Chapter 13
Frequency Response

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Linear elements

- Superposition:

$$i_1 \to v_1$$
$$i_2 \to v_2$$
$$i_1 + i_2 \to v_1 + v_2$$

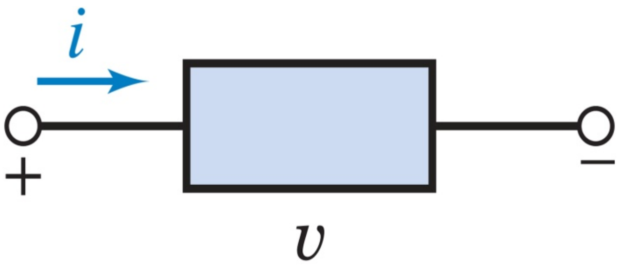

FIGURE 2.2-1

- Homogeneity:

$$i \to v$$

$$ki \to kv$$

Resistor	$\mathbf{V}_{\mathrm{R}}(\omega) = R\,\mathbf{I}_{\mathrm{R}}(\omega)$
Capacitor	$\mathbf{V}_{\mathrm{C}}(\omega) = \dfrac{1}{j\omega C}\mathbf{I}_{\mathrm{C}}(\omega)$
Inductor	$\mathbf{V}_{\mathrm{L}}(\omega) = j\omega L\mathbf{I}_{\mathrm{L}}(\omega)$

Linear circuit

- Superposition:

$$v_{in,1} \rightarrow v_{out,1}$$

$$v_{in,2} \rightarrow v_{out,2}$$

$$v_{in,1} + v_{in,2} \rightarrow v_{out,1} + v_{out,2}$$

- Homogeneity:

$$v_{in} \rightarrow v_{out}$$

$$kv_{in} \rightarrow kv_{out}$$

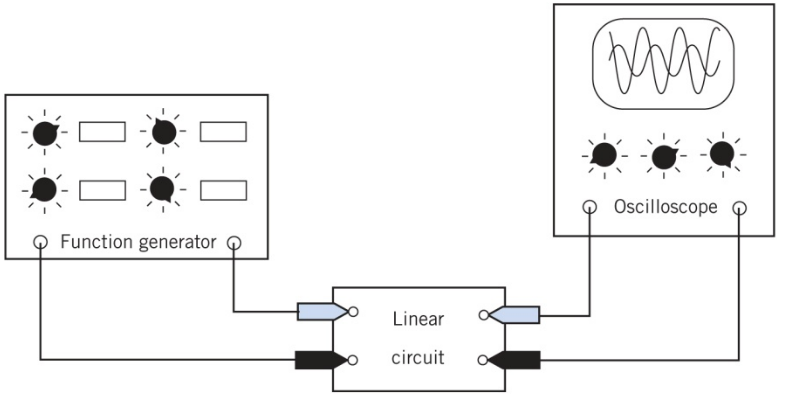

FIGURE 13.1-1

At any fixed frequency

- Between the input and the output sinusoid, these are constant.
  - The ratio of the amplitude
  - The difference of the phase angles

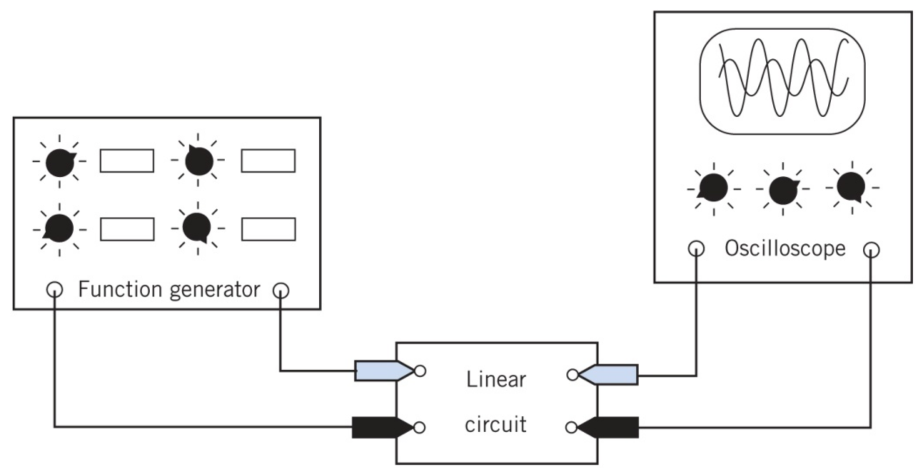

FIGURE 13.1-1

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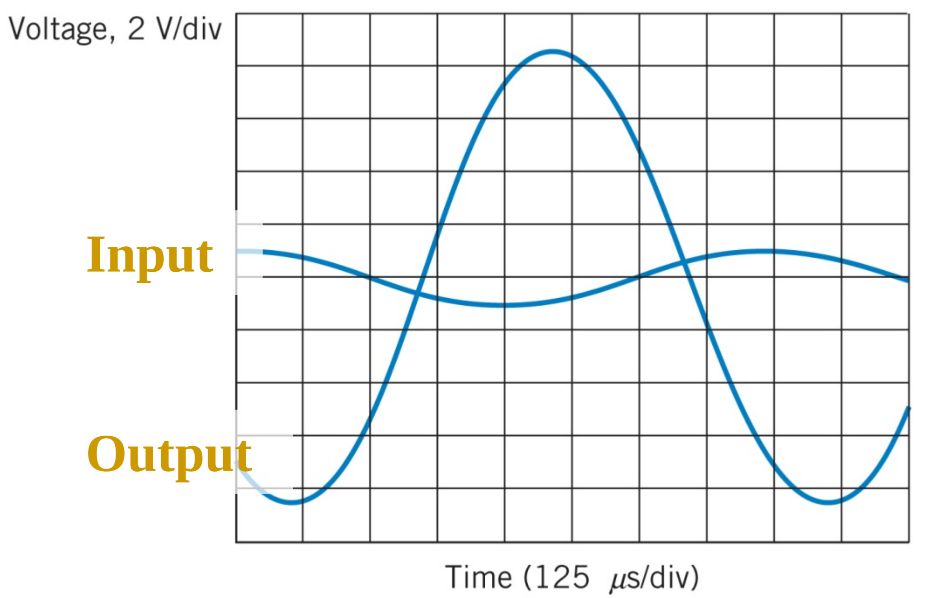


FIGURE 13.2–2

At any fixed frequency

- Between the input and the output sinusoid, these are constant.
  - The ratio of the amplitude
  - The difference of the phase angles

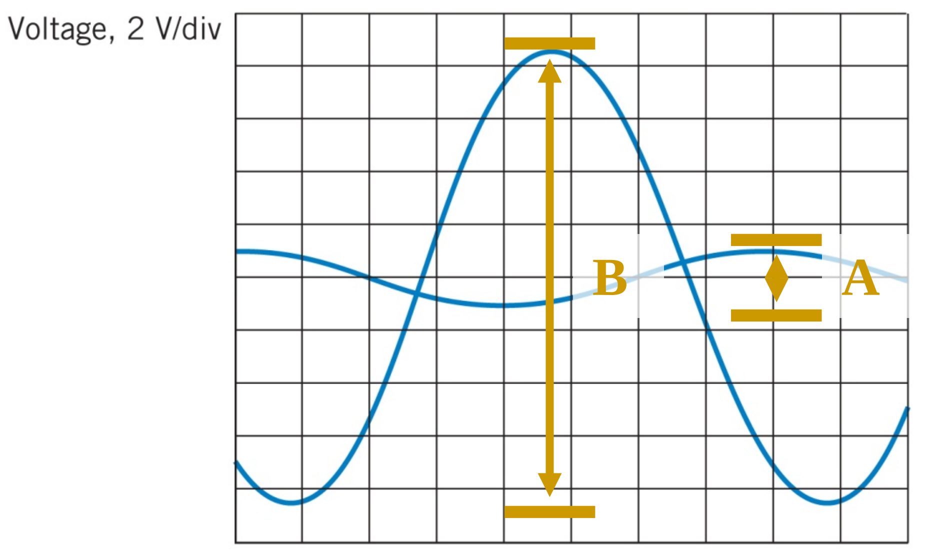


FIGURE 13.2−2

$$\text{gain} = \frac{B}{A}$$

At any fixed frequency

- Between the input and the output sinusoid, these are constant.
  - The ratio of the amplitude
  - The difference of the phase angles

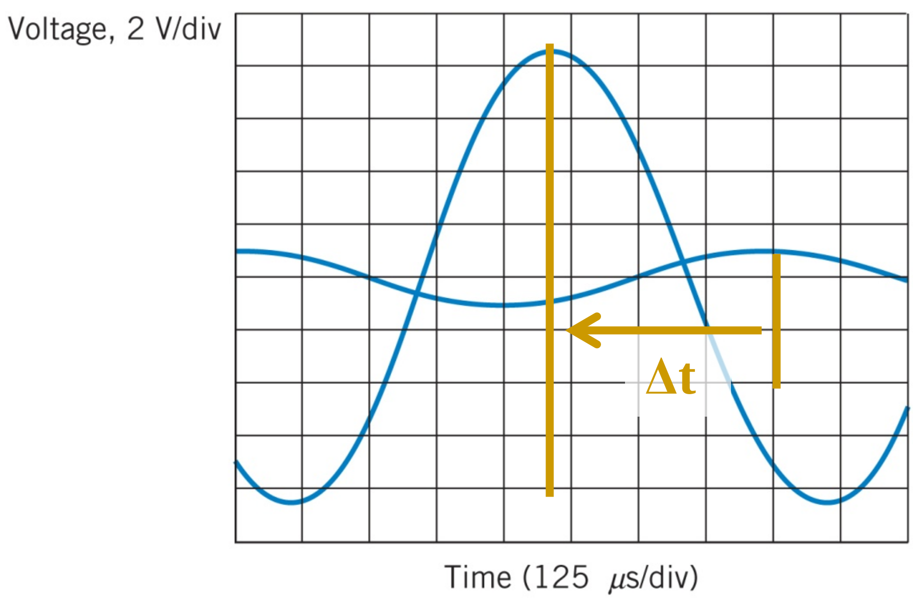


FIGURE 13.2−2

phase shift = $\omega \Delta t$

Network function

- Changing the **frequency** of the input changes the gain and phase shift.
  - The gain and phase shift is the function of the frequency
  - When X is the input and Y is the output, we define **a Network function at steady state** as follows:

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

$$\text{gain} = |\mathbf{H}(\omega)| = \frac{|\mathbf{Y}(\omega)|}{|\mathbf{X}(\omega)|}$$

$$\text{phase shift} = \angle \mathbf{H}(\omega) = \angle \mathbf{Y}(\omega) - \angle \mathbf{X}(\omega)$$

Network function - example

$$\frac{V_{in}(\omega)}{R_1} + \frac{V_{out}(\omega)}{\mathrm{R}_2} + \mathrm{j}\omega\mathrm{CV}_{\mathrm{out}}(\omega) = 0$$

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{-R_2}{R_1 + j\omega C R_1 R_2}$$

$$\boldsymbol{gain} = |\boldsymbol{H}(\omega)| = \frac{\frac{\boldsymbol{R_2}}{\boldsymbol{R_1}}}{\sqrt{\mathbf{1} + \omega^2 \boldsymbol{C}^2 \boldsymbol{R}_2^2}}$$

$$\boldsymbol{phase\ shift} = \angle \boldsymbol{H}(\omega) = \mathbf{180}^{\circ} - \boldsymbol{tan}^{-1}(\omega \boldsymbol{C}\boldsymbol{R}_2)$$

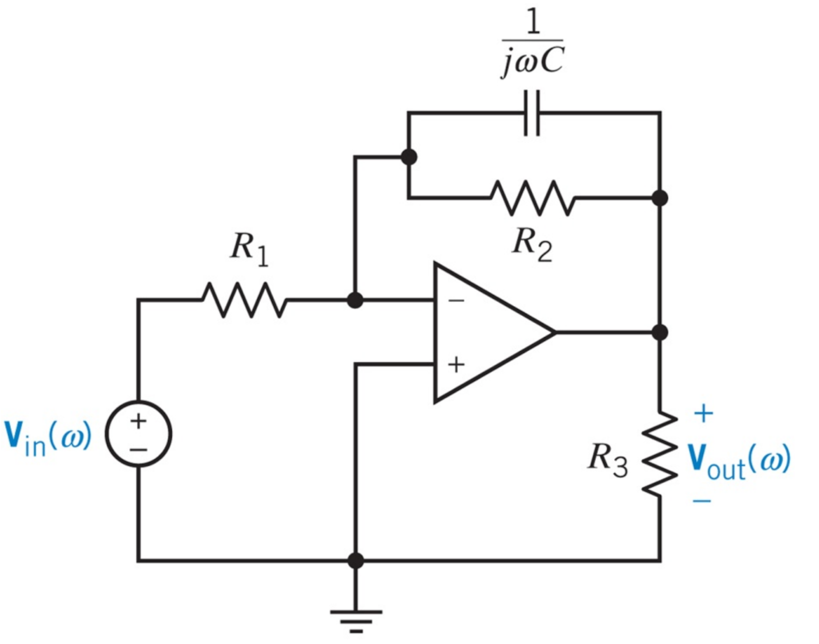


FIGURE 13.2–3

Network function - example

$$when\ R_1 = 5k\Omega, R_2 = 50k\Omega, C = 2nF$$

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{-10}{1 + j\omega/10000}$$

$$\boldsymbol{gain} = |\boldsymbol{H}(\omega)| = \frac{\mathbf{10}}{\sqrt{\mathbf{1} + \omega^{\mathbf{2}}/\mathbf{10^8}}}$$

$$\boldsymbol{phase\ shift} = \angle \boldsymbol{H}(\omega) = \mathbf{180^o} - \boldsymbol{tan^{-1}}(\omega/\mathbf{10000})$$

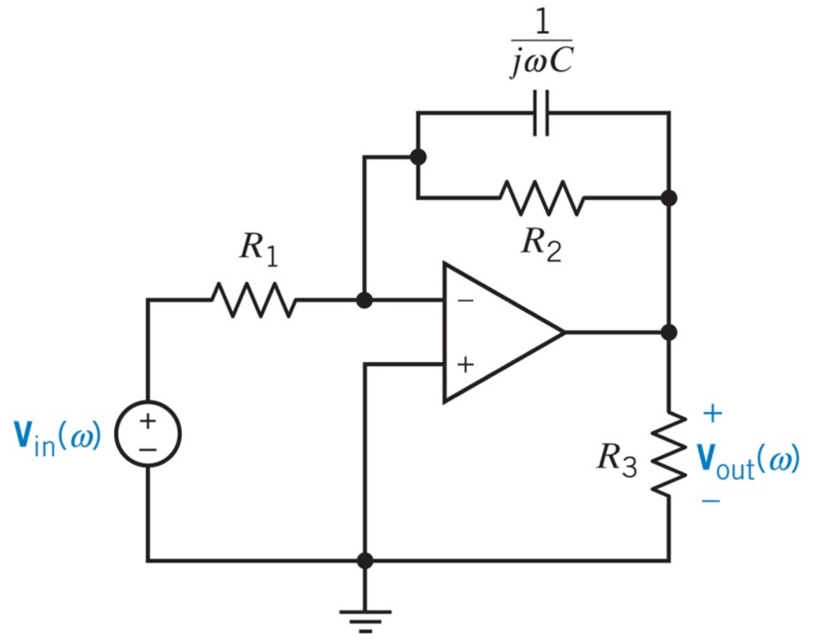


FIGURE 13.2–3

Network function

- Frequency response
  - Equations that represent the gain and phase shift as functions of frequency
  - The same information can be represented by a table or by graphs

- The network function really does represent the behavior of the circuit.
  - Suppose that $v_{in}(t) = 0.4\cos(5000t + 45°)\,V$

$$H(\omega) = \frac{-10}{1 + j5000/10000} = 8.94\angle 153°$$

$$\boldsymbol{V_{out}}(\omega) = \boldsymbol{H(\omega)V_{in}(\omega)} = (\mathbf{8.94\angle 153°})\ (\mathbf{0.4\angle 45°}) = \mathbf{3.58\angle 198°}$$

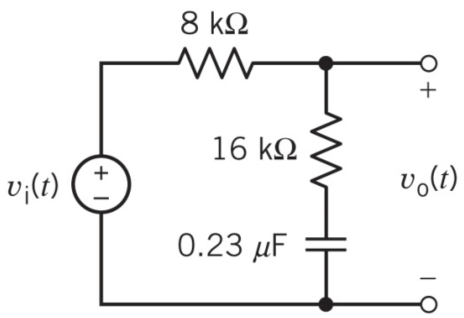
  - Back in the time domain, the steady-state response is

$$\boldsymbol{v_{out}(t) = 3.58\cos(5000t + 198°)\,V}$$

Example 13.2-1 Network Function of a Circuit

- Consider the circuit shown in Figure 13.2-4a. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage $v_o(t)$ across the series connection of the capacitor and the 16-kΩ resistor. The network function that represents this circuit has the form

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$



(a)

- The network function depends on two parameters, z and p. The parameter z is called the zero of the circuit and the parameter is called the pole of the circuit. Determine the values of z and of p for the circuit in Figure 13.2-4a

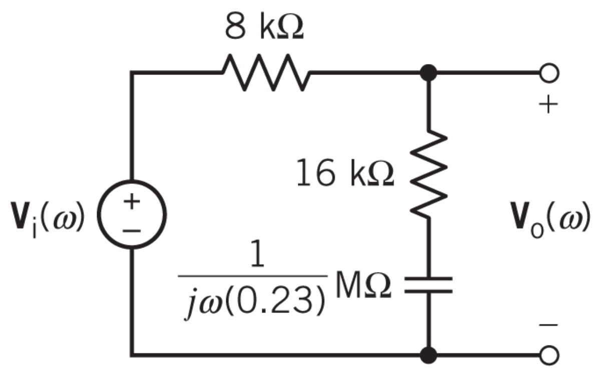
Solution

- The impedances of the capacitor and the 16-kΩ resistor are connected in series. The equivalent impedance of this combination is

$$Z_e(\omega) = 16k\Omega + \frac{1}{j(0.23\mu F)\omega}$$

- The equivalent impedance is connected in series with the 8-kΩ resistor. $V_i(\omega)$ is the voltage across the series impedances, and $V_o(\omega)$ is the voltage across the equivalent impedance $Z_e(\omega)$. Apply the voltage division principle to get

$$\boldsymbol{V_o(\omega) = \frac{16k\Omega + \frac{1}{j(0.23\mu F)\omega}}{8k\Omega + 16k\Omega + \frac{1}{j(0.23\mu F)\omega}} V_i(\omega)}$$



(b)

Solution

- Doing some algebra gives

$$V_o(\omega) = \frac{1 + 16k\Omega(0.23\mu F)\omega}{1 + j(8k\Omega + 16k\Omega)(0.23\mu F)\omega} V_i(\omega)$$

- Equating to the network function $\boldsymbol{H}(\omega) = \frac{1+j\frac{\omega}{z}}{1+j\frac{\omega}{p}}$ gives

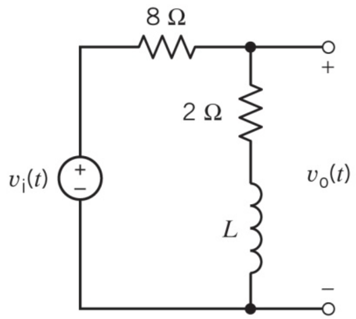
$$\boldsymbol{p = \frac{1}{(8k\Omega + 16k\Omega)(0.23\mu F)} = \frac{1}{\tau} = \frac{1}{0.00552} = 181.16\,\frac{rad}{s}}$$

$$\boldsymbol{z = \frac{1}{8k\Omega(0.23\mu F)} = \frac{8k\Omega + 16k\Omega}{8k\Omega} p = \frac{1}{0.00368} = 271.74\, rad/s}$$

Example 13.2-2 Network Function of a Circuit

- Consider the circuit shown in Figure 13.2-5a. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage $v_o(t)$ across the series connection of the inductor and the 2Ω resistor. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = 0.2\frac{1 + j\dfrac{\omega}{5}}{1 + j\dfrac{\omega}{25}}$$



(a)

- Determine the value of the inductance L.

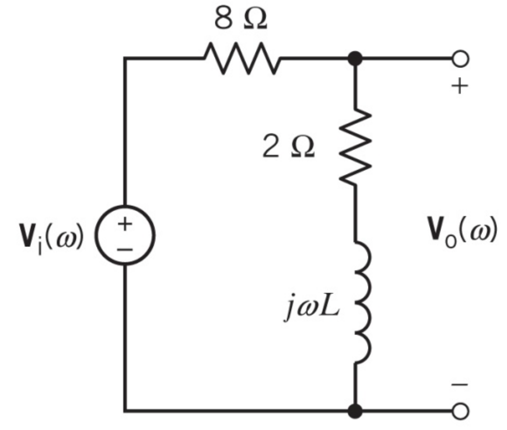
Solution

- The impedances of the inductor and the 2-Ω resistor are connected in series. The equivalent impedance is

$$Z_e(\omega) = 2 + j\omega L$$

- The equivalent impedance is connected in series with the 8-Ω resistor. $V_i(\omega)$ is the voltage across the series impedances, and $V_o(\omega)$ is the voltage across the equivalent impedance $Z_e(\omega)$. Apply the voltage division principle to get

$$\boldsymbol{V_o(\omega) = \frac{2\Omega + j\omega L}{8\Omega + 2\Omega + j\omega L} V_i(\omega)}$$



(b)

Solution

- Doing some algebra gives

$$\boldsymbol{H}(\omega) = \frac{1}{5}\ \frac{1 + j\omega\dfrac{L}{2\Omega}}{1 + j\omega\dfrac{L}{8\Omega + 2\Omega}}\ V_i(\omega)$$

- Where

$$p = \frac{L}{8\Omega + 2\Omega} = \frac{1}{\tau}, \qquad z = \frac{\mathrm{L}}{2\Omega} = \frac{8\Omega + 2\Omega}{2\Omega}p$$

- Equating to the network function $\boldsymbol{H}(\omega) = 0.2\,\frac{1+j\frac{\omega}{5}}{1+j\frac{\omega}{25}}$ gives

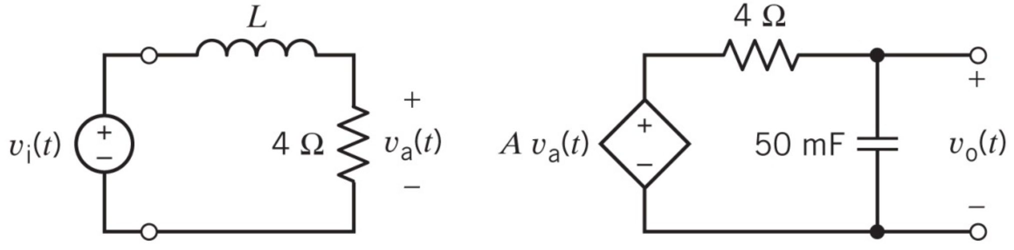
$$\frac{L}{2} = \frac{1}{5}, \qquad \frac{L}{10} = \frac{1}{25}$$

$$\boldsymbol{L = 0.4H}$$

Example 13.2-3 Network Function of a Circuit

- Consider the circuit shown in Figure 13.2-6. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage across the capacitor, $v_o(t)$. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_\mathrm{o}(\omega)}{\mathbf{V}_\mathrm{i}(\omega)} = \frac{3}{\left(1 + j\dfrac{\omega}{2}\right)\left(1 + j\dfrac{\omega}{5}\right)}$$



- Determine the value of the inductance L and of the gain A of the voltage-controlled voltage source (VCVS).

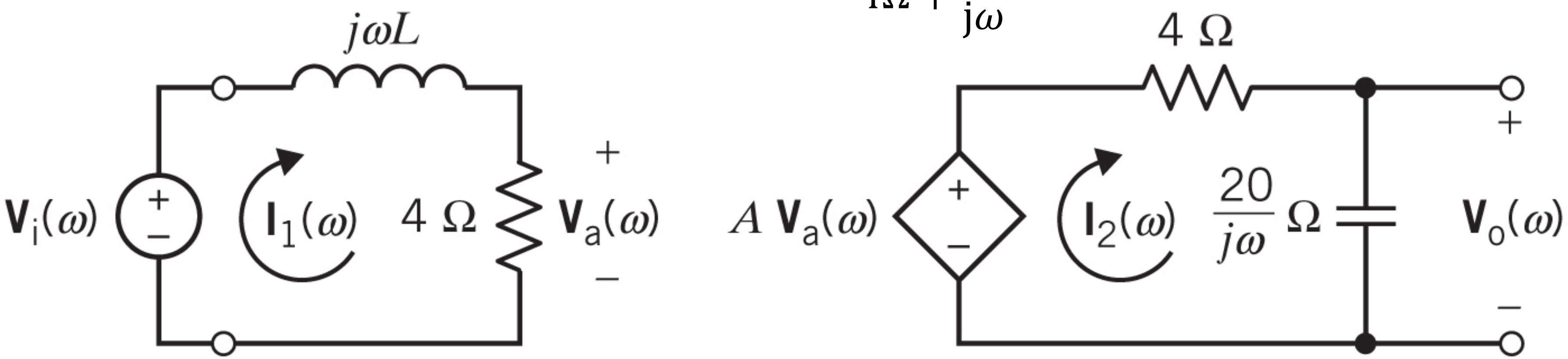
Solution

- The circuit consists of two meshes. In the left-hand mesh, the resistor and the inductor form the voltage divider.

$$V_a(\omega) = V_i(\omega)\frac{4\Omega}{j\omega L + 4\Omega}$$

- The resistor and the capacitor form the voltage divider in the right-hand mesh.

$$V_o(\omega) = AV_a(\omega)\frac{\frac{20}{j\omega}}{4\Omega + \frac{20}{j\omega}}$$



Solution

- Two meshes are cascaded. With some algebra,

$$\boldsymbol{V_o}(\omega) = \boldsymbol{V_i}(\omega)\, A \frac{4\Omega}{j\omega L + 4\Omega} \frac{\frac{20}{j\omega}}{4\Omega + \frac{20}{j\omega}} = \boldsymbol{V_i}(\omega) \frac{A}{1 + j\omega \frac{L}{4\Omega}} \frac{1}{1 + j\omega \frac{4\Omega}{20}}$$

- Where the poles are,

$$p_1 = \frac{4\Omega}{L} = \frac{1}{\tau_L}, \qquad p_2 = \frac{20}{4\Omega} = \frac{1}{\tau_C}$$

- Comparing the given network functions gives A = 3V/V and L=2H.

