Chapter 1

Signals & Systems

Chapter 1. Signals and systems.

1.1. Continuous-time and discrete-time signals

- signal: a function of independent variables

  t: indep. variable in time

  $f(t)$: signal.

continuous-time (~~or continuous~~) signal: → t as indep variable

ex)

discrete-time signal: → n as indep variable, n: integer

ex)

Q: Digital signal?

Q: How to change CT → DT

- Signal Energy and Power.

total energy over $t_1 \le t \le t_2$: $\int_{t_1}^{t_2} |x(t)|^2 dt$ (can be complex)

$n_1 \le n \le n_2$: $\sum_{n=n_1}^{n_2} |x[n]|^2$

time averaged ~~energy~~ power: $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$

$\frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$

$$E_\infty \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$E_\infty \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

can be infinite.

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$
$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Classes of signals:

① $E_\infty < \infty \rightarrow P_\infty = 0$ — $x(t) = 1$ for $0 \le t \le 1$, 0 otherwise

② $P_\infty \ne 0$, $< \infty \rightarrow E_\infty = \infty$ — $x(n) = 4$

③ $P_\infty = \infty$ & $E_\infty = \infty$ — $x(t) = t$

1.2. transformations of Indep. Variables

- Time Shift: $x(t - t_0)$, $x[n - n_0]$
- Time reversal: $x(-t)$, $x[-n]$
- Time scaling: $x(2t)$ → fast; $x(t/2)$ → slow video or audio.

$x(at + b)$

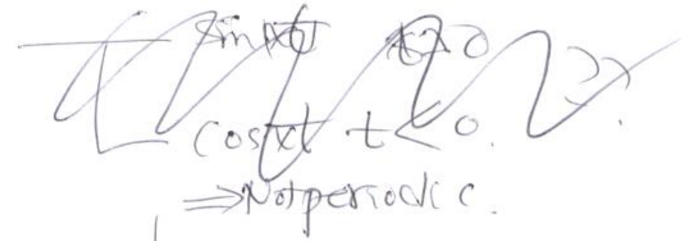
우리선생님이 잘 알기르쳐주세요

- periodic signals

$x(t) = x(t+T)$ → period "fundamental period"

$x(n) = x(n+N]$

e.g.) $\sin(t)$



- Even & odd signals

even: $x(-t) = x(t)$

odd: $x(-t) = -x(t)$

A real signal is sum of even & odd signals

→ Demonstrate this (your HW)

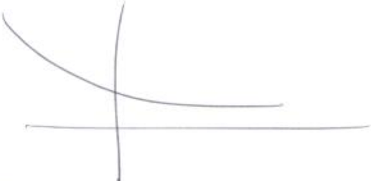
1.3. Exponential & Sinusoidal signals

- Complex exp. : $x(t) = Ce^{at}$

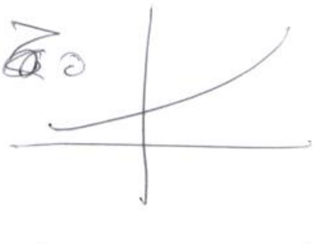
ok we understand this mathematically.
Let's try to understand this ~~physically~~ or intuitively.

If a & C are real.

$a < 0$



$a > 0$

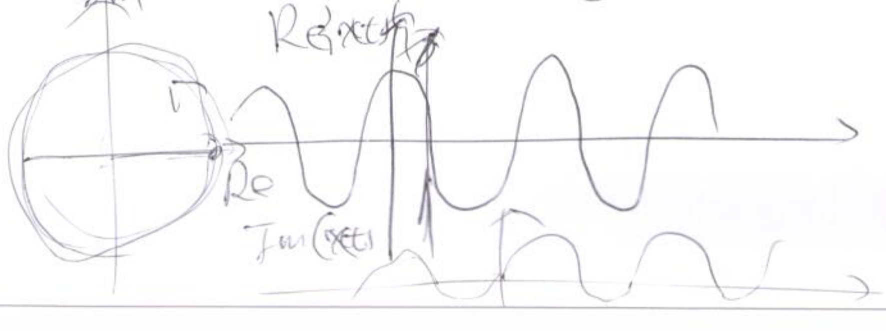


If a is imaginary → real

$x(t) = e^{j2\pi f_0 t} = \cos 2\pi f_0 t + j \sin 2\pi f_0 t$

Euler's relation

Rotation!



~~periodici~~

periodicity: $e^{j2\pi f_0 t} = e^{j2\pi f_0 (t+T)}$

When $e^{j2\pi f_0 T} = 1$

$2\pi f_0 T = 2\pi n$ (n, non zero integer)

$T_0 = \frac{1}{f_0}$ fundamental period

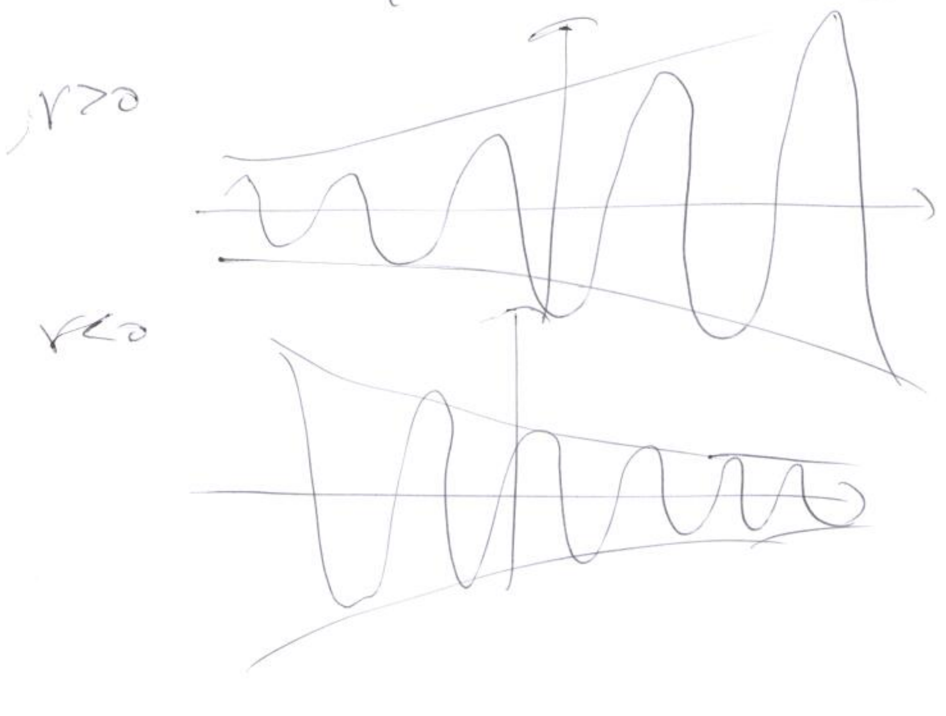
If a&C are both Complex.

Lets write $C = |C| e^{j\theta}$

$a = r + j2\pi f_0$

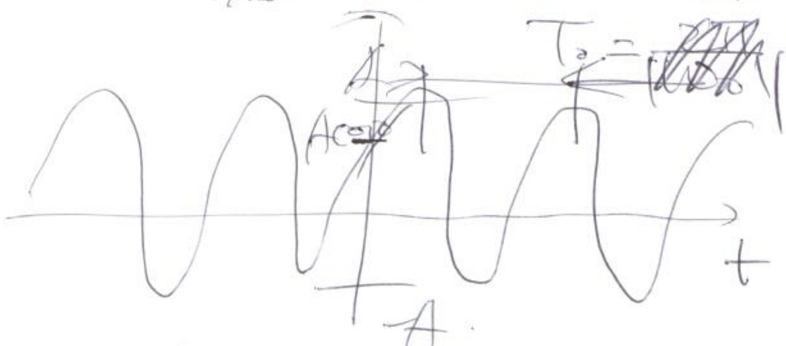
$C e^{at} = |C| e^{j\theta} e^{(r+j2\pi f_0)t} = |C| e^{rt} e^{j(2\pi f_0 t + \theta)}$

θ → offset



- sinusoidal signal $A\cos(\omega_0 t + \phi)$

$$x(t) = A\cos(2\pi f_0 t + \phi)$$



$f_0 \downarrow \; T_0 \uparrow$

$f_0 \uparrow \; T_0 \downarrow$

$t \to$ sec, $\phi \to$ radian $\qquad \omega_0 \to$ radian/sec

$\omega_0 = 2\pi f_0$ where f_0 cycle/sec or hertz (Hz)

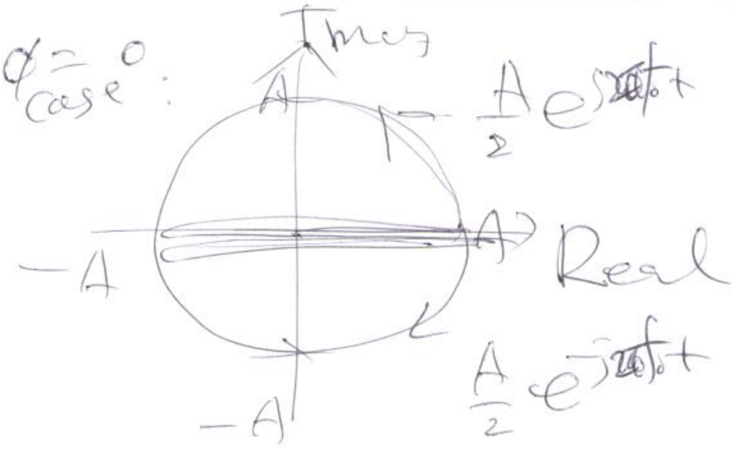
$\Rightarrow$ very similar to complex exponential signal.

$$e^{j2\pi f_0 t} = \cos 2\pi f_0 t + j\sin 2\pi f_0 t$$

Ex)

$$A\cos(2\pi f_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\phi} e^{-j2\pi f_0 t}$$

$\phi = 0$ case:



$$A\cos(2\pi f_0 t + \phi) = A\,\mathrm{Re}\{e^{j(2\pi f_0 t + \phi)}\}$$

$$A\sin(2\pi f_0 t + \phi) = A\,\mathrm{Im}\{e^{j(2\pi f_0 t + \phi)}\}.$$

- Complex Exp & sinusoidal signals: infinite total energy, finite average power.

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underbrace{|e^{j2\pi f_0 t}|^2}_{=1} dt$$

$$= \lim_{T\to\infty} \frac{2T}{2T} = 1.$$

- Harmonically related complex exponentials

→ sets of periodic exponential with a common period T_0

$$e^{j2\pi f T_0} = 1$$

$$2\pi f T_0 = 2\pi k, \quad k = \text{integer}.$$

$$f_0 = \frac{1}{T_0}$$

$$\phi_k(t) = e^{jk2\pi f_0 t}, \quad k = \text{integer}$$

⇒ forms basis functions (remember linear algebra?) in Chapter 3.

Good time to stop.

- Discrete-time complex exponential & sinusoidal signal!

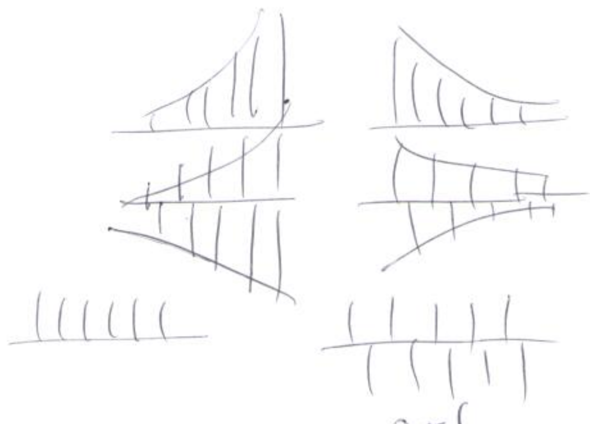
$$x[n] = C\alpha^n = Ce^{\beta n} \quad \text{where } \alpha = e^{\beta}$$

If C and α are real.

$|\alpha| > 1$ vs $|\alpha| < 1$

& $\alpha > 0$ vs $\alpha < 0$

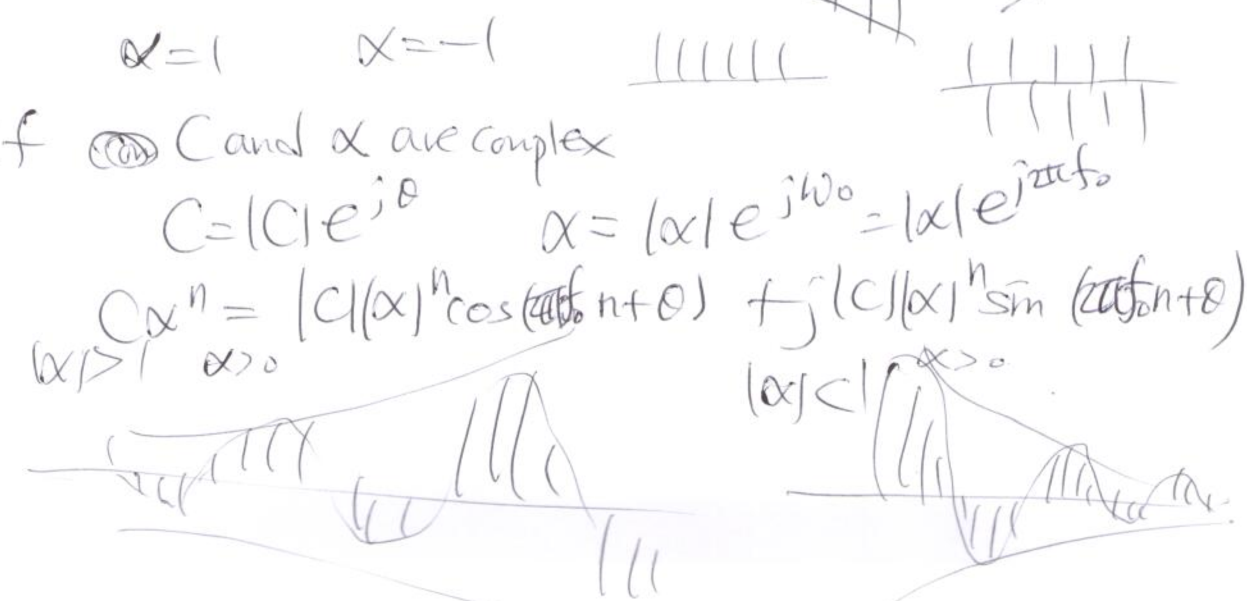
$\alpha = 1$ $\quad \alpha = -1$



If C and α are complex

$$C = |C|e^{j\theta} \qquad \alpha = |\alpha|e^{j\omega_0} = |\alpha|e^{j2\pi f_0}$$

$$C\alpha^n = |C||\alpha|^n \cos(2\pi f_0 n + \theta) + j|C||\alpha|^n \sin(2\pi f_0 n + \theta)$$



- Sinusoidal Signal

β is pure imaginary (i.e. $|\alpha| = 1$)

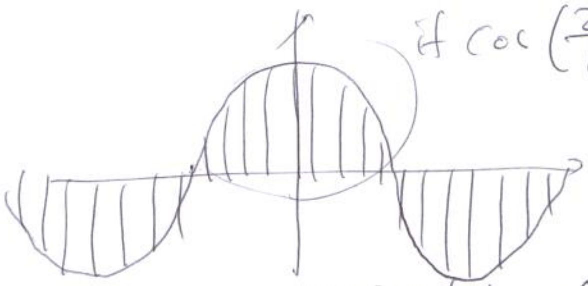
$$x(n) = C e^{j\omega_0 n} = |C| e^{j\theta} e^{j2\pi f_0 n} = |C| e^{j(2\pi f_0 n + \theta)}$$

$$= |C| \{ \cos(2\pi f_0 n + \theta) + j \sin(2\pi f_0 n + \theta) \}$$

In sinusoidal fn.

$$x(n) = A \cos(2\pi f_0 n + \theta) = \frac{A}{2} \{ e^{j\theta} e^{j2\pi f_0 n} + e^{-j\theta} e^{-j2\pi f_0 n} \}$$

Again $E_x = \infty$, $P_x < \infty$.



if $\cos\left(\frac{2\pi n}{12}\right)$ How many points?

- periodicity ~~of~~ property ~~Comp exp. & Sinusoids~~ of DT Comp. exp.

In discrete-time domain

property 1

$$e^{j\omega_0 n} = e^{j2\pi f_0 n} = e^{j2\pi f_0 n} \underbrace{e^{j2\pi n}}_{1} = e^{j2\pi f_0 n} \underbrace{e^{j2\pi k n}}_{1}$$

i.e. $f_0 = f_0 + k$ k: integer

or $\omega_0 = \omega_0 + 2\pi k$

Not always periodic. Shown in property 2

$\Rightarrow 0 \le f_0 < 1$ or $-1/2 \le f_0 < 1/2$

$0 \le \omega_0 < 2\pi$ $-\pi \le \omega_0 \le \pi$

$f_0 = 1/2$ → highest freq $(-1)^n$

property 2

periodicity

$$e^{j2\pi f_0 (n+N)} = e^{j2\pi f_0 n}$$

$$e^{j2\pi f_0 N} = 1$$

$$f_0 N = m \quad (N: \text{integer},\ m: \text{integer})$$

$\therefore e^{j2\pi f_0 n}$ fn with ($f_0 = \frac{m}{N}$: a rational #) is periodic.

show Example in Figure 1.25

ex. $f_0 = 1/12$ perodic

$f_0 = \frac{1}{12\pi}$ aperodic.

property 3 Fundamental Period: from property 2 (8) (integer)
(i.e. $f_0 N = m$), $N = \frac{m}{f_0}$
(assuming m, N ~~have~~ no common factor)
→ This has m times longer period than $1/f_0$ which is the case for CT exp.

