Markov Process

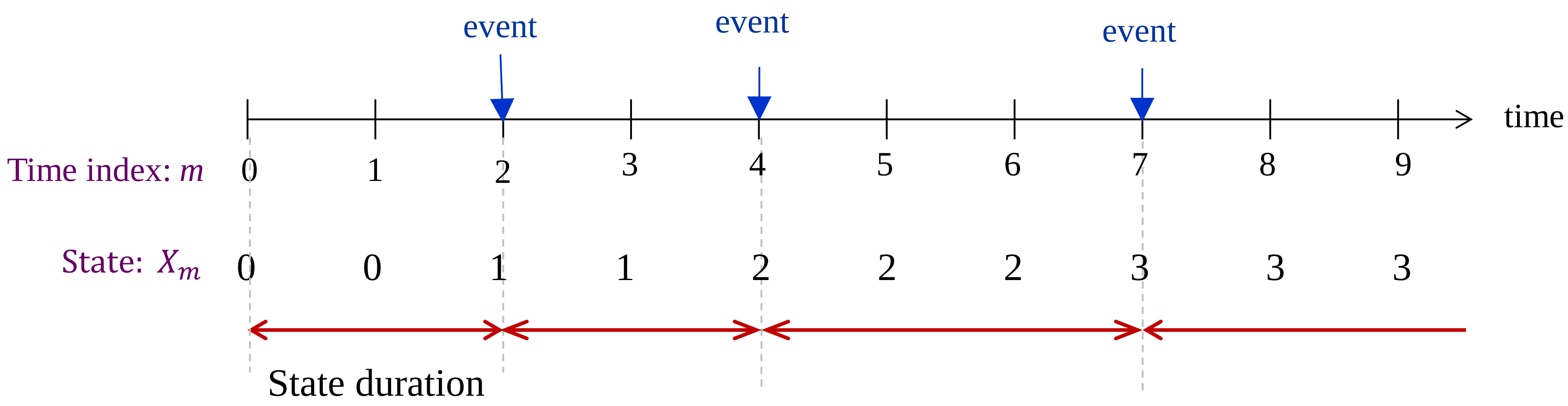
Wha Sook Jeon
Mobile Computing & Communications Lab.
Seoul National University

Markovian Property

- Markovian Property
  - Given past states and present state, conditional distribution of any future state is independent of past states and depends only on the present state.
- Markov Process
  - A stochastic process that satisfies the Markovian property.
- Types
  - Discrete time Markov chain (DTMC)
  - Continuous time Markov chain (CTMC)
  - Embedded Markov chain

Distribution of state duration:	DTMC	$\subseteq$	CTMC	$\subseteq$	Embedded MC
	geometric		exponential		general

