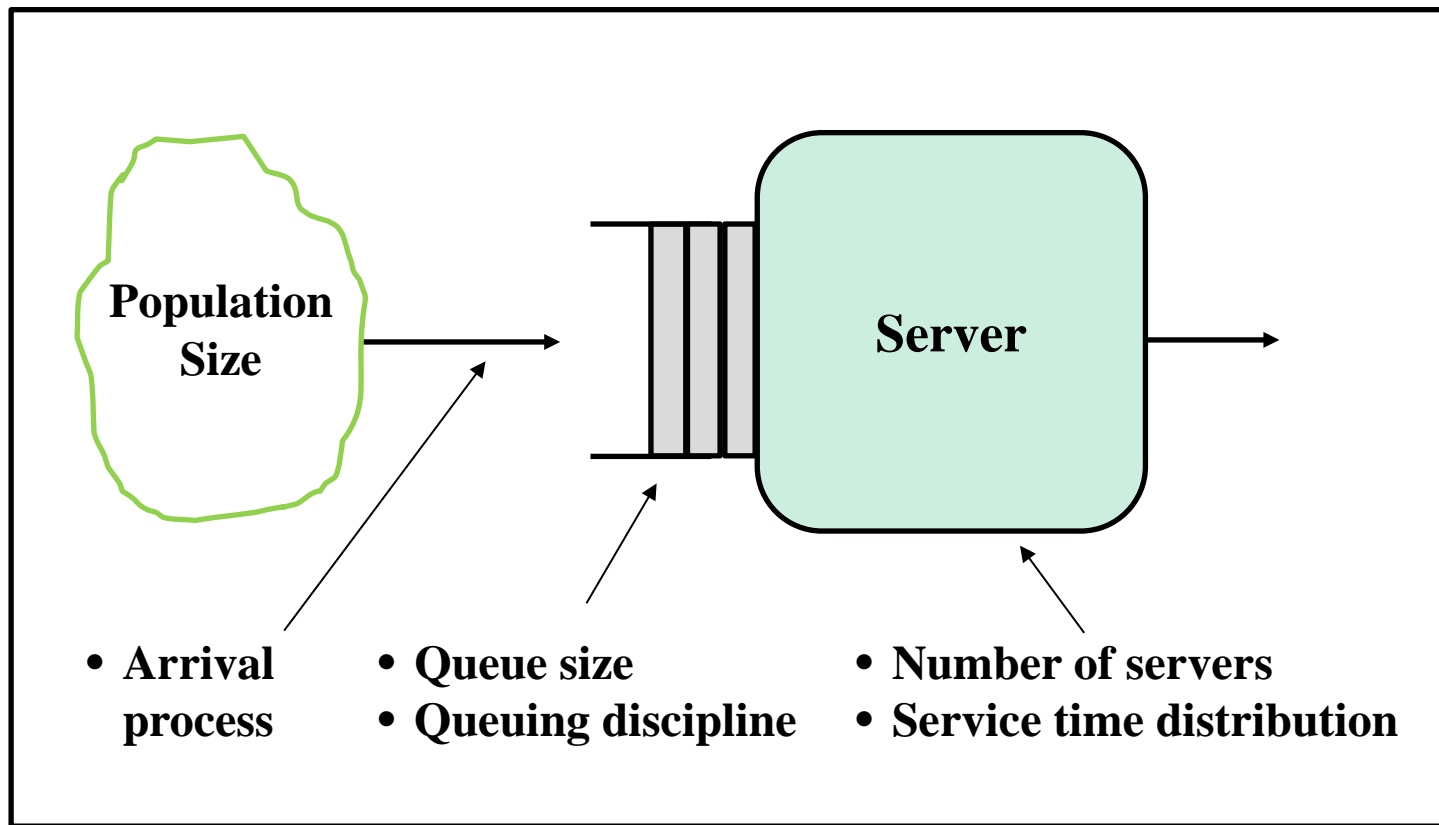# Single Queue System I

Wha Sook Jeon

Mobile Computing & Communications Lab.

Seoul National University

Kendall's Notation (1)

- One way to describe queuing system

Queuing System

Diagram labels: Population Size, Server

- **Arrival process**
- **Queue size**
- **Queuing discipline**
- **Number of servers**
- **Service time distribution**

Kendall's Notation (2)

$$\underbrace{\text{Arrival process} \Big/ \text{Service time distribution} \Big/ \text{Number of servers}}_{\text{mandatory}} \Big/ \text{Size of queue} \Big/ \text{Size of population} \Big/ \text{Queuing discipline}$$

- Arrival process/service time distribution
  - M: Poisson arrivals or exponential service time distribution
  - E_k: k-stage Erlang distribution
  - D: deterministic distribution
  - G: general distribution (GI - general & independent arrival)
- Size of queue (including jobs in the server): ∞ (default)
- Size of population: ∞ (default)
- Queuing discipline
  - FCFS (default)
  - LCFS
  - RS (random selection)
  - PR (priority): preemptive or non-preemptive

Kendall's Notation (3)

- Kendall's notation is only for continuous time stochastic process
- Examples of Kendall's notation
  - M/M/1
  - M/G/1
  - E_k/M/1
  - M/M/c/c : (no queueing)
  - M/M/c/K: K is typically larger than c
  - M/M/∞ : no waiting
  - M/G/∞
  - GI/M/1

M/M/1 (1)

- Poisson arrival process (exponentially distributed inter-arrival time)
- exponentially distributed service time
- One server

- A special case of Birth & Death process
  - Birth & Death process with fixed arrival rate λ and fixed service rate μ, which are independent of system state

- Stochastic process representation of M/M/1:
  - $\{N(t), t \geq 0\}$
    - $N(t)$ is a random variable that represents the number of jobs in the system at time t

M/M/1 (2)