Example ~~Show~~ ~~fund~~ $\cos\left(\frac{8\pi n}{31}\right)$ $f_0 = \frac{4}{31}$ $N = \frac{31}{4} \cdot m = 31$ (m = 4)
→ period 31
$\cos\left(\frac{8\pi}{31} t\right)$ $f_0 = \frac{4}{31}$ $T_0 = \frac{31}{4}$ "

Show Fig 1.25(b)
Fundamental frequency (f_{fund}) = $\frac{1}{N}$

Ex) $\cos\left(2\pi \frac{1}{12} n\right)$ $N = 12$
$\cos\left(2\pi \frac{1}{12} t\right)$ $T_0 = 12$

$\cos(8\pi n/31)$ $N = 31$
$\cos(8\pi t/31)$ $T_0 = 31/4$

$\cos(n/6)$ $N = \text{None}!$
$\cos(t/6)$ $T_0 = 12\pi$

- Harmonically periodic Exponentials

$$\phi_k(n) = e^{jk\left(\frac{2\pi}{N}\right)n}, \quad k: \text{integer}$$

periodic exp. with period of N

$$\phi_{k+N}(n) = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\frac{2\pi}{N}n}$$

→ only N distinct periodic exponentials

$\phi_0(n) = 1$ $\phi_1(n) = e^{j2\pi\frac{n}{N}}$ $\phi_2(n) = e^{j4\pi\frac{n}{N}}$

$\cdots \phi_{N-1}(n) = e^{j2\pi(N-1)\frac{n}{N}}$ ⇒ distinct!

try this for N=32 & see if they are orth.

1.4 Unit Impulse and unit step functions

- Discrete ~~time~~

Unit Impulse $\quad \delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$ → Kronecker δ

Unit step $\quad u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

$\delta(n) = u(n) - u(n-1)$ (Important) — we will come back to δ in Chap 2.

$u(n) = \sum_{m=-\infty}^{n} \delta(m) = \sum_{k=0}^{\infty} \delta(n-k)$

⇒ Show figures

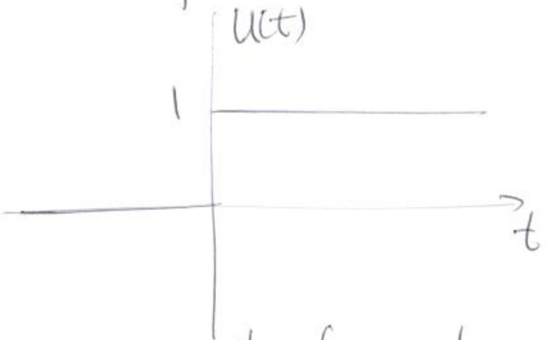
property: $x(n)\delta(n) = x(0)\delta(n)$

Q: why do we need this?

or $x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0)$

- Continuous time

Unit ~~Impulse~~ step $\quad u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

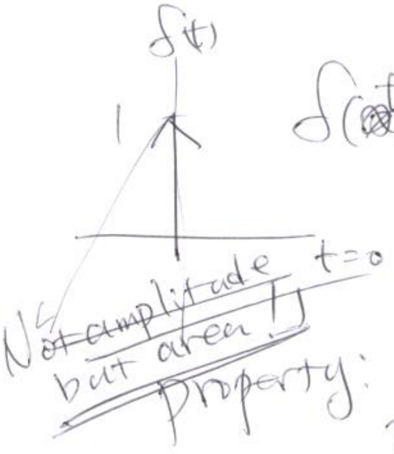


$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

Unit Impulse $\delta(t) = \frac{du(t)}{dt}$ (one way to define it)

problem $u(t)$ is not defined @ $t=0$.

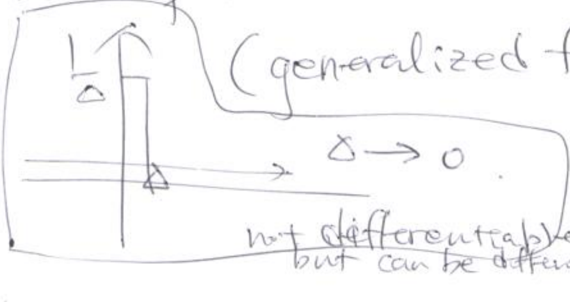
☆ Read page 33 – 35 for your reference.



$$\delta(x) = \begin{cases} +\infty & x=0 \\ 0 & x\neq 0 \end{cases}$$

Dirac delta ~~Direct~~ fn. (generalized fn)

$$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$



not differentiable but can be differentiated.

Property:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$K u(t) = \int_{-\infty}^{t} K\delta(\tau)d\tau$$

$\Rightarrow$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau$$

illustrate this in time domain

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

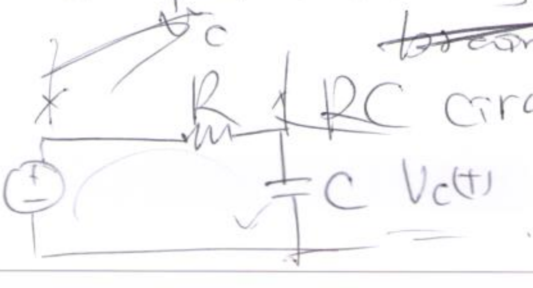
1.5 Continuous-time & Discrete-time Systems

$x(t) \longrightarrow$ [CT system] $\longrightarrow y(t)$

$x[n] \longrightarrow$ [DT system] $\longrightarrow y[n]$

Show some examples : ~~vending machine~~ ~~bacon~~.

RC circuit.

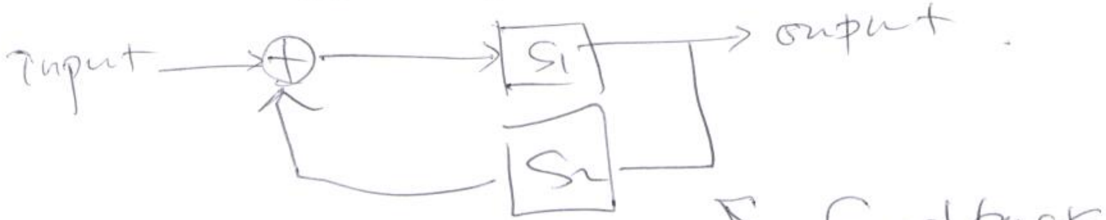


$v_c(t) =$

$v_s(t)$

in this class we will focus on a ~~class~~ type of system that we can easily analyze.
This type of system is called linear time invariant system

· Interconnections of systems



↖ feedback Interconnection.

ex) Cruise controll in a car.

1.6. Basic system properties.

property 1: memoryless system: output only depends on current input.

ex) $y(n) = (2x(n) + x^2(n))^2$

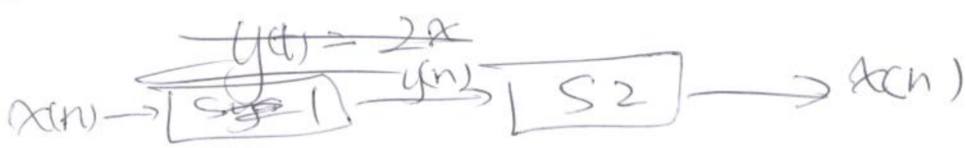
system w memory

ex) $y(n) = \sum_{k=-\infty}^{n} x(k)$

$y(n) = x(n-1)$ ← delay!

capacitor.

property 2: Invertible system



ex) $y(n) = \sum_{k=-\infty}^{n} x(k)$ $\quad w(n) = y(n) - y(n-1)$

$y(t) = x^2(t)$ ← not invertible (no sign info)

encoding decoding.

property 3: Causality

output only depends on input at the present time

(No future data needed)

ex) noncausal $y(n) = x(n+1)$

collected data → no need to be causal.

property 4: Stability.

small input lead to responses that do not diverge.

ex) ball dropped. unstable mic/audio in ≤ 24 kHz positive feed back

$$y[n] = \sum_{k=-\infty}^{n} u(k)$$

property 5 Time invariance

time shift in input results in an identical time shift in output.

$$x(n) \xrightarrow{S} y(n)$$

$$x(n-n_0) \xrightarrow{S} y(n-n_0)$$

ex). time variant $y(n) = n x(n)$

property 6 Linear.

— scaling $x(t) \xrightarrow{S} y(t)$, $a x(t) \xrightarrow{S} a y(t)$

— superposition $x_1(t) \xrightarrow{S} y_1(t)$

$x_2(t) \xrightarrow{S} y_2(t)$

$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$

$$x(n) = \sum_k a_k x_k(n)$$

$$y(n) = \sum_k a_k y_k(n)$$

Quiz 1.

What is linear

What is time invariant

What is definition of $\delta(t)$.

Chapter 2

LTI

Chapter 2. LTI Systems

2.0 LTI?

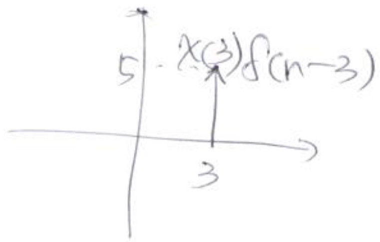
What is linear
What is time invariant
examples of LTI system
examples of linear but not TI
examples of nonlinear but TI
examples of nonlinear, non-TI

2.1 Discrete-time LTI systems: Convolution sum

Fig 2.1

$$x(n) = \cdots + x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots$$

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



Sifting property
↳ 체로 거르다.

→ ~~x(n) is~~ a discrete time signal can be represented by sum of shifted $\delta(n)$'s

x(n) → [h] → y(n)

$$y(n) = h\left[\, x(n) \,\right] \quad \text{~~Not so~~}$$

$$= h\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \right]$$

($x(k)$: constant) ~~if system is~~

If $h(n)$ is linear

using superposition property

$$= \sum_{k=-\infty}^{\infty} h\big(x(k)\delta(n-k)\big)$$

using scaling property

$$= \sum_{k=-\infty}^{\infty} x(k)\, h\big(\delta(n-k)\big) = \sum_{k=-\infty}^{\infty} x(k)\, h_k(n)$$

If $h(n)$ is linear & time invariant.

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

where $h(n)$ is unit impulse response function.

i.e. $\delta(n) \rightarrow \boxed{h} \rightarrow h(n)$

$\delta(n-k) \rightarrow \boxed{h} \rightarrow h(n-k)$ ∵ time invariant.

. In LTI systems

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

// very important

so we call this as "convolution"

$$y(n) = x(n) * h(n)$$

(More importantly, we can say that LTI system is completely characterized by an "impulse".

$h(n) \rightarrow$ impulse response function.

for any known input, we can calculate the output of the LTI system if we know the impulse response function.

- Show one or two examples of How to perform convolution. (may use computer simulation)

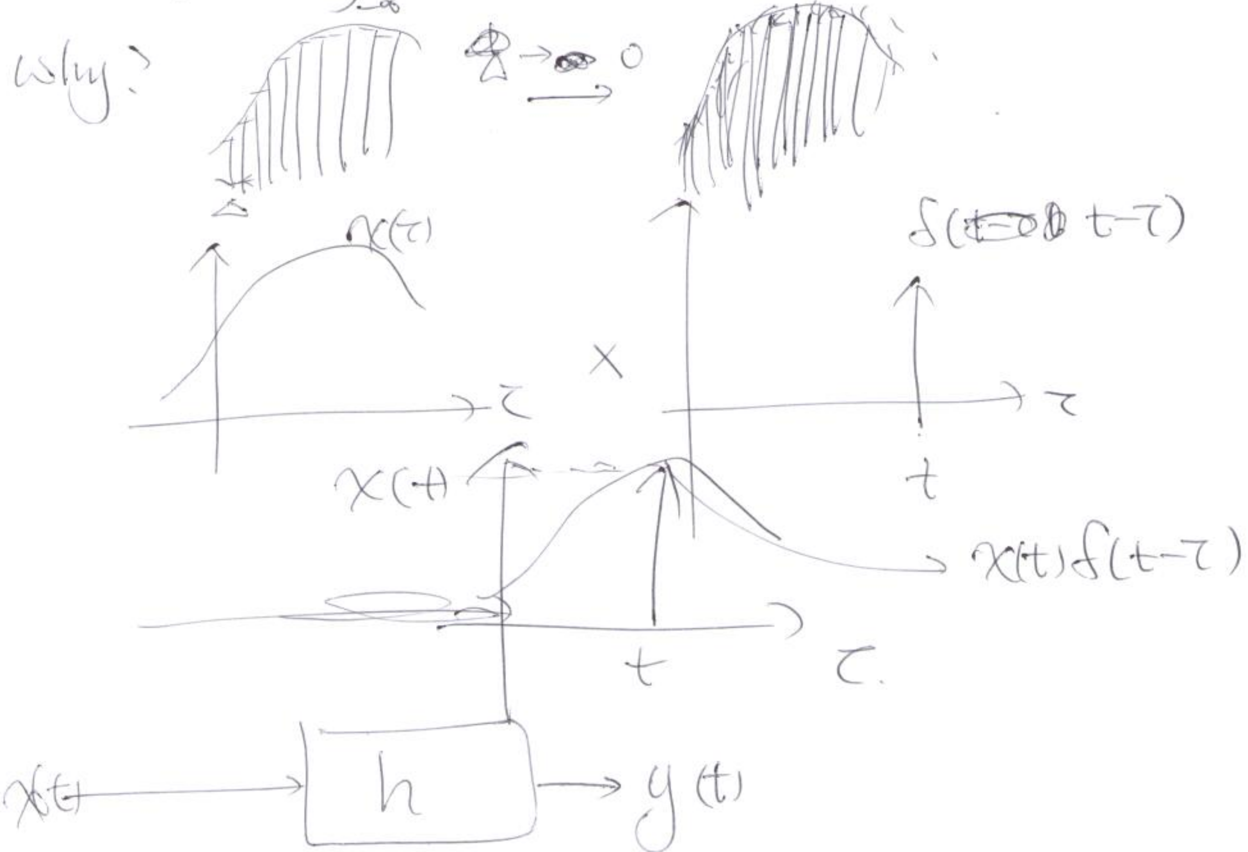
This is ~~every~~ very important !!! Questions?

2.2 Continuous-time LTI system: Convolution integral.

similar to DT case.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

why?



$$y(t) = h\left(x(t) \right)$$

$$= h\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \right)$$

Superposition →

$$= \int_{-\infty}^{\infty} h(x(\tau)\delta(t-\tau))d\tau$$

scaling →

$$= \int_{-\infty}^{\infty} x(\tau)\, h(\delta(t-\tau))\, d\tau$$

time-invariance →

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

$$= x(t) * h(t)$$ Convolution integral !!!

Show one or two examples. (Fig 2.17, 19)

2.3 Properties of LTI systems

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

※ Impulse response fully characterize the system only in LTI.

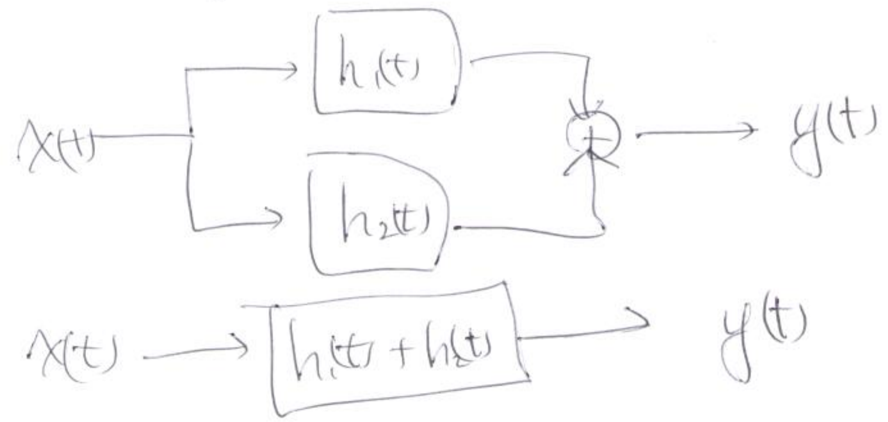
2.3.1 Commutative Property

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

→ This means we can choose what to reverse/shifts in convolution.

2.3.2. Distributive property

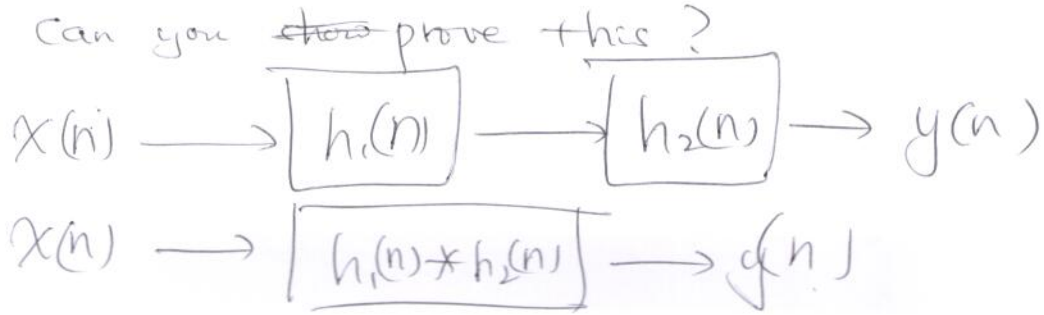
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



2.3.3. Associative Property

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

Can you ~~show~~ prove this?



Commutative property.

$x(n) \rightarrow$ [$h_2(n) * h_1(n)$] $\rightarrow y[n]$ ⑤

$x(n) \rightarrow$ [$h_2(n)$] $\rightarrow$ [$h_1(n)$] $\rightarrow y(n)$

when? → LTI
what happens if a system is nonlin

2.3.4 Memory.

memoryless $h(n) = k\delta(n)$

$h(t) = k\delta(t)$

otherwise, the system has memory.

2.3.5 Invertibility.

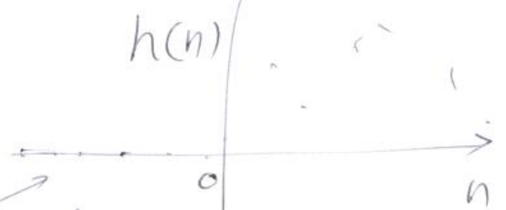
Identity system: $\delta(n)$ or $\delta(t)$

A system is invertible if ~~x(t)~~ $h_1(t)$ that satisfy $h(t) * h_1(t) = \delta(t)$ exists.

Show Ex 2.12.

2.3.6 Causality.

$h(n) = 0$ for $n < 0$



initial rest → only for LTI

In causal system

$$y(n) = \sum_{k=-\infty}^{n} x(k)\,h(n-k)$$

$$= \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

In Continuous LTI system

$h(t) = 0$ for $t < 0$

$$y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau = \int_{0}^{\infty} h(\tau) x(t-\tau) d\tau$$

2.3.7 Stability

BIBO: Bounded Input → Bounded output.

$|x(n)| < B$ for all n

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k) x(n-k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)| |x(n-k)|$$

$$\leq B \sum_{k=-\infty}^{\infty} |h(k)|$$

$\Rightarrow$ if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty \Rightarrow$ system is stable.

sufficient condition.