Discrete Time Markov Chain

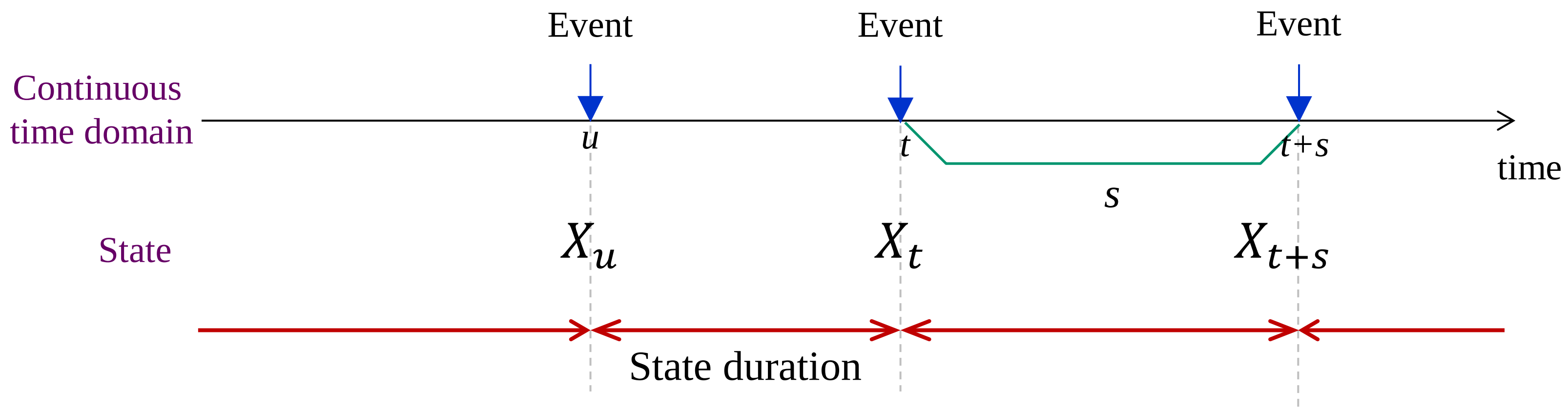


- The state duration has a geometric distribution.
- $p_{ij}(m) = \Pr\{X_{m+1} = j \mid X_0 = i_1,\ X_1 = i_2,\ \dots,\ X_m = i\}$

  $= \Pr\{X_{m+1} = j \mid X_m = i\}$

  - $p_{ij}(m)$: one-step transition probability from state *i* to state *j* at the *m*-th time index

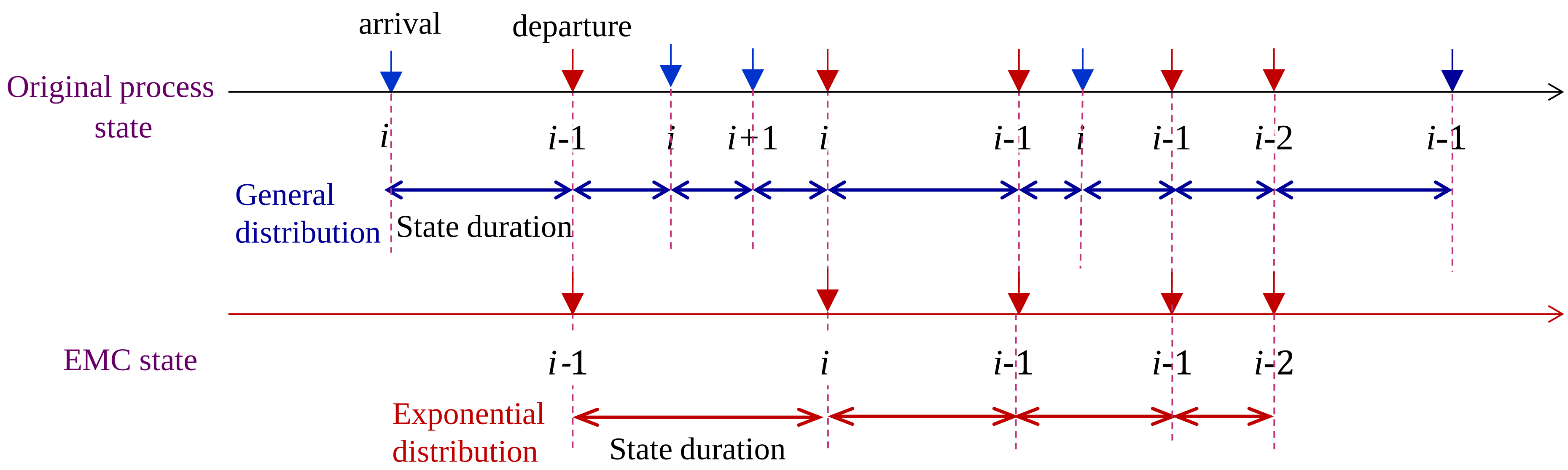
Continuous Time Markov Chain



- The state duration has an exponential distribution.
- $p_{ij}(s) = \Pr\{X_{t+s} = j \mid X_t = i,\ X_u = x_u,\ 0 \le u < t\}$

$$= \Pr\{X_{t+s} = j \mid X_t = i\}$$

Embedded Markov Chain



- The state duration of original process has general distribution; not Markov process.
- When observing the system only at departure epochs , the process has Markovian property. Then, the process at observation times is called *Embedded Markov chain*.
- The original process and the embedded Markov chain have the same statistical properties.