- Derivation of state probability distribution

  Let $P_n(t) \triangleq \Pr\{N(t)=n\}$

  Since M/M/1 is a birth and death process,

  $$P_0(t + h) = P_0(t)(1 - \lambda h) + P_1(t)\mu h$$
  $$P_n(t + h) = P_{n-1}(t)\lambda h + P_{n+1}(t)\mu h + P_n(t)(1 - \lambda h - \mu h), \quad \text{for } n \geq 1$$

  $$\lim_{h \to 0} \frac{P_0(t+h) - P_0(t)}{h} = P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad \cdots\cdots (1)$$

  $$\lim_{h \to 0} \frac{P_n(t + h) - P_n(t)}{h} = P_n'(t) = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - \lambda P_n(t) - \mu P_n(t)$$

  $$\text{for } n \geq 1 \qquad \cdots\cdots (2)$$
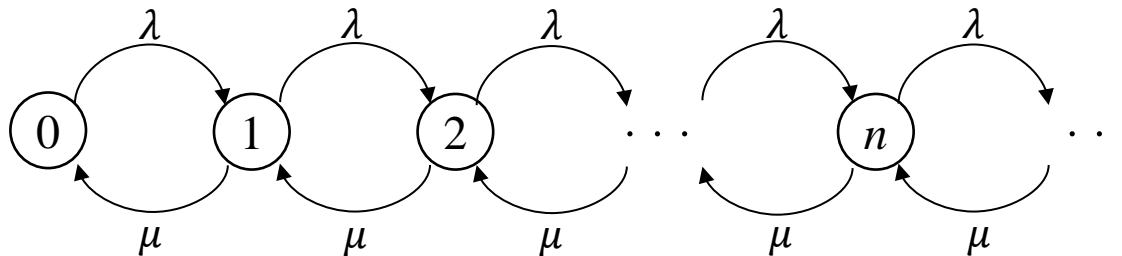
M/M/1 (3)

- In steady state, since $P_n'(t) = 0$, $P_n(t) = P_n$
- From (1) and (2),

    $$-\lambda P_0 + \mu P_1 = 0$$

    $$\lambda P_{n-1} + \mu P_{n+1} - \lambda P_n - \mu P_n = 0, \quad \text{for } n \geq 1$$

    $$\Rightarrow \quad \lambda P_n = \mu P_{n+1}, \quad \text{for } n \geq 0 \quad \cdots \cdots (3)$$

- From (3), state transition rate diagram is

M/M/1 (4)

- From the state rate transition diagram

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = \mu P_2$$
$$\vdots$$

$\times \quad \lambda P_{n-1} = \mu P_n$

$$\lambda^n P_0 P_1 \cdots P_{n-1} = \mu^n P_1 P_2 \cdots P_{n-1} \, P_n \qquad \Rightarrow \lambda^n P_0 = \mu^n P_n$$

- Let $\rho = \dfrac{\lambda}{\mu}$. For stable system, $\lambda < \mu \quad (\rho < 1)$

- $P_n = \rho^n P_0$

- Since $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 = 1 - \rho \ \text{ from } \ P_0(1 + \rho + \rho^2 + \rho^3 + \cdots) = 1$$

- P_0: the probability that the server is idle
- ρ : the probability that the server is busy

- $P_n = \rho^n (1 - \rho)$

M/M/1 (5)

- Performance Measures
  - $\bar{N}$: the mean number of jobs in the system
    - $\bar{N} = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = (1-\rho)\sum_{n=0}^{\infty} n\rho^n$

      $= \frac{\rho}{1-\rho}$ (since $\sum_{n=0}^{\infty} n\rho^n = \frac{\rho}{(1-\rho)^2}$)
  - $\bar{M}$: the mean number of jobs in the server
    - $\bar{M} = 1 \cdot \rho + 0 \cdot (1-\rho) = \rho$
  - $\bar{Q}$: the mean number of jobs in the queue
    - $\bar{Q} = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)\rho^n(1-\rho)$

      $= \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^n = \frac{\rho^2}{1-\rho}$
    - By Little's Law, $\bar{Q} = \bar{N} - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho}$

M/M/1 (6)

- $\bar{T}$: the mean sojourn time of a job in the system

    - $\bar{T} = \sum_{n=0}^{\infty}(n+1)\frac{1}{\mu}P_n = \frac{1}{\mu}(\sum_{n=0}^{\infty}nP_n + \sum_{n=0}^{\infty}P_n)$

        $= \frac{1}{\mu}\left(\frac{\rho}{1-\rho} + 1\right) = \frac{1}{\mu}\frac{1}{1-\rho}$

    - By Little's Law, $\bar{T} = \frac{1}{\lambda}\bar{N} = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu}\frac{1}{1-\rho}$

- $\bar{W}$: the mean waiting time of a job in the queue

    - $\bar{W} = \sum_{n=0}^{\infty}\frac{1}{\mu}nP_n = \frac{1}{\mu}\frac{\rho}{1-\rho}$

    - mean service time: $\bar{S} = \frac{1}{\mu}$

    - By Little's Law, $\bar{W} = \bar{T} - \bar{S} = \frac{1}{\mu}\frac{1}{1-\rho} - \frac{1}{\mu} = \frac{1}{\mu}\frac{\rho}{1-\rho}$
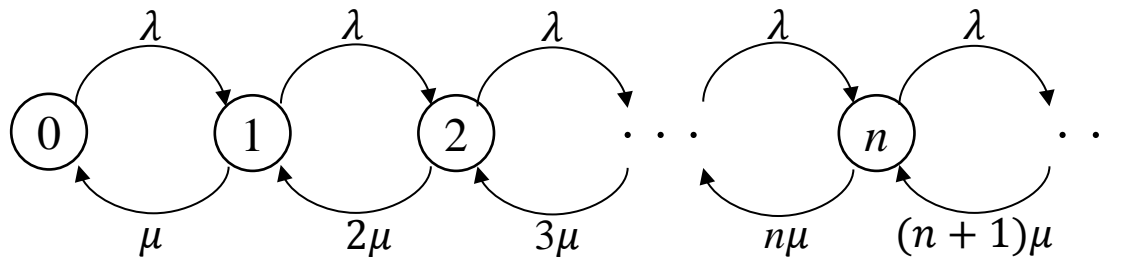
M/M/1 (7)