& necessary condition (Prob 2.49)

2.3.8. Unit step Response.

< voltage applied to a circuit.

$$s(n) = u(n) * h(n)$$

$$= h(n) * u(n)$$

$$= \sum_{k=-\infty}^{n} h(k)$$

The system response fn can be recovered by

$$h(n) = s(n) - s(n-1)$$

In Continuous time

$$s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{t} h(\tau) d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

2.4 Causal LTI system described by differential & difference equations.

2.4.1 Linear constant coeff. Differential equations

Here is a system:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

← $x(t)$ force, $y(t)$: velocity of a vehicle.

How do we solve this?

→ need auxillary conditions (or initeal conditions)

→ $y(t) = y_p(t) + y_h(t)$

$y_p(t)$ → particular solution for an input.

$y_h(t)$ → homogeneous solution or natural response, when there is no input! i.e. $\frac{dy(t)}{dt} + 2y(t) = 0$

ask students Solve Ex 2.14 as a HW.

→ different auxillary condition lead to different relationship between input & output

→ one option for auxillary condition is ~~we will mostly use~~ "initial rest"

→ in LTI "initial rest" means causal

$x(t) = 0$ for $t \le t_0$

$y(t_0) = 0$

General Nth order linear constant coeff. differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

for initial rest (i.e. causal),

$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$ for $x(t)=0$ $t \le t_0$

if $N = 0$

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

(How do we solve this?) Come back in Chapt 4 & 9.

2.4.2 Linear Constant-Coefficient Difference Eq.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)\right\}$$

⇒ $y(n)$ can be solved successively.
"recursive equation"

When $N=0$

$$y(n) = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x(n-k)$$

↳ nonrecursive equation.
→ no need for auxiliary conditions

$$h(n) = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

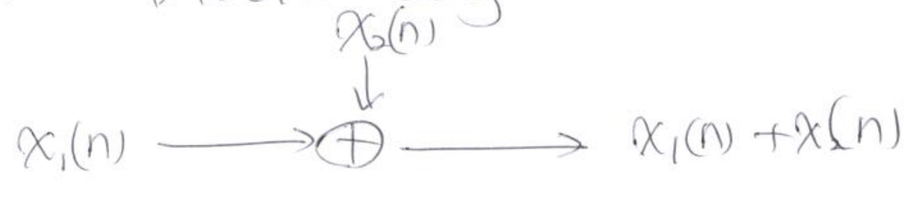
→ finite impulse response system
(FIR) → causal LTI system

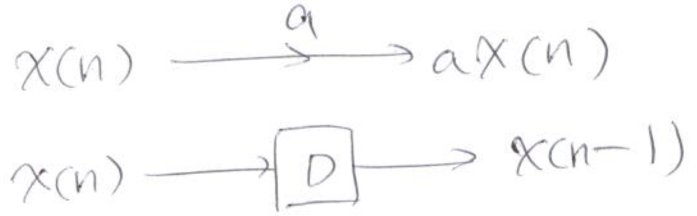
When $N \ge 1$, the causal LTI system has an impulse response of infinite duration

→ infinite impulse response system
(IIR)

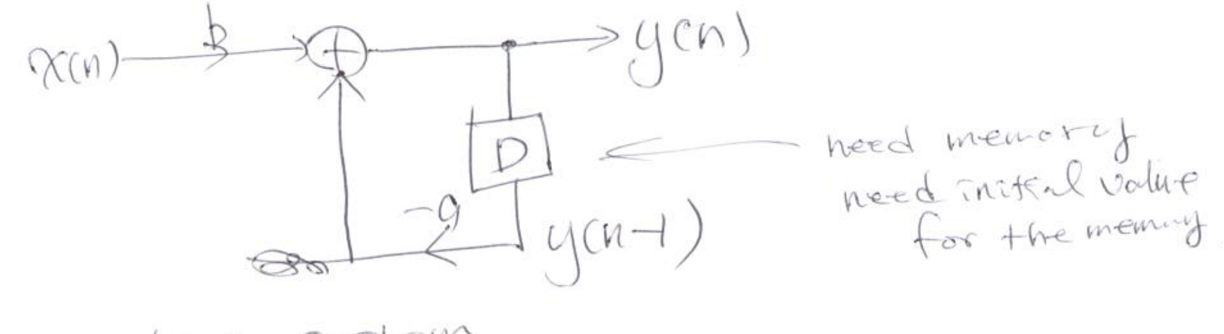
Again wait for Chapters 5 & 10 for solutions

2.4.3 Block diagram.

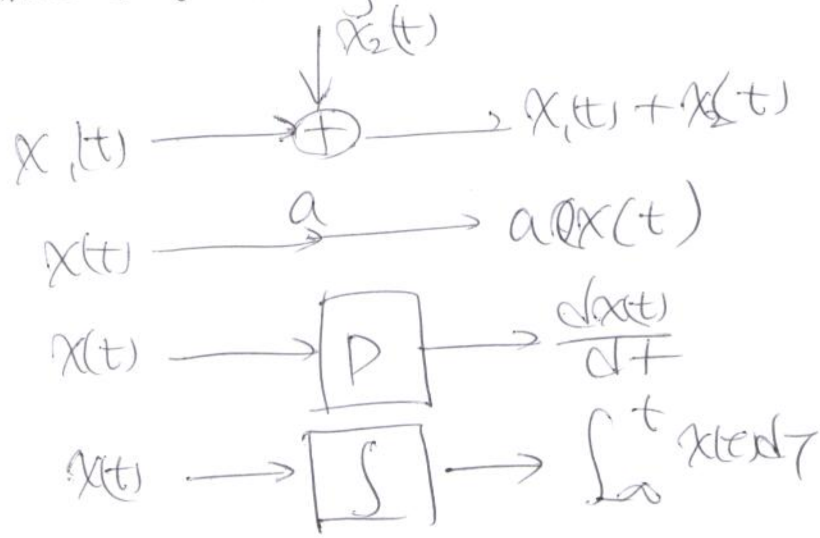




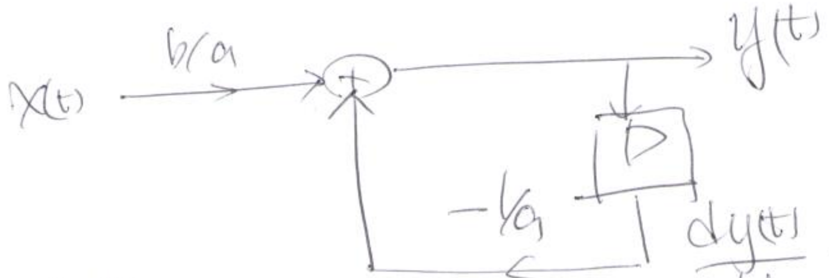
Ex) $y(n) + a\,y(n-1) = b\,x(n)$



Continuous-time system.



Ex) $\frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$



differentiator is difficult to implement in analog

$y(t) = \int_{-\infty}^{t} (b\,x(\tau) - a\,y(\tau))\,d\tau$

10

2.5 Singularity fns. $\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases}$ $\int_{-\infty}^{\infty} \delta(t)dt = 1$

2.5.1 Unit impulse

Sifting property says: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$x(t) = x(t) * \delta(t)$$

Then $\delta(t) = \delta(t) * \delta(t)$ ← behave like an impulse

$\Delta \to 0$?

anything $\delta(t) * \delta(t) * \delta(t)$ ...

2.5.2 Unit impulse through Convolution

- $x(t) * \delta(t) = x(t)$
- $1 = x(t) \Rightarrow x(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau$

$$= \int_{-\infty}^{\infty} 1\,\delta(\tau)d\tau$$

operational definition

$$\therefore \int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

- $g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau)d\tau$

for $t = 0$ $g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$

- $f(t)\delta(t) = f(0)\delta(t)$
- $\int_{-\infty}^{\infty} f(\tau)\delta(\tau)d\tau = \int_{-\infty}^{\infty} f(0)\delta(\tau)d\tau = f(0)$
- $\delta(ax) = \frac{1}{|a|}\delta(x)$

2.5.3 Unit Doublets & etc.

$$y(t) = \frac{dx(t)}{dt}$$

→ unit impulse response of this system.

→ $u_1(t)$: unit doublet.

$$\frac{dx(t)}{dt} = x(t) * u_1(t)$$

$\frac{d\delta(t)}{dt}$ first derivative of δ

(11)

$$\frac{d^2 x(t)}{dt^2} = x(t) * u_2(t)$$

, where $u_2(t) = u_1(t) * u_1(t)$

$$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{k \text{ times}}$$

operational definition using δ

for $x(t) = 1$

$$0 = \frac{dx(t)}{dt} = x(t) * u_1(t) = \int_{-\infty}^{\infty} u_1(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} u_1(\tau) d\tau$$

unit doublet has zero area.

---

unit step $y(t) = x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau = \int_{-\infty}^{t} x(\tau) d\tau$

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

operational definition

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

$$= t\, u(t)$$

unit ramp function.

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau$$

$$u_{-k}(t) = u(t) * \cdots * u(t) = \int_{-\infty}^{t} u_{-(k-1)}(\tau) d\tau$$

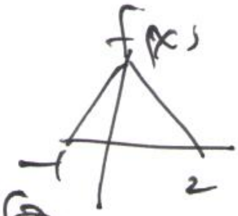
$$= \frac{t^{k-1}}{(k-1)!} u(t).$$

Sometimes, better to define

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

Then $u_k(t) * u_r(t) = u_{k+r}(t)$



Signals

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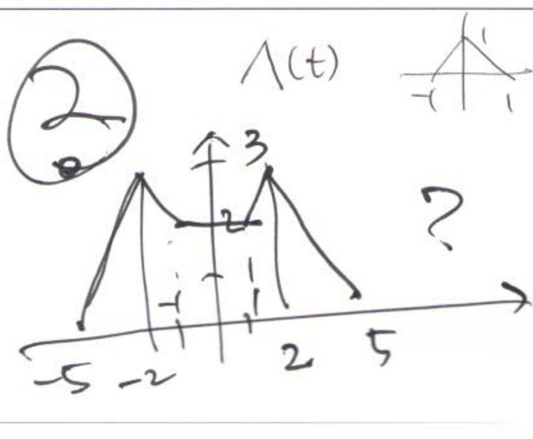
Time domain	
$\delta(t)$	
$\delta(at)$	
$e^{i2\pi f_0 t}$	
$rect(t) \triangleq \begin{cases} 1, & \text{if } \lvert t\rvert \le 1/2 \\ 0, & \text{otherwise} \end{cases}$	
$sinc(t) \triangleq \sin(\pi t)/\pi t$	
$e^{-\pi t^2}$	
$sin2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)-\exp(-i2\pi f_0 t)}{2i}$	
$cos2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)+\exp(-i2\pi f_0 t)}{2}$	
When $a > 0$, $\begin{cases} e^{-at}, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$	
$\frac{1}{a + j2\pi t}$	
$\text{III}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k)$	
$\frac{1}{T}\text{III}\left(\frac{t}{T}\right)$	
$f(at)$	



Properties of Symmetry

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A real function, f(t), is

"even function"	if	$f(t) = f(-t)$
"odd function"	if	$f(t) = -f(-t)$

A real function can be divided into even and odd parts of the function

$$f_{even}(t) = \{f(t) + f(-t)\}/2$$
$$f_{odd}(t) = \{f(t) - f(-t)\}/2$$

A function, f(x), is

"real function"	if	$f(t) = f^*(t)$
"imaginary function"	if	$f(t) = -f^*(t)$

A function, f(x), is

"Hermitian function"	if	$f^*(t) = f(-t)$
"Anti-hermitian function"	if	$f^*(t) = -f(-t)$

Hermitian means real part of the function is even and imaginary part is odd

$$f(t) = a(t) + ib(t)$$

where $a(t)$ and $b(t)$ are real functions

$$a(t) = a(-t)$$
$$b(t) = -b(-t)$$

Fourier transform of a real function, $h(t)$, is Hermitian

$$H^*(f) = H(-f)$$

And

$$h(t) = h_{even}(t) + h_{odd}(t)$$
$$FT\{h_{even}(t)\} = Re\{H(f)\} = Re\{H(-f)\}$$
$$FT\{h_{odd}(t)\} = Im\{H(f)\} = -Im\{H(-f)\}$$

Quiz 2

1) Write down mathematical formula for convolution.

2) What is the name of the output of ~~an~~ impulse for an LTI system.

3) Why do we love LTI systems? (The answer is related to 2).

Quiz 3.

1. What is the condition for causality in LTI systems

2. ~~How do~~ " Stability "

Quiz 4.

1. $f(t)\delta(t)$
2. $\int_{-\infty}^{\infty} f(t)\delta(t)d\tau$
3. $\delta(ax)$
4. plot $III(\frac{t}{T})$

Chapter 3

Fourier Series

Chapter 3. Fourier Series Representation of (D) Periodic Signals

3.2 Response of LTI systems to Complex exp.

- Complex exponential functions (e^{st} or z^n) are "magic function".

Why?

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}_{\text{"not dependent on } t\text{"}}$$

$$= H(s) e^{st}$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$.

* the output of complex exponential fn is the same e^{st} w modified magnitude (and phase) by $H(s)$

This <u>type of</u> (magic) fn is called "eigen function" and $H(s)$ is called "eigen value"

This same is true for z^n.

$$y(n) = \left(\sum_{k=-\infty}^{\infty} h(k) z^{-k}\right) z^n = H(z) z^n$$

where $H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$.

(Great to have complex exp. !)

- → makes life easy!

What if our input is in e^{st} shape?

i.e. $x(t) = \sum_k a_k e^{s_k t}$

great! output will be

$$y(t) \sum_k a_k H(s_k) e^{s_k t}$$

(only true for LTI system) simple.

- (Question). Can we represent our signals in complex exponentials?
- One point s or z is too general so let's confine ourselves to $s = j\omega$ and $z = e^{j\omega}$

we will come back to s & z later (Chaper 9 & 10)

3.2. Fourier Series representation of continuous-time periodic signals.

"$e^{j\omega_0 t}$" looks periodic or is periodic so we may be able to represent periodic signal using our magic/eigen fun.

our magic function

Let's consider a periodic signal.

$$x(t) = x(t+T)$$

$$\omega_0 = \frac{2\pi}{T} \quad \text{(fundamental freq.)}$$

~~our magic fn~~ eigen

$$e^{jk\omega_0 t} = e^{j\frac{2\pi}{T}t}$$

Also $e^{jk\omega_0 t}$ ~~has the same~~ becomes harmonically related complex exp.

so, it is likely that a periodic signal $x(t)$ with ω_0 fun. freq. can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

$k=0 \longrightarrow$ constant.

$k=\pm 1 \longrightarrow$ fundamental freq. or first harmonic comp.

$k=\pm N \longrightarrow$ Nth harmonic.

Read page 188-189 ~~for~~ when $x(t)$ is a "real" case

- Question 1 ~~with~~ How many ~~x(t)~~ fn or ~~type~~ what of fns can be represented in this way?

  Answer: A lot ....

- Question 2. What will be a_k?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) e^{-jn\omega_0 t} = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) e^{-jn\omega_0 t}$$

periodic!

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt$$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t} dt \right]$$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T & k=n \\ 0 & k \neq n \end{cases}$$

~~$T\delta(k-n)$~~

Then $\int_0^T x(t) e^{-jn\omega_0 t} dt = T a_n$

$$a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt$$

This is called Fourier Series representation of x(t)

- To summarize

if x(t) (periodic with $T = \frac{2\pi}{\omega_0}$) has a FS representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

What does this do?

x(t) is synthesized by complex exp. fun.

a_k analyze x(t). How much $e^{jk\omega_0 t}$ exist in x(t).

$a_0 \rightarrow$ DC

FIS

Example 3.3

$$x(t) = \sin 2\pi f_0 t$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi f_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk2\pi f_0 t}\, dt$$

$$\sin 2\pi f_0 t = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

$\therefore$ $a_0 = 0$, $a_1 = \frac{1}{2j}$, $a_{-1} = -\frac{1}{2j}$

$a_k = 0$ for ~~$k \neq \pm 1$~~ $|k| \geq 2$.

Example 3.5.

$$= \text{rect}\left(\frac{t}{2T_1}\right) * \frac{1}{T}Ш\left(\frac{t}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot e^{-j2\pi k f_0 t}\, dt$$

$$= \frac{1}{T}\,\frac{e^{-j2\pi k f_0 t}}{-j2\pi k f_0}\Bigg|_{-T_1}^{T_1}$$

$$= \frac{e^{-j2\pi k f_0 T_1} - e^{+j2\pi k f_0 T_1}}{T(-j2\pi k f_0)} = \frac{\sin 2\pi k f_0 T_1}{\pi k (f_0 T)_{=1}} = \frac{\sin 2\pi k f_0 T_1}{\pi k} \text{ when } k \neq 0$$

$$a_0 = 2T_1/T$$

When $T = 8T_1$ plot it using matlab.

Try it for different $T = \alpha T_1$, α values.

✓ Show FT approach to demon FIS.

3.4 Convergence.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

$$\underbrace{E_N}_{\text{energy}} = \int_T |e_N(t)|^2 dt.$$

- If $x(t)$ ~~converges~~ has FS representation, $E_N \to 0$ as $N \to \infty$
- every continuous periodic signal has a FS. ↓ Condition
  Also true ~~at~~ a lot of discontinuous signals

※ But remember we are not say FS and the orignal signal is the same in all t.
We say no energy difference btw the two

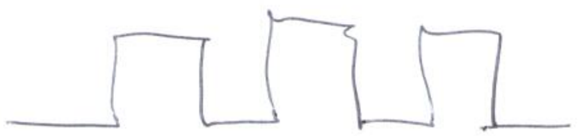
- $\int_T |x(t)|^2 dt < \infty$ ⇒ i.e. finite energy over single period
  Converge,
  → very strong condition that satisfies most of realistic signals.
- Dirichelet conditions (1, 2, 3) page 197-198.
  guarantees that $x(t)$ = FS represen.
  except @ isolated values of t for discontinuity.
  even this value is the same as the average of the values on either side of discontinuities.

(5)

3.4 Congergence [Read!]

→ Question of ~~How many~~ what type of periodic fn. can be represented by FS.

— A: a lot. or most of them.

Example of  .. fn.

& Gibb's ringing. over shoot 9%

still converge

(area → 0 as n → ∞)

3.5 Properites of Continuous-time Fourier Series

$$x(t) \xleftrightarrow{FS} a_k$$

- Linearity
- Time shifting $x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$

  phase shift! magnitude the same

- Time reversal

  $$x(-t) \xleftrightarrow{FS} a_{-k}$$

- Time scaling

  $$x(at) \xleftrightarrow{FS} a_k$$

  → differen meaning of fig

  what does this mean?