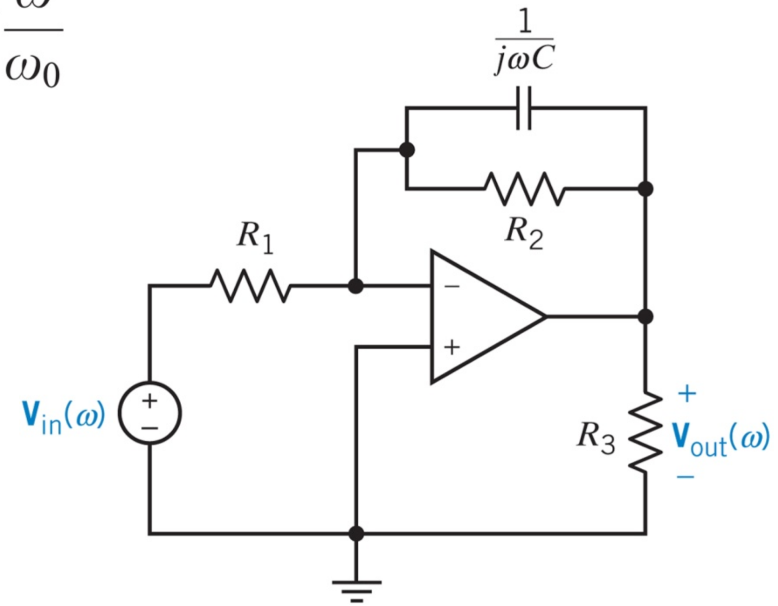
$$\frac{3}{\left(1 + j\frac{\omega}{2}\right)\left(1 + j\frac{\omega}{5}\right)} = \boldsymbol{H}(\omega) = \frac{A}{1 + j\omega \frac{L}{4\Omega}} \frac{1}{1 + j\omega \frac{4\Omega}{20}}$$

Low-pass filter whose network function is $\boldsymbol{H}(\omega) = \frac{H_0}{1+j\frac{\omega}{\omega_0}}$

- First-order low-pass filters have network functions of the form

$$\mathbf{H}(\omega) = \frac{H_0}{1 + j\dfrac{\omega}{\omega_0}}$$

- The gain and phase shift of the filter are

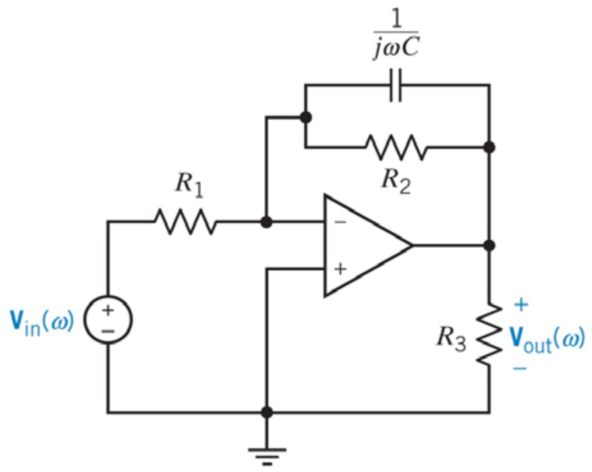


$$\text{gain} = \frac{|H_0|}{\sqrt{1 + \dfrac{\omega^2}{\omega_0^2}}}$$

$$\text{phase shift} = \angle H_0 - \tan^{-1}(\omega/\omega_0)$$

- $|H_0|$ is called the dc gain, and ω_0 is called the corner frequency, the 3-dB frequency, or the half-power frequency

Low-pass filter whose network function is $\boldsymbol{H}(\omega) = \frac{H_0}{1+j\frac{\omega}{\omega_0}}$



- H_0 is called the DC gain as $\boldsymbol{H}(0) = H_0$
- ω_0 is called the half power frequency.
  - At low frequencies, power delivered to R_3 is,

$$P = \frac{1}{2}\frac{|v_{out}(\omega)|^2}{R_3} = \frac{1}{2}\frac{|H_0 v_{in}(\omega)\,|^2}{R_3} = \frac{|\boldsymbol{H_0}|^2}{\boldsymbol{2}}\frac{\boldsymbol{v_{in}(\omega)^2}}{\boldsymbol{R_3}}$$

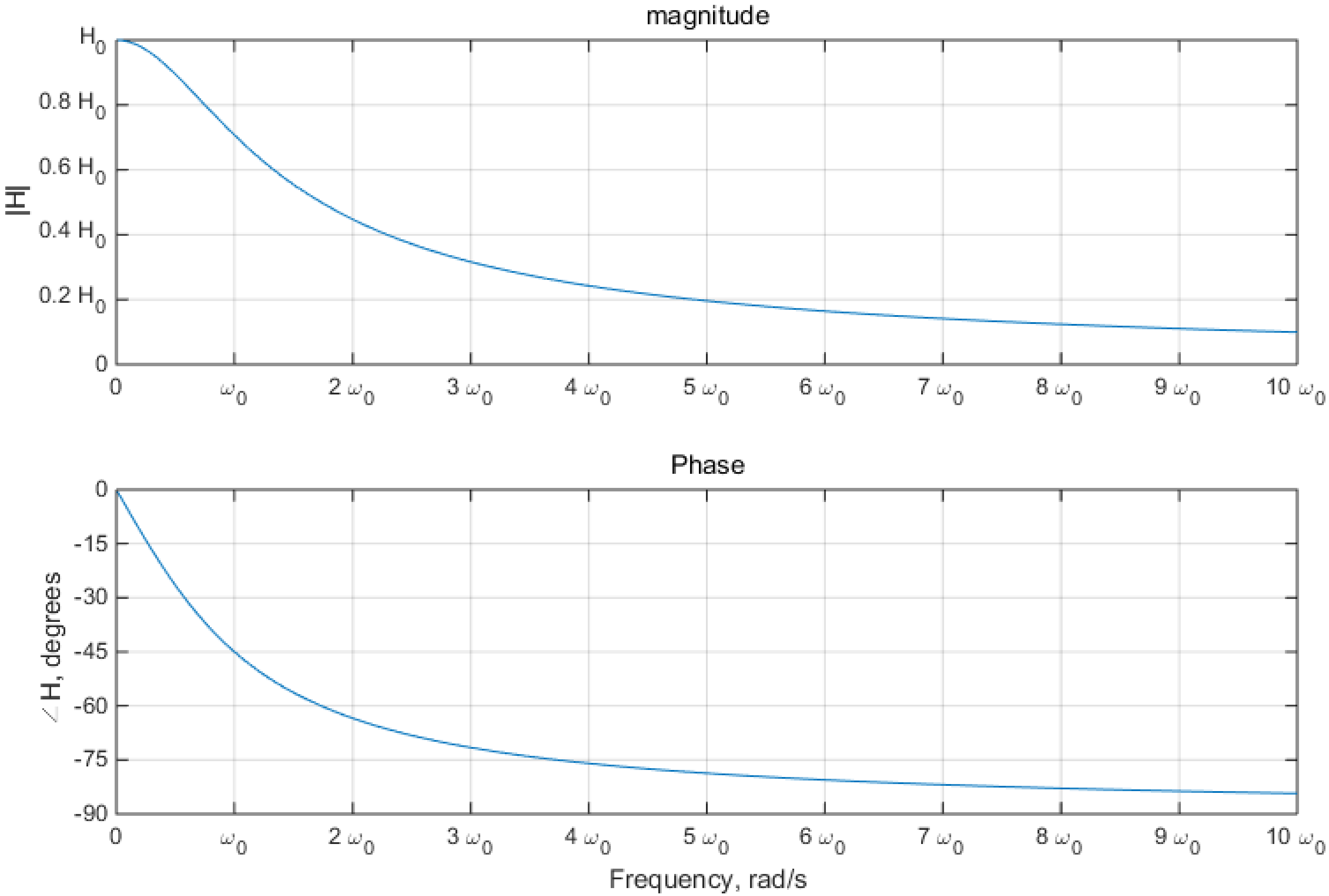
  - When $\omega = \omega_0$, power delivered to R_3 is,

$$P = \frac{1}{2}\frac{|v_{out}(\omega)|^2}{R_3} = \frac{1}{2}\frac{[H(\omega_0)v_{in}(\omega_0)\,]^2}{R_3} = \frac{|\boldsymbol{H_0}|^2}{\boldsymbol{4}}\frac{\boldsymbol{v_{in}(\omega_0)^2}}{\boldsymbol{R_3}}$$

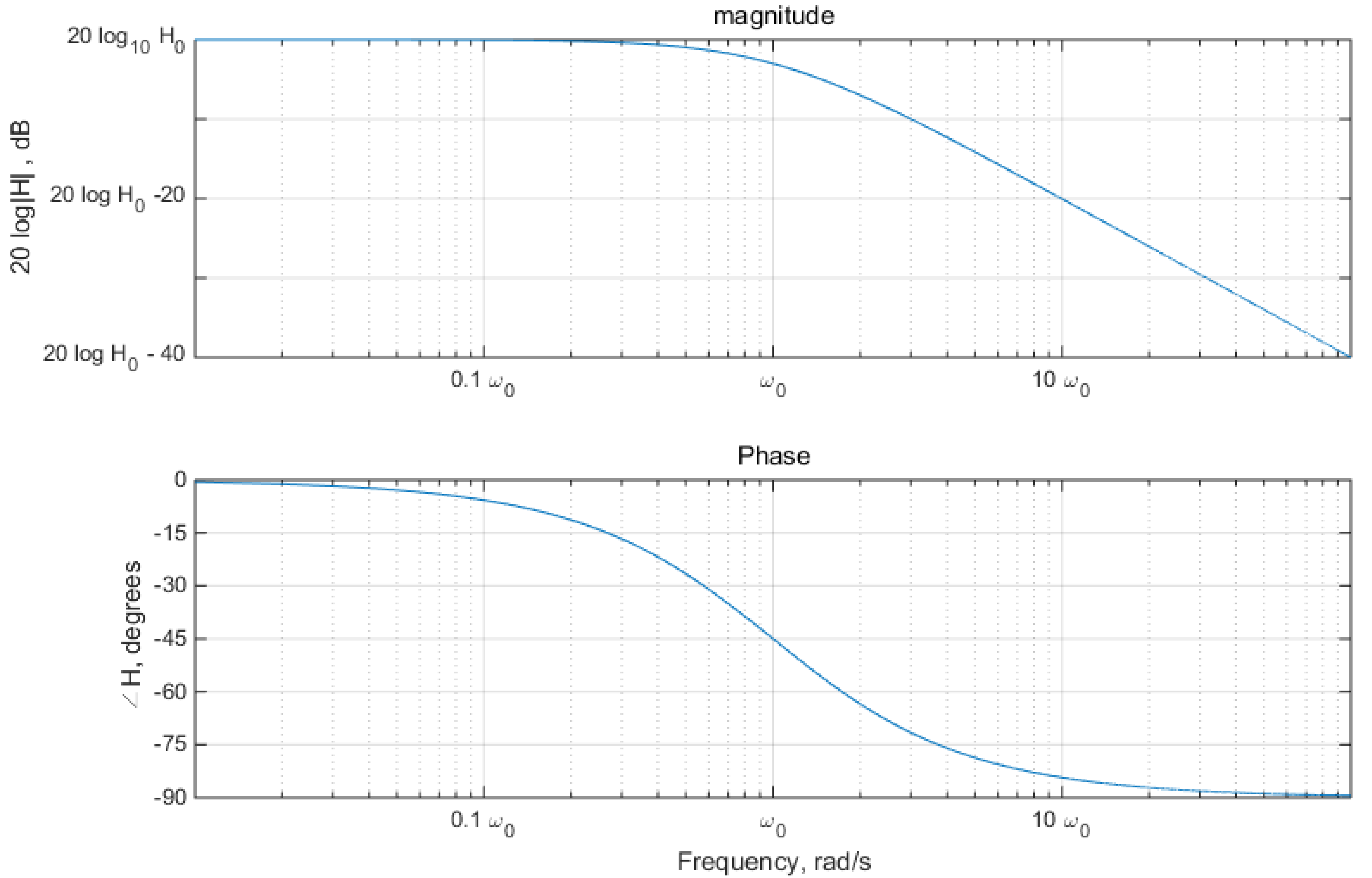
Low-pass filter whose network function is $\boldsymbol{H}(\omega) = \frac{H_0}{1+j\frac{\omega}{\omega_0}}$

- The gain and phase of the network function changes as frequency changes.
  - When $\omega \ll \omega_0$, $1 + j\frac{\omega}{\omega_0} \approx 1$, $\quad \boldsymbol{gain} = |\boldsymbol{H_0}|$, $\quad$ **phase = 0°**
  - When $\omega = \omega_0$, $1 + j\frac{\omega}{\omega_0} = 1 + j$, $\quad \boldsymbol{gain} = \frac{|\boldsymbol{H_0}|}{\sqrt{\boldsymbol{2}}}$, $\quad$ **phase = −45°**
  - When $\omega \gg \omega_0$, $1 + j\frac{\omega}{\omega_0} \approx j\frac{\omega}{\omega_0}$, $\quad \boldsymbol{gain} = \frac{\boldsymbol{\omega_0}}{\boldsymbol{\omega}}|\boldsymbol{H_0}|$, $\quad$ **phase = −90°**

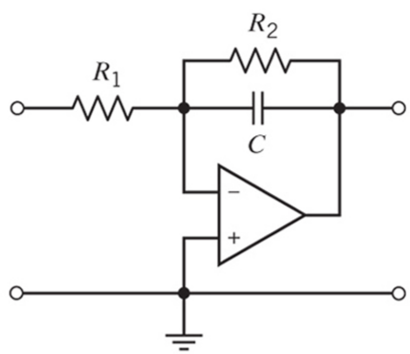
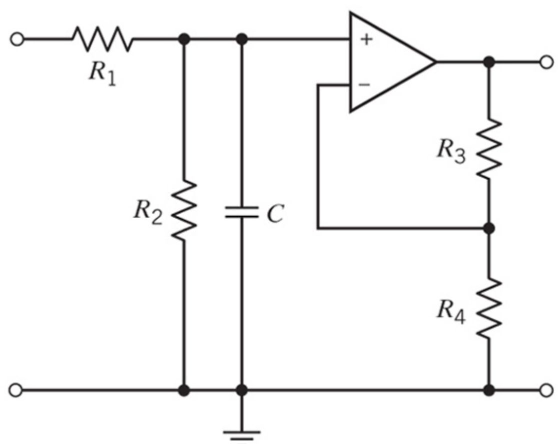
Low-pass filter whose network function is $\boldsymbol{H}(\omega) = \frac{H_0}{1+j\frac{\omega}{\omega_0}}$



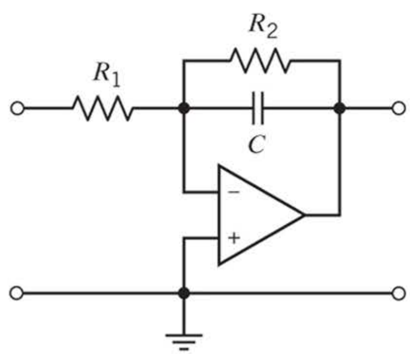
Low-pass filter whose network function is $\boldsymbol{H}(\omega) = \frac{H_0}{1+j\frac{\omega}{\omega_0}}$



Low-pass filter circuits

PHASE SHIFT	FIRST-ORDER LOW-PASS FILTER CIRCUIT	DESIGN EQUATIONS
$90^\circ \leq$ phase shift $\leq 180^\circ$		$H_0 = -\frac{R_2}{R_1}$ $\omega_0 = \frac{1}{R_2 C}$
$-90^\circ \leq$ phase shift $\leq 0^\circ$		$H_0 = \frac{R_2}{R_1 + R_2}\left(1 + \frac{R_3}{R_4}\right)$ $\omega_0 = \frac{R_1 + R_2}{R_1 R_2 C}$

Low-pass filter circuits

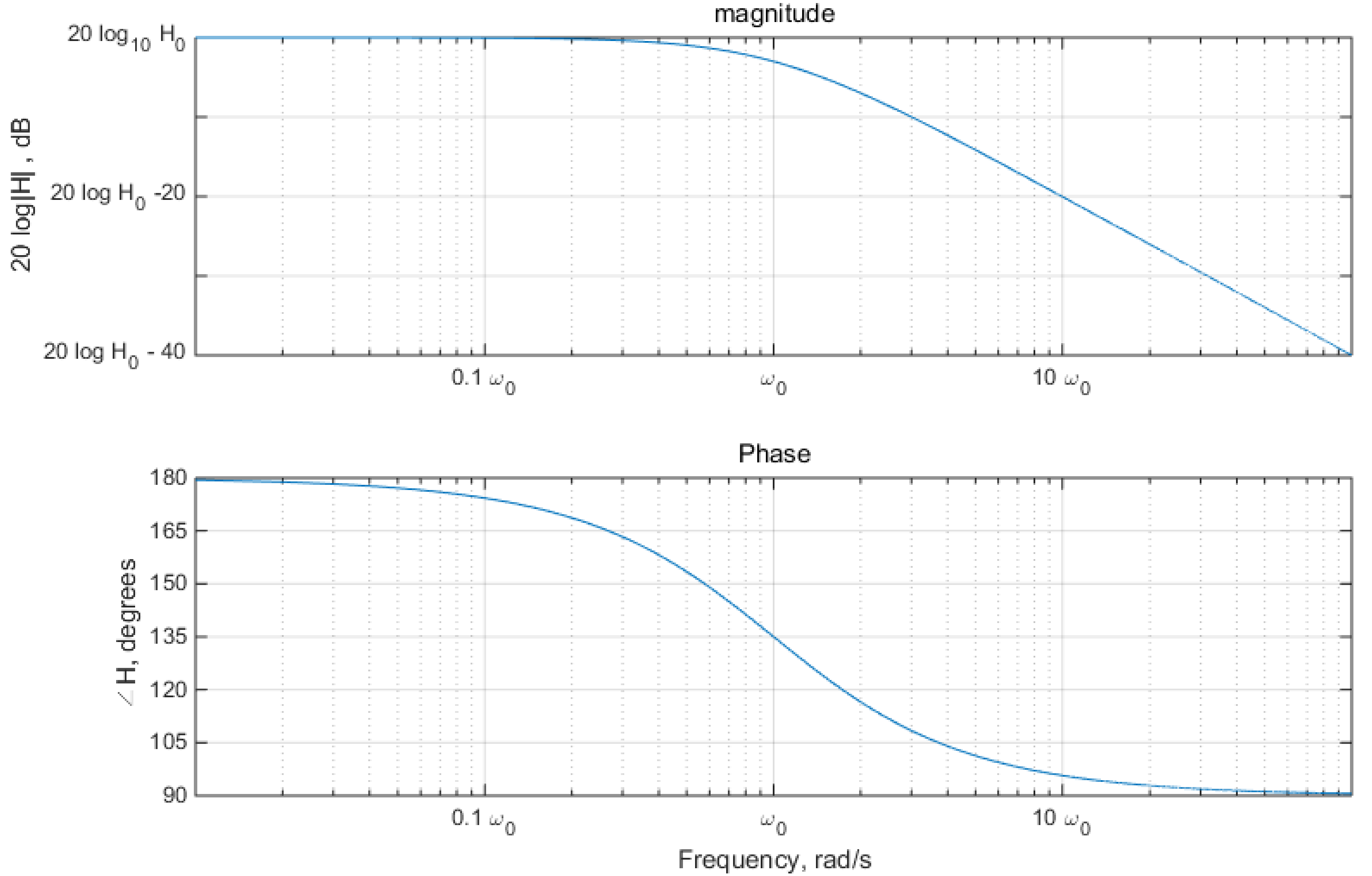
PHASE SHIFT	FIRST-ORDER LOW-PASS FILTER CIRCUIT	DESIGN EQUATIONS
$90^\circ \le$ phase shift $\le 180^\circ$		$H_0 = -\frac{R_2}{R_1}$ $\omega_0 = \frac{1}{R_2 C}$

$$\boldsymbol{V}_- = \boldsymbol{V}_+ = 0$$

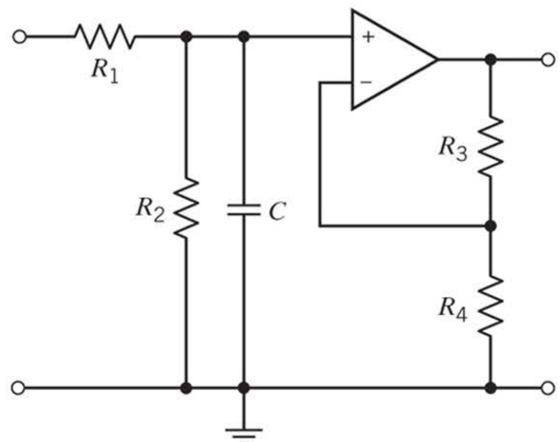
$$\frac{\boldsymbol{V}_{\boldsymbol{in}}}{R_1} = -\frac{\boldsymbol{V}_{\boldsymbol{out}}}{R_2 || \frac{1}{j\omega C}} = -\left(\frac{1}{R_2} + j\omega C\right)\boldsymbol{V}_{\boldsymbol{out}} = -\frac{1 + j\omega R_2 C}{R_2}\boldsymbol{V}_{\boldsymbol{out}}$$

$$\boldsymbol{H}(\omega) = \frac{\boldsymbol{V}_{\boldsymbol{out}}}{\boldsymbol{V}_{\boldsymbol{in}}} = -\frac{R_2}{R_1}\frac{1}{1 + j\omega R_2 C}$$

Low-pass filter circuits



Low-pass filter circuits

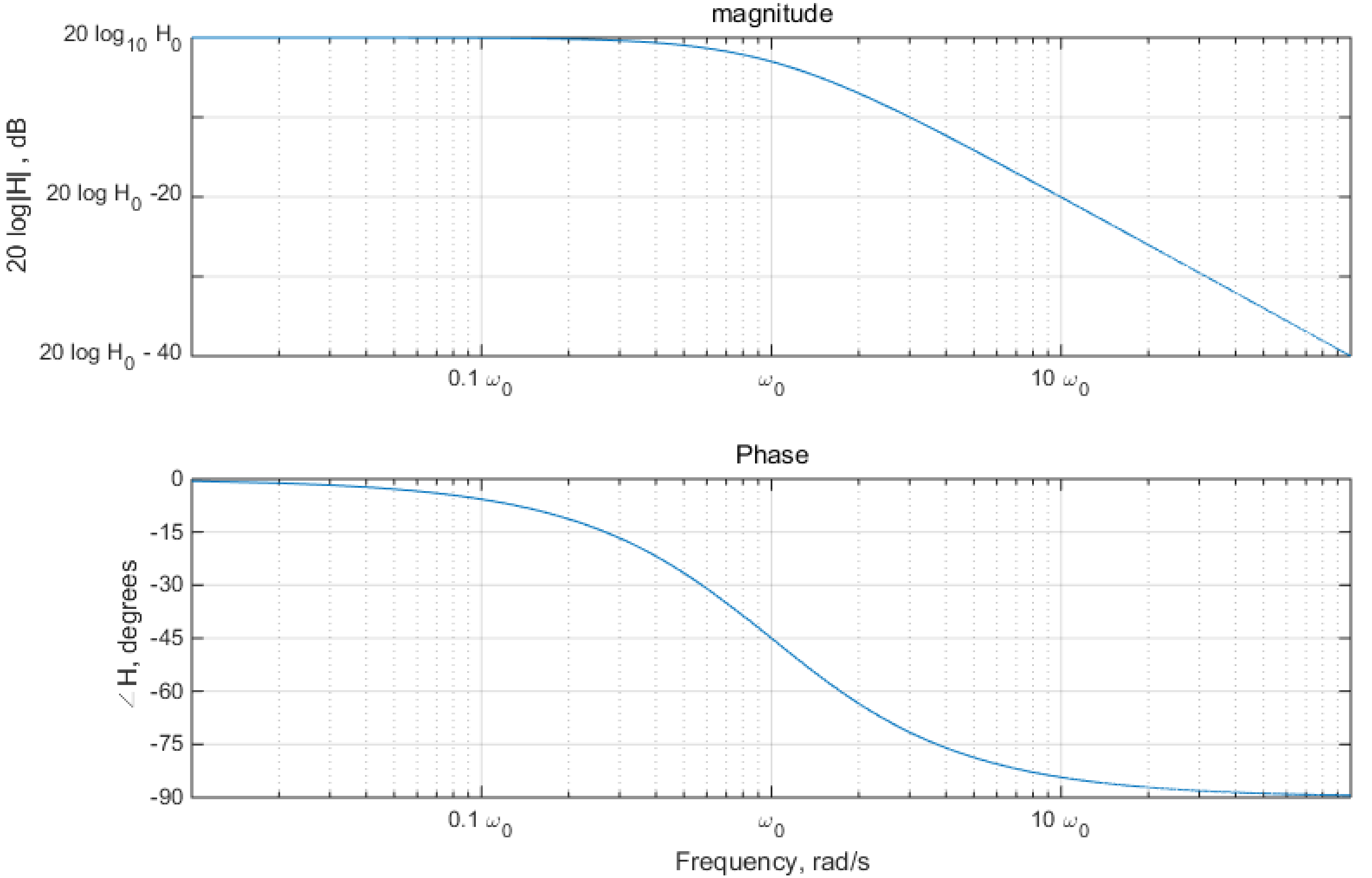
PHASE SHIFT	FIRST-ORDER LOW-PASS FILTER CIRCUIT	DESIGN EQUATIONS
$-90° \leq$ phase shift $\leq 0°$		$H_0 = \frac{R_2}{R_1 + R_2}\left(1 + \frac{R_3}{R_4}\right)$ $\omega_0 = \frac{R_1 + R_2}{R_1 R_2 C}$

$$\frac{\boldsymbol{V_+}}{\boldsymbol{V_{in}}} = \frac{R_2 \| \frac{1}{j\omega C}}{R_2 \| \frac{1}{j\omega C} + R_1} = \frac{\frac{R_2}{1 + j\omega C R_2}}{\frac{R_2}{1 + j\omega C R_2} + R_1} = \frac{R_2}{R_1 + R_2 + j\omega C R_1 R_2} = \frac{R_2}{R_1 + R_2} \frac{1}{1 + \frac{j\omega R_1 R_2}{R_1 + R_2}}$$

$$\frac{\boldsymbol{V_{out}}}{\boldsymbol{V_+}} = \frac{\boldsymbol{V_{out}}}{\boldsymbol{V_-}} = \frac{R_3 + R_4}{R_4} = 1 + \frac{R_3}{R_4}$$