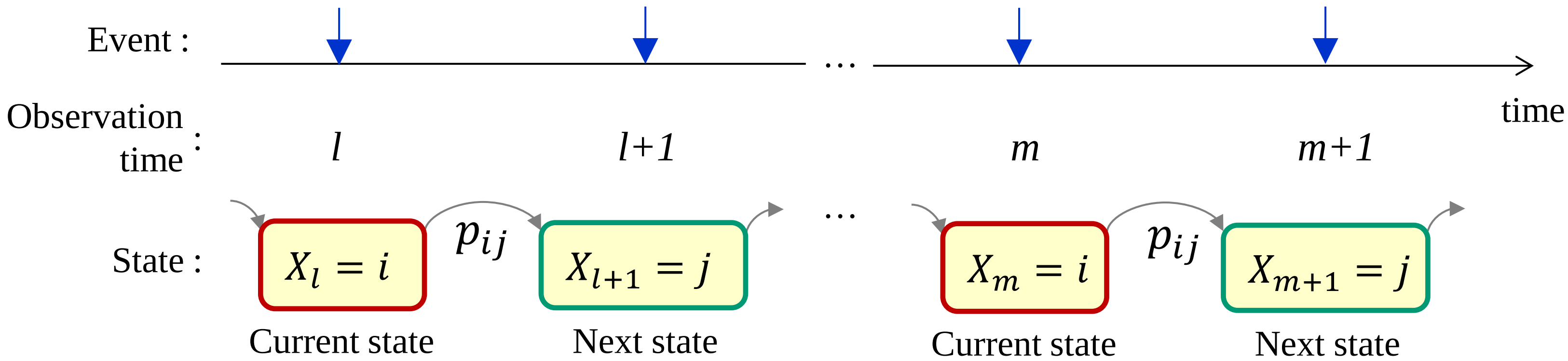
Mathematically analyzable process

- Markovian property
- Time homogeneity
- Ergodicity
  - Irreducible
  - Positive recurrent
  - Aperiodic

Homogeneous
Ergodic
Markov Process

Time homogeneity (1)

- If the conditional probability, $\Pr\{X_{m+1} = j \mid X_m = i\}$, is independent of *m*, the DTMC is said to be homogeneous.



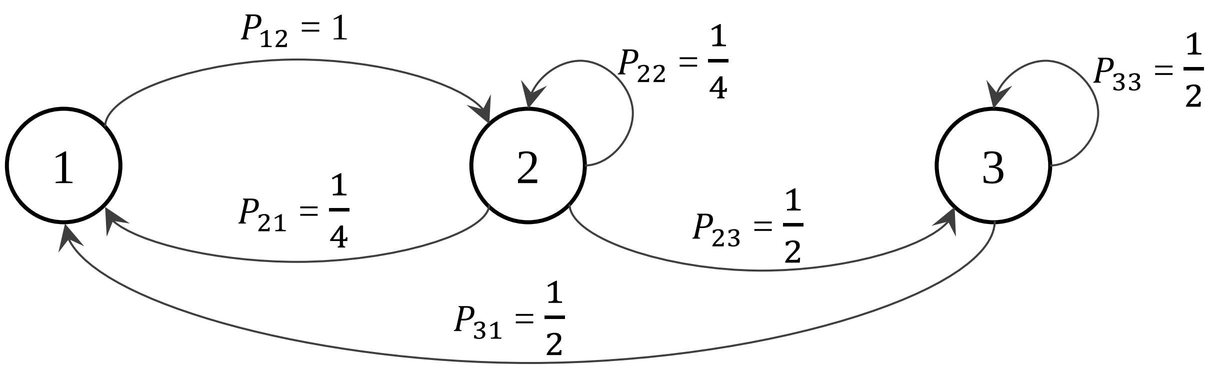
– $p_{ij} = \Pr\{X_{l+1} = j \mid X_l = i\} = \Pr\{X_{m+1} = j \mid X_m = i\}$ without respect to time index *l*, *m*

– The next state depends only on the current state and is independent of observation times.