- Multiplication.

  $$x(t)y(t) \xleftrightarrow{FS} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

  → HW

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- Conjugate symmetry

$$x^*(t) \xleftrightarrow{FS} a^*_{-k}$$

if $x(t)$ real $a_{-k} = a_k^*$ (Hermitian)

- Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

3.6. Fourier Series Representation of discrete-Time periodic Signals.

$$x(n) = x(n+N) \quad \text{with } f_0 = \frac{1}{N}$$

Fundamental frequency (ω)

Harmonic function

$$\phi_k(n) = e^{j2\pi k f_0 n}$$

※ Different from Continuous Case k is not infinite because $2\pi k f_0 = 2\pi(k+N) f_0$! ($f_0 = \frac{1}{N}$)

→ N distinct k.

i.e. $\phi_0(n), \phi_1(n), \ldots, \phi_{N-1}(n)$

$$x(n) = \sum_{k=0}^{N-1} a_k e^{j2\pi k f_0 n}$$

→ Not infinite (different from CT)

$$= \sum_{k=\langle N\rangle} a_k e^{j2\pi k f_0 n}.$$

→ Still don't know if we can represent $x(n)$ (periodic) this way but we has ver strong belief now!

→ so we even ~~call~~ name ~~ak as~~ this relationship as DT-FS and a_k as FS coeff (1)

Now, we need to ~~find~~ determine a_k.

$$x(0) = \sum_{k=\langle N\rangle} a_k$$

$$x(1) = \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}$$

$$\vdots$$

$$x(N-1) = \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}$$

N eq. N unknown. High school math!
more exciting'y we learn linear algebra

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j2\pi/N} & \cdots & e^{j2\pi(N-1)/N} \\ \vdots & & & \vdots \\ 1 & e^{j2\pi(N-1)/N} & \cdots & e^{j2\pi(N-1)(N-1)/N} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

This is just a side / story (but exciting)

Is this full rank?

How can you show this?

Ok. an ~~better~~ easier way?

$$\sum_{n=\langle N\rangle} x(n) e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)(\omega_0)n}$$

$$= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\omega_0 n}$$

$$= N a_r$$ ~~δ(k−r)~~

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n) e^{-jk\omega_0 n}.$$

Remember synthesis & analysis!

Gibbs rings ~~note~~ ? (see page 220) Nope! $N\times N$ equation solved!!
always converge. ⟹ why ---?

3.7 properties of DT FS

- Multiplication.

$$x(n)y(n) \xleftrightarrow{FS} \sum_{l=\langle N\rangle} a_l b_{k-l}..$$

→ periodic convolution.

- First difference.

$$x(n) - x(n-1) \xleftrightarrow{FS} (1 - e^{-jk\omega_0})a_k.$$

- Parseval's Relation

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \sum_{k=\langle N\rangle}|a_k|^2.$$

3.8 FS and LTI

If you still remember Chapter 3, complex exponentials has very important property in LTI system.

i.e. if $x(t) = e^{st}$ ~~$H(s) =$~~ $y(t) = H(s)e^{st}$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

or for discrete case

$x(n) = z^n \quad y(n) = H(z) z^n$

where $H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$

When s & z are general complex number

H(s) & H(z) are called "system functions"

when $s = j\omega$ $z = e^{j\omega}$ → evaluation of system function when amplitude is 1 (i.e. $|z|=1$ $|e^{j\omega}|=1$)

Then the system function is called "frequency response".

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

Let's come back to LTI system again

so if your (or our) input can be represented by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

or $$x(n) = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

(and we know all periodic x(n) & most of periodic x(t) can be represented this way)

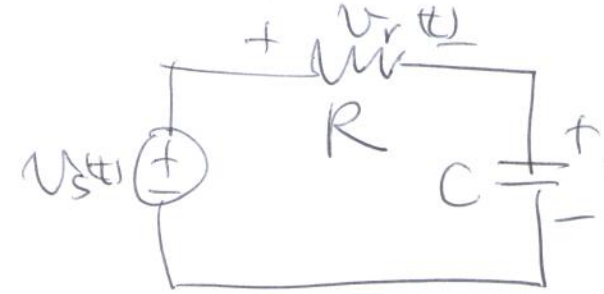
Our output becomes

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$y(n) = \sum_{k=\langle N \rangle} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$$

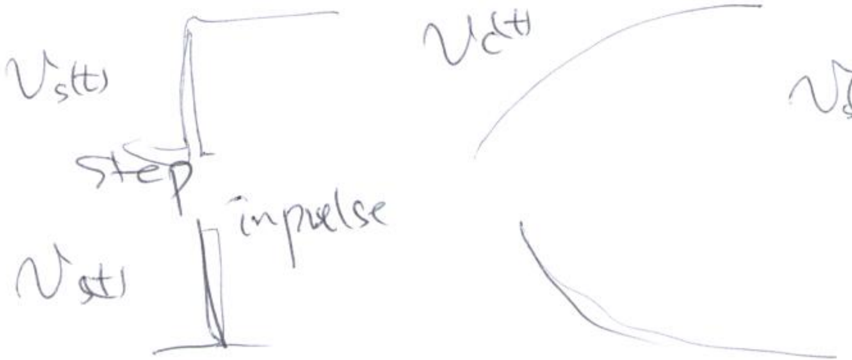
Ask students to read 3.9

Then do 3.10.1 in class as an example.



initial rest
i.e. No ~~voltage~~ Charge
@ Capacitor

1) intuitive solution.



$t>0$

$v_s(0)\left[1-e^{-\frac{t}{RC}}\right]$

$\tau = RC$

slow slow

$v_s(0)\ \frac{1}{RC} e^{-\frac{t}{RC}}$ $t>0$

→ ~~deliveration~~ $\frac{d(\cdot)}{dt}$

2) solution using frequency response.

$$v_s(t) = e^{j\omega t}$$

$$v_c(t) = H(j\omega)e^{j\omega t}$$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

$$RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$H(j\omega) = \frac{1}{1+RCj\omega}$$

plot this

"low pass filter"

RC large vs RC small

→ What do they mean physically!

very important sanity check

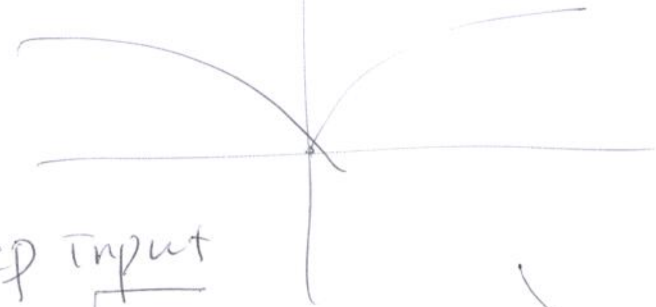
$v_r(t)$?

$v_s(t) = v_r(t) + v_d(t)$

$e^{j\omega t} = \{H(j\omega) + G(j\omega)\} e^{j\omega t}$

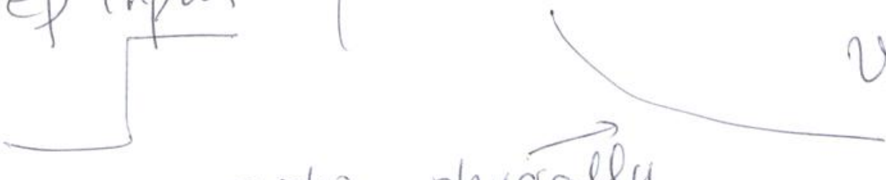
$\{H(j\omega) + G(j\omega)\} = 1.$

$\therefore G(j\omega) = 1 - H(j\omega) = \dfrac{RCj\omega}{1+RCj\omega}$.



high pass filter.

step input



$v_s(0)e^{-t/RC}$.

explain physically

Also as a view point of high pass filter

Do the counterpart in 3.11 for DT Difference equation (

$$y(n) = x(n-1) + x(n) + x(n+1)$$

$$y(n) = (\delta(n-1) + \delta(n) + \delta(n+1)) * x(n)$$

$$\therefore h(n) = \delta(n-1) + \delta(n) + \delta(n+1)$$

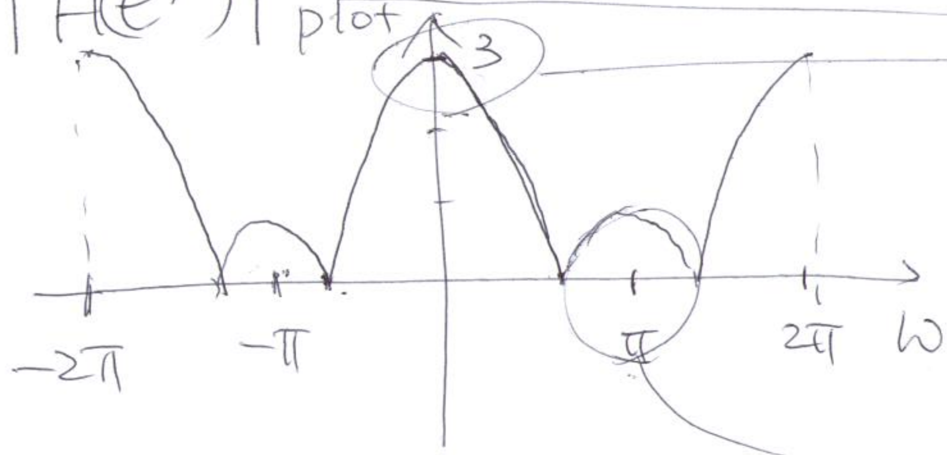
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (\delta(n-1) + \delta(n) + \delta(n+1)) e^{-j\omega n}$$

$$= e^{-j\omega} + 1 + e^{j\omega}$$

$$= 1 + 2 \cdot \cos\omega$$

$|H(e^{j\omega})|$ plot



"low ~~pass~~ frequency region has higher gain (or magnitude) than high frequency region"

$\therefore$ This is a low-pass filter

(2)

Quiz 5

~~What are~~ Write down the eigenfunctions for CT & DT ~~for~~ for LTI systems.

Evaluatate $\int_0^T e^{j\pi(k-n)\frac{t}{T}} dt$

Quiz 6.

Write down FS ~~eq~~ equations.

explain physical meaning of each Eq.

Quiz 7.

Write down discrete-time FS pairs

Why it was important to have a FS representation for an input fn in LTI system

Chapter 4

Continuous Time Fourier Transform

Chapter 4 CT Fourier Transform ①

※ Very Important ※

※ As we mentioned at the beginning of the course we will use "f" instead of "ω"

"f" → good guy (Hz) "ω" → bad guy (rad/s)

Good to write f form of equations in your text book.

OK here we go ~

- We want to develop a (Fourier) transform for aperiodic signal (cover larger volume of signals)

Do you remember FS for periodic signal?

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\,dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

→ periodic signal

→ over a period

- Let's define a transform.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\,dt$$

x(t): aperiodic signal

→ over entire time

used to be $X(j\omega)$ or $X(\omega)$ or $X(2\pi f)$

Then $x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$ ②

- Why?

$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\, d\tau\, e^{j2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(\tau - t)}\, df\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f(\tau - t)}\, df}_{\delta(\tau - t)}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(\tau - t)\, d\tau = x(t) \quad // \text{Done.}$$

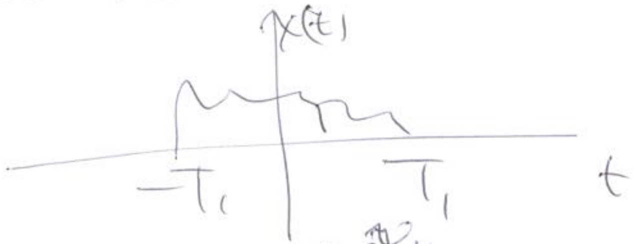
Let's rewrite this as it is so important

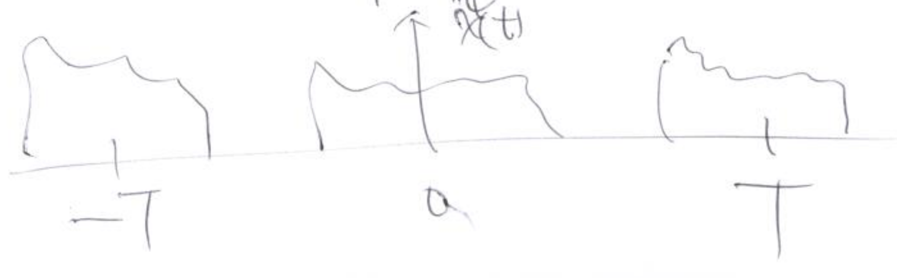
- Fourier transform $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$ ← spectrum. → ~~decompose~~ analyze (basis function)

Inverse Fourier transform $x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$ → synthesis

- Another way of looking at this (via F.S)

Aperiodic signal $x(t)$ can be considered as a periodic signal ($\tilde{x}(t)$) w infinite period.





$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi f_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk2\pi f_0 t} dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk2\pi f_0 t} dt$$

envelope

$$X(f) = T a_k = \int_{-\infty}^{\infty} x(t) e^{-j\overset{\omega}{2\pi f} t} dt$$

here $a_k = \frac{1}{T} X(kf_0)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(kf_0) e^{jk2\pi f_0 t} \qquad \frac{1}{T} = f_0$$

$$= \sum_{k=-\infty}^{\infty} X(kf_0) e^{jk2\pi f_0 t} f_0$$

$T \to \infty \quad f_0 \to 0 \qquad kf_0 = f \quad f_0 = df$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

so

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Read 4.1.1. as an example

4.1.2 Convergence.

When does FT exist?

if $x(t)$ has finite energy $\left(\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty\right)$

$X(f)$ is finite. $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$

$\Rightarrow$ $x(t)$ and $x(t)$ from FT only @ individual values

Sufficient condition for FT exist (4)

1. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
2. finite # of maxima & minima within finite interval
3. finite # of discontinuity "

4.1.3. Examples of CT Fourier Transform

"fill your bucket list"

$x(t) = e^{-at} u(t)$ $a > 0$

$X(f) = \frac{1}{a + j2\pi f}$ $|X(f)| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}}$

↗ Lorentzian fn.

Do Example 2.

$x(t) = \delta(t)$

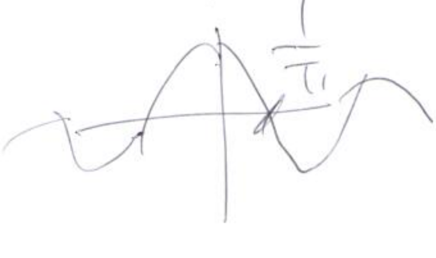
$X(f) = 1$

Scaling property $x(at) \leftrightarrow \frac{1}{|a|} X(\frac{f}{a})$

rect$(\frac{t}{T_1})$

different definition than the book.

$T_1 \text{ sinc}(T_1 f)$



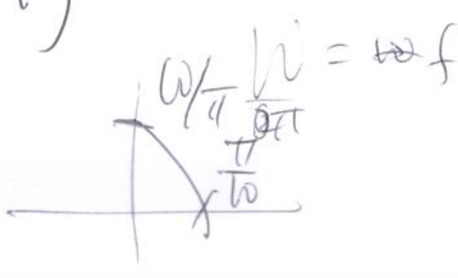
Do Example 4.5

~~$X(f) = \text{rect}(\frac{f}{2W})$~~

~~$x(t) = 2W \text{sinc}(2Wt)$~~

$X(f) = \text{rect}(\frac{\pi f}{W})$

$x(t) = \frac{W}{\pi} \text{sinc}(\frac{W}{\pi} t)$



Signals

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Time domain	Fourier domain
$\delta(t)$	
$\delta(at)$	
$e^{i2\pi f_0 t}$	
$rect(t) \triangleq \begin{cases} 1, & \text{if } \lvert t\rvert \le 1/2 \\ 0, & \text{otherwise} \end{cases}$	
$\Lambda(t) = rect(t) * rect(t)$	
$sinc(t) \triangleq \sin(\pi t)/\pi t$	
$e^{-\pi t^2}$	
$sin2\pi f_0 t = \frac{\exp(+i2\pi f_0 t) - \exp(-i2\pi f_0 t)}{2i}$	
$cos2\pi f_0 t = \frac{\exp(+i2\pi f_0 t) + \exp(-i2\pi f_0 t)}{2}$	
When $a > 0$, $\begin{cases} e^{-at}, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$	
$\frac{1}{a + j2\pi t}$	
$\text{III}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k)$	
$\frac{1}{T}\text{III}\left(\frac{t}{T}\right)$	
$f(at)$	

Fourier Transform Pairs

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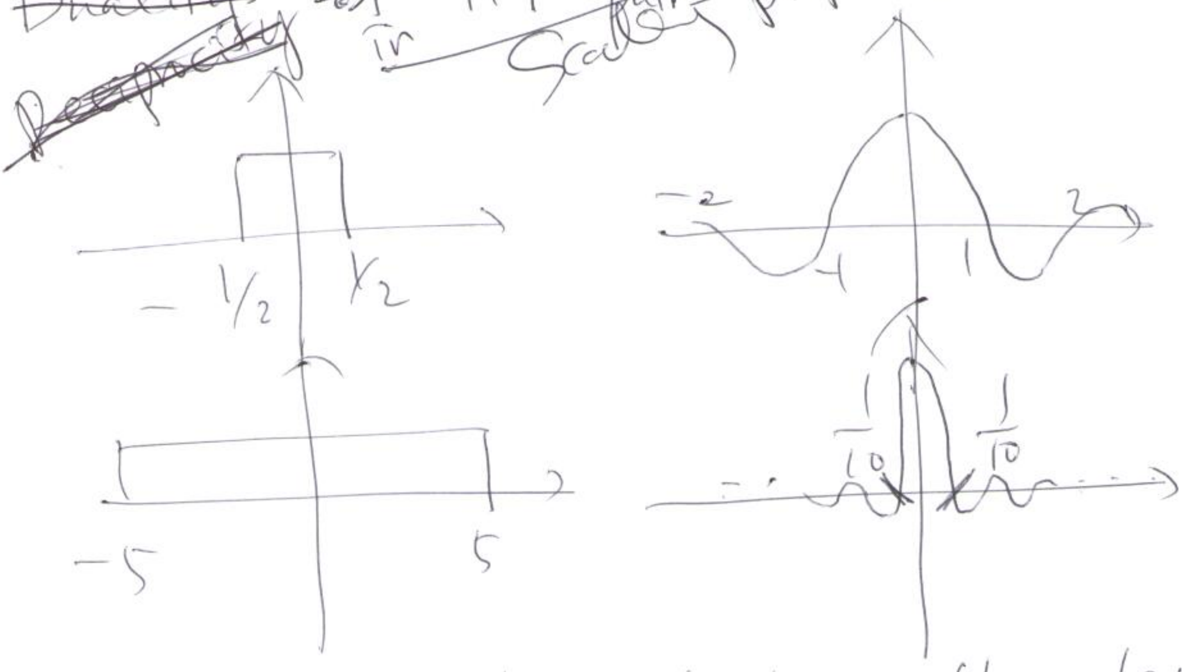
Time/Space domain	Fourier domain
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-i2\pi f t_0}$
1	$\delta(f)$
$e^{i2\pi f_0 t}$	$\delta(f-f_0)$
$rect(t) \triangleq \begin{cases} 1, & \text{if } \lvert t\rvert \le 1/2 \\ 0, & \text{otherwise} \end{cases}$	$\frac{sin\pi f}{\pi f} \triangleq sinc(f)$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$sin2\pi f_0 t$	$\frac{\delta(f-f_0)-\delta(f+f_0)}{2i}$
$cos2\pi f_0 t$	$\frac{\delta(f-f_0)+\delta(f+f_0)}{2}$
When $a > 0$, $\begin{cases} e^{-at}, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1}{a+j2\pi f}$ $Magnitude: \frac{1}{\sqrt{a^2+(2\pi f)^2}}$ $Phase: -\tan^{-1}(\frac{2\pi f}{a})$
$\text{III}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k)$	$\text{III}(f)$
$f(at)$	$\frac{1}{a}F(\frac{f}{a})$

· ~~Duality of~~ FT ~~pairs~~ ~~Reciprocity~~ in Scale property ·



4.2 → We will come back to this later.