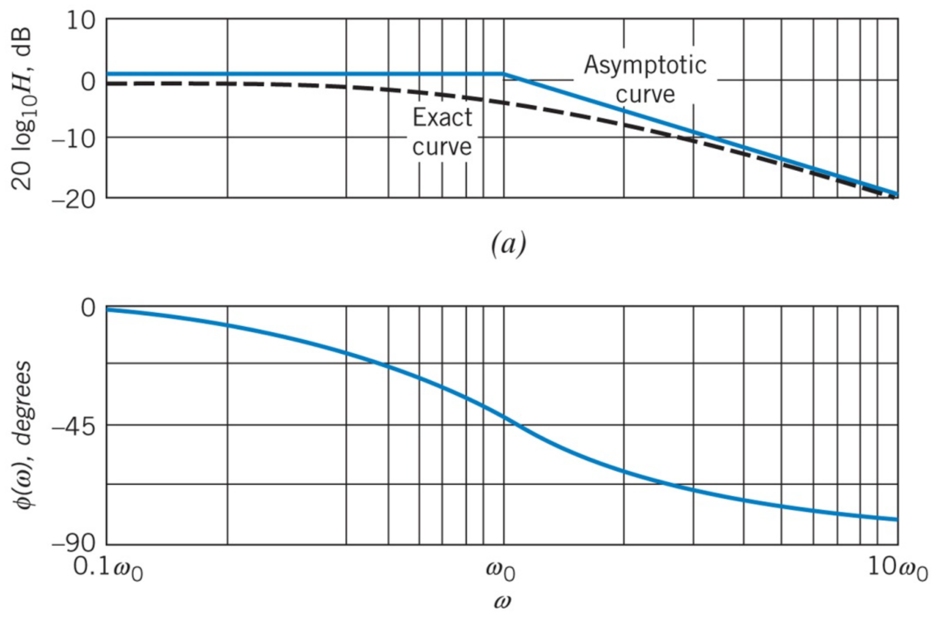
$$\boldsymbol{H(\omega)} = \frac{\boldsymbol{V_{out}}}{\boldsymbol{V_{in}}} = \frac{\boldsymbol{V_{out}}}{\boldsymbol{V_+}} \frac{\boldsymbol{V_-}}{\boldsymbol{V_{in}}} = \frac{R_2}{R_1 + R_2}\left(1 + \frac{R_3}{R_4}\right) \frac{1}{1 + j\omega \frac{R_1 R_2 C}{R_1 + R_2}}$$

Low-pass filter circuits



Bode plot

- A chart of gain in decibels and phase in degrees versus the logarithm of frequency.
  - The use of logarithms expands the range of frequencies portrayed on the horizontal axis.



Bode plot when network function $\boldsymbol{H} = H\angle\Phi = He^{j\Phi}$

- Logarithmic gain
  - $\text{logarithmic gain} = 20\log_{10} \mathrm{H}$
  - The unit is decibel (dB)
  - Also called gain in dB
- Phase
  - $\text{phase} = \Phi$
  - The unit is degrees (°)

MAGNITUDE, *H*	20 log *H* (dB)
0.1	−20.00
0.2	−13.98
0.4	−7.96
0.6	−4.44
1.0	0.0
1.2	1.58
1.4	2.92
1.6	4.08
2.0	6.02
3.0	9.54
4.0	12.04
5.0	13.98
6.0	15.56
7.0	16.90
10.0	20.00
100.0	40.00

Bode plot when network function $\boldsymbol{H}(\omega) = \frac{1}{1+j\frac{\omega}{\omega_0}}$

- The logarithmic gain is

$$\mathbf{20\ log_{10}H = 20\ log_{10}\frac{1}{\sqrt{1+(\omega/\omega_0\ )^2}} = -20\ log_{10}\sqrt{1+(\omega/\omega_0)^2}}$$

  - For small frequencies, $\omega \ll \omega_0,\ 1+\left(\frac{\omega}{\omega_0}\right)^2 \cong 1$

$$20\ log_{10}H = -20\ log_{10}\sqrt{1} = \mathbf{0\ dB}$$

  - For large frequencies, $\omega \gg \omega_0,\ 1+\left(\frac{\omega}{\omega_0}\right)^2 \cong \left(\frac{\omega}{\omega_0}\right)^2$

$$20\ log_{10}H = -20\ log_{10}\sqrt{(\omega/\omega_0)^2}$$

$$= -20\ log_{10}(\omega/\omega_0)$$

$$\mathbf{= 20\log_{10}\omega_0 - 20\log_{10}\omega\ \ dB}$$

Bode plot when network function $\boldsymbol{H}(\omega) = \frac{1}{1+j\frac{\omega}{\omega_0}}$

- The value of the slope of the low-frequency is **0 dB/decade**, because

$$20\, log_{10}|H(\omega)| = 20 \log_{10} 1$$

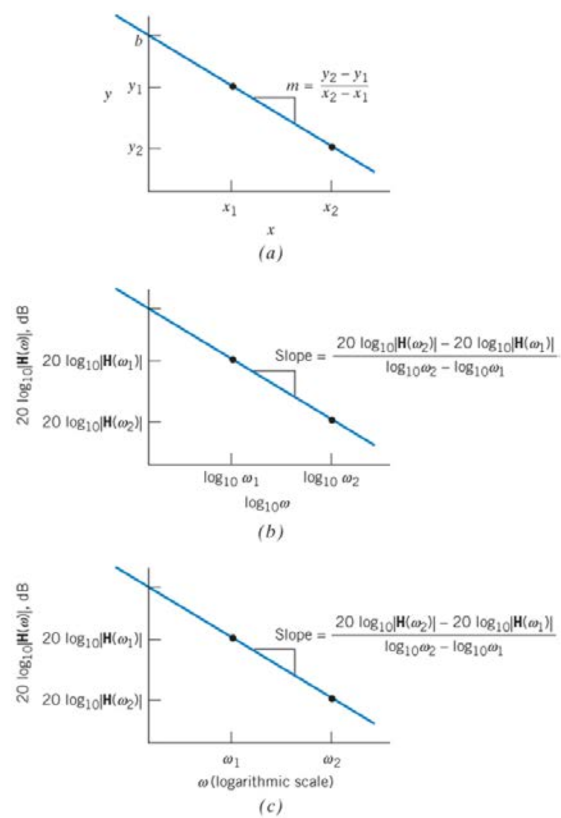
- The value of the slope of the high-frequency is

$$\frac{20\, log_{10}|H(\omega_2)| - 20\, log_{10}|H(\omega_1)|}{log_{10}\omega_2 - log_{10}\omega_1}$$

$$= \frac{20\, log_{10}\,(\omega_0/\omega_2)\; - 20\, log_{10}(\omega_0/\omega_1)}{log_{10}\omega_2 - log_{10}\omega_1}$$

$$= \frac{-20\, log_{10}\,(\omega_2/\omega_1)}{log_{10}(\omega_2/\omega_1)}$$

$$= \boldsymbol{-20\, dB/decade}$$



Bode plot when network function $\boldsymbol{H}(\omega) = \frac{1}{1+j\frac{\omega}{\omega_0}}$

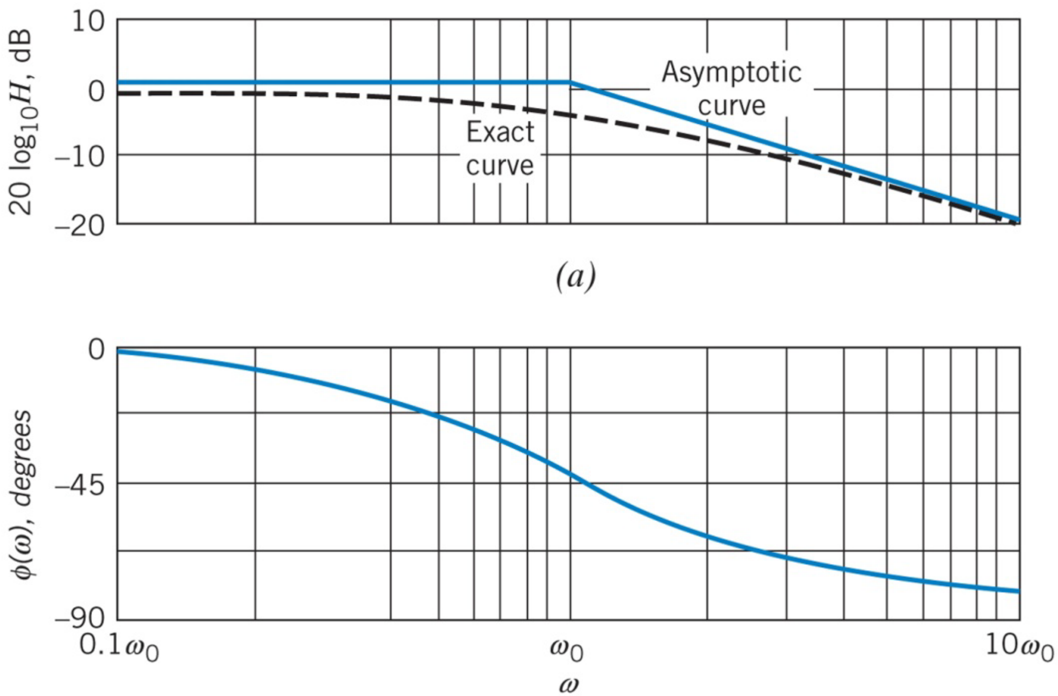
- Asymptotic Bode plot
  - Low-frequency asymptote for $\omega < \omega_0$, and high-frequency asymptote for $\omega > \omega_0$.
  - These asymptotes are good approximation to the Bode plot when $\omega \ll \omega_0$ or $\omega \gg \omega_0$. Near $\omega = \omega_0$, the asymptotic Bode plot deviates from the exact Bode plot.
- **Corner frequency**
  - **At $\boldsymbol{\omega = \omega_0}$**, the value of the asymptotic Bode plot is 0 dB, whereas the value of the exact Bode plot is

$$\mathbf{20}\ \boldsymbol{log_{10}}|\boldsymbol{H}(\boldsymbol{\omega_0})| = \mathbf{20}\ \boldsymbol{log_{10}}\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = -\mathbf{3.01}\boldsymbol{dB}$$

  - The low- and high-frequency asymptotes form a corner where they intersect. Because the asymptotes intersect at frequency $\omega = \omega_0$, ω_0 is called the **corner frequency. (same name as 3-dB frequency or half-power frequency)**

Bode plot when network function $\boldsymbol{H}(\omega) = \frac{1}{1+j\frac{\omega}{\omega_0}}$

$$\mathbf{H} = \frac{1}{1 + j\dfrac{\omega}{\omega_0}} = \frac{1}{\sqrt{1 + (\omega/\omega_0)}} \angle \tan^{-1}(\omega/\omega_0) = H \angle \phi$$

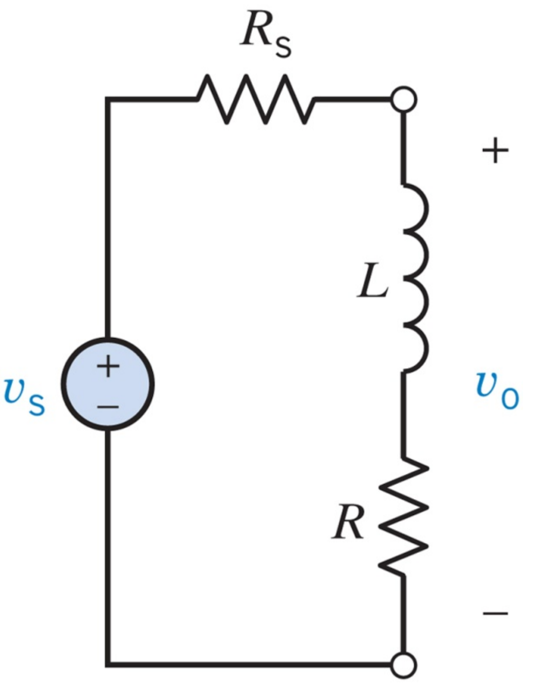


Bode plot when network function $\boldsymbol{H}(\omega) = k\dfrac{1+j\frac{\omega}{\omega_1}}{1+j\frac{\omega}{\omega_2}}$

- Bode plots of the circuit consisting of voltage source, L, and R.

- The network function of this circuit is

$$\boldsymbol{H} = \frac{\boldsymbol{V_o}}{\boldsymbol{V_s}} = \frac{\boldsymbol{R + j\omega L}}{\boldsymbol{R_s + R + j\omega L}}$$



- Put this network function into the form

$$\boldsymbol{H = k\frac{1 + j\dfrac{\omega}{\omega_1}}{1 + j\dfrac{\omega}{\omega_2}}}$$

  - k: dc gain, ω_1: zero, ω_2: pole

Bode plot when network function $\boldsymbol{H}(\omega) = k\frac{1+j\frac{\omega}{\omega_1}}{1+j\frac{\omega}{\omega_2}}$

- The network function can be expressed as $\boldsymbol{H} = \left(\frac{R}{R+R_s}\right)\frac{1+j\frac{\omega}{R/L}}{1+j\frac{\omega}{(R+R_s)/L}}$
  - dc gain $k = \frac{R}{R+R_s}$
  - zero and pole frequencies are related by $\omega_1 = \frac{R}{L} < \frac{R+R_s}{L} = \omega_2$
- The gain corresponding to a network function of this form is,

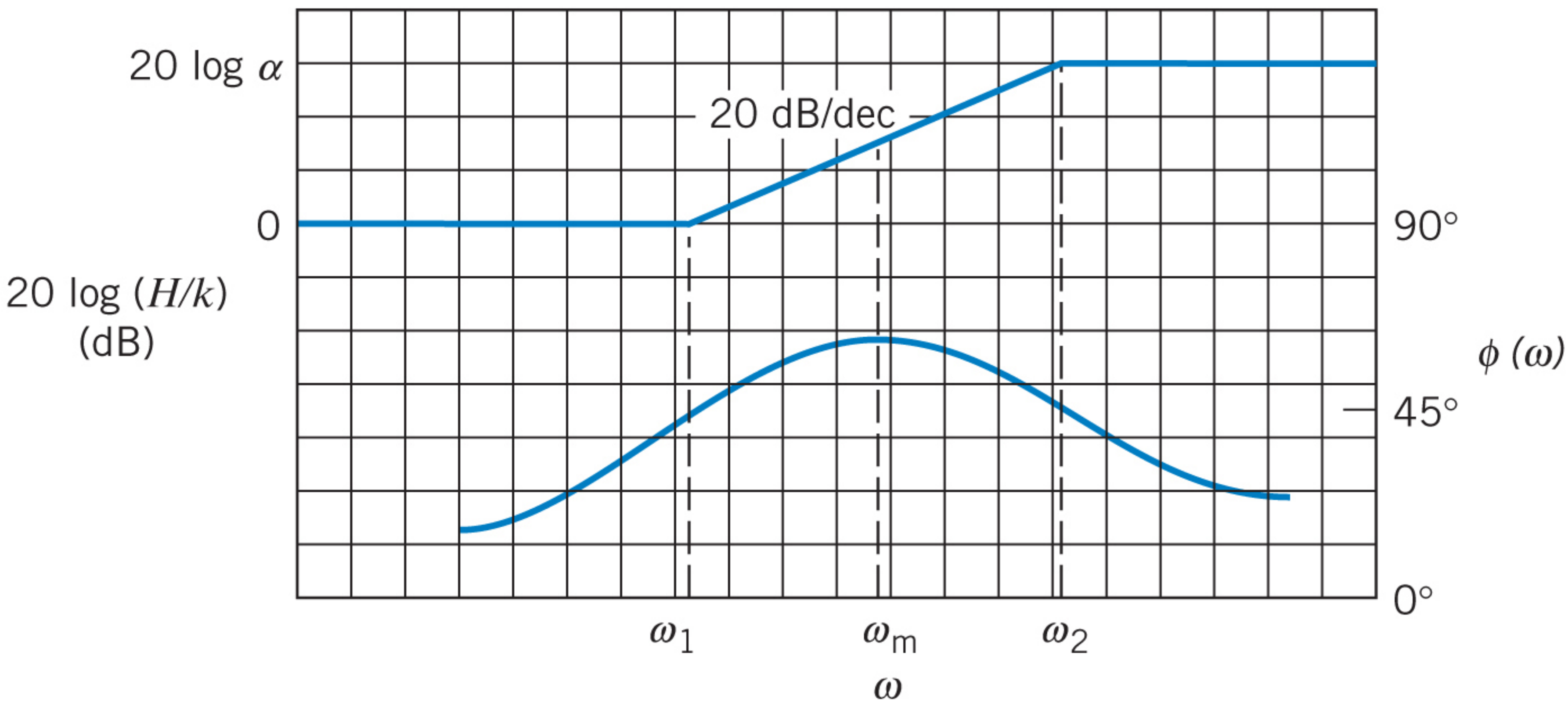
$$H = k\frac{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}{\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2}} \approx \begin{cases} k & \omega < \omega_1 \\ \frac{k\omega}{\omega_1} & \omega_1 < \omega < \omega_2 \\ \frac{k\omega_2}{\omega_1} & \omega_2 < \omega \end{cases}$$

- The phase angle of **H** is

$$\Phi = \angle k + \angle\left(1+j\frac{\omega}{\omega_1}\right) - \angle\left(1+j\frac{\omega}{\omega_2}\right) = \tan^{-1}\frac{\omega}{\omega_1} - \tan^{-1}\frac{\omega}{\omega_2}$$

Bode plot when network function $\boldsymbol{H}(\omega) = k\frac{1+j\frac{\omega}{\omega_1}}{1+j\frac{\omega}{\omega_2}}$

- The phase Bode plot and the asymptotic magnitude Bode plot



Example 13.3-1 Bode Plot

- Find the asymptotic magnitude Bode plot of

$$\boldsymbol{H}(\omega) = K\frac{j\omega}{1 + j\dfrac{\omega}{p}}$$

Solution

- Network function $\boldsymbol{H}(\omega) = K \frac{J\omega}{1+j\frac{\omega}{p}}$
- Approximate $\left(1 + j\frac{\omega}{p}\right)$ by 1 when $\omega < p$, and by $j\frac{\omega}{p}$ when $\omega > p$ to get

$$\boldsymbol{H}(\omega) \cong \begin{cases} K \cdot j\omega & \omega < p \\ K \cdot p & \omega > p \end{cases}$$

- The logarithmic gain is

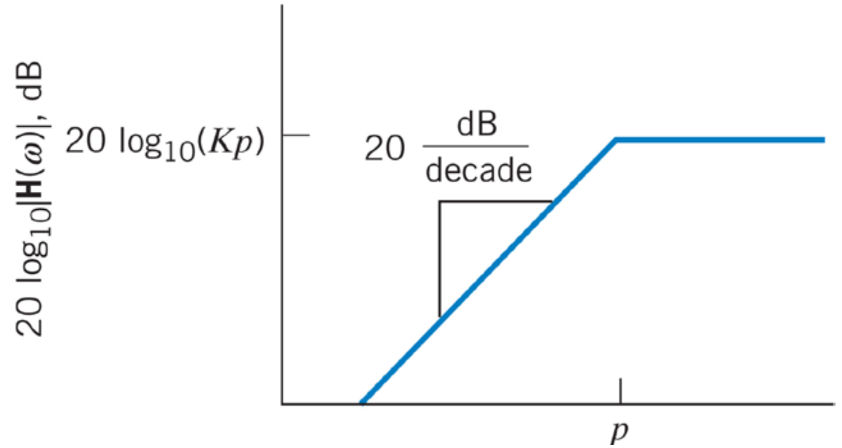
$$20\log_{10}|\boldsymbol{H}(\omega)| \cong \begin{cases} 20\log_{10} K + 20\log_{10}\omega & \omega < p \\ 20\log_{10} Kp & \omega > p \end{cases}$$

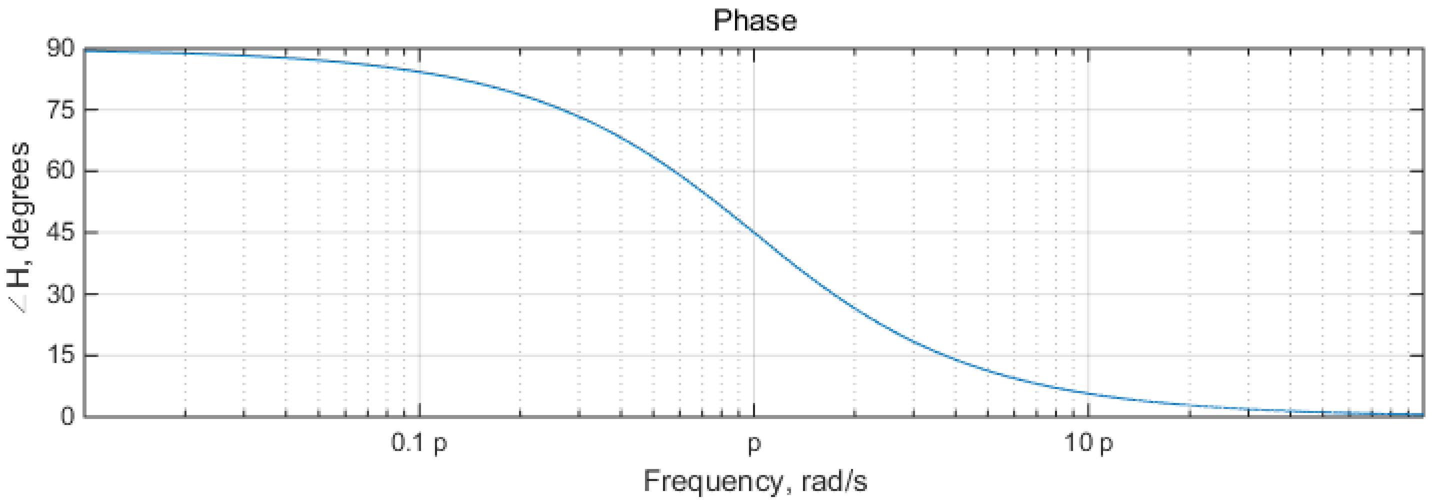
- The $j\omega$ factor in the numerator of $\boldsymbol{H}(\omega)$ causes the low-frequency asymptote to have a slope of 20 dB/decade. The slope of the asymptotic magnitude Bode plot decrease by 20 dB/decade (from 20 dB/decade to zero) as the frequency increases past $\omega = p$.
- The phase is

$$\angle\boldsymbol{H}(\omega) \cong \begin{cases} 90^{\circ} & \omega \ll p \\ 45^{\circ} & \omega = p \\ 0^{\circ} & \omega \gg p \end{cases}$$

Solution

- The asymptotic magnitude bode plot and the phase bode plot is,



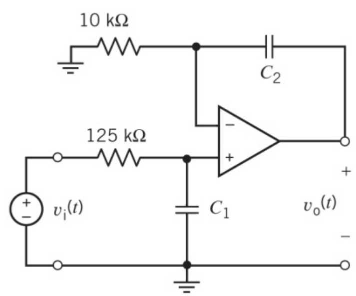


Example 13.3-2 Bode Plot of a Circuit

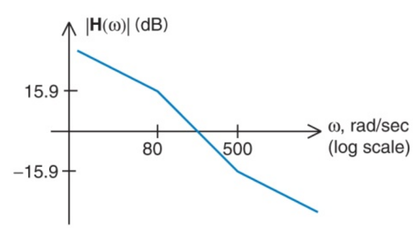
- Consider the circuit shown in Figure 13.3-6a. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the node voltage at the output terminal of the op amp $v_o(t)$. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

- The corresponding magnitude Bode plot is shown in Figure 13.3-6b. Determine the values of the capacitances C_1 and C_2.