- Throughput: mean departure rate

$$\Lambda = \frac{1}{\text{mean departure time}} = \frac{1}{\rho\frac{1}{\mu}+(1-\rho)(\frac{1}{\mu}+\frac{1}{\lambda})} = \lambda$$

- Sever utilization: ρ

  : Server busy probability

  : Mean number of jobs in the server

M/M/∞ (1)

- arrival rate: $\lambda(n) = \lambda$
- load dependent service rate: $\mu(n) = n\mu$

- State transition rate diagram

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$\vdots$$
$$\times \quad \lambda P_{n-1} = n\mu P_n$$
$$\overline{\lambda^n P_0 P_1 \cdots P_{n-1}} = n! \, \mu^n P_1 P_2 \cdots P_{n-1} \, P_n$$

$$\Rightarrow \quad P_n = \frac{1}{n!} \rho^n P_0, \quad \text{where } \rho = \frac{\lambda}{\mu}$$

M/M/∞ (2)

- Since $\sum_{n=0}^{\infty} P_n = 1$,

    $P_0 \sum_{n=0}^{\infty} \frac{\rho^n}{n!} = 1 \qquad \Rightarrow \quad P_0 = e^{-\rho}$

- $P_n = \frac{\rho^n}{n!} e^{-\rho} = \frac{(\lambda \frac{1}{\mu})^n}{n!} e^{-\lambda \frac{1}{\mu}}$

    pdf of Poisson distribution

    M/G/∞ :
    $$P_n = \frac{(\lambda \mathrm{E}[s])^n}{n!} e^{-\lambda \mathrm{E}[S]}$$

- Performance Measures

  - Mean number of jobs in system

    $\bar{N} = \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} e^{-\rho} = \rho e^{-\rho} \sum_{n=0}^{\infty} \frac{\rho^n}{n!} = \rho$

    - When using Little's Law, $\bar{N} = \bar{T} \times \lambda = \frac{1}{\mu} \times \lambda = \rho$

  - System utilization: $1 - P_0 = 1 - e^{-\rho}$
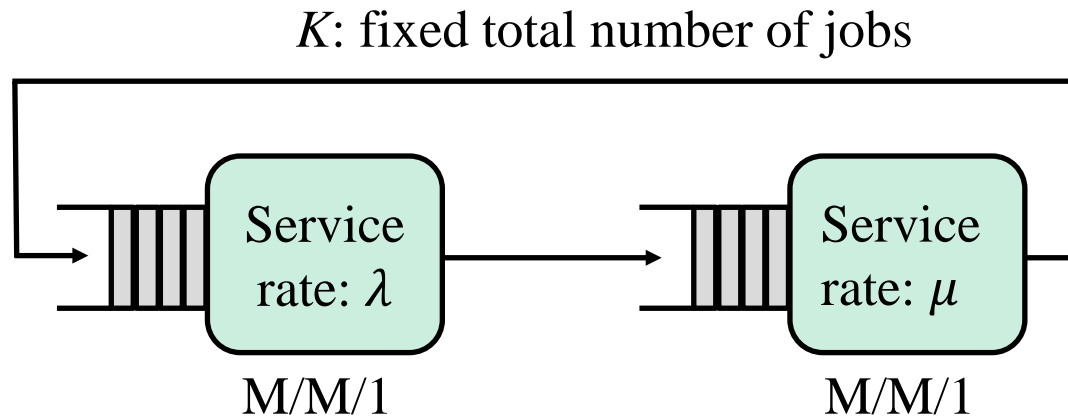
  - Mean response time: $\bar{T} = \frac{1}{\mu}$

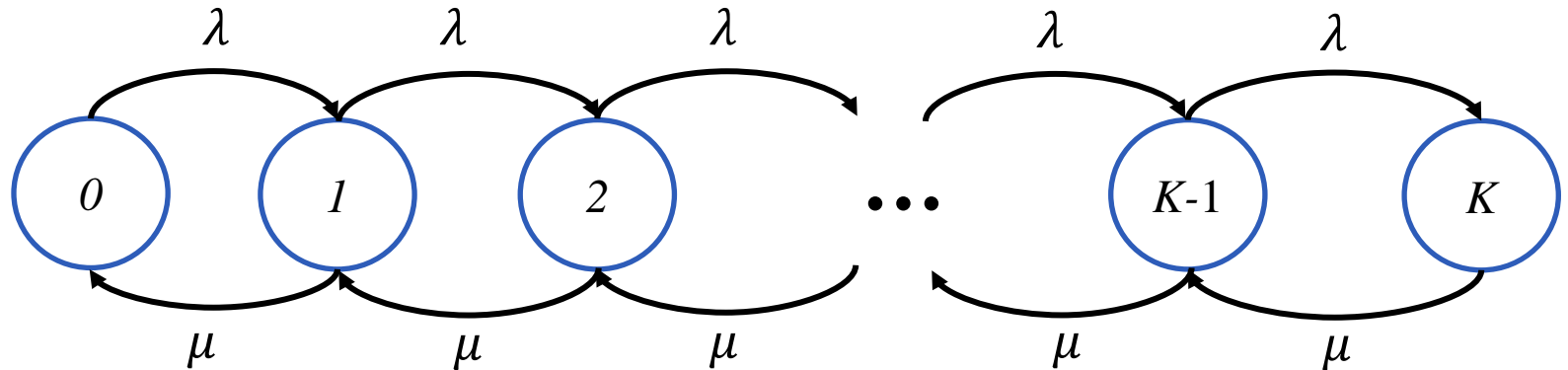 - Throughput: λ

M/M/1/K (1)