4.3 Properties

$$x(t) \xleftrightarrow{F_1} X(f)$$

or $X(f) \triangleq F_1\{x(t)\}$

Linearity : $ax(t) + by(t) \xleftrightarrow{F_1} aX(f) + bY(f)$

Time shifting $x(t-t_0) \xleftrightarrow{F_1} e^{-j2\pi f t_0} X(f)$

magnitude the same
linear phase shift

Conjugation & symmetry $x^*(t) \xleftrightarrow{F_1} X^*(-f)$

if $x(t)$ is real $X(-f) = X^*(f)$ (Hermitian)

" $Even\{x(t)\} \xleftrightarrow{F_1} Real\{X(f)\}$

$Odd\{x(t)\} \xleftrightarrow{F_1} j\,Imag\{X(f)\}$

Differentiation & Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j2\pi f X(f)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$

Time & frequency scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$x(-t) \xleftrightarrow{\mathcal{F}} X(-f)$$

Duality.

Very important Let's do it again

Fig. 4.17.

examples

$$-j2\pi t\, x(t) \longleftrightarrow \frac{dX(f)}{df}$$

$$e^{j2\pi f_0 t} x(t) \longleftrightarrow X(f-f_0)$$

Check.

$$-\frac{1}{j2\pi t} x(t) + \frac{1}{2} x(0)\delta(t) \longleftrightarrow \int_{-\infty}^{f} X(\tau)d\tau$$

Parseval's ~~theo~~ Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

total energy is the same.

energy density spectrum

Properties of Symmetry

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A real function, $f(t)$, is

"even function"	if	$f(t) = f(-t)$
"odd function"	if	$f(t) = -f(-t)$

A real function can be divided into even and odd parts of the function

$$f_{even}(t) = \{f(t) + f(-t)\}/2$$
$$f_{odd}(t) = \{f(t) - f(-t)\}/2$$

A function, $f(x)$, is

"real function"	if	$f(t) = f^*(t)$
"imaginary function"	if	$f(t) = -f^*(t)$

A function, $f(x)$, is

"Hermitian function"	if	$f^*(t) = f(-t)$
"Anti-hermitian function"	if	$f^*(t) = -f(-t)$

Hermitian means real part of the function is even and imaginary part is odd

$$f(t) = a(t) + ib(t)$$

where $a(t)$ and $b(t)$ are real functions

$$a(t) = a(-t)$$
$$b(t) = -b(-t)$$

Fourier transform of a real function, $h(t)$, is Hermitian

$$H^*(f) = H(-f)$$

And

$$h(t) = h_{even}(t) + h_{odd}(t)$$
$$FT\{h_{even}(t)\} = Re\{H(f)\} = Re\{H(-f)\}$$
$$FT\{h_{odd}(t)\} = Im\{H(f)\} = -Im\{H(-f)\}$$

4.4 Convolution property

Remember convolution?

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$Y(f) = \mathcal{F}(y(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi f t} dt\, d\tau$$

time shift

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} H(f) d\tau$$

$$= H(f) \cdot X(f)$$

Awesome!

$y(t) = h(t) * x(t)$ ⟵ complex

$Y(f) = H(f) \cdot X(f)$ ⟵ simplex

Very Very important!!

Show an example in matlab.

How simple it is.

LTI system. $H(f)$ fully characterize system.

$$H_1(f) \cdot H_2(f) = H_2(f) H_1(f)$$

If LTI system is stable, → Q: what does this mean?

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

+ Dirichlet conditions ⟶ $\exists H(f)$

for unstable LTI system → Laplace transform.

Do Example 4.19 & 4.20

4.5 Multiplication or modulation property

$$s(t) \cdot p(t) = \int_{-\infty}^{\infty} S(\theta) P(f-\theta) d\theta$$

$$= S(f) * P(f)$$

→ AM modulation

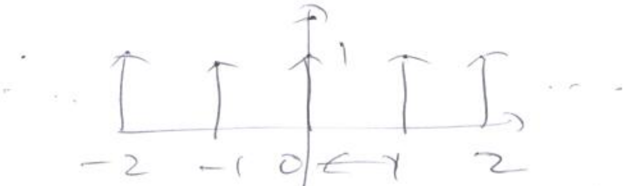
Do example 4.21, 4.22.

4.6 Table : Handout FT pairs in f.

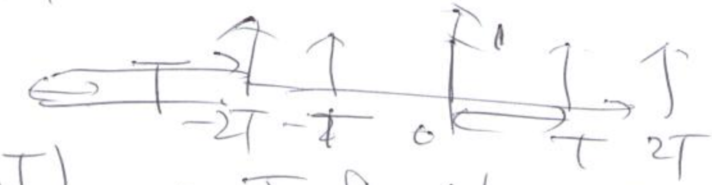
OK. Now we learned enough. So it's time to revisit 4.2 FT for periodic signals

Before we start, let's define $Ш(t)$

$$Ш(t) = \Sigma \delta(t-n)$$



What about



$$\Sigma \delta(t-nT) = \Sigma \delta(T(\frac{t}{T}-n))$$

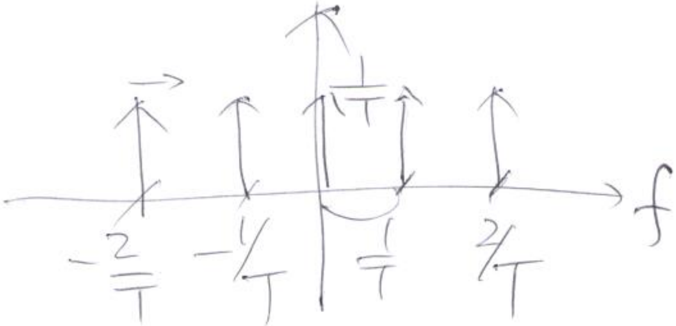
$$= \frac{1}{T} \Sigma \delta(\frac{t}{T}-n) = \frac{1}{T} Ш(\frac{t}{T})$$

Let's memorize this

$\frac{1}{T} Ш(\frac{t}{T})$

What is FT of $\frac{1}{T}Ш(\frac{t}{T})$?

$$= Ш(Tf) = \sum \delta(Tf-n)$$
$$= \sum \delta(T(f-\tfrac{n}{T}))$$
$$= \frac{1}{T}\sum \delta(f-\tfrac{n}{T})$$



for $T=1$, every thing is good!

$$Ш(t) \overset{FT}{\longleftrightarrow} Ш(f)$$

This will be used extensively in sampling! (will revisit)
& is very important.

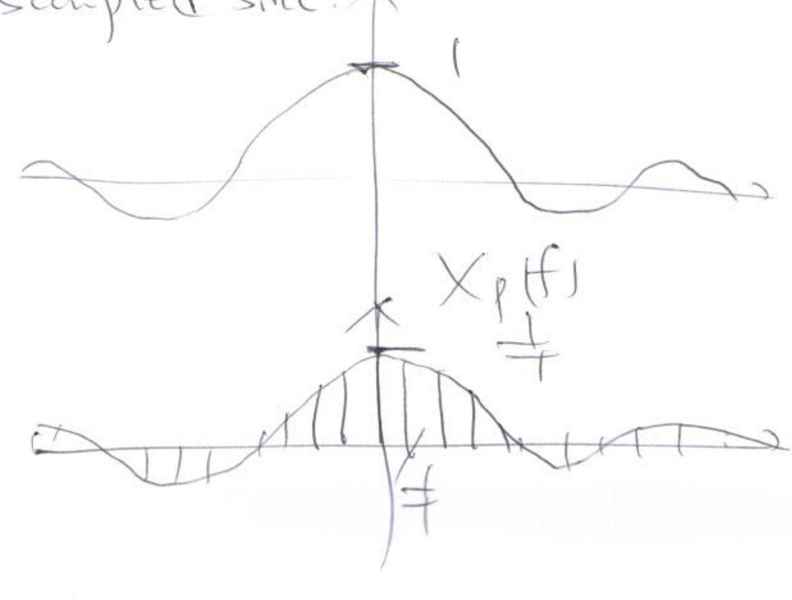
- FT for periodic signals

$$x_p(t) = x(t) * \frac{1}{T}Ш(\frac{t}{T})$$

$$\mathcal{F}\{x_p(t)\} = \mathcal{F}\{x(t) * \frac{1}{T}Ш(\frac{t}{T})\}$$

$$X_p(f) = X(f)\cdot Ш(Tf)$$

Example if $X(f)$ sinc, $X_p(f)$ is scaled & sampled sinc.



Very simple!

(10)

$$X_p(f) = X(f)\cdot \text{Ш}(Tf)$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \cdot \text{Ш}(Tf)$$

$$= \int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt \cdot \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T})$$

$$= \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt}\cdot \sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T})$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta(f-\frac{k}{T})$$

$$x_p(t) = \int_{-\infty}^{\infty} X_p(f)e^{j2\pi ft}df$$

$$= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_k\delta(f-\frac{k}{T})\, e^{j2\pi ft}df$$

$$= \sum_{k=-\infty}^{\infty} a_k\int_{-\infty}^{\infty}\delta(f-\frac{k}{T})e^{j2\pi ft}df$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi\frac{k}{T}t} = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$ //

4.7. LTI system in differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

FT for each term.

$$\sum_{k=0}^{N} a_k (j2\pi f)^k Y(f) = \sum_{k=0}^{M} b_k (j2\pi f)^k X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{k=0}^{M} b_k(2\pi f)^k}{\sum_{k=0}^{N} a_k(2\pi f)^k}.$$

Solve 4.25 4.26 → rational fn.

2D Fourier Transform Pairs

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Space domain	Fourier domain
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-i2\pi(k_x x_0 + k_y y_0)}$
1	$\delta(k_x,\ k_y)$
$rect(x, y) \triangleq rect(x)rect(y)$	$sinc(k_x)sinc(k_y)$
$e^{-\pi(x^2 + y^2)}$	$e^{-\pi(k_x{}^2 + k_y{}^2)}$
$sin2\pi(k_1 x + k_2 y)$	$\dfrac{\delta(k_x - k_1, k_y - k_2) - \delta(k_x + k_1, k_y + k_2)}{2i}$
$cos2\pi(k_1 x + k_2 y)$	$\dfrac{\delta(k_x - k_1, k_y - k_2) + \delta(k_x + k_1, k_y + k_2)}{2}$
$rect(r)$	$\dfrac{J_1(\pi\rho_k)}{2\rho_k} \triangleq jinc(\rho_k)$
$f(ax, by)$	$\dfrac{1}{\lvert ab \rvert} F(\dfrac{k_x}{a}, \dfrac{k_y}{b})$

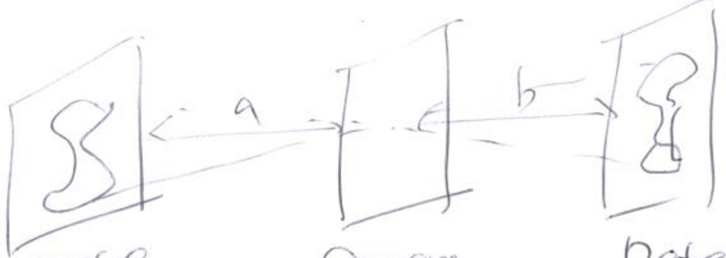
~~$X(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2)\, e^{-j2\pi f_1 t_1}$~~

$$A(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x, y)\, e^{-j(2\pi k_x x + 2\pi k_y y)}\, dx\, dy$$

$$a(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(k_x, k_y)\, e^{j(2\pi k_x x + 2\pi k_y y)}\, dk_x\, dk_y$$

seperable function $a(x, y) = a_1(x)\, a_2(y)$

A pinhole Camera.



Let's simplify the system to 1D

if $a = b$

$i_d(x_d) = i_s(-x_d)$

if $a \neq b$

$i_d(x_d) = i_s\left(-\frac{x_d}{b/a}\right)$

$m = -b/a$

$i_d(x_d) = i_s(+x_d/m)$

Well, In 2D case $i_d(x_d, y_d) = i_s(+x_d/m, +y_d/m)$

Is this linear shift invariant system?

Let's further complicate the system.

Images are added! (Not always true but let's assume it)

$\delta(x_c) \rightarrow \delta(x_c - x')$

$h(x_d; x') = i_s\left(\frac{x_d - Mx'}{m}\right)$

what will be M? $M = \frac{a+b}{a}$

$\therefore h(x_d; x') = \frac{1}{m} i_s\left(\frac{x_d - Mx'}{m}\right)$

$i_d(x_d) = \frac{K}{m}\int h(x_d; x')\, C(x')\, dx'$

$= \frac{K}{m}\int i_s\left(\frac{x_d - Mx'}{m}\right) C(x')\, dx'$

$$Mx' = x''$$

$$i\,I_d(x_d) = \frac{K}{mM}\int i\,I_s\left(\frac{x_d - x''}{m}\right)\cdot C\left(\frac{x''}{M}\right)dx''$$

$$= \frac{K}{mM}\, i\,I_s\left(\frac{x_d}{m}\right) * C\left(\frac{x_d}{M}\right)$$

In 2D

$$i\,I_d(x_d, y_d) = \frac{K}{m^2M^2}\, i\,I_s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) * C\left(\frac{x_d}{M}, \frac{y_d}{M}\right)$$

2DFT

$$I_d(k_x, k_y) = K\, I_s(m x_d, m y_d) \cdot C(M x_d, M y_d)$$

Quiz 8

① Write CTFT pairs

② $\delta(t) \overset{FT}{\longleftrightarrow}$?

③ $x(t) \overset{FT}{\longleftrightarrow} X(f)$

$x(at) \longleftrightarrow$?

Quiz 9.

① $x(t) * y(t) \overset{F}{\longleftrightarrow} X(f) \cdot Y(f)$

② $Ш(t) \overset{F}{\longleftrightarrow} Ш(f)$

③ $e^{-\pi t^2} \overset{F}{\longleftrightarrow} e^{-\pi f^2}$

Correction. $-j2\pi t\, x(t) \longleftrightarrow \dfrac{dX(f)}{dt}$

$-\dfrac{1}{j2\pi t}x(t) + \dfrac{1}{2}x(0)\delta(t) \longleftrightarrow \int_{-\infty}^{f} X(f)\,df$

Quiz 10

① $x(t) \cdot p(t) \overset{F}{\longleftrightarrow}$

② plot $\frac{1}{T}Ш\left(\frac{t}{T}\right)$ in time & freq domain

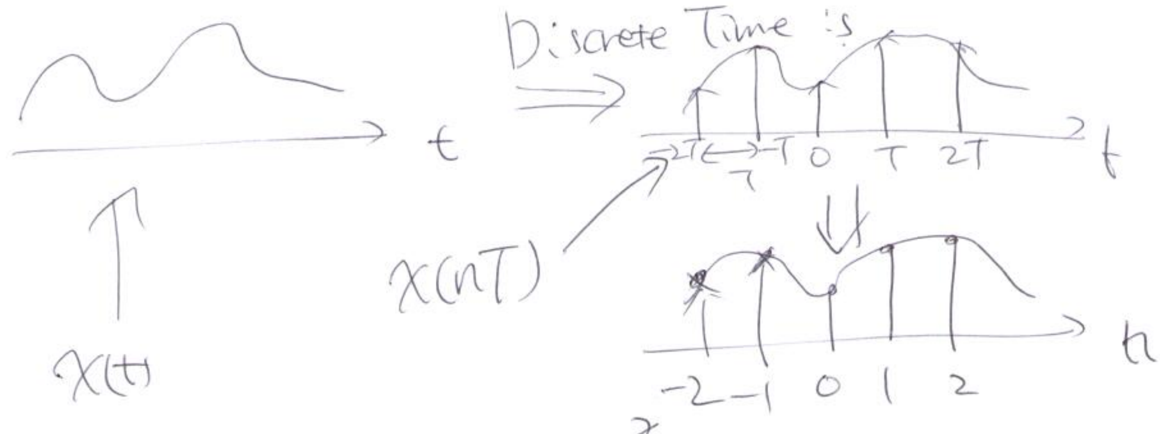
Chapter 5

Discrete-Time Fourier Transform

Chapter 5

5.1 Discrete-Time Fourier Transform

Let's start from CT-FT



$$x(nT) = x(n) = x(t)\cdot \frac{1}{T}Ш\left(\frac{t}{T}\right)\Big|_{t=nT} = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)\Big|_{t=nT}$$

$$= \sum_{k=-\infty}^{\infty}\delta(t-kT)\,x(kT)\Big|_{t=nT}$$

$$X(f) = \int_{-\infty}^{\infty} x(nT)\, e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)\Big|_{t=nT} e^{-j2\pi f t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\int_{-\infty}^{\infty}\delta(t-kT)\,e^{-j2\pi f t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\, e^{-j2\pi f kT}$$

Discrete-time, discrete-time, $kT = n$

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n}$$

Another approach of looking at this is DT-FS with infinite ~~cycle~~ period $(N \to \infty)$

see 5.1.1 for your reading.