(a)



(b)

Solution

- Step 1: Finding the network function corresponding to the Bode plot
  - Two corner frequencies, at 80 and 500 rad/s. The corner frequency at 80 rad/s is a pole because the slope of the Bode plot decreases at 80 rad/s. The corner frequency at 500 rad/s is a zero because the slope increases at 500 rad/s
  - The corner frequencies are separated by $\log_{10}\left(\frac{500}{80}\right) = 0.796$ decades. The slope of the Bode plot is $\frac{-15.9-15.9}{0.796} = -40$ dB/decade between the corner frequencies.
  - At low frequencies – that is, at frequencies smaller than the smallest corner frequency – the slope is -1 x 20 dB/decade, so the network function includes a factor $(j\omega)^{-1}$

- Consequently, the network function corresponding to the Bode plot is

$$\boldsymbol{H}(\omega) = k(j\omega)^{-1}\left(\frac{1 + j\dfrac{\omega}{500}}{1 + j\dfrac{\omega}{80}}\right) = k\frac{1 + j\dfrac{\omega}{500}}{j\omega\left(1 + j\dfrac{\omega}{80}\right)}$$

- Where k is a constant that is yet to be determined

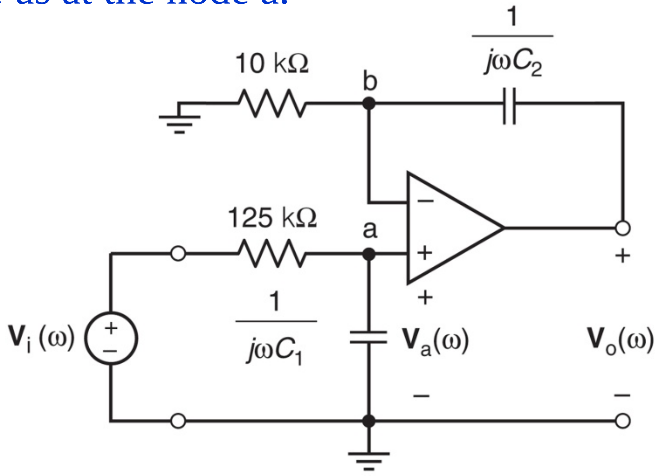
Solution

- Step 2: Analyzing the circuit to determine its network function.
  - We first analyze the node labeled as node a. The current entering the noninverting input of the op amp is zero, so the two currents in this node equation, the currents in the impedances corresponding to 125-kΩ resistor and capacitor C_1, have same magnitude.

$$\frac{V_a(\omega)}{V_i(\omega)} = \frac{\frac{1}{j\omega C_1}}{125\text{k}\Omega + \frac{1}{j\omega C_1}} = \frac{1}{1 + \text{j}\omega C_1(125k\Omega)}$$

  - Next, we analyze the node labeled as node b. The currents in the impedances corresponding to 10-kΩ resistor and capacitor C_2, have same magnitude as at the node a.

$$\frac{V_b(\omega)}{V_o(\omega)} = \frac{10k\Omega}{10\text{k}\Omega + \frac{1}{j\omega C_2}} = \frac{\text{j}\omega C_2(10k\Omega)}{1 + \text{j}\omega C_2(10k\Omega)}$$



  - Equating $V_b(\omega)$ *and* $V_a(\omega)$ and gives,

$$\frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 + \text{j}\omega C_1(125k\Omega)} \frac{1 + \text{j}\omega C_2(10k\Omega)}{\text{j}\omega C_2(10k\Omega)}$$

Solution

- Step 3: The network functions given in Step 1 and Step 2 must be equal.

$$k\frac{1+j\frac{\omega}{500}}{j\omega\left(1+j\frac{\omega}{80}\right)}=\boldsymbol{H}(\omega)=\frac{1}{1+\mathrm{j}\omega C_1(125k\Omega)}\;\frac{1+\mathrm{j}\omega C_2(10k\Omega)}{\mathrm{j}\omega C_2(10k\Omega)}$$

  - Equating coefficients gives

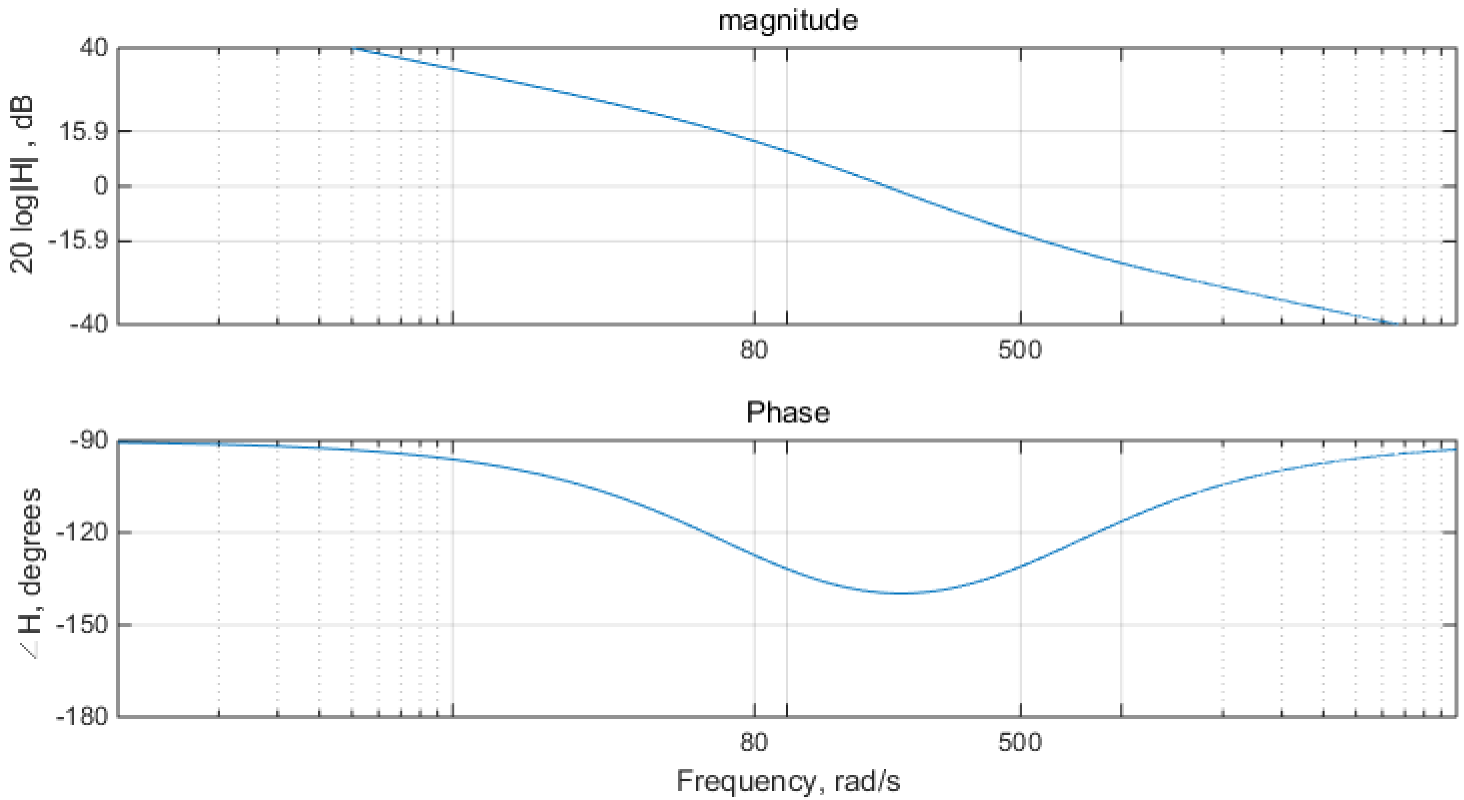
$$\frac{1}{80}=C_1(125k\Omega),\qquad \frac{1}{500}=C_2(10k\Omega),\qquad and\ k=\frac{1}{C_2(10k\Omega)}=500$$

  - so

$$C_1=\frac{1}{80*125*10^3}=0.1\mu F\ and\ C_2=\frac{1}{500(10*10^3)}=0.2\mu F$$

Solution

- Bode plot of $\boldsymbol{H}(\omega) = 500\dfrac{1 + j\dfrac{\omega}{500}}{j\omega\left(1 + j\dfrac{\omega}{80}\right)}$

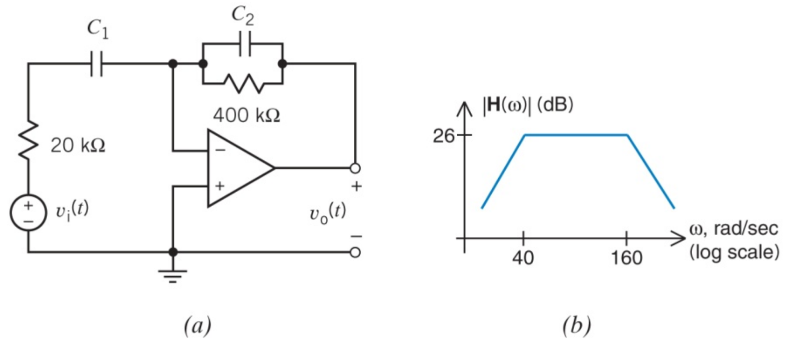


Example 13.3-3 Bode Plot of a Circuit

- Consider the circuit shown in Figure 13.3-8a. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the node voltage at the output terminal of the op amp $v_o(t)$. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_\mathrm{o}(\omega)}{\mathbf{V}_\mathrm{i}(\omega)}$$

- The corresponding magnitude Bode plot is shown in Figure 13.3-8b. Determine the values of the capacitances C_1 and C_2.



Solution

- Step 1: Finding the network function corresponding to the Bode plot
  - Two corner frequencies, at 40 and 160 rad/s. Both corner frequencies are poles because the slope of the Bode plot decreases at both the corner frequencies.
  - Between the corner frequencies, the gain is $|H(\omega)| = 26\,dB = 10^{26/20} = 20V/V.$

    $|H(\omega)| \cong \dfrac{k\,(j\omega)}{j\dfrac{\omega}{40}} = 40k = 20,$ so |k| = 0.5

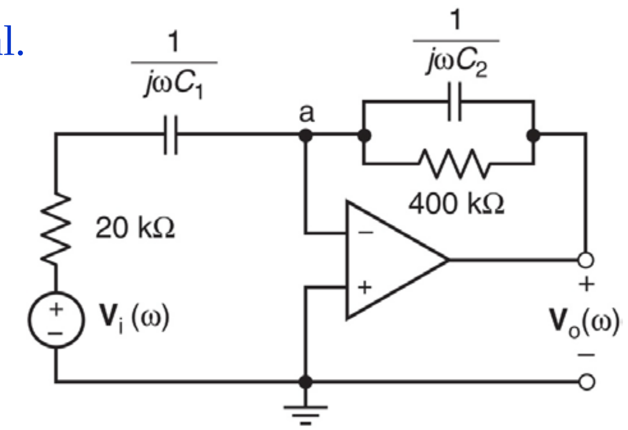
  - At low frequencies – that is, at frequencies smaller than the smallest corner frequency – the slope is 1 x 20 dB/decade, so the network function includes a factor $(j\omega)^1$

- Consequently, the network function corresponding to the Bode plot is

$$\boldsymbol{H}(\omega) = \frac{\pm 0.5(j\omega)}{\left(1 + j\dfrac{\omega}{40}\right)\left(1 + j\dfrac{\omega}{160}\right)}$$

Solution

- Step 2: Analyzing the circuit to determine its network function.
  - We will write a node equation at the node labeled as node a. In doing so, we will treat the series impedance, 20kΩ and $\frac{1}{j\omega C_1}$, as a single equivalent impedance equal to $20k\Omega + \frac{1}{j\omega C_1}$. Also, we will treat the parallel impedance, 400kΩ and $\frac{1}{j\omega C_2}$, as a single equivalent impedance equal to $\frac{1}{j\omega C_2 + \frac{1}{400k\Omega}}$. The node voltage at node a is zero volts because the voltages at the input nodes of an ideal op amp are equal.



$$\frac{V_i(\omega)}{20k\Omega + \frac{1}{j\omega C_1}} + V_o(\omega)\left(j\omega C_2 + \frac{1}{400k\Omega}\right) = 0$$

  - Doing some algebra gives,

$$\boldsymbol{H}(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = -\frac{1}{20k\Omega + \frac{1}{j\omega C_1}}\ \frac{1}{j\omega C_2 + \frac{1}{400k\Omega}} = \frac{-j\omega C_1(400k\Omega)}{\big(1 + j\omega C_1(20k\Omega)\big)\big(1 + j\omega C_2(400k\Omega)\big)}$$

Solution

- Step 3: The network functions given in Step 1 and Step 2 must be equal.

$$\frac{\pm 0.5(j\omega)}{\left(1+j\frac{\omega}{40}\right)\left(1+j\frac{\omega}{160}\right)} = \boldsymbol{H}(\omega) = \frac{-j\omega C_1(400k\Omega)}{\left(1+j\omega C_1(20k\Omega)\right)\left(1+\mathrm{j}\omega C_2(400k\Omega)\right)}$$

  - Equating coefficients gives

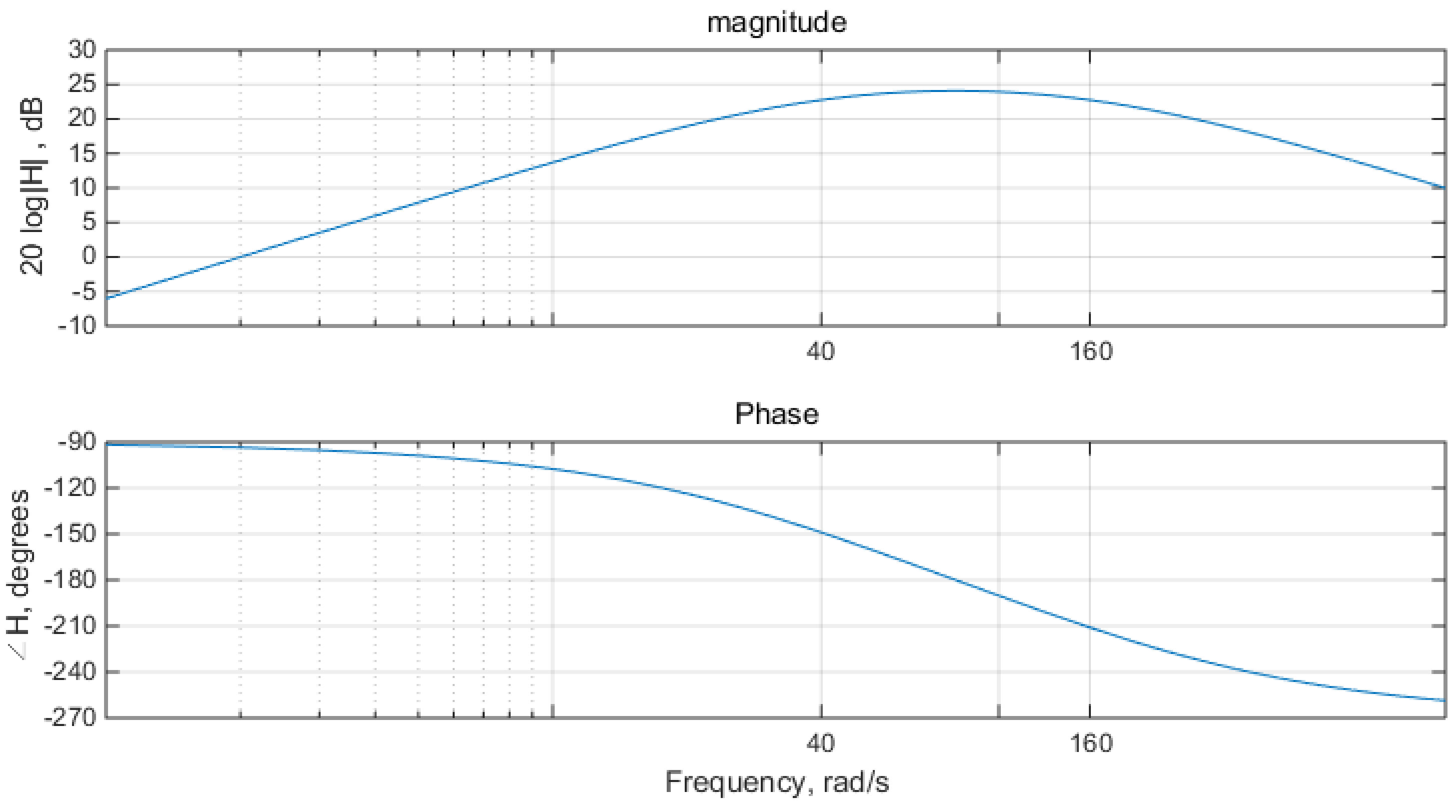
$$\frac{1}{40} = C_1(20k\Omega), \qquad \frac{1}{160} = C_2(400k\Omega), \qquad -0.5 = -C_1(400k\Omega)$$

  - so

$$C_1 = \frac{1}{40(20k\Omega)} = 1.25\ \mu F, \qquad C_2 = \frac{1}{160(400k\Omega)}, = 15.625\mu F$$

Solution

- Bode plot of $\boldsymbol{H}(\omega) = \dfrac{-0.5(j\omega)}{\left(1 + j\dfrac{\omega}{40}\right)\left(1 + j\dfrac{\omega}{160}\right)}$



Example 13.3-4 Network Function with Complex Poles

- The network function of a second-order low-pass filter can have the form

$$\boldsymbol{H(\omega) = \frac{k\omega_0^2}{(j\omega)^2 + j2\zeta\omega_0\omega + \omega_0^2}}$$

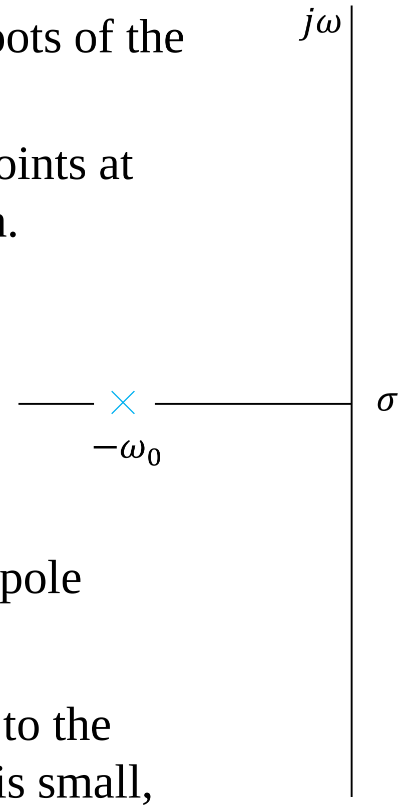
- This network function depends on three parameters: the dc gain k; the corner frequency ω_0; and the damping ratio ζ. For convenience, we consider the case where k = 1. Then, using $j^2 = -1$, we can write network function as

$$\boldsymbol{H(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega}}$$

- Determine the asymptotic magnitude Bode plot of the second-order low-pass filter when the dc gain is 1.

Complex poles

- Consider the complex plane or the s-plane
  - In chapter 9, we used the s-plane to show the location of the roots of the characteristic equation of circuits.
  - The natural response of a circuit was in the form of e^{st}. The points at the vertical axis represents pure sinusoid functions in this form.
  - We use this plane to describe the network function with poles.



- Consider the network function with a pole, $H(\omega) = \frac{H_0}{1+j\frac{\omega}{\omega_0}}$
  - $H(\omega) = H_0 \frac{\omega_0}{\omega_0 + j\omega}$ is a reciprocal of relative position from the pole to $j\omega$ multiplied by DC gain and pole.
  - The magnitude of the network function inversely proportional to the distance between the pole and $j\omega$. The distance is ω_0 when ω is small, ω when ω is large, and $\sqrt{2}\omega_0$ when ω equals to ω_0.
  - The phase is the angle at the pole with negative sign.

Complex poles

- When we have two independent energy storage elements in the circuit, two roots of the characteristic equation exists, which can be rewritten as,

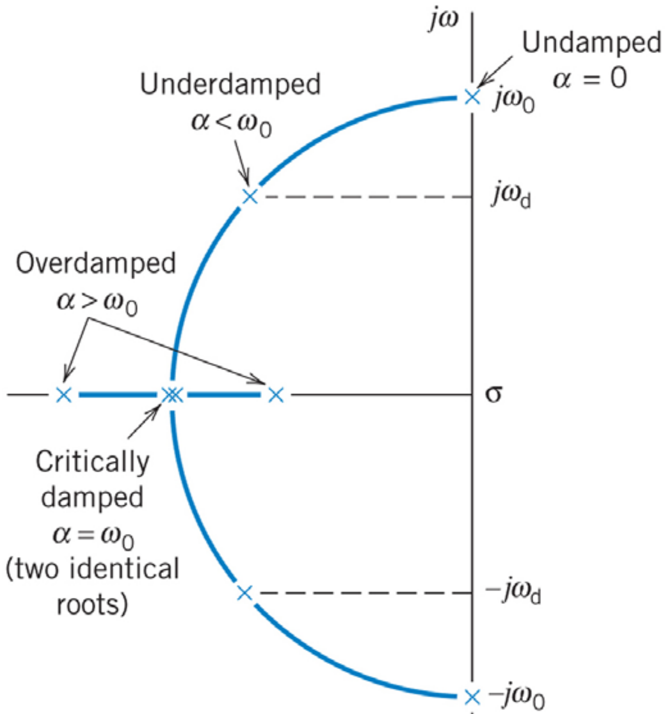
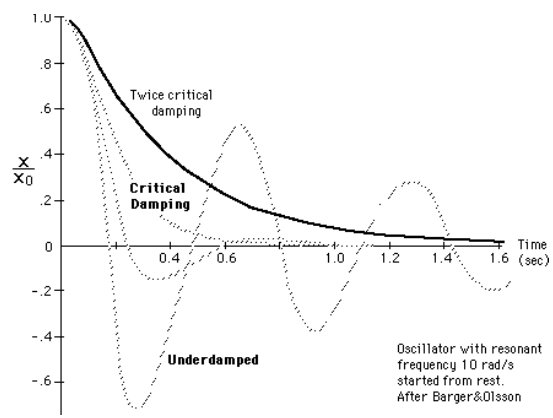


Fig_9-10-1

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

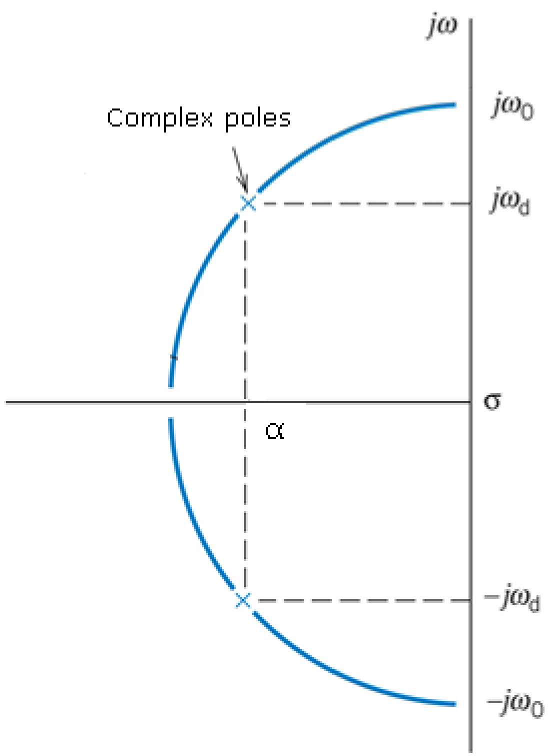
The roots of the characteristic equation assume three possible conditions:

1. Two real and distinct roots when → *overdamped*
2. Two real equal roots when → *critically damped*
3. Two complex roots when → *underdamped*



Complex poles

- When we have complex poles at $\alpha \pm j\omega_d$,



$$\boldsymbol{H}(\omega_0) = H_0 \frac{\alpha - j\omega_d}{j\omega + (\alpha - j\omega_d)} \frac{\alpha + j\omega_d}{j\omega + (\alpha + j\omega_d)}$$

$$= H_0 \frac{\alpha^2 + \omega_d^2}{(j\omega)^2 + j\omega\, 2\alpha + (\alpha^2 + \omega_d^2)}$$

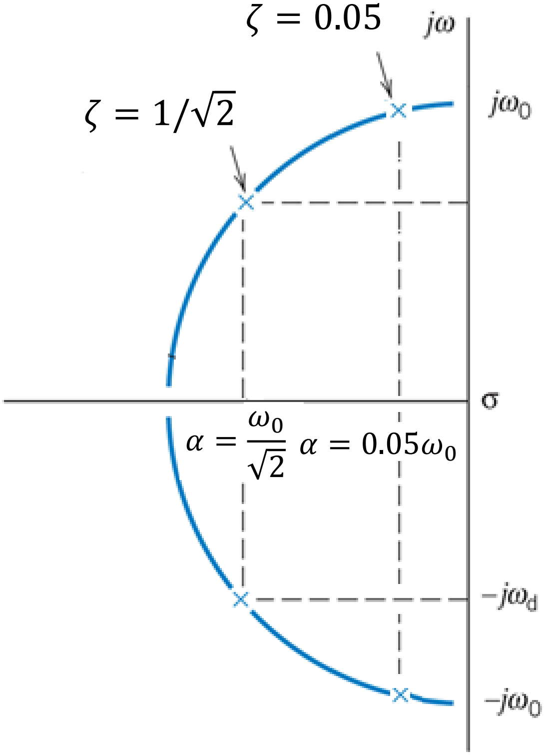
Let $\zeta = \frac{\alpha}{\omega_0}$ and $\omega_0 = \sqrt{\alpha^2 + \omega_d^2}$, then

$$\boldsymbol{H}(\omega_0) = H_0 \frac{\omega_0^2}{(j\omega)^2 + j2\zeta\omega_0\omega + \omega_0^2}$$

Any pole on the semi circle has same ω_0, differs only ζ

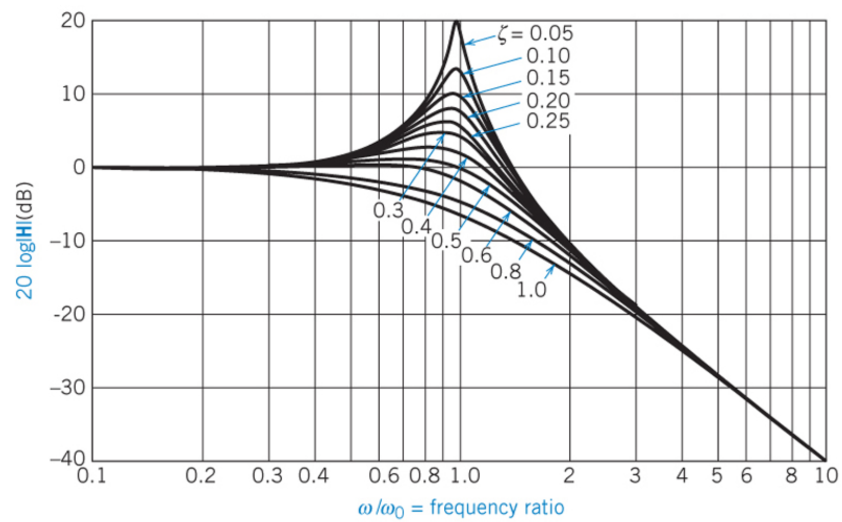
Complex poles

- As frequency approaches to the upper pole, the distance between them decreases and the magnitude of the network function peaks.
  - We can find out the peak frequency by differentiating the magnitude.