Time homogeneity (2)

- The homogeneous DTMC is described with the state space, S, and one-step transition probability matrix, $\mathbb{P} = [p_{ij}]$,
  or state transition probability diagram.

- Example
  - State space: $S = \{1, 2, 3\}$
  - One-step transition probability matrix: $\mathbb{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$
  - State transition Probability diagram :



Time homogeneity (3)

- One-step transition probability

  $p_{ij} = \Pr\{X_{m+1} = j \mid X_m = i\}$

- n-step transition probability

  ${P_{ij}}^{(n)} = \Pr\{X_{m+n} = j \mid X_m = i\}$

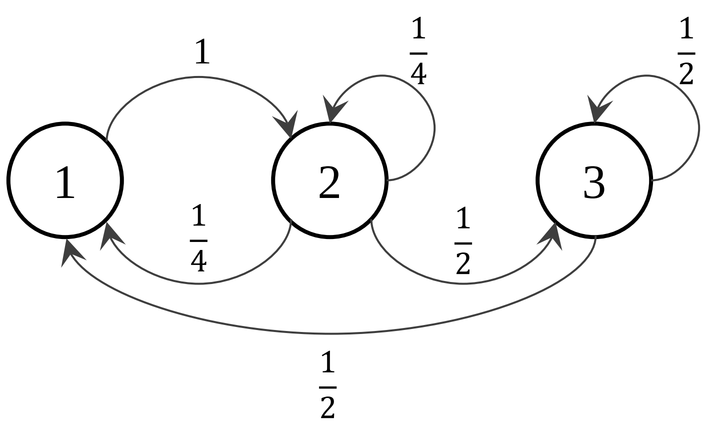
- Chapman-Kolmogorov equation

  $${P_{ij}}^{(m+n)} = \sum_{k \in S} {P_{ik}}^{(m)} {P_{kj}}^{(n)}$$

  $$\mathbb{P}^{(m+n)} = \mathbb{P}^{(m)} \times \mathbb{P}^{(n)}$$

An Ergodic Markov Chain (1)

- An ergodic Markov chain has a limiting distribution.
  - *State transition probability to state j is converge to only one value without respect to an initial state.*
  - $\lim_{n\to\infty} P_{ij}{}^{(n)} = q_j$
  - After a long period of time, an ergodic Markov chain has a distribution independent of the starting condition (limiting distribution).



$$\mathbb{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \xrightarrow{n \to \infty} \begin{bmatrix} \frac{2}{9} & \frac{4}{9} & \frac{3}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} \end{bmatrix}$$

An Ergodic Markov Chain (2)

- Ensemble average distribution
  - Let ${\pi_j}^{(n)}$ be the unconditional probability that DTMC is in state j at the n-th time index, i.e., ${\pi_j}^{(n)} \triangleq \Pr\{X_n = j\}$
  - ${\pi_j}^{(n)} = \sum_{i \in S} {\pi_i}^{(0)} {P_{ij}}^{(n)}$

    state space
  - $\pi_j \triangleq \lim_{n\to\infty} {\pi_j}^{(n)}$

    $= \lim_{n\to\infty} \sum_{i \in S} {\pi_i}^{(0)} {P_{ij}}^{(n)}$

    $= \sum_{i \in S} {\pi_i}^{(0)} \lim_{n\to\infty} {P_{ij}}^{(n)}$
- The ensemble average distribution is the same as the limiting distribution
  - Since $\lim_{n\to\infty} {P_{ij}}^{(n)} = q_j$, $\pi_j = q_j \sum_{i \in S} {\pi_i}^{(0)} = q_j \quad \Rightarrow \quad \underline{\underline{\pi_j = q_j}}$

An Ergodic Markov Chain (3)

– $\pi_j{}^{(n)} = \sum_{i \in S} \pi_i{}^{(n-1)} P_{ij}$

$$\lim_{n\to\infty} \pi_j{}^{(n)} = \lim_{n\to\infty} \sum_{i \in S} \pi_i{}^{(n-1)} P_{ij}$$