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn}.$$ (2)

Then inverse FT becomes

$$x(n) = \int_1 X(e^{j2\pi f}) e^{j2\pi fn} df.$$

Let's prove this.

$$x(n) = \int_0^1 \sum_{k=-\infty}^{\infty} x(k) e^{-j2\pi fk} e^{j2\pi fn} df$$

$$= \sum_{k=-\infty}^{\infty} x(k) \underline{\int_0^1 e^{-j2\pi f(k-n)} df}$$

$$= \delta(k-n). \text{ why?}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \delta(k-n) = x(n).$$

if $k=n$ ans $= 1$

if $k \neq n$ $\dfrac{e^{-j2\pi(k-n)} - 1}{-j2\pi(k-n)} = 0$

So Here is DTFT pair

$$x(n) = \int_1 X(e^{j2\pi f}) e^{j2\pi fn} df \rightarrow \text{synthesis}$$

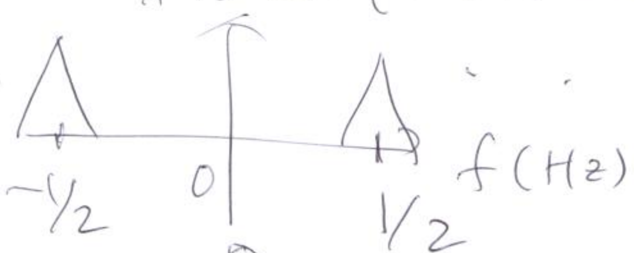
$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn} \longrightarrow \text{analysis}$$

decompose. linear combination of complex exp.

Spectrum

periodic "integer" → in phase it is 2π. ($\because 2\pi f = 2\pi \cdot 1$)

High frequency ...



Low frequency

• Careful.

$e^{j2\pi fn}$ vs. $e^{j2\pi ft}$ looks very benign

but $n = 1 \cdot 2 \cdot 3$ the same $t \rightarrow \sim \cdots$ very diff.

Do Example 5.1

5.1.3 Convergence

In DT-FT, we have infinite sum.
so, for convergence

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

or finite energy

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

synthesis eq. has no issue with convergence
(finite interval)

No gibbs ringing.

5.2 FT of periodic signals

If $x(n)$ is a periodic signal

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

FT

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n} e^{-j2\pi f n}$$

$$= \sum_{k=\langle N \rangle} a_k \sum_{n=-\infty}^{\infty} e^{jk(\frac{2\pi}{N})n} e^{-j2\pi f n}$$

$$= \sum_{k=\langle N \rangle} a_k \delta(f - \frac{k}{N})$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta(f - \frac{k}{N})$$

($\because e^{j2\pi f}$ is periodic with 2π).

5.3 Properties $x(n) \xleftrightarrow{F_1} X(e^{j2\pi f})$ (4)

- periodicity $X(e^{j(2\pi f + 2\pi)}) = X(e^{j2\pi f})$
- linearity
- Time shift $x(n-n_0) \xleftrightarrow{F_1} e^{-j2\pi f n_0} X(e^{j2\pi f})$
- freq. shift $e^{j2\pi f_0 n} x(n) \xleftrightarrow{F_1} X(e^{j2\pi(f-f_0)})$

Ef 5.1 → MRI, OS

- Conjugate & Conjugate symmetry

$$x^*(n) \xleftrightarrow{F_1} X^*(e^{-j2\pi f})$$

if $x(n)$ real $X(e^{j2\pi f}) = X^*(e^{-j2\pi f})$
→ Hermitian

Then ⇒ $Re\{X(e^{j2\pi f})\}$: even fn.
$Im\{X(e^{j2\pi f})\}$: odd fn.

- Difference

$$x(n) - x(n-1) \xleftrightarrow{F_1} (1-e^{-j2\pi f})X(e^{j2\pi f})$$

- Accumulation

$$y(n) = \sum_{m=-\infty}^{n} x(m) \xleftrightarrow{F_1} \frac{1}{1-e^{-j2\pi f}} X(e^{j2\pi f}) + \frac{1}{2}X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(f-k)$$

- Time Reversal

$$x(-n) \xleftrightarrow{F_1} X(e^{-j2\pi f})$$

- Time Expansion

CT case, $x(at) \xleftrightarrow{F_1} \frac{1}{|a|} X(\frac{f}{a})$

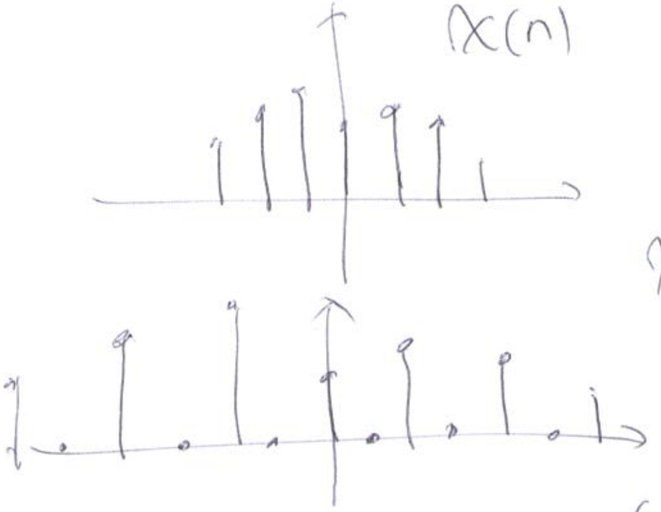
DT case $x_{(k)}(n) \xleftrightarrow{F_1} X(e^{j2\pi f k})$

See next page

- Differentiation in freq.

$$n x(n) \longleftrightarrow \frac{j}{2\pi} \frac{dX(e^{j2\pi f})}{df}$$

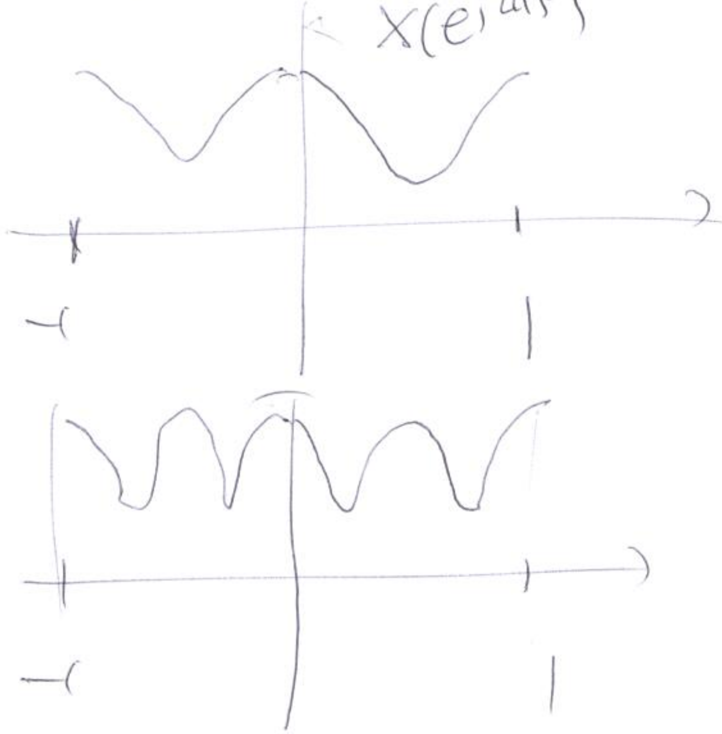
$$x_{(k)}(n) = \begin{cases} x(n/k) & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise.} \end{cases}$$



wide ⟷

in frequency domain

$$x_{(k)}(n) \overset{FT}{\longleftrightarrow} X(e^{j2\pi kf})$$



narrow ⟷

"Zero filling"

- Parseval's Relation.

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \int_1 |X(e^{j2\pi f})|^2 df \quad \rightarrow \text{example}$$

5.4. Convolution property, Do example 5.14

$$y(n) = x(n) * h(n)$$

$$Y(e^{j2\pi f}) = X(e^{j2\pi f}) H(e^{j2\pi f})$$

5.5 Multiplication property.

$$y(n) = x_1(n) \cdot x_2(n)$$

$$Y(e^{j2\pi f}) = \int X_1(e^{j2\pi F_1}) X_2(e^{j2\pi (f-F_1)}) dF_1$$

5.7. Duality. ⇐ Read.

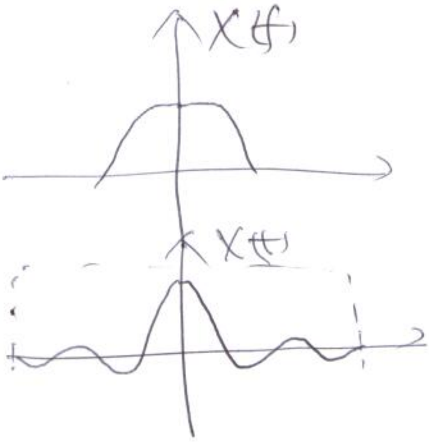
~~Over~~ Summary of CT & DT
FT & FS.

- Summary of FS & FT

CT - FT

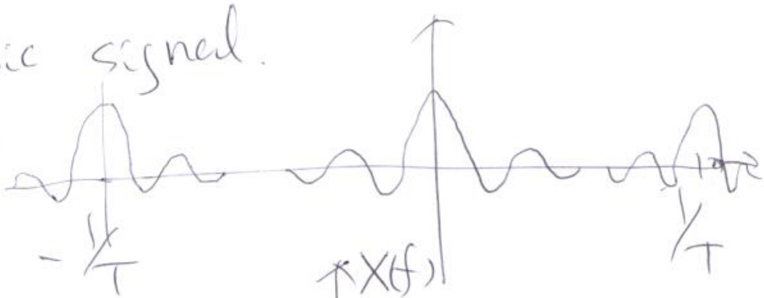
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$



CT - FS : periodic signal.

$\rightarrow x(t) * Ш\left(\frac{t}{T}\right)$



$X(f)\, Ш(Tf)$

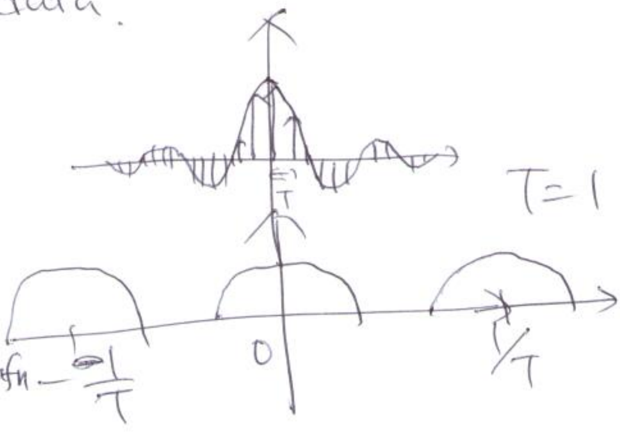
$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j2\pi k f_0 t}\, dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi k f_0 t}$$

DT - FT : sampled data.

$\rightarrow x(t) \cdot Ш\left(\frac{t}{T}\right)$

$X(f) * Ш(Tf)$



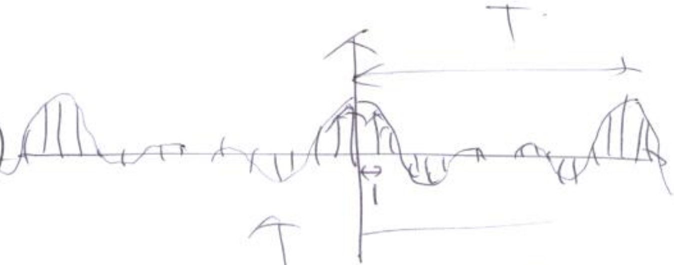
$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi fn}$$

$$x(n) = \int_0^1 X(e^{j2\pi f})\, e^{j2\pi fn}\, df$$

DT-FS

① From CT-FS

$\{x(t) * Ш(\frac{t}{T_0})\} \cdot Ш(\frac{t}{1})$



$\{X(f) \cdot T Ш(Tf)\} * Ш(f)$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-j2\pi f_0 n k} = \frac{1}{N}\sum_{\langle N\rangle} x(n)\, e^{-jk\frac{2\pi}{N}n}$$

$$x(n) = \sum_{k=\langle N\rangle} a_k\, e^{j2\pi f_0 n k} = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}$$

② From DT-FT

$\{x(t) \cdot Ш(\frac{t}{1})\} * Ш(\frac{t}{T})$

$\{X(f) * Ш(f)\} \cdot T Ш(Tf)$

Discrete Fourier Transform (DFT)

→ FT for computer.

??? Not DT-FT? Not DT-FS?

For a signal, $x(n)$, ~~tat~~ of finite duration. (N_1)

e.g. $x(n) = 0$ outside of $0 \le n \le N_1 - 1$

We know DT-FT is

$$X(e^{j2\pi f}) = \sum x(n) e^{-j2\pi f n}$$

→ Continuous in f.
→ Not suitable ~~fo~~ for Computers.
→ Discretize in frequency
→ DT-FS!

D~~T~~-FS. Choose $N \ge N_1$

Then construct $\tilde{x}(n)$ that is periodic with N and is equal to $x(n)$ for N_1.

Let's assume $N = N_1$

Then

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}(n) e^{-jk(\frac{2\pi}{N})n}$$

We replace a_k to $X(k)$ then call this transform as "DFT".

→ Discrete both in time & ~~Four~~ Frequency.

→ Yes still different from DT-FS because the transform is defined ~~over~~ for ~~a~~ ~~period~~ a finite duration signal.

~~$x(n)$~~

DFT

$$X(k) = \frac{1}{N}\sum_{n=0}^{n-1} x(n)\, e^{-jk(\frac{2\pi}{N})n} \qquad k=0,\cdots,N-1$$

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{jk(\frac{2\pi}{N})n} \qquad n=0,\cdots,N-1$$

→ Can be viewed as sampled & scaled version of DT-FT in frequency.

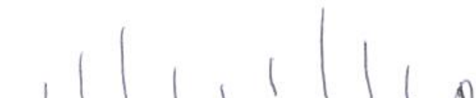
$$X(e^{j2\pi f}) = \sum x(n)\, e^{-j2\pi f n}$$

$k=0,\cdots,N-1$

i.e. $X(k) = \frac{1}{N} X(e^{j2\pi \frac{k}{N}})$

Question. what if we use $N > N_1$?

$x(n)$

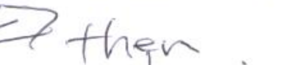


0 ... 9

$N_1 = 10$

$X(k)$ ⇒ 10개 coeff.

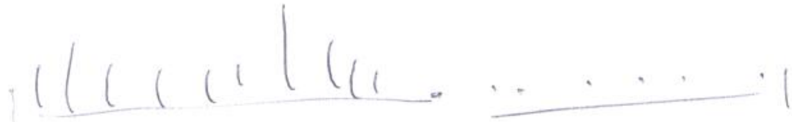
if $N = N_1 = 10$ ⇒ then



10 points

if $N = 20$

$x(n)$



10 10

$X(k)$ ⇒ 20개 Coeff!

"Zero padding"

remember zero filling???

From FT $X(k) = \frac{1}{20} X(e^{j2\pi\frac{k}{20}})$ ⇐

5.8. linear constant coeff. difference Eq. (8)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$H(e^{j2\pi f}) = \frac{Y(e^{j2\pi f})}{X(e^{j2\pi f})}$$

$$= \frac{\sum_{k=0}^{M} b_k e^{-j2\pi fk}}{\sum_{k=0}^{N} a_k e^{-j2\pi fk}}$$

//.

Quiz 11

Write down DT-FT pairs.

Quiz 12

4 FT/FS pairs

Chapter 7

Sampling

Chapter 7. Sampling ①

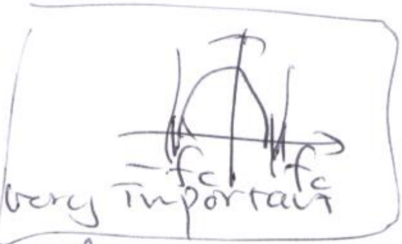
Show examples of samples look continuous

→ Image, color, movie.

→ sound → CD

Why: digitaize (not digital yet)

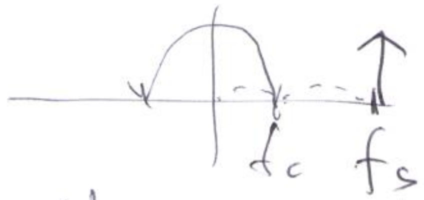
7.1. Sampling theorem



If a signal is band limited, ↳ very important

We can ~~regenerate~~ perfectly reconstruct the original signal ~~when we sample the signal~~ from the samples of the signal. ~~when we~~

To do so, we need to ~~the~~ samples the signal more than twice the frequency of the maximum frequency of the signal



⇒ Nyquist sampling theorem. (Will be in your Quiz!)

7.1.1 Sampling function.

- Remember Ш?