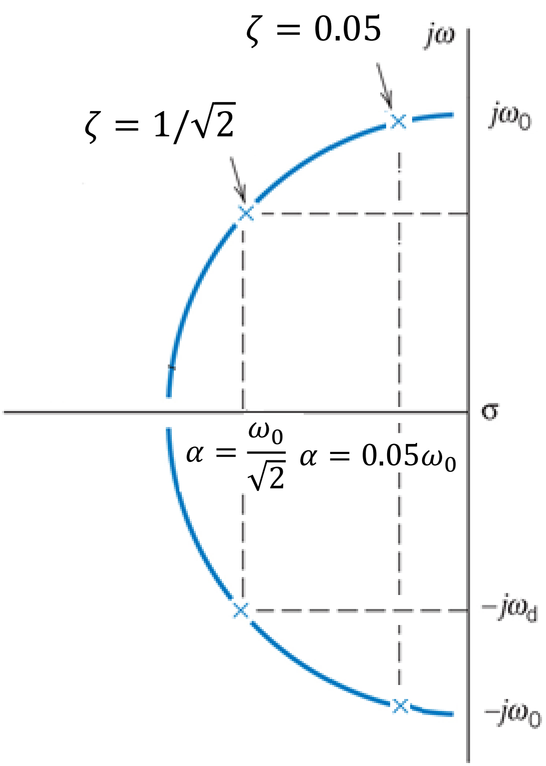
$$\frac{d|\boldsymbol{H}(\omega_0)|}{d\omega} = -|H_0|\omega_0^2 \frac{2(\omega^2-\omega_0^2)(2\omega)+(2\zeta\omega_0)^2(2\omega)}{2\left((\omega^2-\omega_0^2)^2+(2\zeta\omega_0\omega)^2\right)^{\frac{3}{2}}} = 0$$

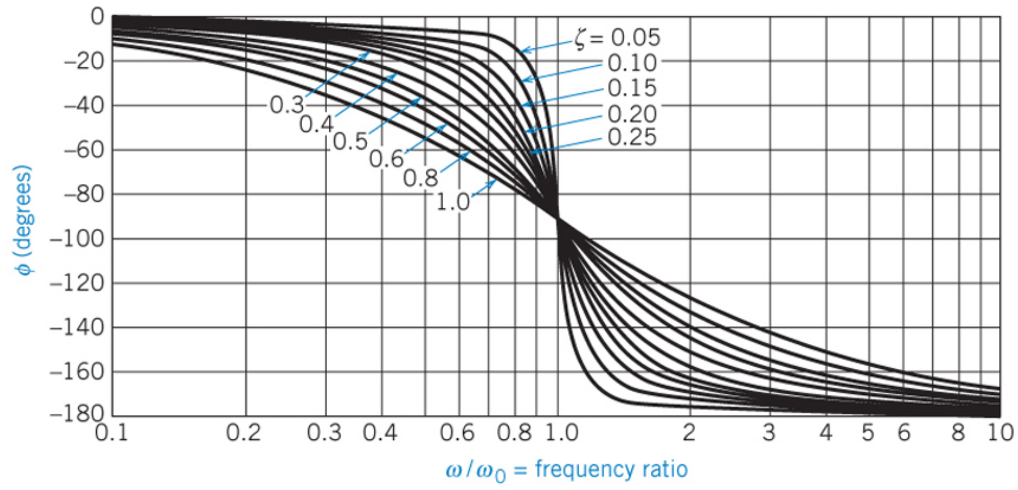
$$\omega_{peak} = \sqrt{\omega_d^2 - \alpha^2}$$



Complex poles

- When the frequency is ω_0, the sum of the angle at both poles equals to the $90°$, so the phase is always $90°$ at ω_0 regardless of ζ.





Solution

- The network function $\boldsymbol{H}(\omega) = \dfrac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega}$

- The denominator of $H(\omega)$ contains a new factor, one that involves ω^2. The asymptotic Bode plot is based on the approximation,

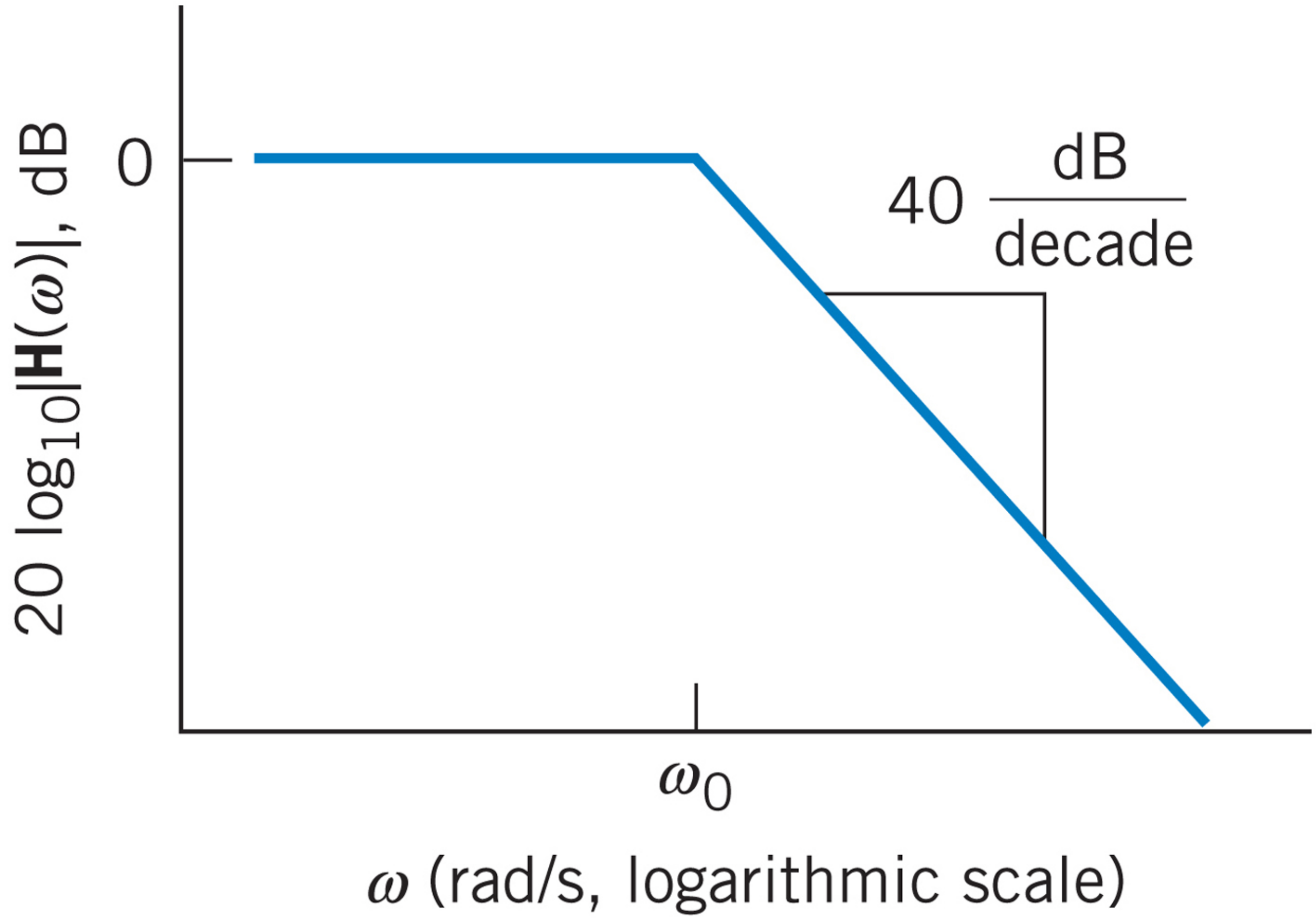
$$(\omega_0^2 - \omega^2) + j2\zeta\omega_0\omega \cong \begin{cases} \omega_0^2 & \omega < \omega_0 \\ -\dfrac{\omega_0^2}{\omega^2} & \omega > \omega_0 \end{cases}$$

- Using this approximation, we can express the logarithmic gain as

$$20\, log_{10}|H(\omega)| \cong \begin{cases} 0 & \omega < \omega_0 \\ 40\, log_{10}\omega_0 - 40\, log_{10}\omega & \omega > \omega_0 \end{cases}$$

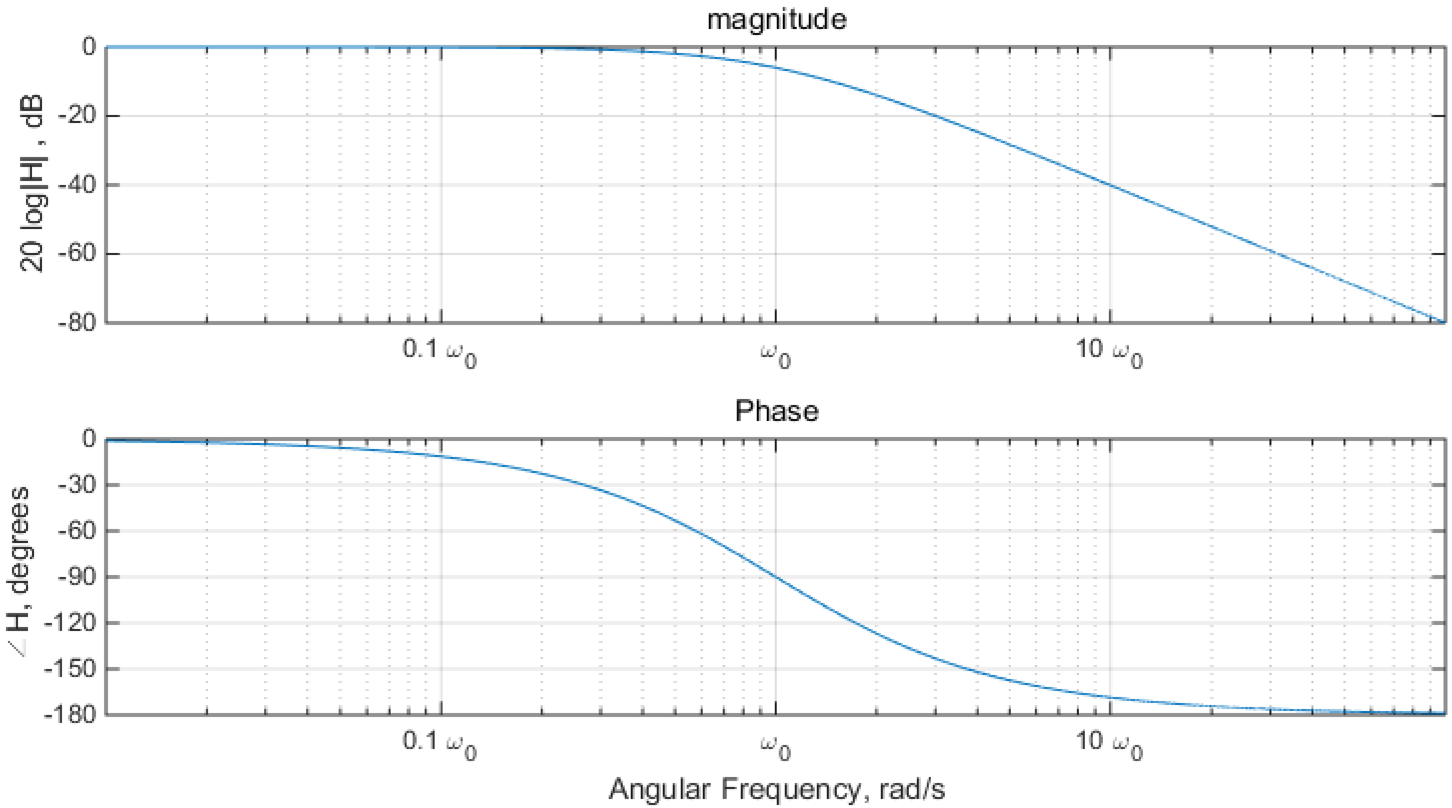
Solution

- The asymptotic magnitude Bode plot of the second-order low-pass filter is,



Solution

- The Bode plot of the second-order low-pass filter with real poles, $\boldsymbol{H}(\omega) = \frac{1}{\left(1+j\frac{\omega}{\omega_0}\right)^2}$,



Solution

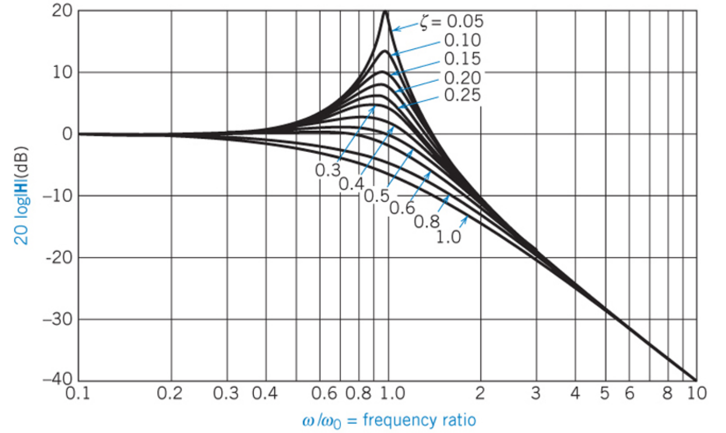
- The asymptotic Bode plot is a good approximation to the actual Bode plot when $\omega \ll \omega_0 \ \ or \ \ \omega \gg \omega_0$. Near $\omega = \omega_0$, the asymptotic Bode plot deviates from the actual Bode plot. At $\omega = \omega_0$, the value of the asymptotic Bode plot is 0 dB, whereas the value of the actual Bode plot is,

$$\boldsymbol{H(\omega_0) = \frac{1}{2\zeta}}$$

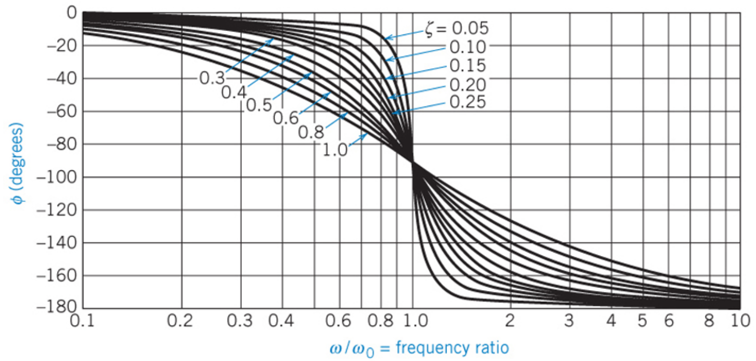
- The deviation between the actual and asymptotic magnitude Bode plot near $\omega = \omega_0$ depends on ζ.

Solution

- Bode diagram of $H(j\omega) = \dfrac{1}{1+\left(\frac{2\zeta}{\omega_0}\right)j\omega+\left(\frac{j\omega}{\omega_0}\right)^2}$ is,



(a)



(b)

- Find the asymptotic magnitude Bode plot of

$$\boldsymbol{H}(\omega) = \frac{5(1 + 0.1j\omega)}{j\omega(1 + 0.5j\omega)\left[1 + 0.6\left(\frac{j\omega}{50}\right) - \left(\frac{\omega}{50}\right)^2\right]}$$

Solution

- The corner frequencies of $H(\omega)$ are z = 10, p = 2, and ω_0 = 50 rad/s. The smallest corner frequency is p = 2. When ω < 2, $H(\omega)$ can be approximated as

$$\boldsymbol{H}(\omega) = \frac{5}{j\omega}$$

- So the equation of the low-frequency asymptote is

$$20\ log_{10}\ \boldsymbol{H} = 20\log_{10} 5\ - 20\log_{10}\omega$$

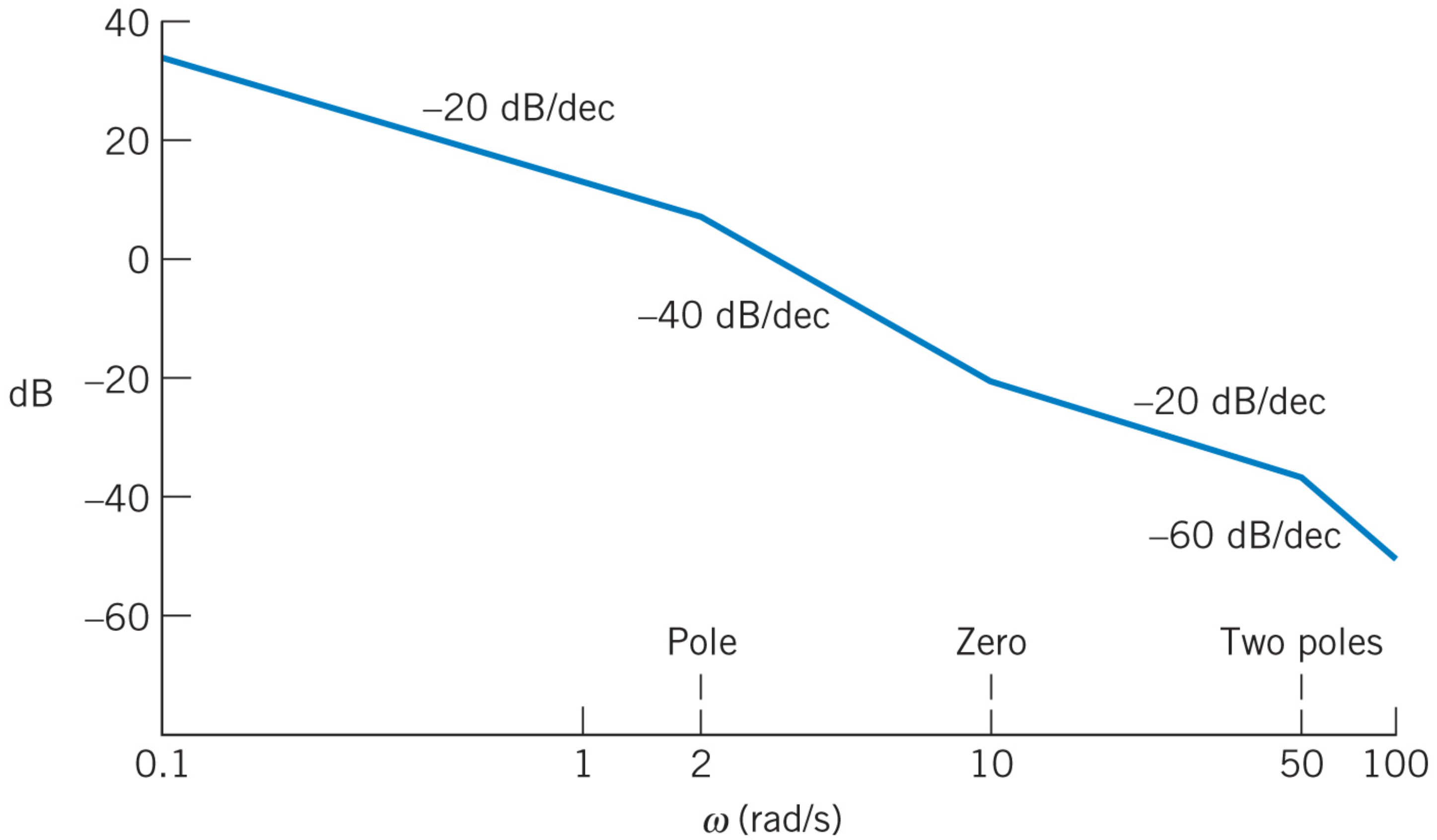
- Let's find a appoint on the low-frequency asymptote. When ω = 1,

$$20\ log_{10}\ \boldsymbol{H} = 20\log_{10} 5\ - 20\log_{10} 1 = 14dB$$

- The low-frequency asymptote is a straight line with a slope of -20 dB/decade passing through the point $\omega = 1\ rad/s, |\boldsymbol{H}| = 14dB.$
- The slope of the asymptotic Bode plot will change as ω increases past each corner frequency. The slope decreases by 20 dB/decade at ω = p = 2 rad/s, then increases by 20 dB/decade at ω = 10 rad/s, and finally decreases by 40 dB/decade at 50 rad/s.

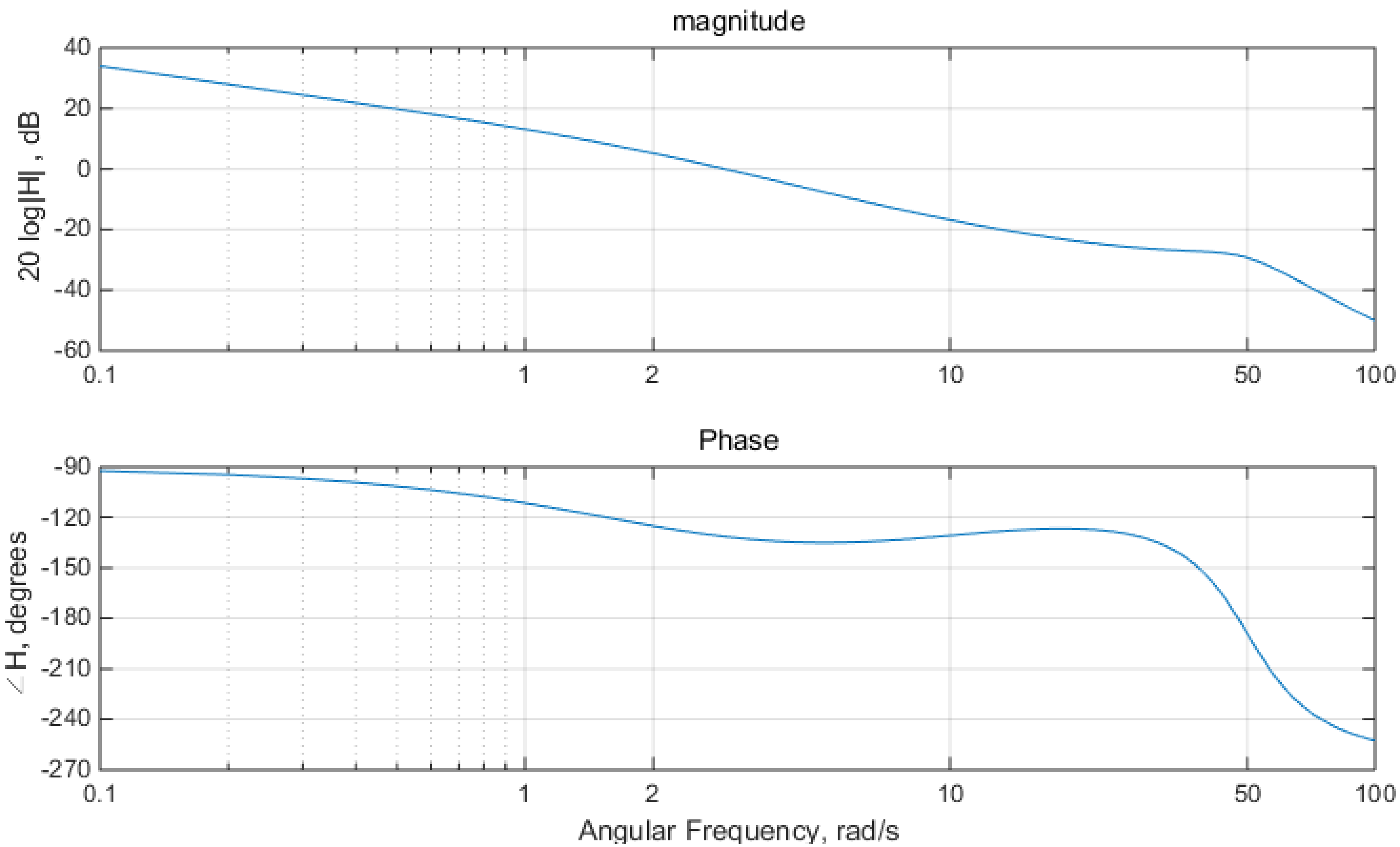
Solution

- The asymptotic magnitude Bode plot is



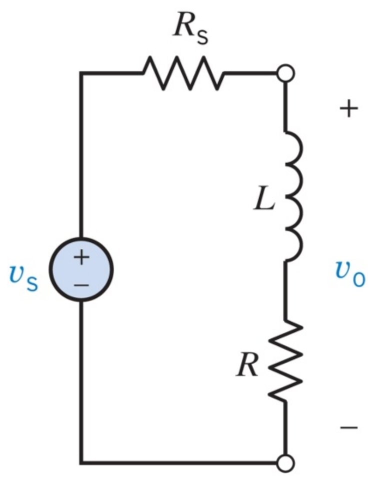
Solution

- The Bode plot of $\boldsymbol{H}(\omega) = \dfrac{5(1+0.1j\omega)}{j\omega(1+0.5j\omega)\left[1+0.6\left(\frac{j\omega}{50}\right)-\left(\frac{\omega}{50}\right)^2\right]}$ is



Example 13.3-6 Designing a Circuit to Have a Specified Bode Plot

- Let's design the circuit shown in Figure 13.3-3 to satisfy the following specifications
  - The low-frequency gain is 0.1
  - The high-frequency gain is 1.
  - The corner frequencies lie in the range of 100 hertz to 2000 hertz



Solution

- Our earlier analysis of this circuit showed that the low-frequency gain is less than 1 and that the high-frequency gain is equal to 1. This circuit can be used only to satisfy specifications that are consistent with these facts. Fortunately, the given specifications are consistent with these facts. **The first specification requires**

$$\mathbf{0.1 = low - frequency\ gain = k = \frac{R}{R + R_s}}$$

- Because the high-frequency gain is 1, **the second specification is satisfied**.
- Now let's turn our attention to the specifications on the corner frequencies. The specified frequency range is given using units of hertz, whereas the corner frequencies have units of rad/s. Because $\omega_1 > \omega_2$, **the third specification requires that**

$$\mathbf{(2\pi)100 < \frac{R}{L} = \omega_1, (2\pi)2000 > \frac{R + R_s}{L} = \omega_2}$$

- Our job is to find values of $R, R_S, and\ L$ that satisfy these three requirements. We have no guarantee that appropriate values exist. Also, it may well not be unique.