- Arrival rate $\lambda(n) = \lambda$,   for $n \leq K$
- Service rate $\mu(n) = \mu$,   for $n \leq K$
- One server
- The number of jobs in system: at maximum K

K: fixed total number of jobs

Service rate: λ →   Service rate: μ

M/M/1   M/M/1

M/M/1/K (2)

- A state transition rate diagram

$$\lambda P_0 = \mu P_1$$

$$\lambda P_1 = \mu P_2$$

$$\lambda P_2 = \mu P_3$$

$$\vdots$$

$$\times \qquad \lambda P_{n-1} = \mu P_n$$

$$\overline{\lambda^n P_0 P_1 \cdots P_{n-1} = \mu^n P_1 P_2 \cdots P_{n-1} \, P_n}$$

$$\Rightarrow \lambda^n P_0 = \mu^n P_n \qquad\qquad \therefore P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad (0 \le n \le K)$$

M/M/1/K (3)

- Since $\sum_{n=0}^{k} P_n = 1$

$$1 \;=\; P_0 + \left(\frac{\lambda}{\mu}\right) P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \left(\frac{\lambda}{\mu}\right)^3 P_0 \;+\; ... \;+\; \left(\frac{\lambda}{\mu}\right)^K P_0$$
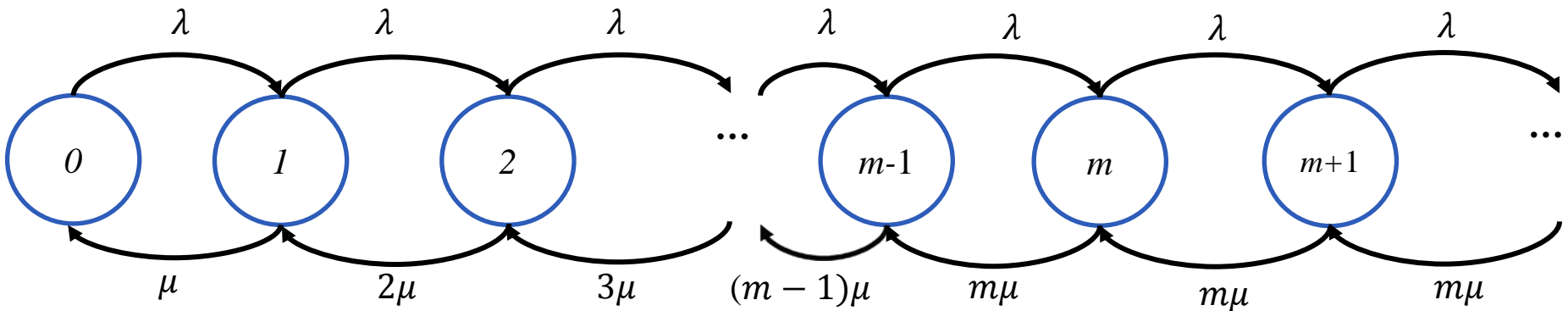
$$= P_0 \left\{ \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 \;+\; ... \;+\; \left(\frac{\lambda}{\mu}\right)^K \right\}$$

$$= P_0 \left(\frac{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}{1 - \frac{\lambda}{\mu}} \right) \qquad \Rightarrow \qquad P_0 = \left(\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} \right)$$

$$\therefore\; P_n = \left(\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} \right) \left(\frac{\lambda}{\mu}\right)^n \qquad (0 \le n \le K)$$

M/M/m (1)

- State transition rate diagram

- If $n \leq m$

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$\vdots$$
$$\times \quad \lambda P_{n-1} = n\mu P_n$$

$$\overline{\lambda^n P_0 \, P_1 P_2 \ldots P_{n-1} = n! \, \mu^n P_1 P_2 P_3 \ldots P_n}$$

$$\Rightarrow \quad \lambda^n P_0 = n! \, \mu^n P_n \qquad\qquad \therefore P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad (n \leq m)$$

M/M/m (2)

- If $n > m$,

$$\lambda P_m = m\mu P_{m+1}$$
$$\lambda P_{m+1} = m\mu P_{m+2}$$
$$\lambda P_{m+2} = m\mu P_{m+3}$$
$$\vdots$$

$\times \quad \lambda P_{n-1} = m\mu P_n$

$$\overline{\lambda^{n-m} P_m P_{m+1} P_{m+2} \dots P_{n-1} = (m\mu)^{n-m} P_{m+1} P_{m+2} P_{m+3} \dots P_n}$$

$$\Rightarrow \quad \lambda^{n-m} P_m = (m\mu)^{n-m} P_n$$

$$\therefore P_n = \left(\frac{\lambda}{m\mu}\right)^{n-m} \frac{1}{m!} \left(\frac{\lambda}{\mu}\right)^m P_0 \quad \text{for } n > m$$

M/M/m (3)

- Since $\sum_{n=0}^{\infty} P_n = 1$

$$1 = \sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=m+1}^{\infty} \left(\frac{\lambda}{m\mu}\right)^{(n-m)} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m P_0$$

$$= P_0 \left\{ \sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \sum_{k=1}^{\infty} \left(\frac{\lambda}{m\mu}\right)^k \right\}$$

$$= P_0 \left\{ \sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \left(\frac{\lambda}{m\mu-\lambda}\right) \right\}$$