To have unit amplitude with spacing of T

$$\frac{1}{T}Ш\left(\frac{t}{T}\right)$$

why? $\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(\frac{t}{T}-n\right)$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(\frac{t-nT}{T}\right) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$$

ex) $x(t)\cdot\frac{1}{T}Ш\left(\frac{t}{T}\right)$

$\frac{1}{T}Ш\left(\frac{t}{T}\right)$ ↓

$x(t)$ → ⊗ → $x_p(t)$

(2)

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

— In frequency domain

$$X_p(f) = \mathcal{F}\{x(t)\} * \mathcal{F}\{\tfrac{1}{T}\text{Ш}(\tfrac{t}{T})\}$$

$$= X(f) * \text{Ш}(Tf)$$

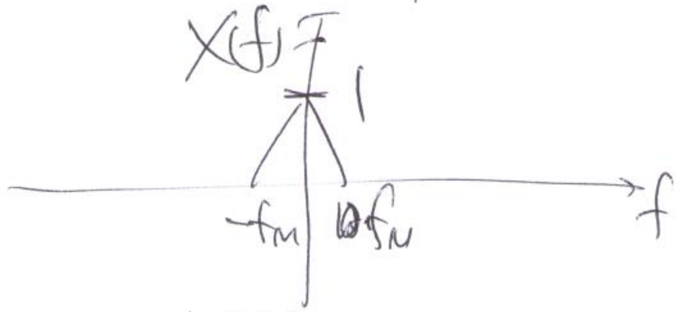
$$= X(f) * \sum \delta(Tf-n)$$

$$= X(f) * \frac{1}{T}\sum \delta(f-\frac{n}{T})$$

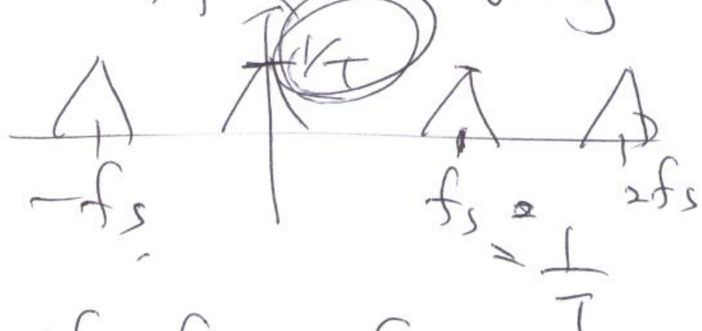
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f-\frac{n}{T})$$

$\frac{1}{T} = f_s$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f-nf_s)$$



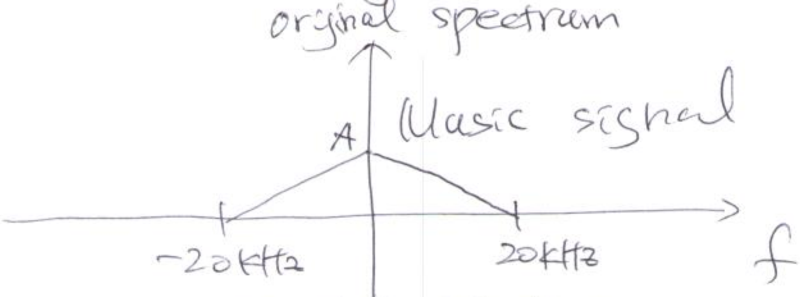
very Important



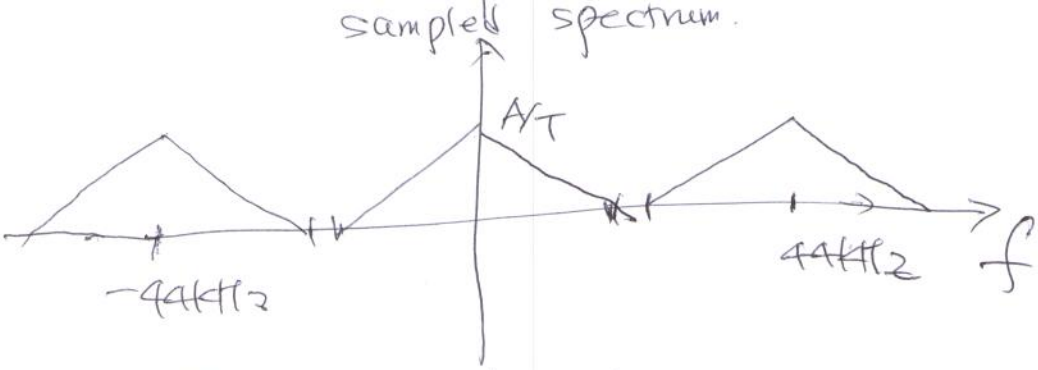
if $2f_M > f_s$

$2f_M$: Nyquist rate (need more than this)

Anti-aliasing Filter.

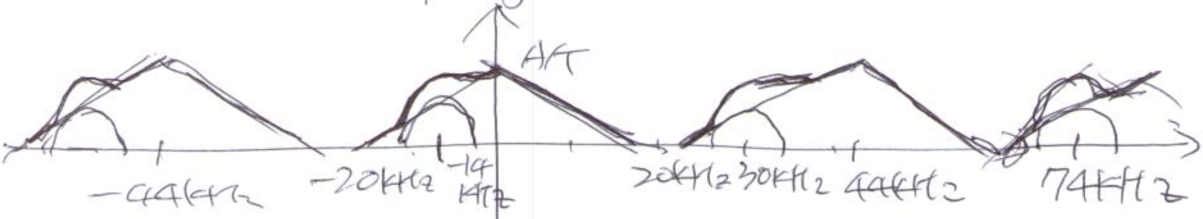


$\otimes \frac{1}{T} Ш\left(\frac{t}{T}\right)$ @ 44kHz

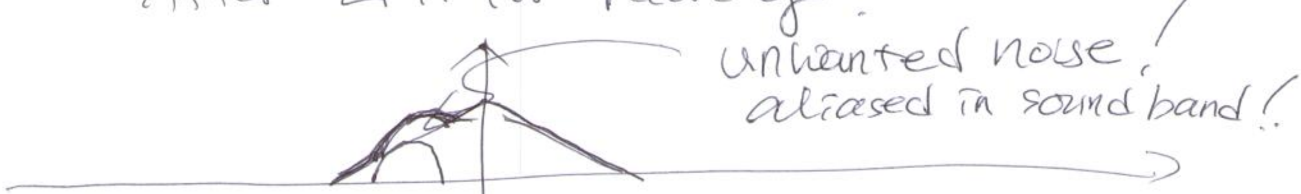


If we had ultra-sound noise @ 30kHz or around

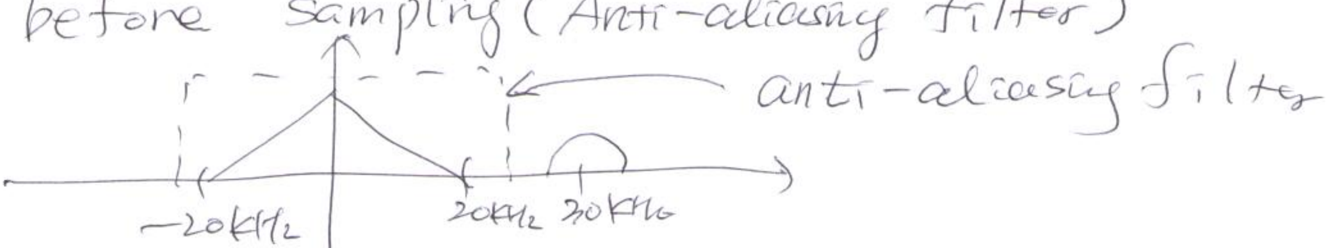
After sampling @ 44kHz

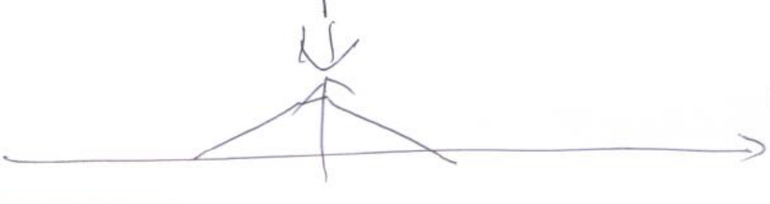


After LPF for recovery.



Hence, we need to filter out the ultrasound noise before sampling (Anti-aliasing filter)



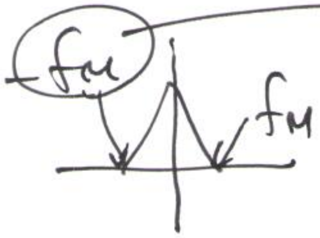


then sample.

Alias → Show fan & iphone sampling
resize image.

Point Question.

→ ???

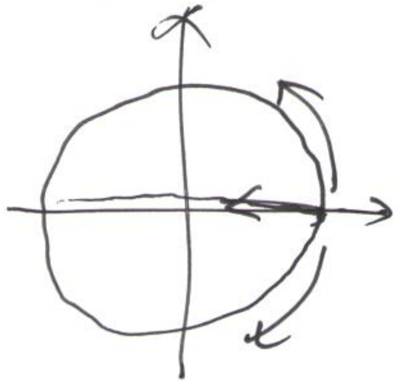


What is negative frequency?

$$\cos 2\pi ft = \frac{e^{+j2\pi ft} + e^{-j2\pi ft}}{2}$$

$e^{+j2\pi ft}$ → positive freq

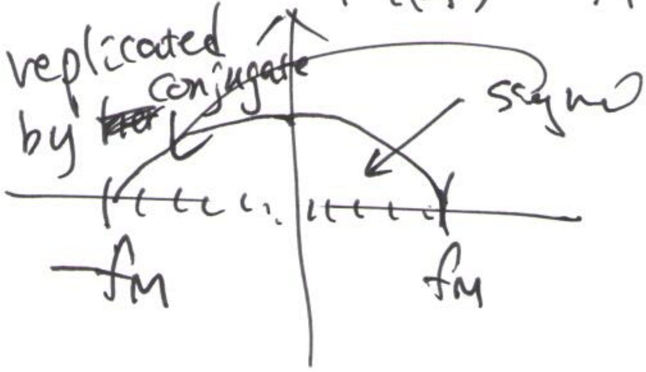
$e^{-j2\pi ft}$ → Negative freq.



What about voice? (or other real signals)

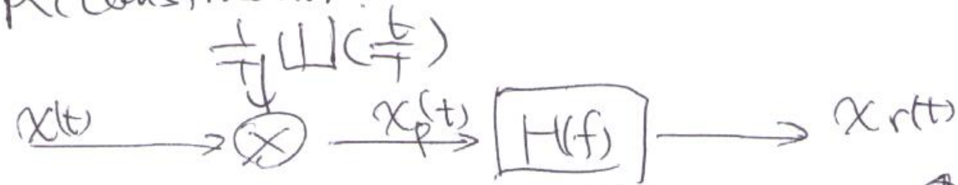
Real → FT : Hermitian

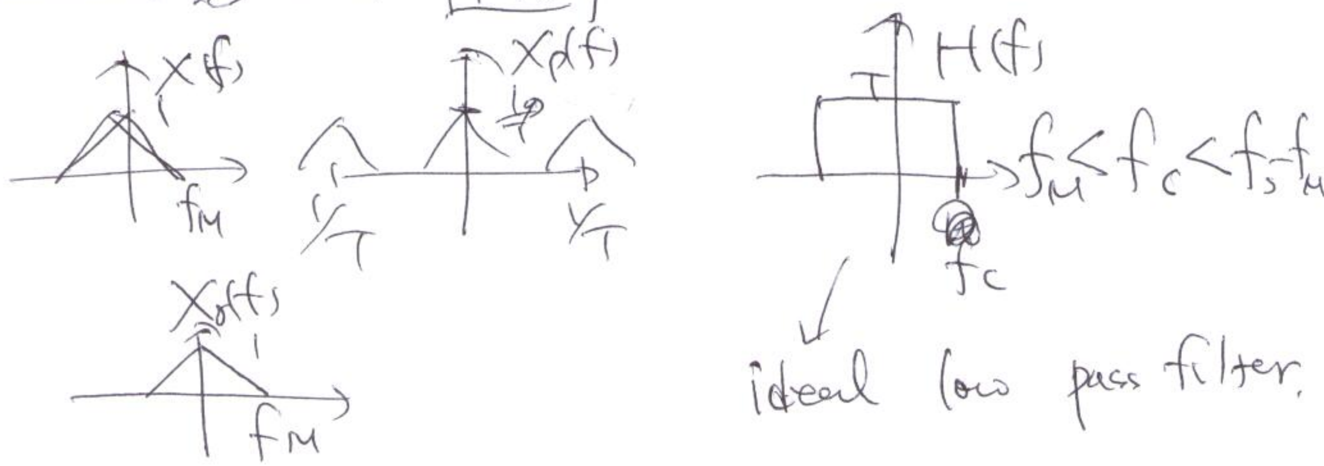
$$H(-f) = H^*(f)$$



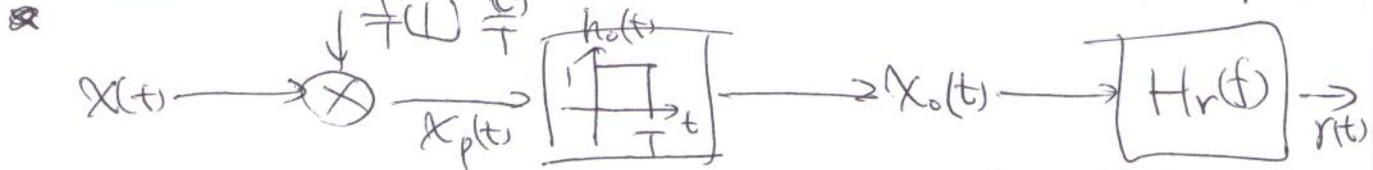
Note that there are complex signals.

Reconstruction





7.1.2 Zero-Order Hold.



$$x_o(t) = \left\{ x(t)\cdot \frac{1}{T}Ш\left(\frac{t}{T}\right) \right\} * \underbrace{rect\left(\frac{t}{T} - \frac{1}{2}\right)}_{h_o(t)}$$

additional term.

Need to remove during reconstruction

$$h_o(t) = rect\left(\frac{t}{T} - \frac{1}{2}\right) = rect\left(\frac{t - \frac{1}{2}T}{T}\right)$$ → delay

$$H_o(f) = e^{-j2\pi f \frac{1}{2}T}\, T\, sinc(Tf)$$

$$= e^{-j\pi f T}\, \frac{\sin(\pi T f)}{\pi f}$$

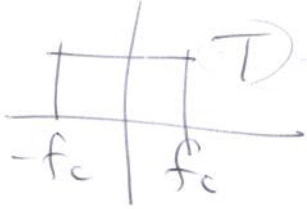
We need to divide this effect during reconstruction.

$|H_0(f)|$

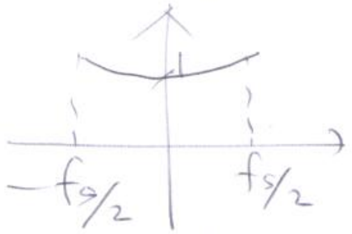
$|H_r(f)|$

but has to be band limited.

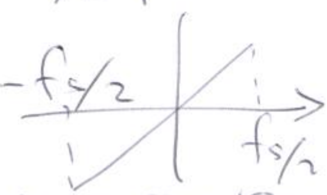
$$H_r(f) = \frac{e^{j\pi f T}\,\pi T f}{\sin(\pi T f)} \cdot$$



$\therefore |H_r(f)|$



$\angle H_r(f)$



→ (May need approximation (time domain!)

7.2 Reconstruction using Interpolation

$$x_r(t) = x_p(t) * h_r(t)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\, h_r(t-nT)$$

- For ideal interpolator $\left(H_r(f) = T\,\mathrm{rect}\left(\frac{f}{2f_c}\right)\right)$

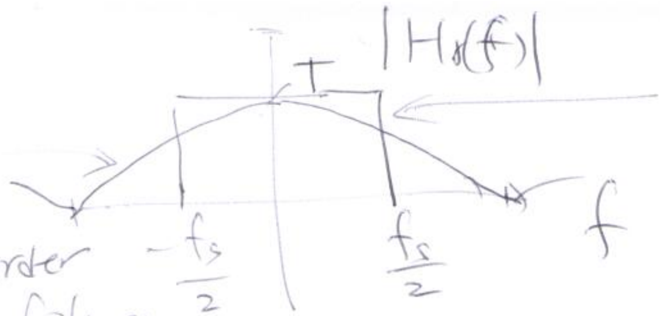
$$h_r(t) = T \cdot 2f_c \,\mathrm{sinc}(2f_c t)$$

~~$= 1$ if $T = \frac{1}{2f_c}$~~

$$\therefore x_r(t) = T 2f_c \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\big(2f_c(t-nT)\big)$$

$$\text{if } T = \frac{1}{2f_c} \qquad T2f_c = 1$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\big(2f_c(t-nT)\big)$$



ideal interpolating filter

zero-order hold filter?

SHow Figure 7.12

- Linear interpolation

What should be an interpolator?

What is $H_r(f)$? Will be in your Quiz

$h_r(t) = \frac{1}{T}\text{rect}\left(\frac{t}{T}\right) * \frac{1}{T}\text{rect}\left(\frac{t}{T}\right)$

$H_r(f) = T\,\text{sinc}^2(Tf)$

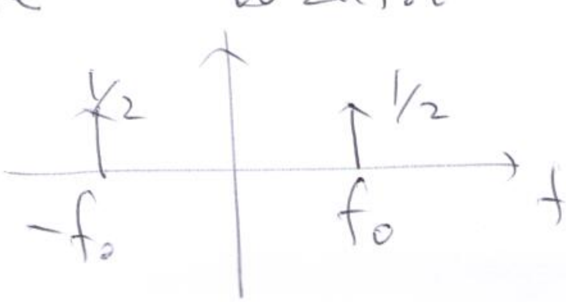
first order hold

SHow figure 7.14

7.3. Undersampling: Aliasing

If $f_s \leq 2f_M$, we cannot reconstruct our original ~~data~~ signal.

Example $\cos 2\pi f_0 t$



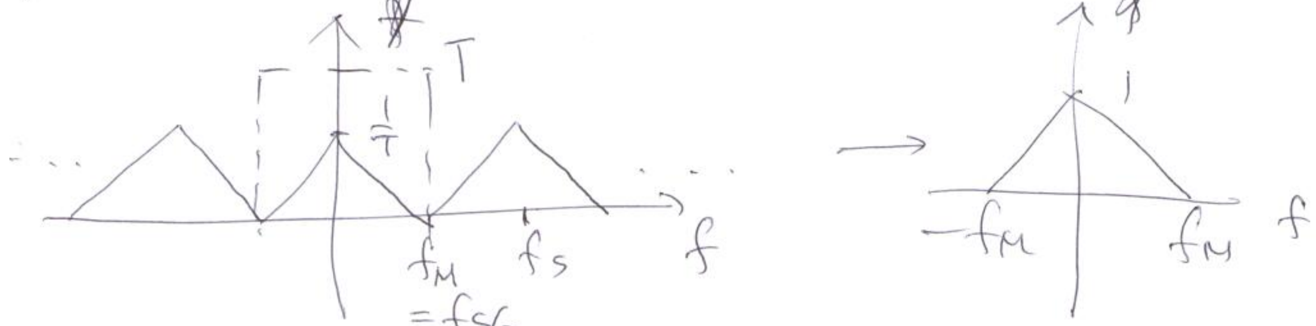
- Reconstruction with ideal LPF, zero order hold, linear interpolator. (5-7)

① Sampling with $\frac{1}{T}Ш(\frac{t}{T})$

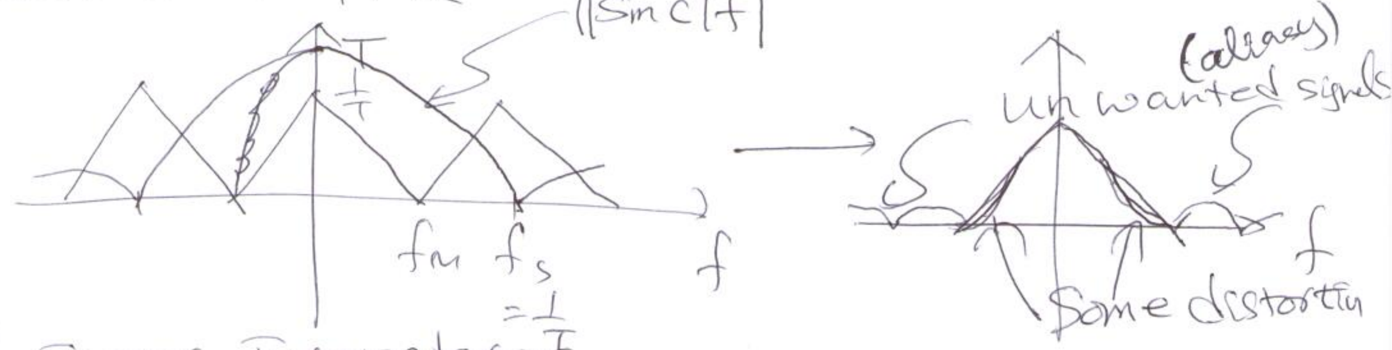
② Recon only with one of them.