Solution

- Let's try

$$\boldsymbol{R = 100\Omega}$$

- The specification on the low-frequency gain requires that

$$\boldsymbol{R_s = 9R = 900\Omega}$$

- The specification on the zero will be satisfied if

$$\boldsymbol{L = \frac{R}{(2\pi)100} = 0.159H}$$

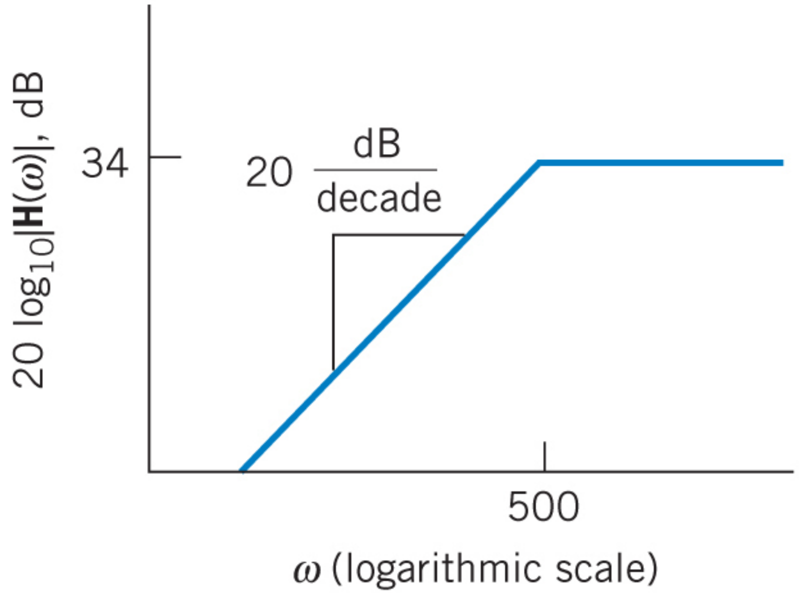
- It remains to verify that these values of $R, R_s, and\ L$ satisfy the specification on the pole frequency. The specification is satisfied because

$$\boldsymbol{\frac{R + R_s}{L} = 6289 < 12{,}566 = (2\pi)2000}$$

- This solution is not unique. Indeed, when R = 100 and R_s = 900, any inductance in the range 0.0796 < L < 0.159H can be used to satisfy these specifications.

Example 13.3-7 Designing a Circuit to Have a Specified Bode Plot

- Design a circuit that has the asymptotic magnitude Bode plot shown in Figure 13.3-13a.



(a)

Solution

- The slope of this Bode plot is 20 dB/decade for low frequencies, that is, $\omega < 500\ rad/s$, so $H(\omega)$ must have a $j\omega$ factor in its numerator. The slope decreases by 20 dB/decade as ω increases past $\omega = 500\ rad/s$, so H(ω) must have a pole at $\omega = 500\ rad/s$. Based on these observations,

$$\boldsymbol{H(\omega) = \pm k \frac{j\omega}{1 + j\frac{\omega}{500}}}$$

- The gain of the asymptotic Bode plot is 34 dB = 50 when $\omega > 500\ rad/s$, so

$$\boldsymbol{50 = \pm k \frac{j\omega}{j\frac{\omega}{500}} = \pm k * 500}$$

- Thus, $k = \pm 0.1$ and

$$\boldsymbol{H(\omega) = \pm 0.1 \frac{j\omega}{1 + j\frac{\omega}{500}}}$$

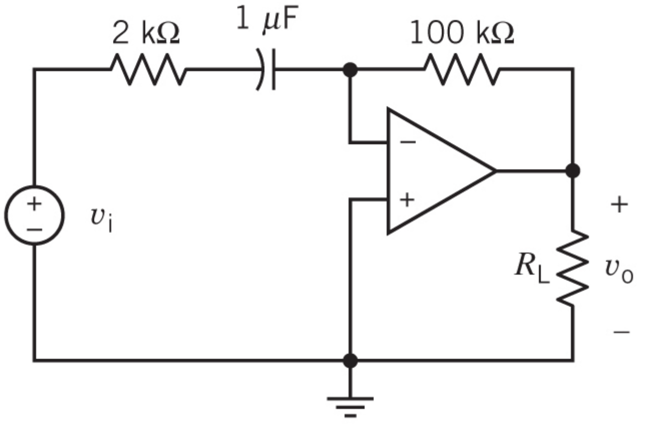
Solution

- The design equations provided in row 4 of Table 13.3-2 indicate that

$$0.1 = R_2 C, \qquad 500 = \frac{1}{CR_1}$$

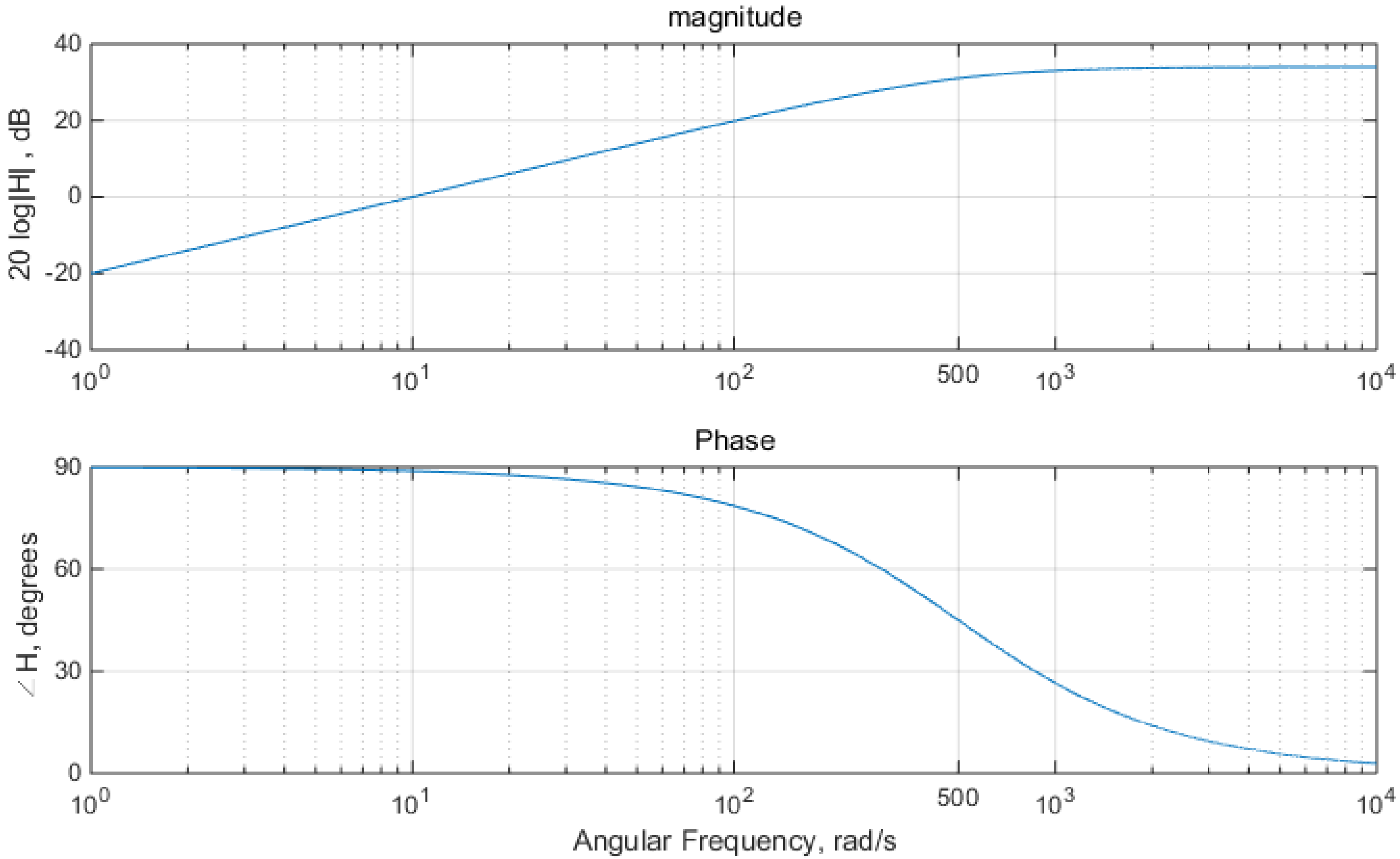
- Pick C = 1μF. Then

$$R_2 = \frac{0.1}{10^{-6}} = 100k\Omega, \qquad R_1 = \frac{1}{500 * 10^{-6}} = 2k\Omega$$



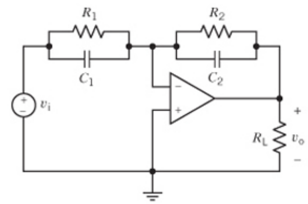
Solution

- The Bode plot of $H(\omega) = 0.1\frac{j\omega}{1+j\frac{\omega}{500}}$ is,



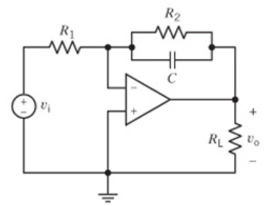
Solution

CIRCUIT | NETWORK FUNCTION



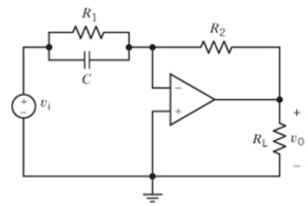
$$\mathbf{H}(\omega) = -k\,\frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$

where $k = \dfrac{R_2}{R_1}$, $z = \dfrac{1}{C_1 R_1}$, $p = \dfrac{1}{C_2 R_2}$



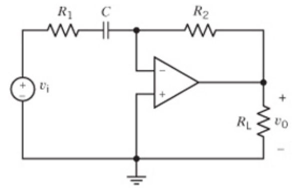
$$\mathbf{H}(\omega) = -\frac{k}{1 + j\dfrac{\omega}{p}}$$

where $k = \dfrac{R_2}{R_1}$, $p = \dfrac{1}{CR_2}$



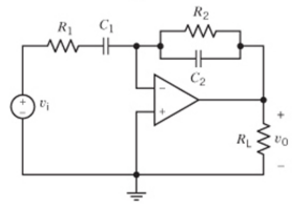
$$\mathbf{H}(\omega) = -k\left(1 + j\frac{\omega}{z}\right)$$

where $k = \dfrac{R_2}{R_1}$, $z = \dfrac{1}{CR_1}$



$$\mathbf{H}(\omega) = -k\,\frac{j\omega}{1 + j\dfrac{\omega}{p}}$$

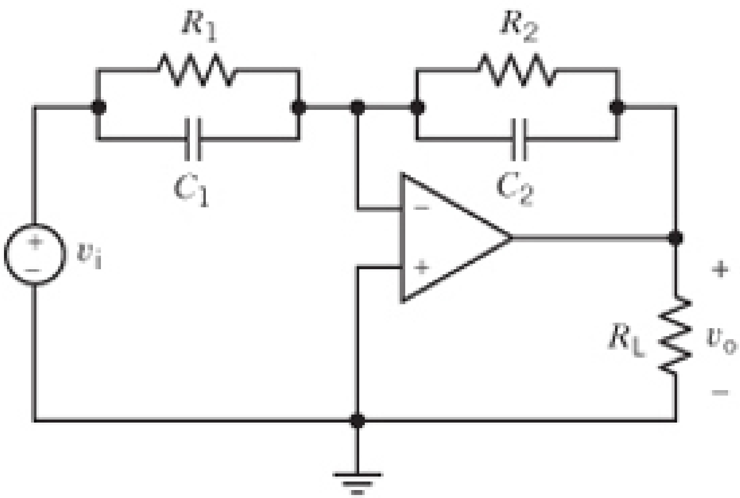
where $k = R_2 C$, $p = \dfrac{1}{CR_1}$



$$\mathbf{H}(\omega) = -\frac{k(j\omega)}{\left(1 + j\dfrac{\omega}{p_1}\right)\left(1 + j\dfrac{\omega}{p_2}\right)}$$

where $k = C_1 R_2$, $p_1 = \dfrac{1}{C_1 R_1}$, $p_2 = \dfrac{1}{C_2 R_2}$

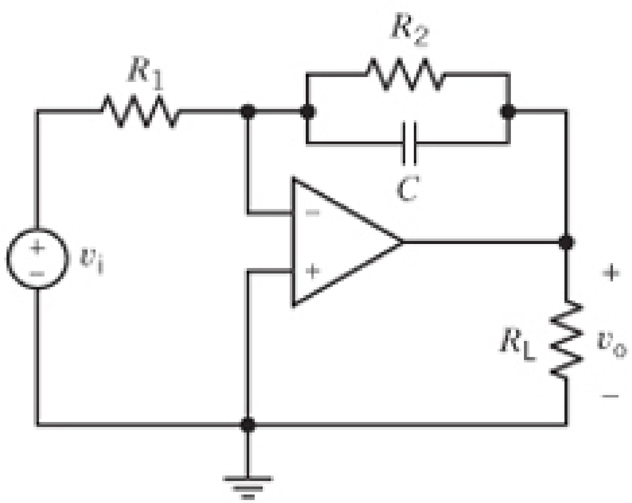
Solution

CIRCUIT	NETWORK FUNCTION
	$\mathbf{H}(\omega) = -k \dfrac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$ where $k = \dfrac{R_2}{R_1}$ $z = \dfrac{1}{C_1 R_1}$ $p = \dfrac{1}{C_2 R_2}$

$$V_o(\omega) = V_i(\omega)\left(\frac{1}{R_1} + j\omega C_1\right) * -\left(R_2 || \frac{1}{j\omega C_2}\right) = -V_i(\omega)\frac{1 + j\omega R_1 C_1}{R_1}\frac{1}{\frac{1}{R_2} + j\omega C_2}$$

$$H(\omega) = -\frac{R_2}{R_1}\;\frac{1 + j\omega R_1 C_1}{1 + j\omega R_2 C_2}$$

Solution

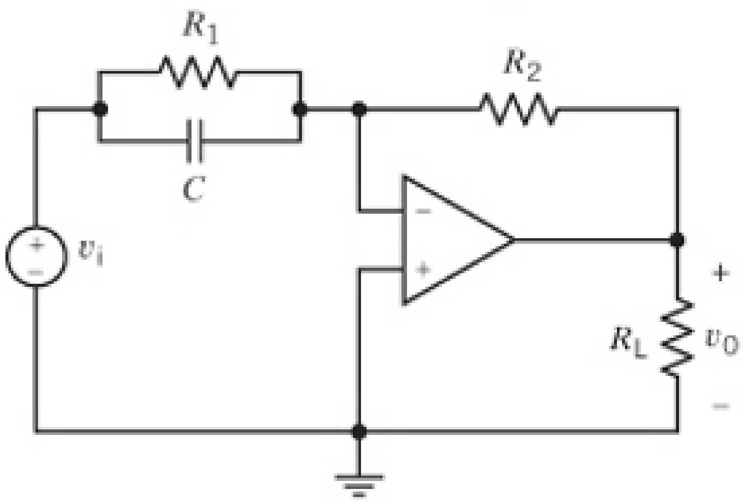
CIRCUIT	NETWORK FUNCTION
	$\mathbf{H}(\omega) = -\dfrac{k}{1 + j\dfrac{\omega}{p}}$ where $k = \dfrac{R_2}{R_1}$, $p = \dfrac{1}{CR_2}$

$$V_o(\omega) = V_i(\omega)\frac{1}{R_1} * -\left(R_2 || \frac{1}{j\omega C}\right) = -V_i(\omega)\ \frac{1}{R_1}\ \frac{1}{\dfrac{1}{R_2} + j\omega C}$$

$$H(\omega) = -\frac{R_2}{R_1}\ \frac{1}{1 + j\omega R_2 C}$$

Solution

CIRCUIT	NETWORK FUNCTION



$$\mathbf{H}(\omega) = -k\left(1 + j\frac{\omega}{z}\right)$$

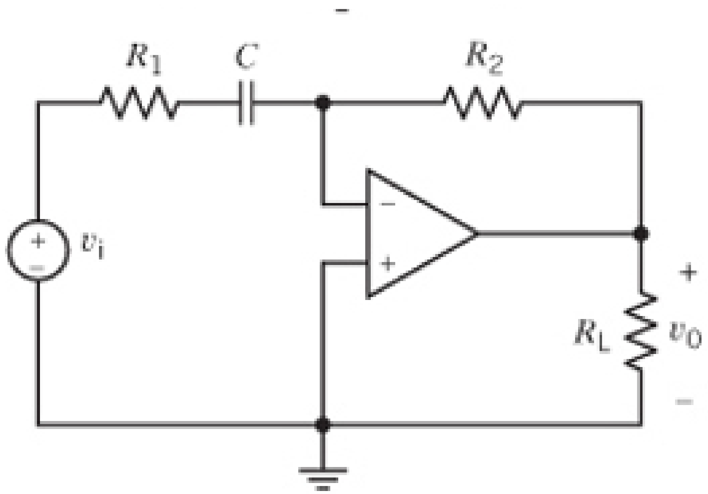
where
$$k = \frac{R_2}{R_1}$$
$$z = \frac{1}{CR_1}$$

$$V_o(\omega) = V_i(\omega)\left(\frac{1}{R_1} + j\omega C\right) * -R_2 = -V_i(\omega)\ R_2\ \frac{1 + j\omega R_1 C}{R_1}$$

$$H(\omega) = -\frac{R_2}{R_1}\ (1 + j\omega R_1 C)$$

Solution

CIRCUIT | NETWORK FUNCTION



$$\mathbf{H}(\omega) = -k\,\frac{j\omega}{1 + j\frac{\omega}{p}}$$

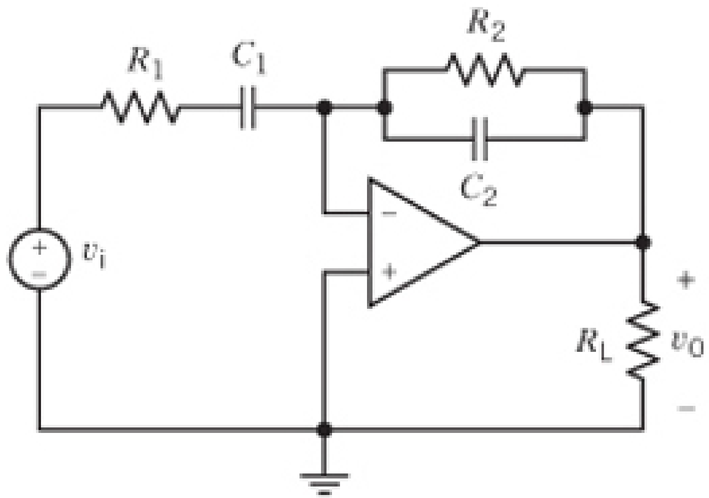
where $k = R_2 C$

$$p = \frac{1}{CR_1}$$

$$V_o(\omega) = V_i(\omega)\frac{1}{\left(R_1 + \frac{1}{j\omega C}\right)} * -R_2 = -V_i(\omega)\ R_2\,\frac{j\omega C}{1 + j\omega R_1 C}$$

$$H(\omega) = -R_2 C\,\frac{j\omega}{1 + j\omega R_1 C}$$

Solution

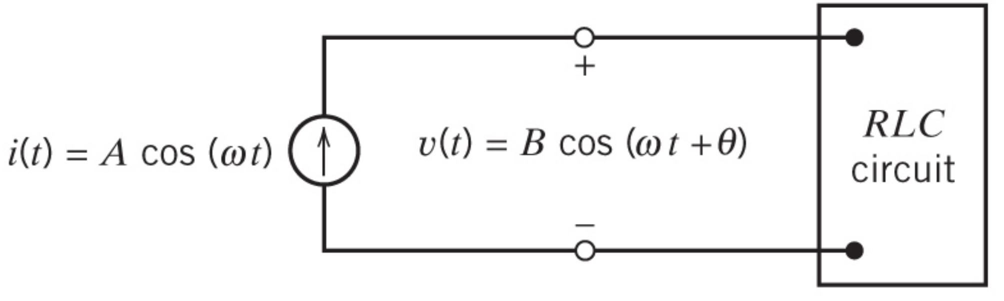
CIRCUIT	NETWORK FUNCTION
	$H(\omega) = -\dfrac{k(j\omega)}{\left(1 + j\dfrac{\omega}{p_1}\right)\left(1 + j\dfrac{\omega}{p_2}\right)}$ where $k = C_1R_2$, $p_1 = \dfrac{1}{C_1R_1}$, $p_2 = \dfrac{1}{C_2R_2}$

$$V_o(\omega) = V_i(\omega)\frac{1}{\left(R_1 + \frac{1}{j\omega C_1}\right)} * -\left(R_2 || \frac{1}{j\omega C_2}\right) = -V_i(\omega)\ \frac{j\omega C_1}{1 + j\omega R_1 C_1}\ \frac{1}{\frac{1}{R_2} + j\omega C_2}$$

$$H(\omega) = -R_2C_1\ \frac{j\omega}{1 + j\omega R_1C_1}\ \frac{1}{1 + j\omega R_2C_2}$$

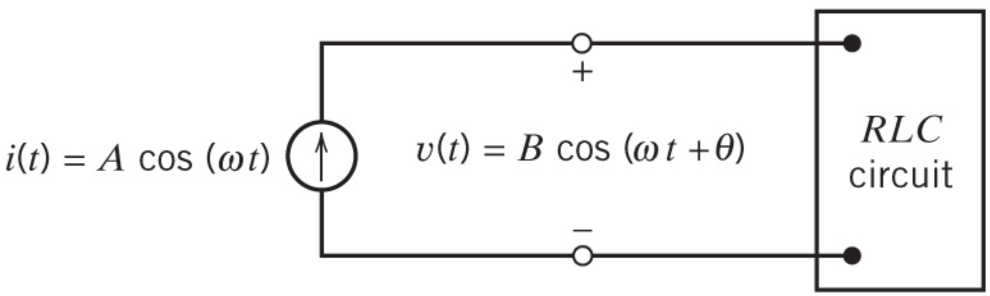
Resonant Circuits

- Consider the situation shown below. The input to this circuit is the current of the current source, and the response is the voltage across the current source. Because the input to the circuit is sinusoidal, we can use phasors to analyze this circuit. We know the network function of the circuit is the ratio of the response phasor to the input phasor. In this case, the network function will be an impedance, $\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{A\angle\theta}{B\angle 0°}$



Resonant Circuits

- Figure (b) shows some data that were obtained by applying an input with an amplitude of 2mA and a frequency that was varied. Row 1 of this table describes the performance of this circuit when $\omega = 200 rad/s$. At this frequency, the impedance of the circuit is $\boldsymbol{Z} = \frac{6.6\angle 48°}{0.002} = 3300\angle 48°\Omega$.



