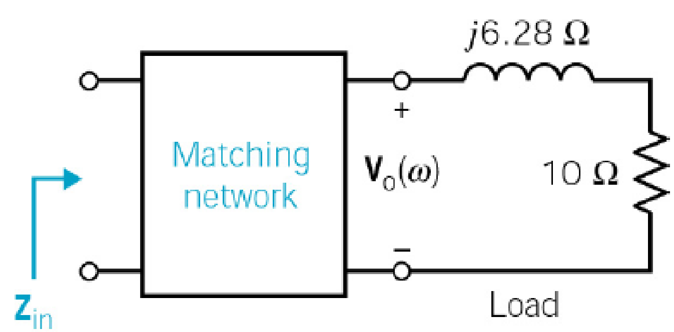
- 실제 **cellular telephone antenna**의 입력 임피던스는 **10+*j* 6.28** Ω 이다.
- 부하에 최대 전력이 전달되도록 회로망을 설계하라.




$$\mathbf{Z}_s = R_s + j\omega L_s = 1 + j2\pi 10^5 \times 10^{-6} = 1 + j0.628$$

- 최대 전력이 전달되는 임피던스는 입력 단에서 보았을 때

$$\mathbf{Z}_{in} = \mathbf{Z}_s^* = 1 - j0.628$$

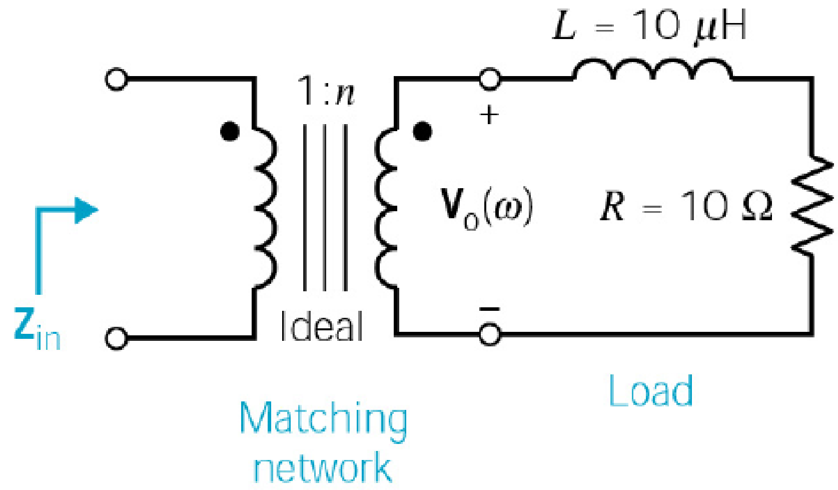


# Maximum Power Transfer_예제(Pf.김용권)

- 변압기를 사용하여 전체적인 계수를 조정

$$\mathbf{Z}_{in} = \frac{1}{n^2}(R + j\omega L) = \frac{1}{n^2}(10 + j6.28)$$

$$n = 3.16$$



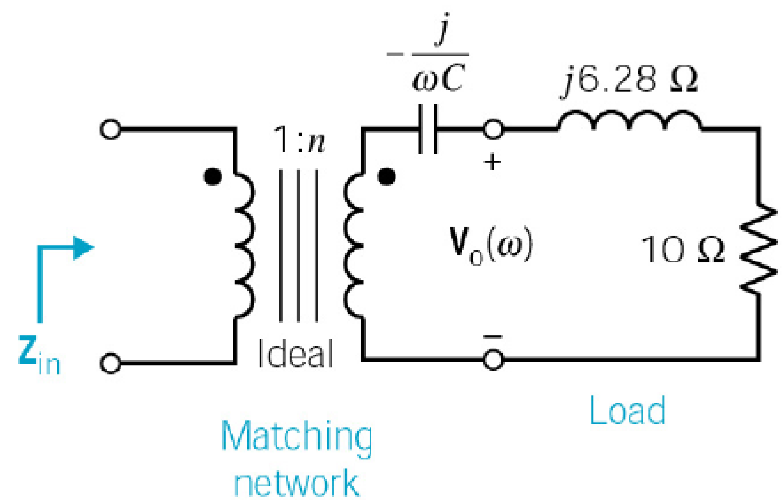

- $n^2$이 **10** 이면 실수부는 최대 전력 전달 조건에 만족이 되나 허수부는 만족이 안된다.
- 캐패시터를 삽입.