$$\therefore P_0 = \frac{1}{\sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \left(\frac{\lambda}{m\mu-\lambda}\right)}$$
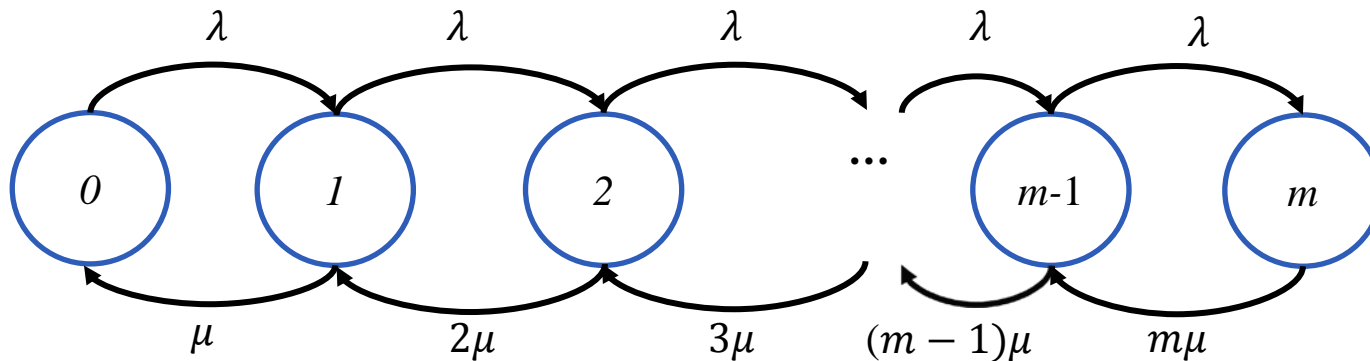
- Erlang's *C* formula
  - The probability that an arriving customer enters the queue

$$\mathrm{Pr}\{queuing\} = \sum_{n=m}^{\infty} P_n = \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \frac{m\mu-\lambda}{m\mu} P_0$$

M/M/m/m (1)

- State transition rate diagram

- $n = 0, 1, 2, \ldots, m$

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$\vdots$$

$\times$ $\quad \lambda P_{n-1} = n\mu P_n$

$$\overline{\lambda^n P_0 \, P_1 P_2 \ldots P_{n-1} = n! \, \mu^n P_1 P_2 P_3 \ldots P_n}$$

$$\Rightarrow \quad \lambda^n P_0 = n! \, \mu^n P_n \qquad\qquad \therefore P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

M/M/m/m (2)

- Since $\sum_{n=0}^{\infty} P_n = 1$

$$\sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 1 \quad \Rightarrow \quad P_0 = \frac{1}{\sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n}$$
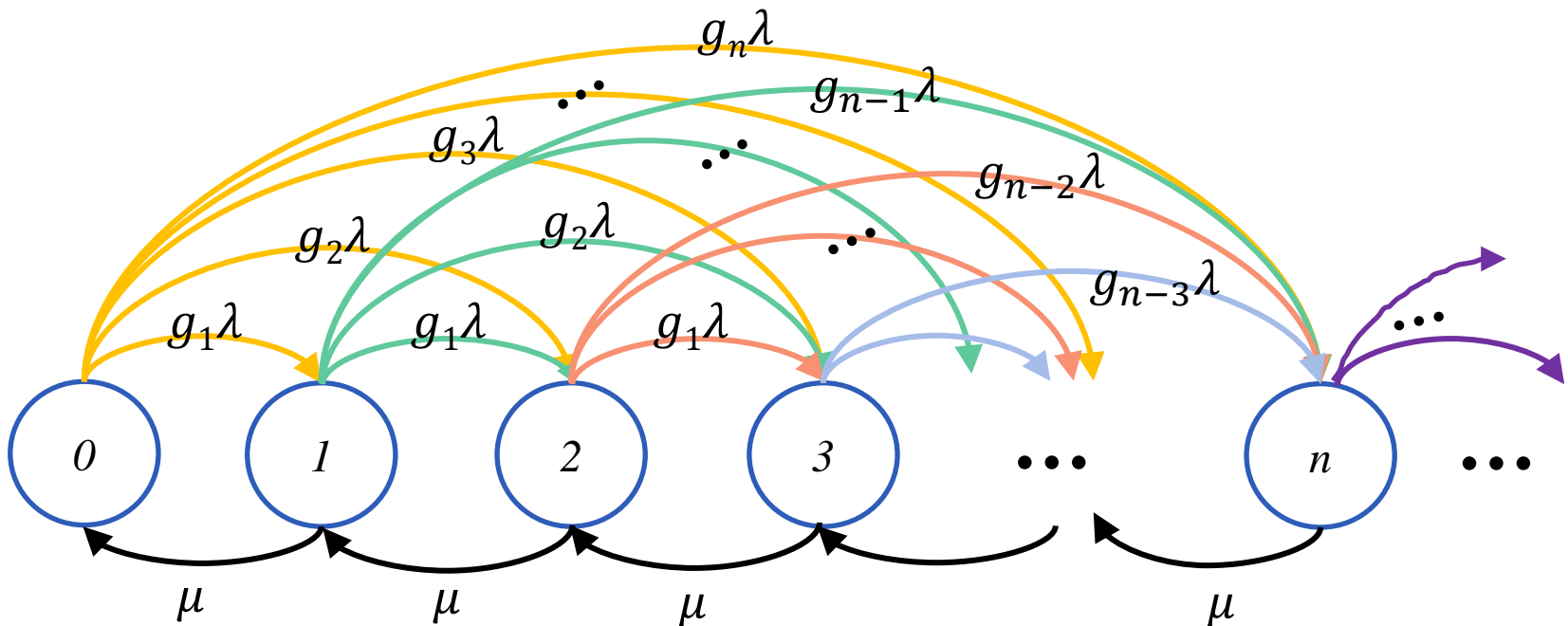
- Erlang's *B* formula
  - Probability that an arriving customer is blocked: Erlang's loss formula