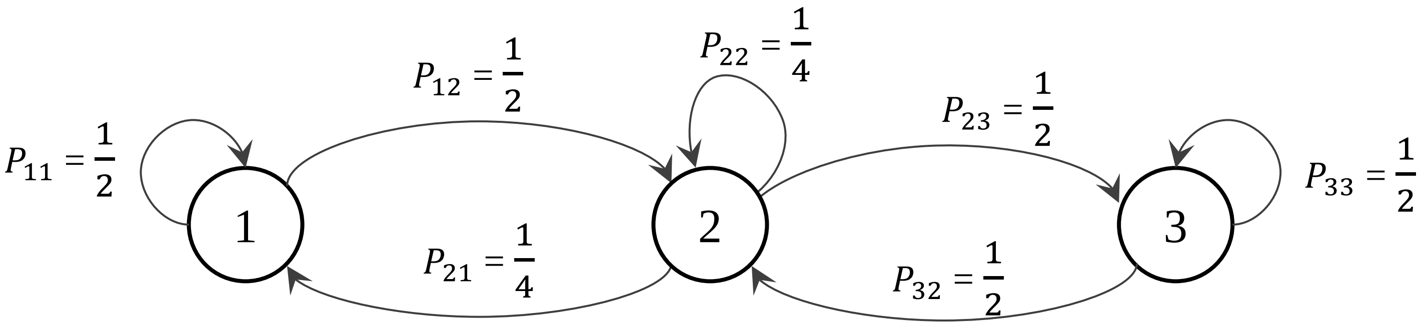
$$= \sum_{i \in S} \left(\lim_{n\to\infty} \pi_i{}^{(n-1)} \right) P_{ij}$$

$\Rightarrow$ $\underline{\underline{\pi_j = \sum_{i \in S} \pi_i P_{ij}}}$

- We can obtain the state distribution of ergodic Markov chain, by solving (1) and (2).
  - $\pi_i = \sum_{j \in S} \pi_j P_{ji}$ for all $i \in S$ … (1)
  - $\sum_{i \in S} \pi_i = 1$ … (2)

An Ergodic Markov Chain (4)

- Example
  - State space $S = \{1, 2, 3\}$
  - State transition Probability diagram :



  - $\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2$

    $\pi_2 = \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3$

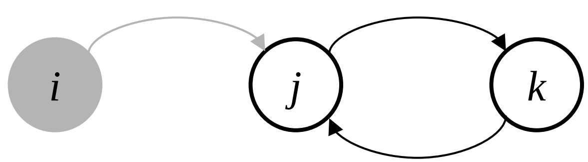
    $\pi_3 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3$
  - $\pi_1 + \pi_2 + \pi_3 = 1$

$$\Rightarrow \quad \pi_1 = \frac{2}{9}, \quad \pi_2 = \frac{4}{9}, \quad \pi_3 = \frac{3}{9}$$

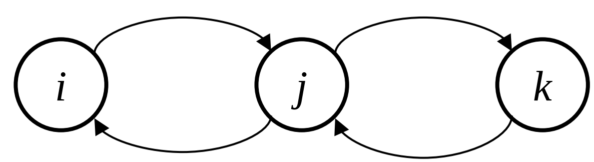
Necessary Conditions for an Ergodic MC (1)

❖ Irreducible

- State j is reachable from state i if there is an integer $n \geq 1$ such that ${P_{ij}}^{(n)} > 0$.
- If state i is reachable from state j and state j is reachable from state i, state i and j are said to communicate.
- If all states in the Markov chain communicate to each other, the Markov chain is called "irreducible".



reducible



irreducible

Necessary Conditions for an Ergodic MC (2)

❖ Positive recurrent

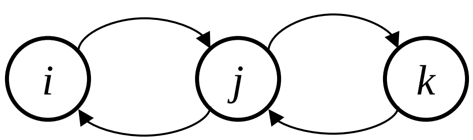
- f_{ij} : the probability of ever making a transition into state j, given that Markov chain is in state i.
- State i is said to be recurrent if $f_{ii} = 1$
- If the mean recurrent time is finite, state i is a positive recurrent state.
- If all states in the Markov chain are positive recurrent, the Markov chain is called "positive recurrent".
- An irreducible Markov chain having the finite number of states is positive recurrent.