$$\mathbf{Z}_{in} = \mathbf{Z}_s^* = 1 - j0.628$$

$$= \frac{1}{10}(10 + j6.28 - j\frac{1}{\omega C})$$

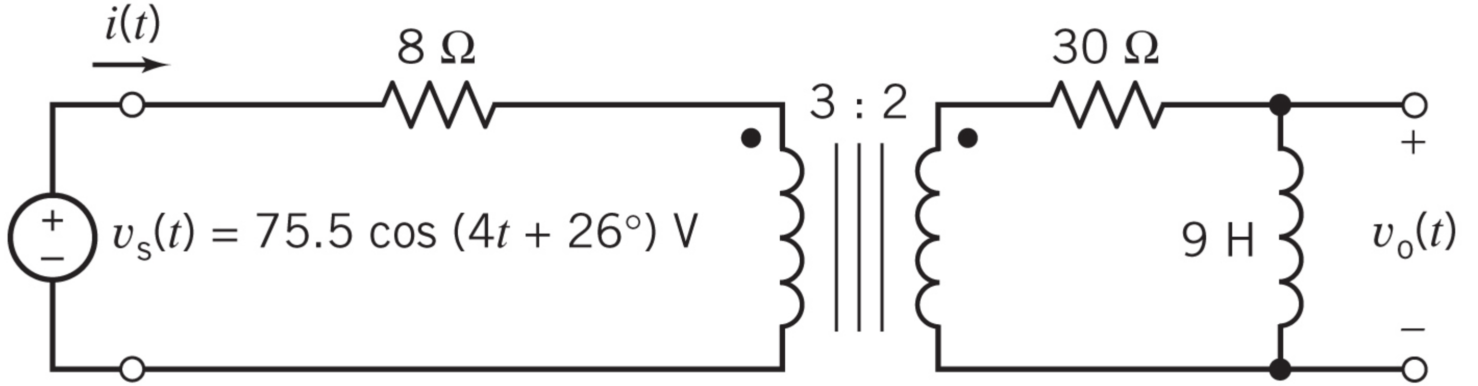
$$C = \frac{1}{10 \times 2\pi \times 10^5 \times 2 \times 0.628}$$

$$= 0.1267\ \mu\mathrm{F}$$



# Example 11.10-2 *Transformer Circuit*

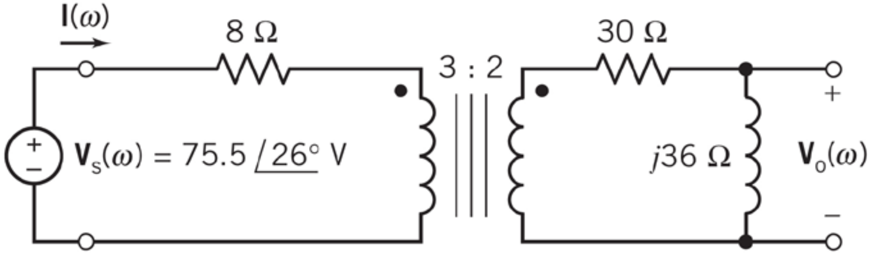
- The input to the circuit shown in figure below is the voltage source $v_S(t)$. The output is the voltage across the 9-H inductor, $v_O(t)$. Determine the output voltage $V_O(t)$.



# Solution

- Represent the circuit in the frequency domain and express the load impedance seen in primary coil.

$$Z_{load\ at\ primary} = \left(\frac{3}{2}\right)^2(30 + j36)$$




- Find the current in the primary coil using mesh equation.

$$Vs = I(8 + Z_{load\ at\ primary}) \rightarrow I = 0.682\angle - 21\ [A]$$

- Calculate the current flowing through secondary coil using current in primary coil.

$$I_2 = -\left(\frac{3}{2}\right)I = -1.023\angle - 21\ [A]$$

- Calculate Vo

$$V_O = -j36I_2 = 36.82\angle 69\ [V] \rightarrow vO(t) = 36.82(4t + 69)\ [V]$$