(a)

A, A	ω, rad/s	B, V	θ
0.002	200	6.6	48°
0.002	220	8.4	33°
0.002	250	10.0	0°
0.002	270	9.3	–21°
0.002	300	7.4	–43°

(b)

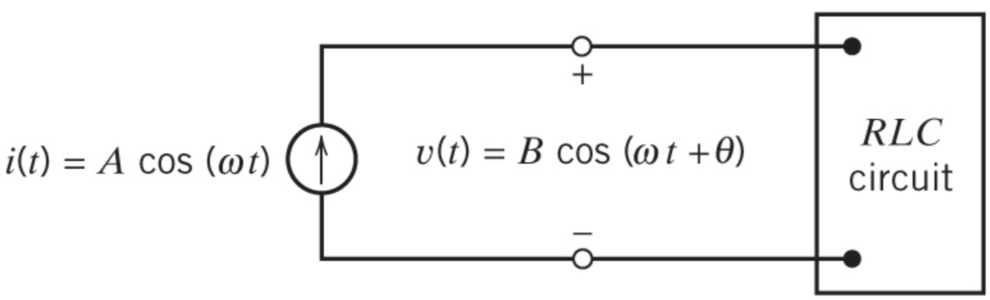
Resonant Circuits

- Let's convert this impedance from polar form to rectangular form:

$$\boldsymbol{Z} = 2208 + j2452\ \Omega.$$

- This looks like the equivalent impedance of a series resistor and inductor. The resistance would be 2208Ω and the inductance would be 12.26H.

- Recall that in rectangular form impedances are represented as $\boldsymbol{Z} = R + jX$ where R is called the resistance and X is called the reactance. When ω is 200 rad/s, we say that the reactance of this circuit is inductive because the reactance is positive and therefore could have been caused by a single inductor.



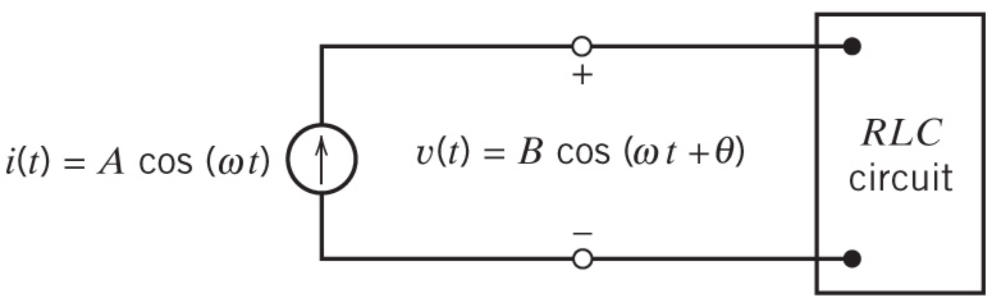
A, A	ω, rad/s	B, V	θ
0.002	200	6.6	48°
0.002	220	8.4	33°
0.002	250	10.0	0°
0.002	270	9.3	−21°
0.002	300	7.4	−43°

Resonant Circuits

- The last row of the table describes the performance of this circuit when ω is 300 rad/s. Now $\boldsymbol{Z} = \frac{7.4\angle{-43^\circ}}{0.002} = 3700\ \angle{-43^\circ} = 2706 - j2523\ \Omega$.
- Because the reactance is negative, it could not have been caused by a single inductor. This impedance looks like the equivalent impedance of a single resistor connected in series with a single capacitor:

$$R - j\frac{1}{\omega C} = 2706 - j2523\Omega$$

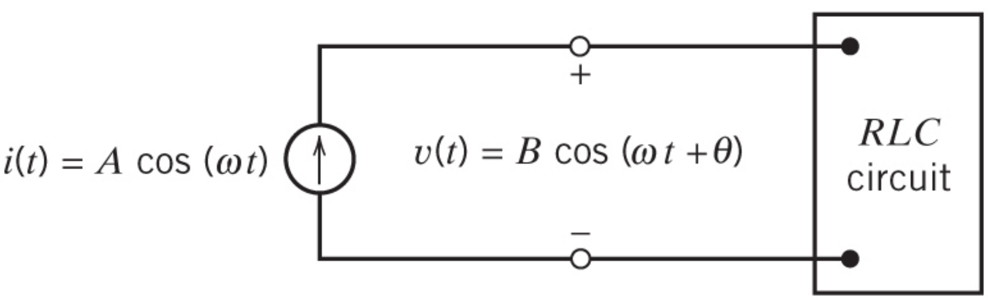
- Equating the real parts shows that the resistance is 2706Ω . Equating the imaginary parts shows that the capacitance is $1.32\mu F$



A, A	ω, rad/s	B, V	θ
0.002	200	6.6	48°
0.002	220	8.4	33°
0.002	250	10.0	0°
0.002	270	9.3	−21°
0.002	300	7.4	−43°

Resonant Circuits

- The reactance of this circuit is inductive at some frequencies and capacitive at other frequencies. We can tell the reactance will be inductive and when it will be capacitive by looking at the last column of the table.
- When the input frequency is less than 250 rad/s, the reactance is inductive, but when the input frequency is greater than 250 rad/s, the reactance is capacitive. This frequency is called resonant frequency and is denoted as ω_0.
- At the resonant frequency, the impedance is purely resistive. Also, the magnitude of the impedance is maximum

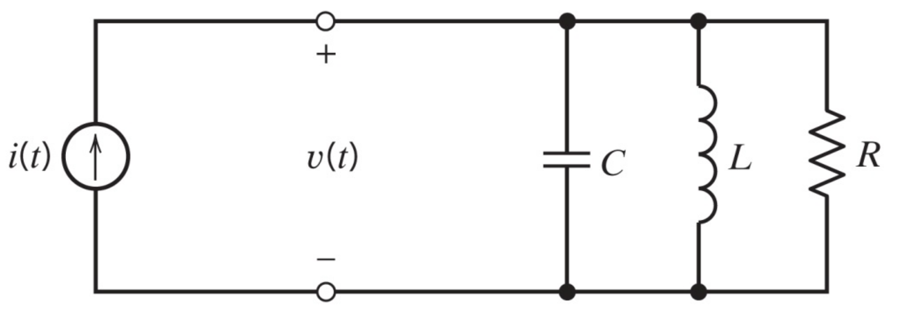


A, A	ω, rad/s	B, V	θ
0.002	200	6.6	48°
0.002	220	8.4	33°
0.002	250	10.0	0°
0.002	270	9.3	−21°
0.002	300	7.4	−43°

Resonant Circuits

- Consider the circuit below. This circuit is called the parallel resonant circuit. The equivalent impedance of the parallel resistor, inductor, and capacitor is

$$\boldsymbol{Z = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}} = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}} \angle - tan^{-1} R\left(\omega C - \frac{1}{\omega L}\right)}$$

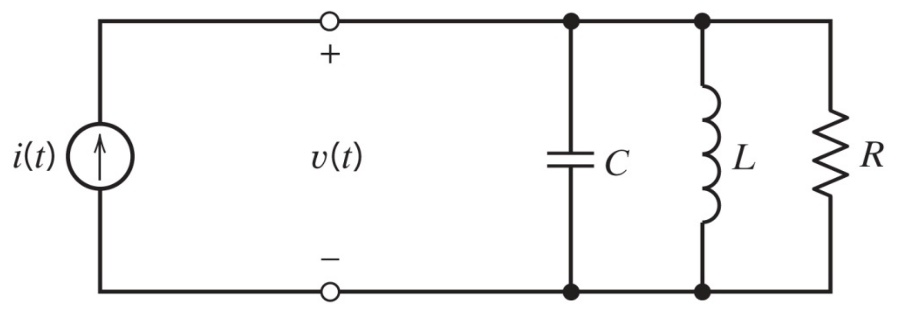


$i(t) = A\cos(\omega t)$
$v(t) = B\cos(\omega t + \theta)$

Resonant Circuits

- The circuit exhibits some familiar behavior. The reactance will be zero when $\omega C - \frac{1}{\omega L} = 0$.
- The frequency that satisfies this equation is the resonant frequency ω_0. Solving this equation gives $\omega_0 = \frac{1}{\sqrt{LC}}$

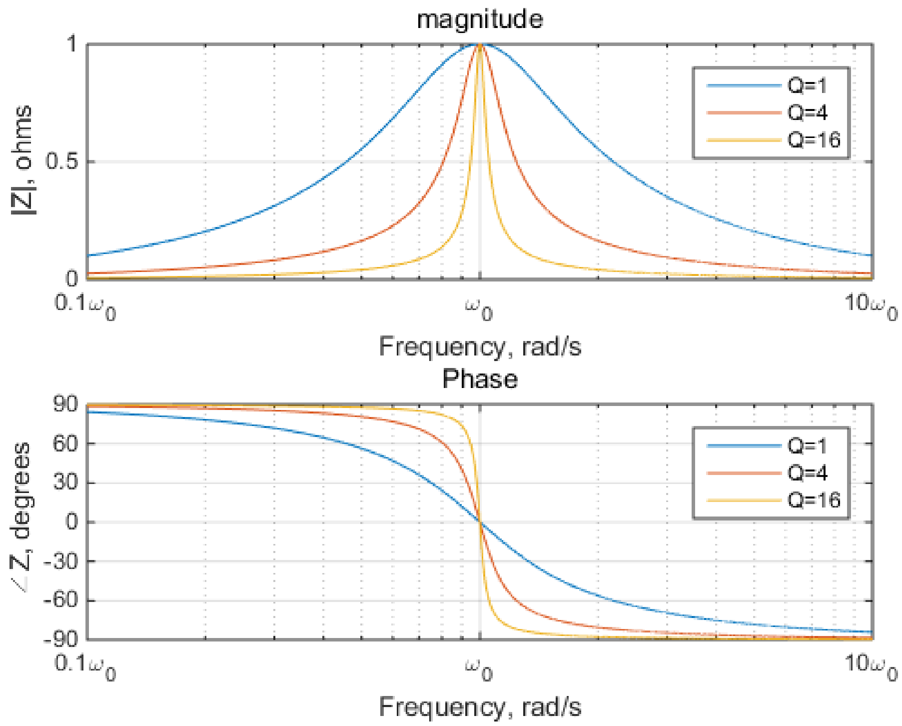
$$\boldsymbol{Z = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}} = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}} \angle - tan^{-1} R\left(\omega C - \frac{1}{\omega L}\right)}$$



$i(t) = A\cos(\omega t)$
$v(t) = B\cos(\omega t + \theta)$

Resonant Circuits

- At $\omega = \omega_0, Z = R$. The magnitude of Z decreases as ω deviates from ω_0. The angle of Z is positive when $\omega < \omega_0$ and negative when $\omega > \omega_0$, so the reactance is inductive when $\omega < \omega_0$ and capacitive when $\omega > \omega_0$



Resonant Circuits

- The impedance can be put in the form

$$\boldsymbol{Z = \frac{k}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}}$$

- where

$$\boldsymbol{k = R, Q = R\sqrt{\frac{C}{L}}, \omega_0 = \frac{1}{\sqrt{LC}}}$$

- The parameters k, Q, and ω_0 characterize the resonant circuit. The resonant frequency ω_0 is the frequency at which the reactance is zero and where the magnitude of the impedance is maximum. k is the maximum value of the impedance. Q is called the quality factor of the resonant circuit. It controls how rapidly |Z| decreases.

Resonant Circuits

- The larger the value of Q, the more sharply peaked is the frequency response plot. We can quantify this observation by introducing the bandwidth of the resonant circuit. To that end, let $\omega_1 < \omega_2$ denote the frequencies where

$$|\boldsymbol{Z}(\boldsymbol{\omega})| = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\boldsymbol{Z}(\boldsymbol{\omega_0})| = \frac{\boldsymbol{k}}{\sqrt{\mathbf{2}}}$$

- There will be two such frequencies, one smaller than ω_0 and the other larger than ω_0. Let $\omega_1 < \omega_0$ and $\omega_2 > \omega_0$. The bandwidth BW of the resonant circuit is defined as $BW = \omega_2 - \omega_1$.
- The frequencies ω_1, ω_2 are solutions of the equation

$$\frac{\boldsymbol{k}}{\sqrt{\mathbf{2}}} = \frac{\boldsymbol{k}}{\sqrt{\mathbf{1} + \boldsymbol{Q}^2\left(\frac{\boldsymbol{\omega}}{\boldsymbol{\omega_0}} - \frac{\boldsymbol{\omega_0}}{\boldsymbol{\omega}}\right)^2}}$$

Resonant Circuits

- Doing some algebra, we get,

$$\pm\mathbf{1} = \boldsymbol{Q}\left(\frac{\boldsymbol{\omega}}{\boldsymbol{\omega_0}} - \frac{\boldsymbol{\omega_0}}{\boldsymbol{\omega}}\right)$$

- It can be rearranged as

$$\boldsymbol{\omega^2 \mp \frac{\omega_0\omega}{Q} - \omega_0^2 = 0}$$

- This equation has two positive solutions

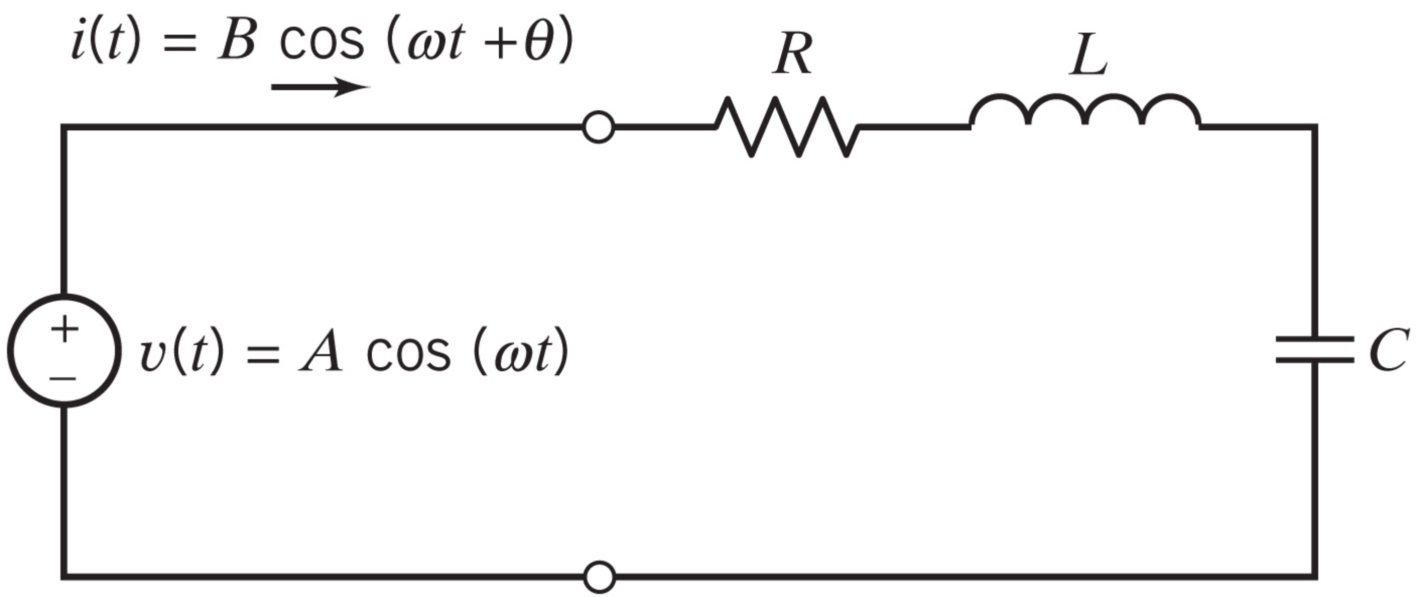
$$\boldsymbol{\omega_1 = -\frac{\omega_0}{2Q} + \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 + \omega_0^2} \quad and \quad \omega_2 = \frac{\omega_0}{2Q} + \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 + \omega_0^2}}$$

- Finally, the bandwidth is

$$\boldsymbol{BW = \omega_2 - \omega_1 = \frac{\omega_0}{Q}}$$

Example 13.4-1 Series Resonant Circuit

- Figure 13.4-4 shows a series resonant circuit. Determine the relationship between parameters k, Q, and ω_0 and the element values R, L, and C for the series resonant circuit.



Solution

- The input to this circuit is the voltage source, and the response is the current in the mesh. The network function is the ratio of the response phasor to the input phasor. In this case, the network function is the equivalent admittance of the series resistor, capacitor, and inductor:

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}}$$

- To identify k, Q, and ω_0, this network function must be rearranged so that it is in the form

$$\mathbf{Y} = \frac{k}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

- Rearranging the equation gives

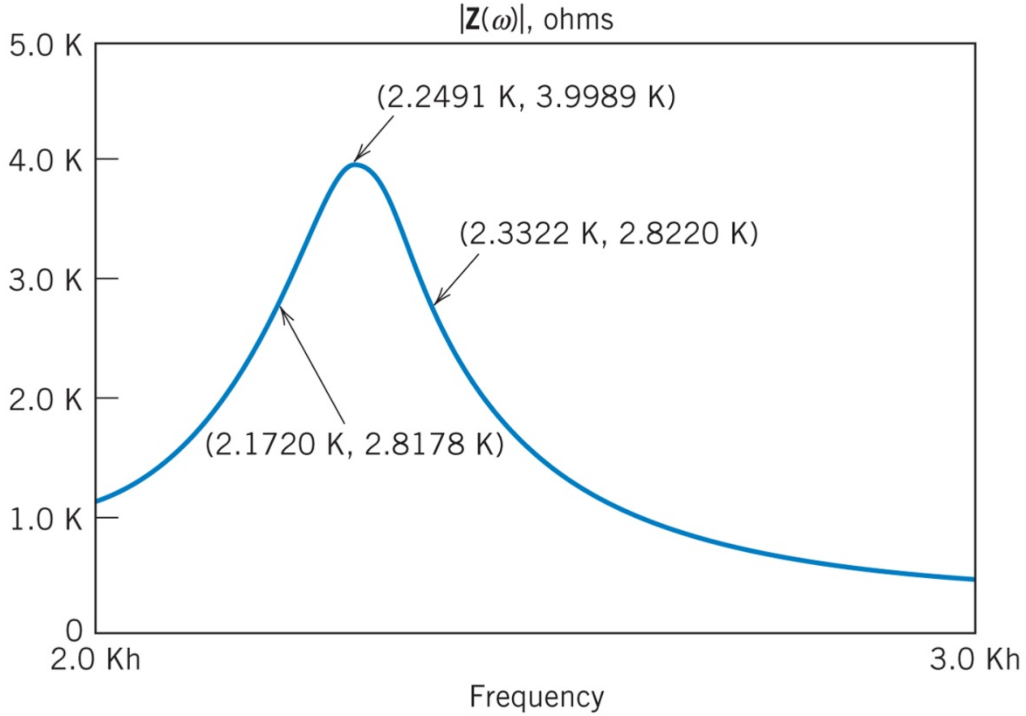
$$\mathbf{Y} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{R + j\sqrt{\frac{L}{C}}\left(\frac{\omega}{\frac{1}{\sqrt{LC}}} - \frac{\frac{1}{\sqrt{LC}}}{\omega}\right)} = \frac{\frac{1}{R}}{1 + j\frac{1}{R}\sqrt{\frac{L}{C}}\left(\frac{\omega}{\frac{1}{\sqrt{LC}}} - \frac{\frac{1}{\sqrt{LC}}}{\omega}\right)}$$

- Comparing two equations gives,

$$k = \frac{1}{R}, Q = \frac{1}{R}\sqrt{\frac{L}{C}}, \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

Example 13.4-2 Frequency Response of a Resonant Circuit

- Figure 13.4-5 shows the magnitude frequency response of a resonant circuit. What are the values of the parameters k, Q, and ω_0?



Solution

- The first step is to find the peak of the frequency response and determine the values of the frequency and the impedance corresponding to that point. This frequency is the resonant frequency ω_0, and the impedance at this frequency is k. The frequency and the impedance is

$$\omega_0 = (2\pi)2249 = 14{,}130\, rad/s$$

$$k = 4000\,\Omega$$

- Next, the frequency ω_1 and ω_2 are identified by finding the points on the frequency response where the value of the impedance is $k/\sqrt{2} = 2828\ \Omega$.

$$\omega_1 = (2\pi)2172 = 13,647\, rad/s \text{ and } \omega_2 = (2\pi)2332 = 14,653\, rad/s$$

- The quality factor Q is calculated as

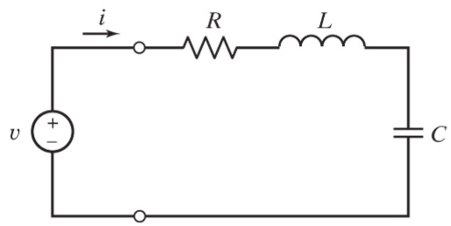
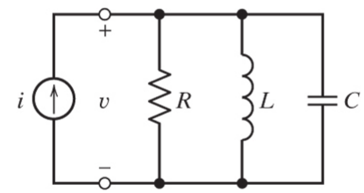
$$Q = \frac{\omega_0}{BW} = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{14{,}130}{14{,}653 - 13{,}647} = 14$$

- Then the network function can be expressed as

$$\boldsymbol{Z}(\omega) = \frac{4000}{1 + j14\left(\frac{\omega}{14{,}130} - \frac{14{,}130}{\omega}\right)}$$

Example 13.4-3 Parallel Resonant Circuit

- Design a parallel resonant circuit that has $k = 4000\ \Omega$, $Q = 14$, and $\omega_0 = 14{,}130\ rad/s$

	SERIES RESONANT CIRCUIT	PARALLEL RESONANT CIRCUIT
Circuit		
Network function	$\mathbf{Y} = \dfrac{k}{1 + jQ\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)}$	$\mathbf{Z} = \dfrac{k}{1 + jQ\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)}$
Resonant frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$	$\omega_0 = \dfrac{1}{\sqrt{LC}}$
Maximum magnitude	$k = \dfrac{1}{R}$	$k = R$
Quality factor	$Q = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$	$Q = R\sqrt{\dfrac{C}{L}}$
Bandwidth	$BW = \dfrac{R}{L}$	$BW = \dfrac{1}{RC}$

Solution

- From Table 13.4-1, we have the relationship between parameters k, Q, and ω_0 and the element values R, L, and C for the parallel resonant circuit. These relationships can be used to calculate R, L, and C from k, Q, and ω_0. First,

$$R = k = 4000\ \Omega$$

$$\frac{1}{\sqrt{LC}} = \omega_0 = 14{,}130\ rad/s$$

and

$$R\sqrt{\frac{C}{L}} = Q = 14$$

- Rearranging these equations gives

$$\frac{14\sqrt{L}}{4000} = \sqrt{C} = \frac{1}{14{,}130\sqrt{L}}$$

- So,

$$L = \frac{4000}{14{,}130(14)} = 20\ mH \text{ and } C = \frac{1}{14{,}130^2(0.002)} = 0.25\ \mu F$$

Frequency Response of Op Amp Circuits

- The gain of an op amp is not infinite; rather, it is finite and decreases with frequency. The gain $A(\omega)$ of the operational amplifier is a function of ω given by

$$\boldsymbol{A(\omega) = \frac{A_o}{1 + j\omega/\omega_1}}$$

- A_o is the dc gain and ω_1 is the corner frequency. The dc gain is normally greater than 10^4 and ω_1 is less than 100 rad/s.

Example 13.5-1 Frequency Response of a Noninverting Amplifier

- Consider the noninverting amplifier in Figure 13.5-2a. Replacing the op amp with a frequency-dependent op amp gives the circuit shown in Figure 13.5-2b. Suppose that $R_2 = 90\ k\Omega$ and $R_1 = 10\ k\Omega$ and that the parameters of the op amp are $A_o = 10^5$ and $\omega_1 = 10\ rad/s$. Determine the magnitude Bode plot for both the gain of the op amp $\mathbf{A}(\omega)$ and the network function of the noninverting amplifier $\mathbf{V}_o/\mathbf{V}_s$.

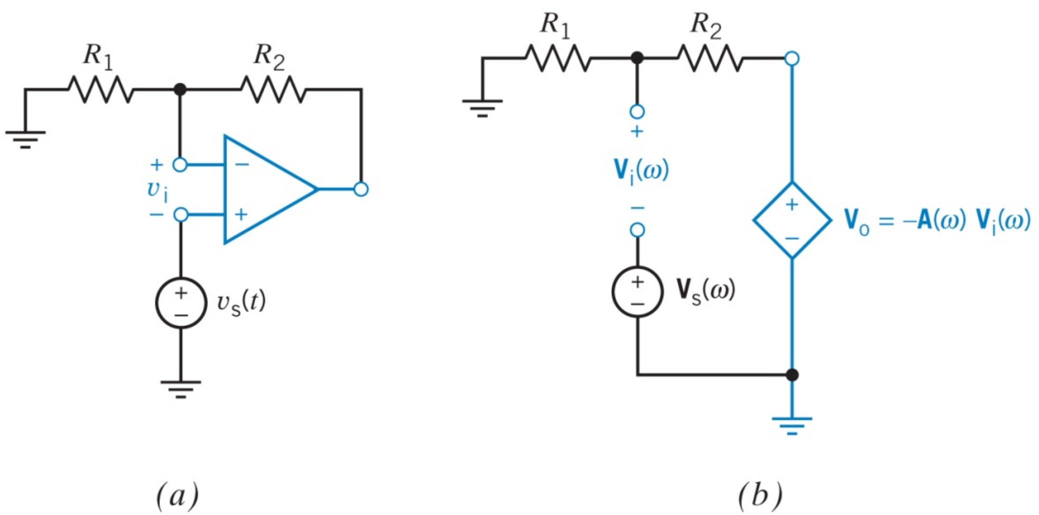


FIGURE 13.5−2

Solution(1/4)

- Since we have $\mathbf{A}(\omega) = \frac{A_o}{1+j\omega/\omega_1}$, the Bode plot of $20\log|\mathbf{A}(\omega)|$ is as shown in Figure 13.5-3. Note that the magnitude is equal to 1 (0 dB) at $\omega = 10^6 rad/s$.