Necessary Conditions for an Ergodic MC (3)

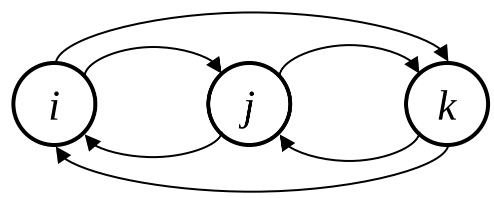
❖ Aperiodic

- State i is said to have a period of d, if $P_{ii}{}^{(n)} = 0$ whenever n is not divided by d and d is the greatest integer with this property.
- A state with period 1 is an aperiodic state.
- If all states in the Markov chain are aperiodic, the Markov chain is called “aperiodic”.

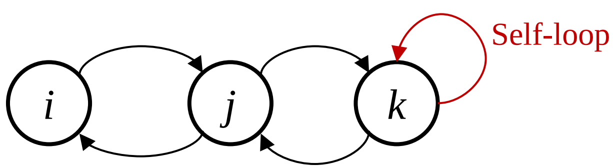
aperiodic if there is
at least one self-loop



Periodic



Aperiodic



Aperiodic

$\text{GCD}_i(2,4,6,8,\ldots) = 2$	$\text{GCD}_i(2,3,4,5,\ldots) = 1$	$\text{GCD}_i(2,4,5,6,7,8,\ldots) = 1$
$\text{GCD}_j(2) = 2$	$\text{GCD}_j(2,3,4,5,\ldots) = 1$	$\text{GCD}_j(2,3,4,5,6,7,\ldots) = 1$
$\text{GCD}_k(2,4,6,8,\ldots) = 2$	$\text{GCD}_k(2,3,4,5,\ldots) = 1$	$\text{GCD}_k(1,2,3,4,5,6,\ldots) = 1$

$\text{GCD}_s(n_1, n_2, \ldots)$: the greatest common divisor of the state transition steps $(n_1, n_2, \ldots)$ for back to the state s.

Time Average and Ensemble Average

- If a system is an ergodic Markov chain, the ensemble average is equal to the time average.
- π_i can be interpreted as two aspects; one is the time average, and the other is the ensemble average.
  - Time average
    - π_i is the long-run time proportion that the Markov Chain is in state *i*. *on any sample path*
  - Ensemble average
    - π_i is the probability that the state of Markov chain is *i* in steady state.
- $\{X(t)\}$ is ergodic in the most general sense if all its measures can be determined or well approximated from a single realization of the process.
- It is often done in analyzing simulation outputs

Stationary DTMC

- ${\pi_j}^{(n)} = \sum_{i \in S} {\pi_i}^{(n-1)} P_{ij}. \quad \Rightarrow \quad \Pi^{(n)} = \Pi^{(n-1)} \mathbb{P}$

- If the initial state distribution $\Pi^{(0)}$ is set to the limiting distribution,

$$\left.\begin{aligned} \Pi^{(1)} &= \Pi^{(0)} \mathbb{P} = \Pi \mathbb{P} = \Pi \\ \Pi^{(2)} &= \Pi^{(1)} \mathbb{P} = \Pi \mathbb{P} = \Pi \\ &\cdots \\ \Pi^{(m)} &= \Pi^{(m-1)} \mathbb{P} = \Pi \mathbb{P} = \Pi \end{aligned}\right\} \Rightarrow \quad \Pi^{(n)} = \Pi, \quad \text{for all } n$$

The state distribution is invariant over time, $\pi_i = \Pr\{X_n = i\}$ for all n

$\Rightarrow$ stationary process

- In summary, DTMC of which the initial state distribution is set to the limiting distribution is stationary, and then the limiting distribution is called the stationary distribution.