$$P_m = \frac{\frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m}{\sum_{n=0}^{m} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n}$$
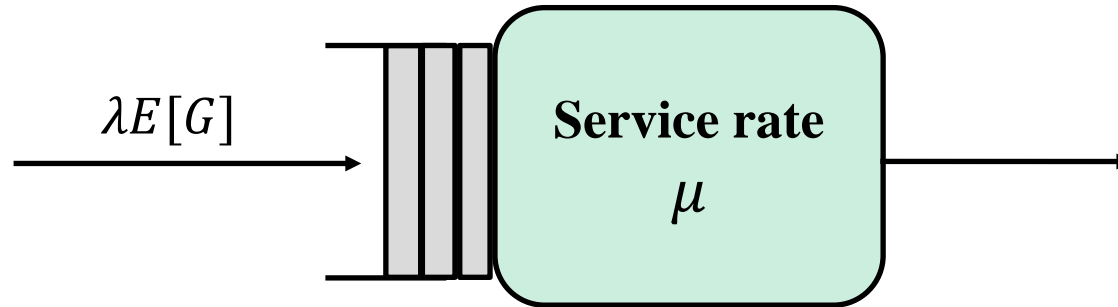
$M_X/M/1$ (1)

- Compound Poisson arrival process (CPP): Batch (Bulk) arrival
- Size of a batch: a random variable G with general distribution

  $\Pr\{G = n\} = g_n$ $(n \geq 1)$

- State transition rate diagram

M$_X$/M/1 (2)

- $\lambda P_0 = \mu P_1$ … (1)

- $(\lambda+\mu)P_1 = \lambda g_1 P_0 + \mu P_2$

  $(\lambda+\mu)P_2 = \lambda g_2 P_0 + \lambda g_1 P_1 + \mu P_3$

  $(\lambda+\mu)P_3 = \lambda g_3 P_0 + \lambda g_2 P_1 + \lambda g_1 P_2 + \mu P_4$

  $$\vdots$$

  $\Rightarrow \quad (\lambda+\mu)P_n = \mu P_{n+1} + \lambda \sum_{k=1}^{n} g_k P_{n-k} \qquad (n \geq 1) \qquad … (2)$

- How to calculate the mean number of jobs in the system, $\overline{N}$

1. Calculate P_0, P_1, P_2 … until $(nP_n < \varepsilon)$ and approximate $\overline{N} = \sum_{k=0}^{n} kP_k$

2. Derive $\overline{N}$ using probability generating function

Mx/M/1 (3)

Mean number of jobs in server: $\rho = \dfrac{\lambda E[G]}{\mu}$
(server busy probability)

1. Calculation of state probability distribution: $\{P_n\}$

$$P_0 = 1 - \rho = 1 - \lambda E[G]/\mu$$

$$P_1 = \frac{\lambda}{u} P_0$$

$$P_2 = \left(1 + \frac{\lambda}{\mu}\right) P_1 - \frac{\lambda}{u} g_1 P_0$$

$$\vdots$$

until $(n P_n < \varepsilon)$

Mx/M/1 (4)

2. Derivation of $\overline{N}$ using probability generating function

- Using the probability generating function, we define
$$Y(z) = E[z^X] = \sum_{n=0}^{\infty} z^n P_n$$
$$B(z) = E[z^G] = \sum_{k=1}^{\infty} z^k g_k$$

- Using (2)

$$\sum_{n=1}^{\infty} (\lambda + \mu) z^n P_n = \sum_{n=1}^{\infty} \left(\mu z^n P_{n+1} + \lambda \sum_{k=1}^{n} z^n g_k P_{n-k} \right)$$

$$(\lambda + \mu) \sum_{n=1}^{\infty} z^n P_n = \frac{\mu}{z} \sum_{n=1}^{\infty} z^{n+1} P_{n+1} + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^{n} z^k g_k z^{n-k} P_{n-k}$$

$$(\lambda + \mu) \left(\sum_{n=0}^{\infty} z^n P_n - P_0 \right)$$

$$= \frac{\mu}{z} \left(\sum_{n=0}^{\infty} z^n P_n - P_0 - z P_1 \right) + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^{n} z^k g_k z^{n-k} P_{n-k}$$

$$\rightarrow (\lambda + \mu)(Y(z) - P_0) = \frac{\mu}{z}(Y(z) - P_0 - z P_1) + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^{n} z^k g_k z^{n-k} P_{n-k}$$

Mx/M/1 (5)

✓ $\sum_{n=1}^{\infty}\sum_{k=1}^{n} z^k g_k z^{n-k} P_{n-k}$

$=$

$z^1 g_1 z^0 P_0$
$+ z^1 g_1 z^1 P_1 \quad | + z^2 g_2 z^0 P_0$
$+ z^1 g_1 z^2 P_2 \quad | + z^2 g_2 z^1 P_1 \quad | + z^3 g_3 z^0 P_0$
$+ z^1 g_1 z^3 P_3 \quad | + z^2 g_2 z^2 P_2 \quad | + z^3 g_3 z^1 P_1 \quad | + z^4 g_4 z^0 P_0$
$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$

①   ②   ③ $\cdots$

① $= z^1 g_1 (z^0 P_0 + z^1 P_1 + z^2 P_2 + \cdots) = z^1 g_1 Y(z)$

② $= z^2 g_2 (z^0 P_0 + z^1 P_1 + z^2 P_2 + \cdots) = z^2 g_2 Y(z)$

③ $= z^3 g_3 (z^0 P_0 + z^1 P_1 + z^2 P_2 + \cdots) = z^3 g_3 Y(z)$

$\vdots$

① $+$ ② $+$ ③ $+ \cdots = Y(z)(z^1 g_1 + z^2 g_2 + z^3 g_3 + \cdots) = \boldsymbol{Y}(z)\,\boldsymbol{B}(z)$

Mx/M/1 (6)

$\Rightarrow (\lambda + \mu)z(Y(z) - P_0) = \mu(Y(z) - P_0 - zP_1) + \lambda zY(z)B(z)$ … (3)

- From (1) and (3), $Y(z) = \dfrac{\mu P_0(1-z)}{\mu(1-z) - \lambda z + \lambda zB(z)}$ … (4)

