Preliminary Study

Transformation

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Transformation (1)

- Probability generating function
  - Nonnegative discrete random variable X

$$G_X(z) = E[z^X] = \sum_{n=0}^{\infty} z^n P_X(n)$$

  - Properties
    - $G_X(1) = 1$
    - $\frac{dG_X(z)}{dz} = G'_X(z) = \sum_{n=0}^{\infty} n\, z^{n-1} P_X(n)$

$$\Rightarrow \quad G'_X(1) = \sum_{n=0}^{\infty} n P_X(n) = E[X]$$

    - $\frac{d^2 G_X(z)}{dz^2} = G''_X(z) = \sum_{n=0}^{\infty} n(n-1)\, z^{n-2} P_X(n)$

$$\Rightarrow \quad G''_X(1) = \sum_{n=0}^{\infty} n(n-1) P_X(n) = E[X^2] - E[X]$$

Transformation (2)

- Laplace transform
  - Nonnegative continuous random variable X

$$F^*{}_X(s) = E[e^{-sX}] = \int_0^\infty f_X(x) e^{-sx} dx$$

  - Properties
    - $F^*{}_X(0) = \int_0^\infty f_X(x)dx = 1$
    - $\lim\limits_{s\to 0} \dfrac{dF_X^*(s)}{ds} = \lim\limits_{s\to 0} \int_0^\infty -xe^{-sx} f_X(x)dx = -\int_0^\infty x f_X(x)dx = -E[X]$
    - $\lim\limits_{s\to 0} \dfrac{d^2F_X^*(s)}{ds^2} = \lim\limits_{s\to 0} \int_0^\infty x^2 e^{-sx} f_X(x)dx = \int_0^\infty x^2 f_X(x)dx = E[X^2]$

Transformation (3)

- Why the transformation?
  1. It is easy to obtain the convolution, which is a distribution of sums of independent random variables

$$G_X(z) = E[Z^{(X_1+X_2+...+X_n)}] = E[Z^{X_1}]E[Z^{X_2}]...E[Z^{X_n}]$$
$$F_X^*(s) = E[e^{-s(X_1+X_2+...+X_m)}] = E[e^{-sX_1}]E[e^{-sX_2}]...E[e^{-X_m}]$$

    - Summation => multiplication
  2. It is easy to calculate the moments of a random variable
    - Integral => differential

Examples:

Probability generating Function (1)

- For Bernoulli r.v. X with parameter p
  - $G_X(z) = \mathrm{E}[z^X]$

    $= z^1 p + z^0(1-p) = zp + 1 - p$
  - $G'_X(z) = p, \quad G''_X(z)=0$
  - $\mathrm{E}[X] = G'_X(1) = p$
  - $\mathrm{Var}[X] = E[X^2] - \{\mathrm{E}[X]\}^2$

    $= G''_X(1) + G'_X(1) - \{G'_X(1)\}^2 = p\,(1-p)$

Examples:
Probability generating Function (2)

- For binomial r.v. Y with parameters n and p
  - Average of Y using probability density function
    - $\mathrm{E}[Y] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

      $= \sum_{k=0}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$

It is simpler to find the average using probability generating function

  - Average of Y using probability generating function
    - $Y = X_1 + X_2 + \cdots + X_n$ (X_i: Bernoulli r.v.)
    - $G_Y(z) = G_{X_1}(z) G_{X_2}(z) \cdots G_{X_n}(z)$

      $= \{zp + 1 - p\}^n$

      $G'_Y(z) = n\{zp + 1 - p\}^{n-1} p$

      $\mathrm{E}[Y] = G'_Y(1) = np$

Examples:
Probability generating Function (3)

- For binomial r.v. Y with parameters n and p
  - Variance of Y using probability generating function

$$G''_Y(z) = n(n-1)\{zp + 1 - p\}^{n-2}p^2$$

$$G''_Y(1) = \mathrm{E}[Y^2] - \mathrm{E}[Y] = n(n-1)p^2$$

$$\begin{aligned}\mathrm{Var}[Y] &= \mathrm{E}[Y^2] - \{\mathrm{E}[Y]\}^2 \\ &= G''_Y(1) + G'_Y(1) - \{G'_Y(1)\}^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np(1-p)\end{aligned}$$

Examples:

Probability generating Function (4)