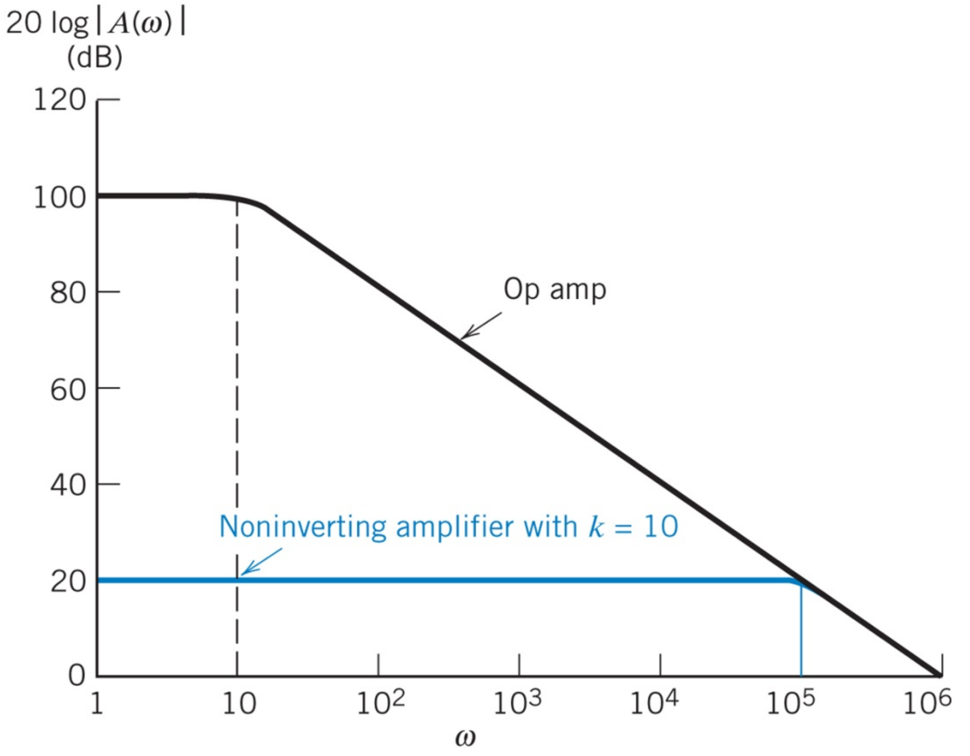


FIGURE 13.5−3

Solution(2/4)

- Then, writing a node equation at in Figure 13.5-2b gives

$$\frac{\mathbf{V}_i + \mathbf{V}_s}{R_1} + \frac{\mathbf{V}_i + \mathbf{V}_s + A(\omega)\mathbf{V}_i}{R_2} = 0$$

- The frequency-dependent model of the op amp is described by

$$\mathbf{V}_o = -\mathbf{A}(\omega)\mathbf{V}_i$$

- Combining these equations gives

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{\mathbf{A}(\omega)}{1 + \frac{\mathbf{A}(\omega)}{k}}$$

where $k = (R_1 + R_2)/R_1$ is the gain of the noninverting amplifier when the op amp is modeled as an ideal op amp.

Solution(3/4)

- Substituting for $\mathbf{A}(\omega)$, we get

$$\frac{V_o}{V_s} = \frac{A_o/(1+j\omega/\omega_1)}{1+(A_o/k)/(1+j\omega/\omega_1)} = \frac{A_o}{1+j\omega/\omega_1 + A_o/k} = \frac{A_c}{1+j\omega/(A_2\omega_1)}$$

where A_c is the gain of the noninverting amplifier defined as $A_c = \frac{A_o}{1+\frac{A_o}{k}}$ and $A_2 = 1 + \frac{A_o}{k}$. Usually, $1 \ll \frac{A_o}{k}$, so $A_c \cong k$ and $A_2 = \frac{A_c}{k}$. Then

$$\frac{V_o}{V_s} \cong \frac{k}{(1+j\omega/\omega_0)}$$

where $\omega_0 = A_o\omega_1/k$ is the corner frequency of the noninverting amplifier. Notice that the product of the dc gain and the corner frequency is

$$\omega_0 k = \frac{A_o\omega_1}{k}k = A_o\omega_1$$

This is called the **gain-bandwidth product**. Notice it depends only on the op amp, not on R_1 and R_2.

Solution(4/4)

- For this example, $k = 10$, and $A_o = 100\ dB$, and, thus, we have $A_c = 10$, $A_2 = 10^4$, and $\omega_1 A_2 = 10^5$.

- Therefore,

$$\frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{10}{1 + j10^{-5}\omega}$$

- This circuit has a magnitude Bode plot as shown in color in Figure 13.5-3. Note that noninverting op amp has a low-frequency gain of 20 dB and a break frequency of 10^5 rad/s. The gain-bandwidth product remains 10^6 rad/s

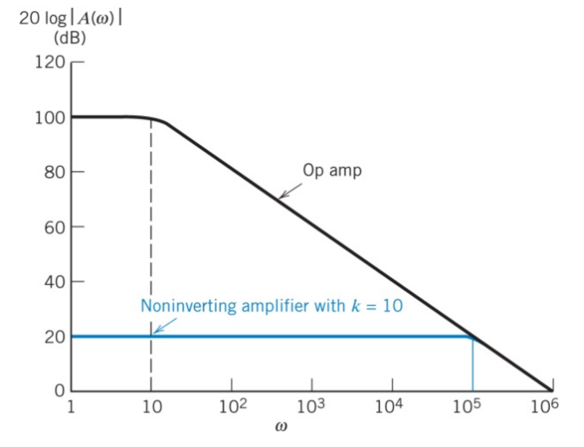


FIGURE 13.5−3

How to find out the network function

- When the circuit with one energy storage element has a sinusoid input,

$$\frac{d}{dt}x(t) + \frac{x(t)}{\tau} = Ke^{j\omega t}$$

- Where $x(t)$ is the current passing through the inductor, or the voltage across the capacitor. Converting this equation to the frequency domain,

$$j\omega\boldsymbol{X}(\omega) + \frac{\boldsymbol{X}(\omega)}{\tau} = K$$

- Then,

$$\boldsymbol{X}(\omega) = \frac{K}{j\omega + \frac{1}{\tau}}$$

- We can get the voltage across the inductor or the current passing through the capacitor by multiplying $j\omega$ to the $\boldsymbol{X}(\omega)$

How to find out the network function

- The network function of the circuit with one energy storage element is $\boldsymbol{H}(\omega) = H_0 \frac{1+\frac{j\omega}{z}}{1+\frac{j\omega}{p}}$, where $p = \frac{1}{\tau}$.

- When the frequency is zero, $H(0) = H_0$, and when the frequency is infinity, $H(\infty) = \frac{p}{z}$. Combining these equations give

$$\boldsymbol{H}(\omega) = \frac{\mathrm{H}_0 + \mathrm{j}\omega\, \tau H(\infty)}{1 + j\omega\, \tau}$$

How to find out the 2nd order network function

- When we have two energy storage elements in the circuit, the network function of the circuit is $\boldsymbol{H}(\omega) = \frac{a_0+a_1(j\omega)+a_2(j\omega)^2}{1+b_1(j\omega)+b_2(j\omega)^2}$.
- The only way for a coefficient ω to occur in a transfer function of a lumped circuit is as a multiplicative factor to a capacitor or an inductor, as in C_is or L_js. Let us initially limit our discussion to just capacitors and then generalize to include the inductors.
- In that case, the b_1 coefficient must be a linear combination of all the capacitors in the circuit. The b_1 term cannot contain a term C_iC_j because such a term must have an ω^2 multiplier. The b_2 coefficient must consist of a linear combination of two-way products of different capacitors. The same argument can be applied to a_k coefficients in the numerator and,

$$\boldsymbol{H}(\omega) = \frac{a_0 + (\alpha_1^1 C_1 + \alpha_1^2 C_2)(j\omega) + \alpha_2^{12} C_1 C_2 (j\omega)^2}{1 + (\beta_1^1 C_1 + \beta_1^2 C_2)(j\omega) + \beta_2^{12} C_1 C_2 (j\omega)^2}$$

How to find out the 2nd order network function

- The network function is determined independently of the specific value of the capacitor and must be valid for all capacitor values including zero and infinity. Let us look at a reduced case when all capacitors, except C_i, have a value of zero. The transfer function reduces to the first-order one.

$$\boldsymbol{H}(\omega) = \frac{a_0 + \alpha_1^i C_i(j\omega)}{1 + \beta_1^i C_i(j\omega)}$$

- The reduced system has a time constant of $\tau_i^0 = R_i^0 C_i$ where R_i^0 is the resistance seen by the capacitor C_i looking into port i with all other reactive elements their zero value, namely open-circuited capacitors and short-circuited inductors, and the independent sources nulled.

- Hence, the first denominator coefficient b_1 is simply given by the sum of these zero-value time constants. $b_1 = \tau_1^0 + \tau_2^0$.
- Same arguments can be applied to an inductor

How to find out the 2nd order network function

- When $C_i \to \infty$ while the other elements are still at zero value, the transfer function from the input to the output reduces to a constant,

$$H^i \equiv H\Big|_{\substack{C_i \to \infty \\ C_j \to 0 \\ i \neq j}} = \frac{\alpha_1^i}{\beta_1^i}$$

- Where H^i is a first-order transfer constant between the input and the output with the single reactive element i at its infinite value and all others zero-valued. We have already determined β_1^i to be R_i^0, which leads to $\alpha_1^i = R_i^0 H^i$. Therefore, $\alpha_1^i C_i = R_i^0 C_i H^i = \tau_i^0$. Thus, we can write

$$a_1 = \tau_1^0 H^1 + \tau_2^0 H^2$$

How to find out the 2nd order network function

- Assume that we set C_i to infinity and consider a capacitor C_j at port j while all other capacitors have a value of zero. This is a first-order system, yet different from the one we used to determine b_1. The time constant of this new first-order system is

$$\tau_j^i = R_j^i C_j$$

- Where R_j^i is the resistance seen at port j with port i infinite valued. Evaluating the network function with $C_i \to \infty$ and all other capacitors other than C_i and C_j at their zero value, we obtain

$$H(\omega)\Big|_{C_i\to\infty} = \frac{C_i(j\omega)\left(\alpha_1^i + \alpha_2^{ij} C_j s\right)}{C_i(j\omega)\left(\beta_1^i + \beta_2^{ij} C_j(j\omega)\right)} = \frac{\alpha_1^i}{\beta_1^i}\frac{1 + \dfrac{\alpha_2^{ij}}{\alpha_1^i} C_j(j\omega)}{1 + \dfrac{\beta_2^{ij}}{\beta_1^i} C_j(j\omega)}$$

- Equating the coefficient of s in the denominator,

$$\beta_2^{ij} = \beta_1^i R_j^i = R_i^0 R_j^i$$

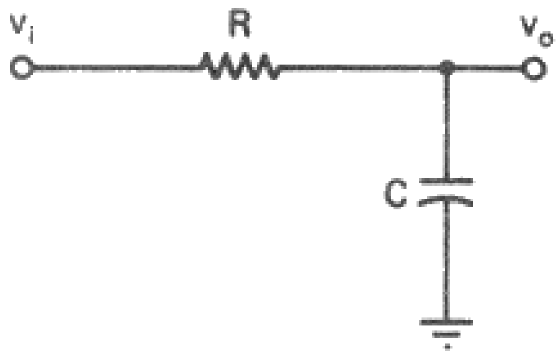
$$b_2 = R_1^0 C_1 R_2^i C_2$$

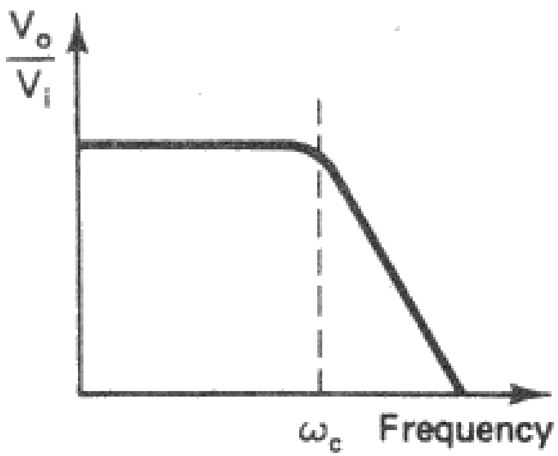
Filters

- Filters
  - Eliminate unwanted signal from the loop
  - Low Pass, High Pass, Band Pass, Notch, …

Passive First-Order Low Pass Filter

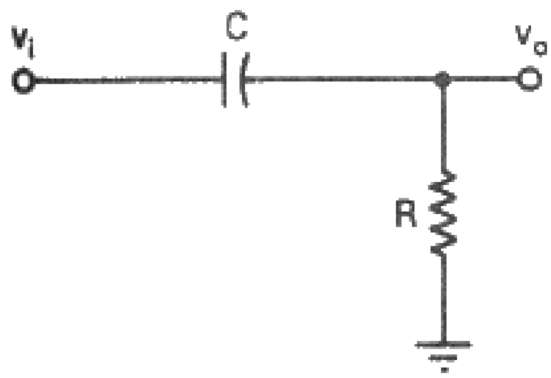
- **Pass desired Audio signal and reject undesired RF signal**

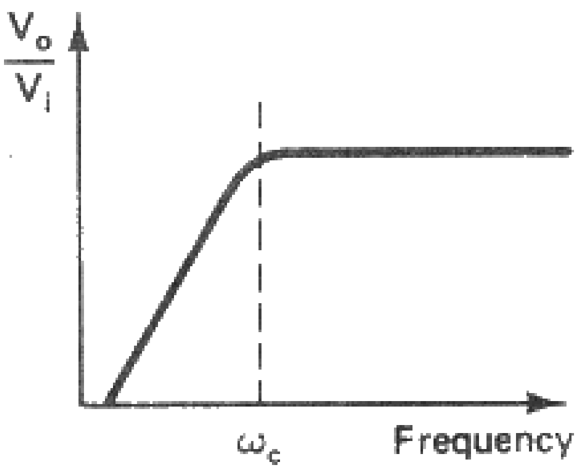




Passive First-Order High Pass Filter

- **Pass desired High frequency signal and reject undesired low frequency signal**





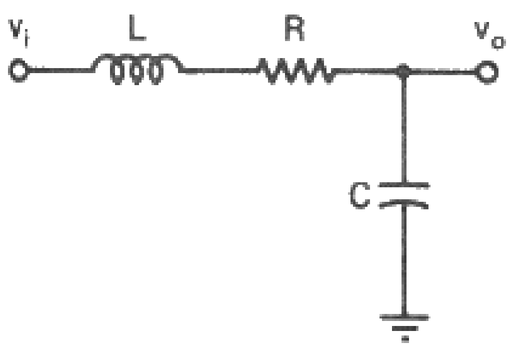
Passive Second-Order Low Pass Filter

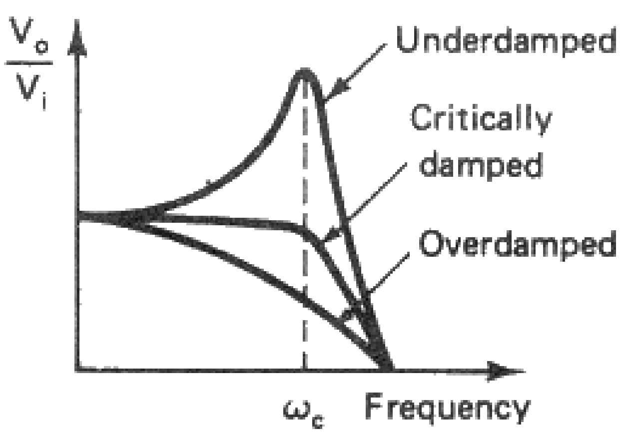
- **To increase the attenuation of transfer function**
- **Order of Filter**
  - ◆ Number of C and L

$$V_o(\omega) = V_i(\omega)\frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

$$= V_i(\omega)\frac{1}{1 + j\omega RC + (j\omega)^2 LC}$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 + j\,2\zeta(\omega/\omega_c) + (j\omega/\omega_c)^2}$$





$\zeta < 1$

$\zeta = 1$

$\zeta > 1$

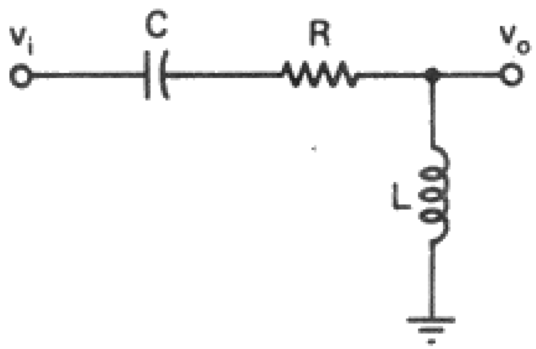
Passive Second-Order High Pass Filter

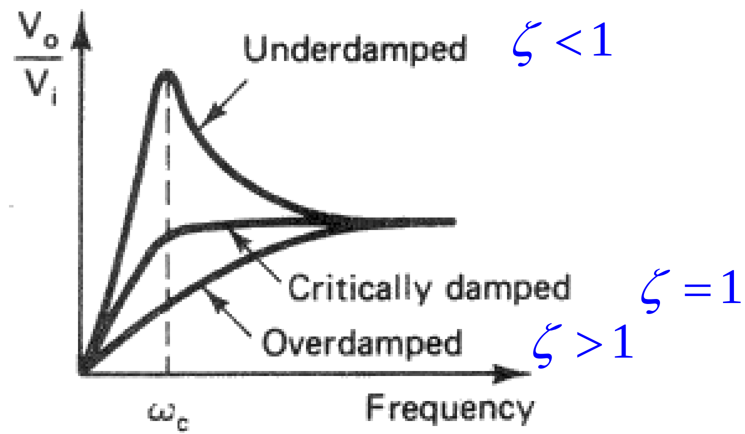
- **To increase the attenuation of transfer function**
- **Order of Filter**
  - Number of C and L

$$V_o(\omega) = V_i(\omega)\frac{j\omega L}{\frac{1}{j\omega C} + R + j\omega L}$$

$$= V_i(\omega)\frac{(j\omega)^2 LC}{1 + j\omega RC + (j\omega)^2 LC}$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{-(\omega/\omega_c)^2}{1 + j\,2\zeta(\omega/\omega_c) + (j\omega/\omega_c)^2}$$

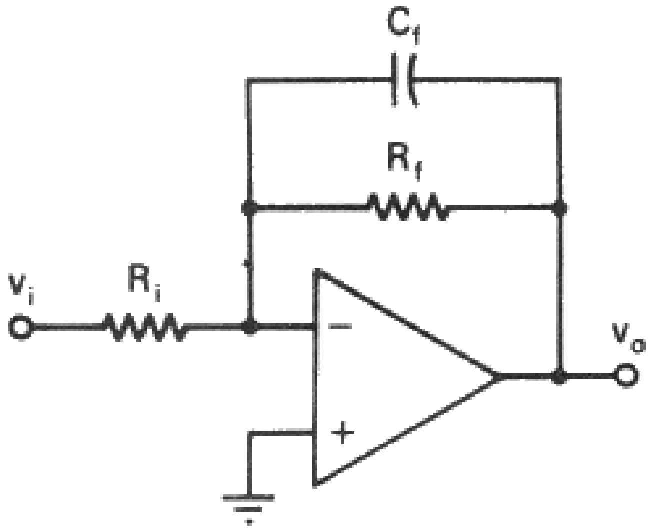


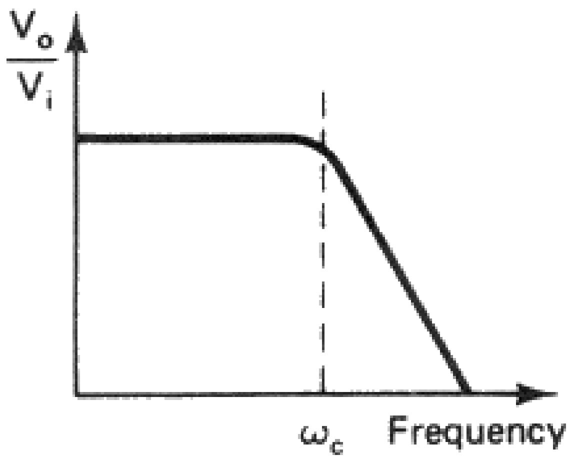


Active First-Order Low Pass Filter

- **Inverting Amp + Feedback Capacitor**

- **Identical frequency response with Passive filter**
- **Very Low Output impedance**
  - **Negligible Loading Effect**

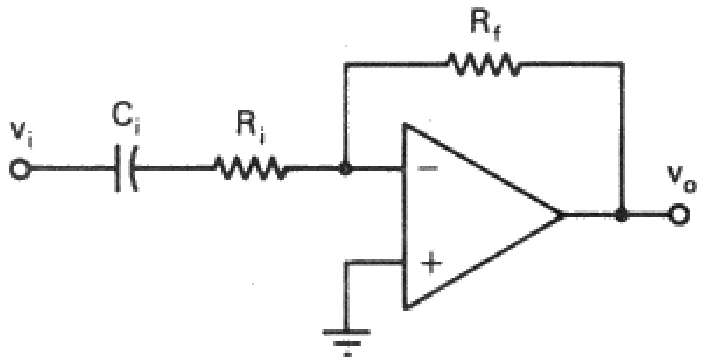


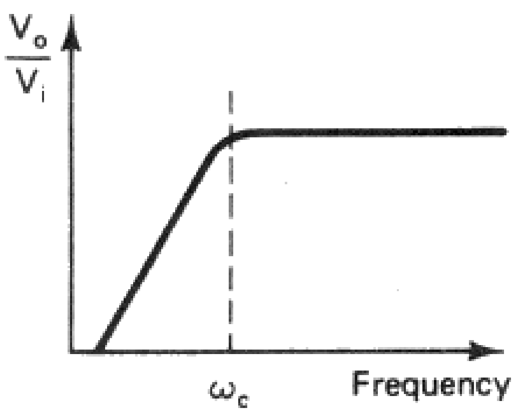


Active First-Order High Pass Filter

- Inverting Amp + Input Capacitor

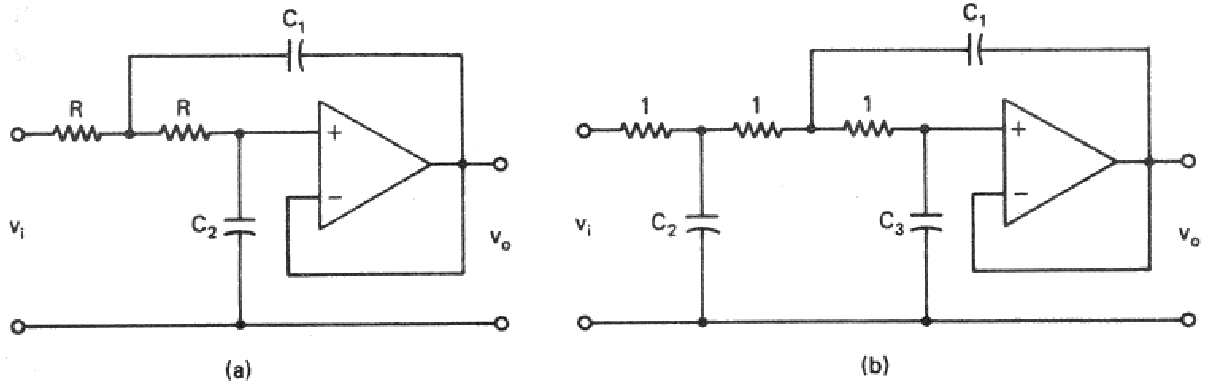
- Identical frequency response with Passive filter
- Very Low Output impedance
  - Negligible Loading Effect



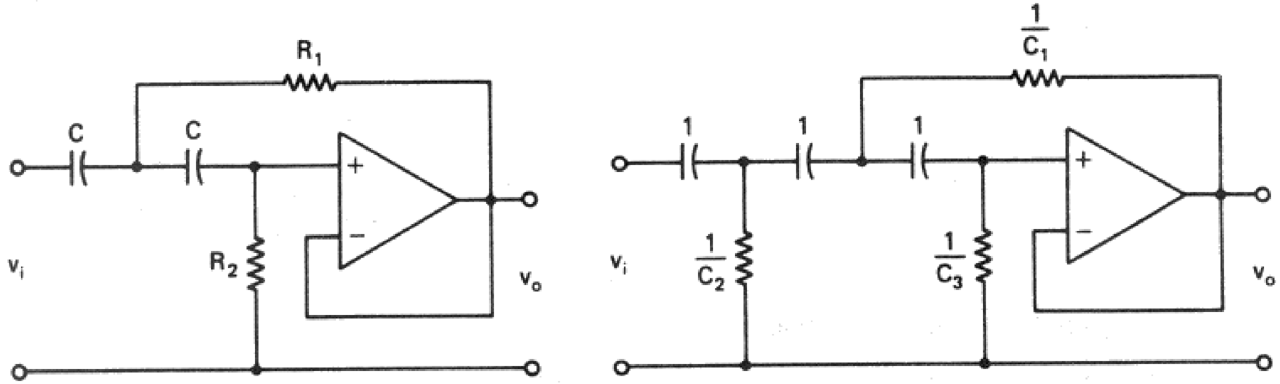


Active High-Order Filters

- **Low Pass Filters**



- **High Pass Filters**



Bandpass and Band-Reject Filters

- Butterworth Filters
  - Maximally Flat Magnitude response in pass band
  - High Attenuation Rate
- Chebyshev Filters
  - Maximum Attenuation Rate
  - Ripple in pass band
- Bessel Filters
  - Maximally flat time delay in response to step input
  - Attenuation Rate is very gradual