- Y(1) = 1 , but Y(1) = $\dfrac{\mu P_0(1-1)}{\mu(1-1) - \lambda + \lambda B(1)} = \dfrac{0}{0}$ from (4)

- By using L'Hopital's rule,

  $Y(z) = \dfrac{(\mu P_0(1-z))'}{(\mu(1-z) - \lambda z + \lambda zB(z))'} = \dfrac{-\mu P_0}{-\mu - \lambda + \lambda B(z) + \lambda zB'(z)}$ … (5)

- Since $B(1) = 1$ and $B'(1) = E[G]$, from (5)

  $Y(1) = \dfrac{-\mu P_0}{-\mu + \lambda E[G]} = 1$ $\therefore P_0 = 1 - \dfrac{\lambda E[G]}{\mu}$

Mx/M/1 (7)

- Performance Measures
    - $\overline{N}$: the mean number of jobs in the system

$$\overline{N} = \mathrm{E}[X] = \mathrm{Y}'(1) = \frac{\lambda(E[G] + E[G^2])}{2(\mu - \lambda E[G])}$$

    - ρ: the mean number of jobs in server

$$\rho = \lambda E[G] E[S]$$

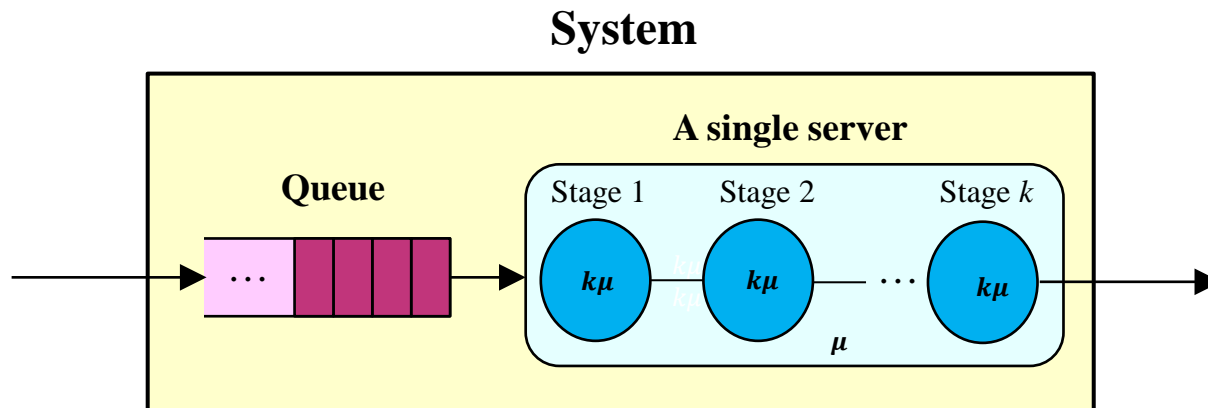   - $\overline{Q}$: the mean number of jobs in the queue

$$\overline{Q} = \overline{N} - \rho$$

    - $\overline{W}$: the mean sojourn time of a job in system

$$\overline{W} = \frac{\overline{N}}{\lambda E[G]} \quad \text{by Little's law}$$
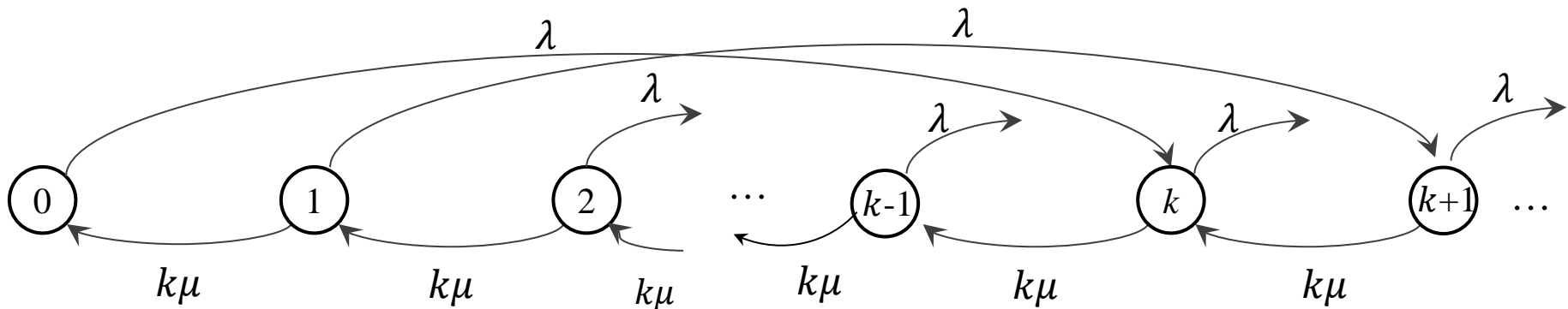
M/E$_k$/1 (1)

- $Y = X_1 + X_2 + \cdots + X_k$
  - Y: k-stage Erlang r.v. with service rate μ
    - mean service time $= \frac{1}{\mu}$
  - X: exponential r.v. with service rate $k\mu$
    - mean service time $= \frac{1}{k\mu}$

System

- Each stage is modeled as exponential sever with service rate $k\mu$

M/E$_k$/1 (2)

- We will take the system state as the total number of service stages left in the system

$$\lambda \pi_0 = k\mu \pi_1$$

$$(\lambda + k\mu)\pi_n = k\mu \pi_{n+1} \qquad (1 \le n \le k-1)$$

$$(\lambda + k\mu)\pi_n = \lambda \pi_{n-k} + k\mu \pi_{n+1} \qquad (n \ge k)$$

M/E$_k$/1 (3)