- For geometric r.v. A with parameter p
  - Average of A using probability density function
    - $\mathrm{E}[A] = \sum_{k=1}^{\infty} k\,(1-p)^{k-1}p$
  - Average of A using probability generating function
    - $G_A(z) = \mathrm{E}[z^A] = \sum_{k=1}^{\infty} z^k\,(1-p)^{k-1}p$

$$= zp\sum_{k=1}^{\infty}\{z(1-p)\}^{k-1}$$

$$= zp[1 + z(1-p) + \{z(1-p)\}^2 + \cdots]$$

$$= \frac{zp}{1-z(1-p)}$$

$$G'_A(z) = \frac{p}{\{1-z(1-p)\}^2} \quad \Rightarrow \ \mathrm{E}[A] = G'_A(1) = \frac{1}{p}$$

Examples:

Probability generating Function (5)

- For geometric r.v. A with parameter p
  - Variance of A using probability generating function
    - $G''_A(z) = \frac{2p(1-p)}{\{1-z(1-p)\}^3}$

$$G''_A(1) = \mathrm{E}[A^2] - \mathrm{E}[A] = \frac{2p(1-p)}{p^3} = \frac{2(1-p)}{p^2}$$

$$\begin{aligned}\mathrm{Var}[A] &= \mathrm{E}[A^2] - \{\mathrm{E}[A]\}^2 \\ &= G''_A(1) + G'_A(1) - \{G'_A(1)\}^2 \\ &= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2}\end{aligned}$$

Examples:
Probability generating Function (6)

- For negative binomial r.v. B with parameter k and p
  - Average of B using p.d.f.
    - $\mathrm{E}[B] = \sum_{n=k}^{\infty} n \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} p$
    - It is not easy to directly calculate the above equation
  - Average of B using probability generating function
    - $B = A_1 + A_2 + \cdots + A_k$ (A_i: geometric r.v.)
    - $G_B(z) = G_{A_1}(z)\, G_{A_2}(z) \cdots G_{A_k}(z)$

$$= \left(\frac{pz}{1-z(1-p)}\right)^k$$

$$G'_B(z) = k\left(\frac{pz}{1-z(1-p)}\right)^{k-1} \times \frac{p}{\{1-z(1-p)\}^2}$$

$$\mathrm{E}[B] = G'_B(1) = \frac{k}{p}$$

Examples: Laplace Transform(1)

- For exponential r.v. X with parameter λ
  - Average of X using p.d.f.
    - $\mathrm{E}[X] = \int_0^\infty x\lambda\, e^{-\lambda x} dx$
    - It is not easy to directly calculate the above equation
  - Average and variance of X using Laplace transform
    - $F^*{}_X(s) = \mathrm{E}[e^{-sX}] = \int_0^\infty e^{-sx}\lambda e^{-\lambda x} dx$

      $= \lambda \int_0^\infty e^{-(\lambda+s)x} dx \; = \lambda \left[\frac{1}{-(\lambda+s)} e^{-(\lambda+s)x}\right]_0^\infty = \frac{\lambda}{\lambda+s}$
    - $\mathrm{E}[X] = -\lim_{s\to 0} \frac{dF^*{}_X(s)}{ds} = -\lim_{s\to 0} \frac{-\lambda}{(\lambda+s)^2} = \frac{1}{\lambda}$
    - $\mathrm{E}[X^2] = \lim_{s\to 0} \frac{d^2F^*{}_X(s)}{ds^2} = \lim_{s\to 0} \frac{2\lambda}{(\lambda+s)^3} = \frac{2}{\lambda^2}$
    - $\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Examples: Laplace Transform (2)

- For k-stage Erlang r.v. Y with parameter k and λ
  - Average of Y using probability density function
    - $f_Y(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!}$
    - $\mathrm{E}[Y] = \int_0^\infty x \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!} dx$
    - It is not easy to directly calculate the above equation
  - Average of Y using Laplace transform
    - $Y = X_1 + X_2 + \cdots + X_k$ (X_i: exponential r.v.)
    - $F^*{}_Y(s) = F^*{}_{X_1}(s) F^*{}_{X_2}(s) \cdots F^*{}_{X_k}(s) = \left(\frac{\lambda}{\lambda+s}\right)^k$
    - $\mathrm{E}[Y] = -\lim_{s\to 0} \frac{dF^*{}_Y(s)}{ds} = -\lim_{s\to 0} k \times \left(\frac{\lambda}{\lambda+s}\right)^{k-1} \times \left(\frac{-\lambda}{(\lambda+s)^2}\right) = \frac{k}{\lambda}$
    - $\mathrm{Var}[Y] = \frac{k}{\lambda^2}$