- **Solution approach 1**

$$\pi_1 = \frac{\lambda}{k\mu}\pi_0$$

$$\pi_n = \frac{\lambda + k\mu}{k\mu}\pi_{n-1} \qquad (2 \le n \le k)$$

$$\pi_n = -\frac{\lambda}{k\mu}\pi_{n-k-1} + \frac{\lambda + k\mu}{k\mu}\pi_{n-1} \qquad (n \ge k + 1)$$

  - If we know π_0, we can calculate π_n based on the above equations
    - Stopping criteria: $\left|1 - \sum_{n=0}^{N}\pi_n\right| \le \epsilon$
  - π_0: the probability that the server is idle

$$\pi_0 = 1 - Sever\ utilization\ = 1 - \frac{\lambda}{\mu}$$

M/E$_k$/1 (4)

- **Solution approach 1**

  - P_n: the probability of n jobs in the system

    $$P_n = \sum_{i=(n-1)k+1}^{nk} \pi_i$$

  - $\overline{N}$: Mean number of jobs in the system

    $$\overline{N} = \sum_{n=0}^{\infty} n P_n$$

  - $\overline{T}$: Mean sojourn time of a job in the system

    $$\overline{T} = \frac{\overline{N}}{\lambda}, \qquad \text{by the Little's law}$$

M/E$_k$/1 (5)

- **Solution approach 2**
  - Using a probability generating function (PGF)

$$A(z) = E[z^{\mathrm{x}}] = \sum_{n=0}^{\infty} z^n \pi_n$$

$$\sum_{n=1}^{k-1} (k\mu + \lambda)\pi_n = \sum_{n=1}^{k-1} k\mu\pi_{n+1}$$

$$+ \left.\right) \frac{\sum_{n=k}^{\infty} (k\mu + \lambda)\pi_n = \sum_{n=k}^{\infty} \lambda\pi_{n-k} + \sum_{n=k}^{\infty} k\mu\pi_{n+1}}{\sum_{n=1}^{\infty} (k\mu + \lambda)\pi_n = \sum_{n=k}^{\infty} \lambda\pi_{n-k} + \sum_{n=1}^{\infty} k\mu\pi_{n+1}}$$

M/E$_k$/1 (6)

- **Solution approach 2**

$$\sum_{n=1}^{\infty}(k\mu + \lambda)\pi_n = \sum_{n=k}^{\infty}\lambda\pi_{n-k} + \sum_{n=1}^{\infty}k\mu\pi_{n+1}$$

$$\sum_{n=1}^{\infty}z^n(k\mu + \lambda)\pi_n = \sum_{n=k}^{\infty}z^n\lambda\pi_{n-k} + \sum_{n=1}^{\infty}z^n k\mu\pi_{n+1}$$

$$(k\mu + \lambda)\sum_{n=1}^{\infty}z^n\pi_n = \lambda z^k \sum_{n=k}^{\infty}z^{n-k}\lambda\pi_{n-k} + \frac{k\mu}{z}\sum_{n=1}^{\infty}z^{n+1}\pi_{n+1}$$

$$(k\mu + \lambda)(A(z) - \pi_0) = \lambda z^k A(z) + \frac{k\mu}{z}(A(z) - \pi_0 - z\pi_1)$$

$$z(k\mu + \lambda)(A(z) - \pi_0) = \lambda z^{k+1}A(z) + k\mu(A(z) - \pi_0 - z\pi_1)$$

M/E$_k$/1 (7)

- **Solution approach 2**

$$z(k\mu + \lambda)(A(z) - \pi_0) = \lambda z^{k+1} A(z) + k\mu(A(z) - \pi_0 - z\pi_1)$$

Since $\lambda\pi_0 = k\mu\pi_1$,

$$A(z)\left(k\mu z + \lambda z - k\mu - \lambda z^{k+1}\right) = k\mu z\pi_0 - k\mu\pi_0$$

$$A(z) = \frac{k\mu(z-1)\pi_0}{k\mu(z-1) - \lambda z(z^k - 1)} = \frac{k\mu\pi_0}{k\mu - \lambda(z + z^2 + z^3 + \cdots + z^k)}$$

Since $A(1) = \dfrac{k\mu\pi_0}{k\mu - \lambda k} = 1$, $\pi_0 = 1 - \lambda/\mu$

$$A'(z) = \frac{k\lambda(\mu - \lambda)(1 + 2z + 3z^2 + 4z^3 \cdots + kz^{k-1})}{\{k\mu - \lambda(z + z^2 + z^3 \cdots + z^k)\}^2}$$

$$A'(1) = \frac{k\lambda(\mu - \lambda)\dfrac{k(k+1)}{2}}{k^2(\mu - \lambda)^2} = \frac{\lambda(k+1)}{2(\mu - \lambda)} \quad \text{: average number of stages } (\overline{N}_{\text{stage}})$$

M/E$_k$/1 (8)

- **Solution approach 2**
  - Mean number of stages in the server

$$0 \cdot \pi_0 + (1 - \pi_0)\left(\frac{1}{k} \cdot 1 + \frac{1}{k} \cdot 2 + \cdots + \frac{1}{k} \cdot k\right)$$

$$= \frac{1 - \pi_0}{k} \cdot \frac{k(k+1)}{2} = \frac{\lambda}{\mu} \cdot \frac{(k+1)}{2}$$

  - Mean number of stages in the queue

$$\bar{Q}_{\text{stage}} = \bar{N}_{\text{stage}} - \frac{\lambda}{\mu} \cdot \frac{(k+1)}{2} = = \frac{\lambda(k+1)}{2(\mu-\lambda)} - \frac{\lambda}{\mu} \cdot \frac{(k+1)}{2}$$

  - Mean number of jobs in the queue

$$\bar{Q}_{\text{job}} = \frac{\bar{Q}_{\text{stage}}}{k}$$

  - Mean number of jobs in the system

$$\bar{N}_{\text{job}} = \bar{Q}_{\text{job}} + \frac{\lambda}{\mu}$$