③ ~~Sampling~~ ④ Nyquist Sampling rate

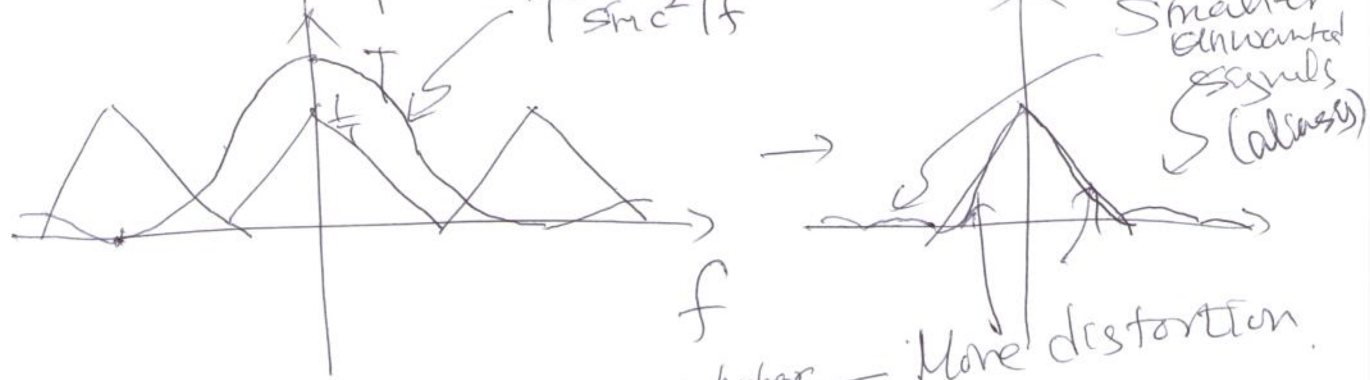
Ideal LPF



Zero order Hold



Linear Interpolator

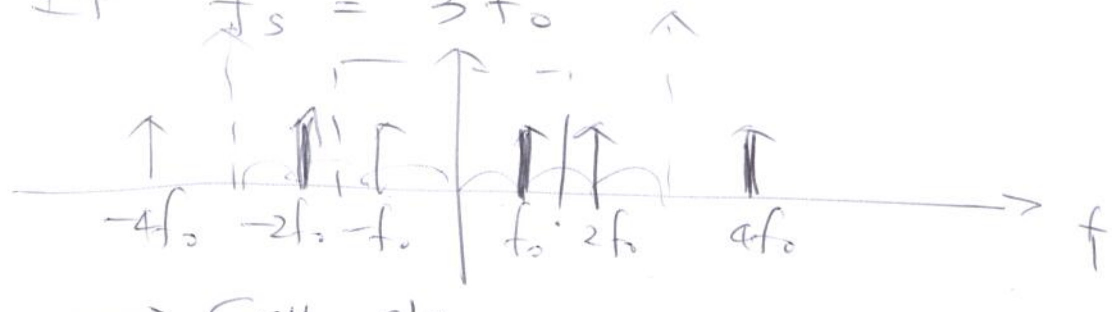


- They do change with higher f_s.
- " with additional filter.

If $f_s = 6 f_0$ (sampling)

→ No aliasing

If $f_s = 3 f_0$



→ Still ok

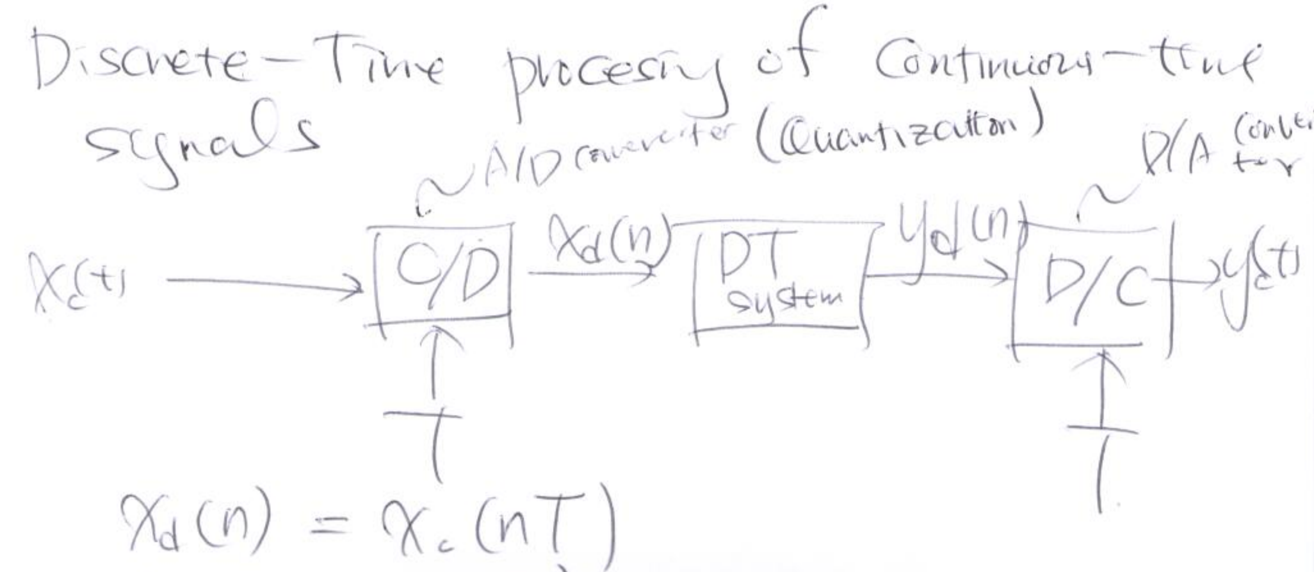
If $f_s = \frac{3}{2} f_0$

→ aliased!

Show Fig. 7.16 for time domain demonstration.

Show example of a movie

7.4 Discrete-Time processing of Continuous-time signals



$$x_d(n) = x_c(nT)$$
$$y_d(n) = y_c(nT)$$

— C/D Conversion

- Let's take a look at freq. domain (7)

$$x_p(t) = x_c(t) \cdot \frac{1}{T} Ш\left(\frac{t}{T}\right)$$

$$X_p(f) = X_c(f) * Ш(Tf)$$

alternatively,

$$x_p(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

$$X_p(f) = \mathcal{F}\{x_p(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j2\pi f nT}$$

~~This is pretty surprising because~~

From discrete time FT

$$X_d(e^{j2\pi F}) = \sum_{n=-\infty}^{\infty} x_d(n)\, e^{-j2\pi F n}$$

(Let's use F for discrete case to distinguish it from f)

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j2\pi F n}$$

If we compare these two equations

~~f T = F~~ (discrete time; continuous time)

$$X_d(e^{j2\pi F}) = X_p(F/T)$$

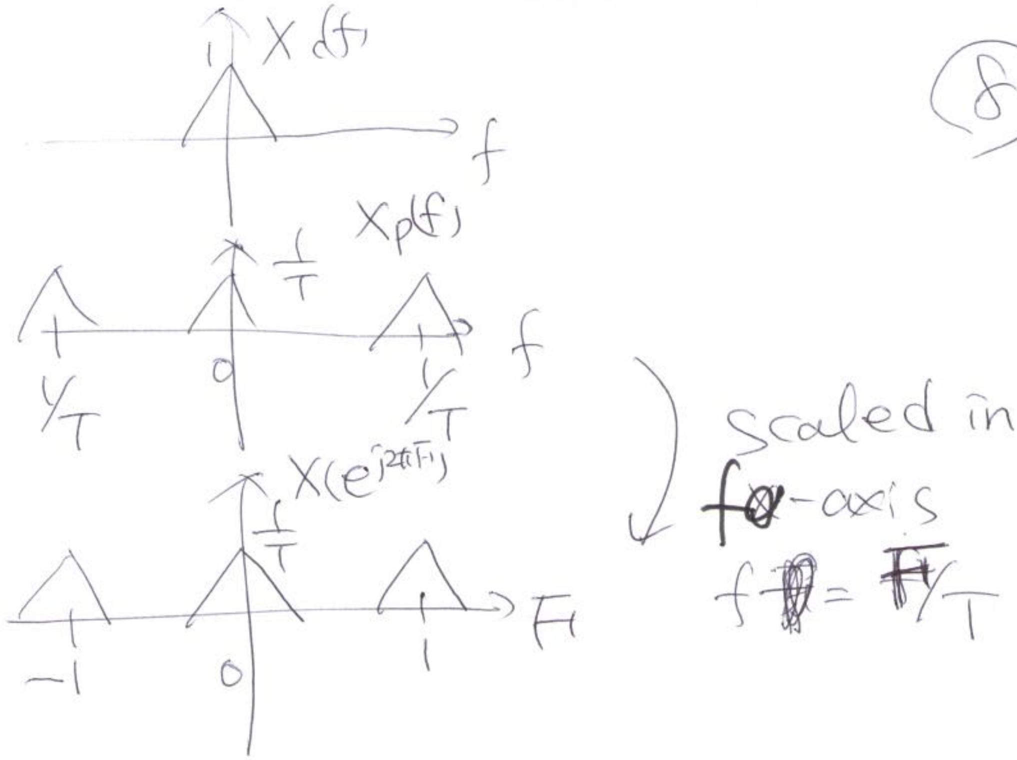
$$= X_c(F/T) * Ш(F)$$

$$= X_c(F/T) \sum_{n=-\infty}^{\infty} \delta\left(T\frac{(F-n)}{T}\right)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(\frac{F-n}{T}\right)$$

~~Show Fig 7.22~~ ~~[illegible]~~ ~~scaled in "F"~~

~~have repetition @ every M~~



- D/C conversion

- Overall system

Show in time domain (Fig 7.24 + example)

show in frequency domain (Fig 7.25)

If input is bandlimited, & sampling meets Nyquist rate, Fig 7.24 becomes equivalent to a continuous-time LTI system

$$H_c(f) = \begin{cases} H_d(e^{j2\pi f T}) & |f| < f_s/2 \\ 0 & |f| > f_s/2 \end{cases}$$

H.W: 7.4.1 & Ex 7.2
7.4.2 & Ex 7.3

7.5 Sampling of Discrete-time Signal

7.5.1 Impulse-Train Sampling

$$x_p(n) = \begin{cases} x(n) & \text{if } n = \text{an integer multiple of } N \\ 0 & \text{otherwise.} \end{cases}$$

$$x_p(n) = x(n)\,p(n) = \sum_{k=-\infty}^{\infty} x(kN)\,\delta(n-kN)$$

in frequency domain

$$X_p(e^{j2\pi f}) = \int_1 P(e^{j2\pi\theta})\,X(e^{j2\pi(f-\theta)})\,d\theta$$

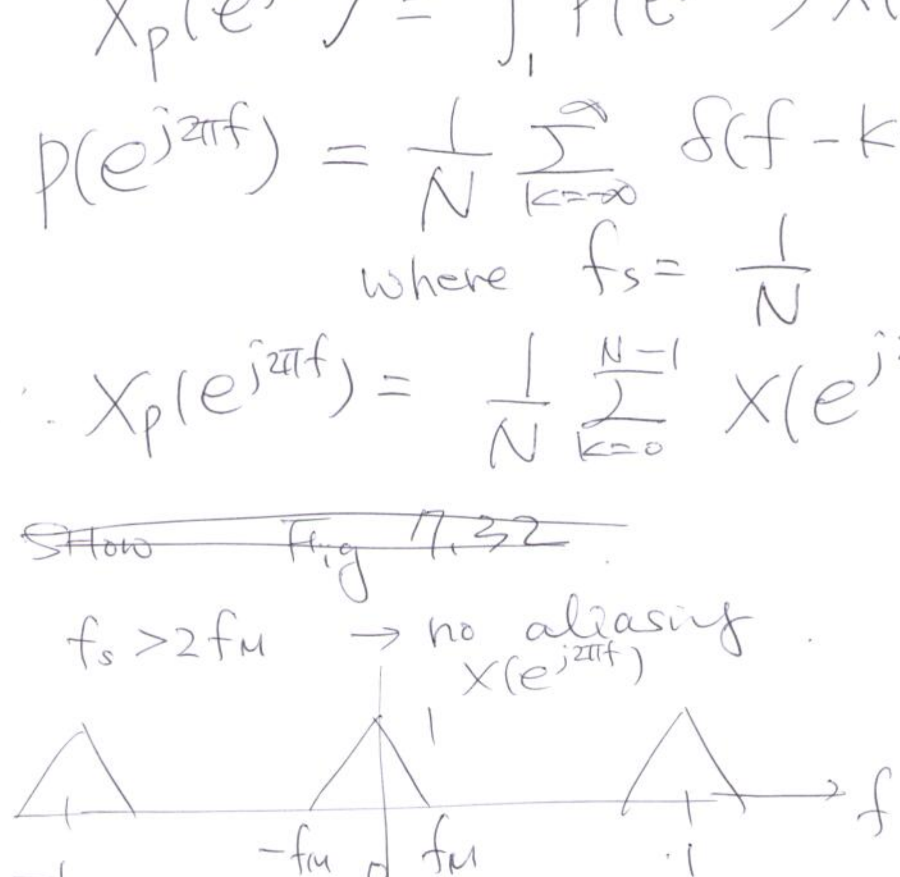
$$P(e^{j2\pi f}) = \frac{1}{N}\sum_{k=-\infty}^{\infty} \delta(f-kf_s)$$

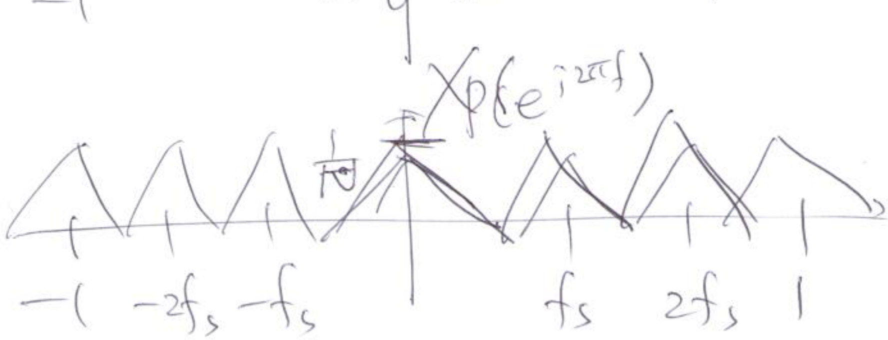
where $f_s = \frac{1}{N}$

$$\therefore X_p(e^{j2\pi f}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j2\pi(f-kf_s)})$$

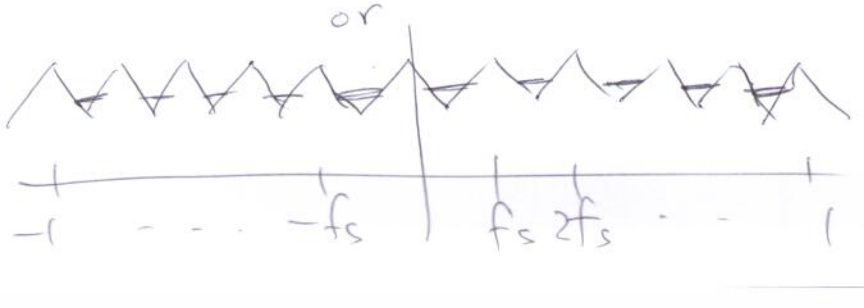
~~Show Fig 7.32~~

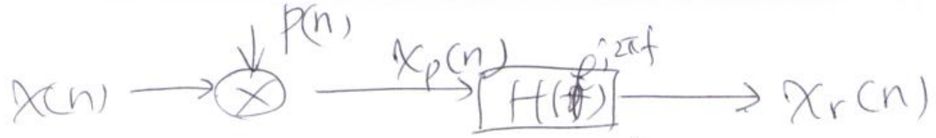
If $f_s > 2f_M$ → no aliasing.

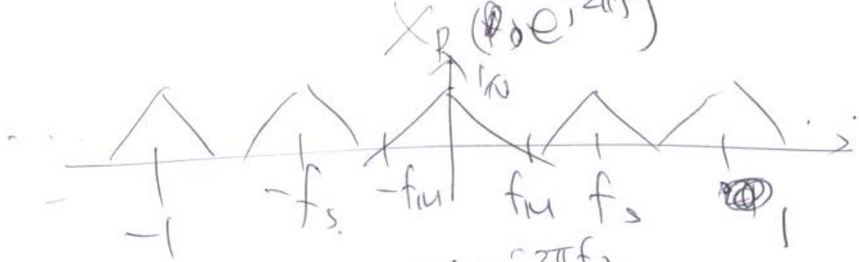


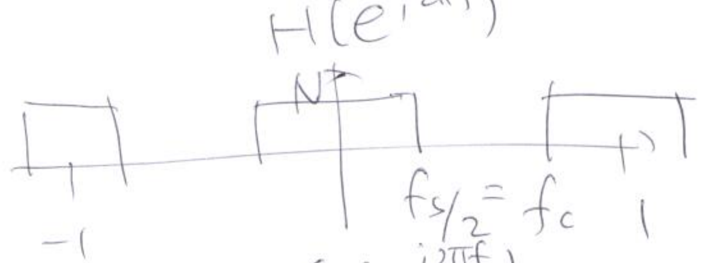


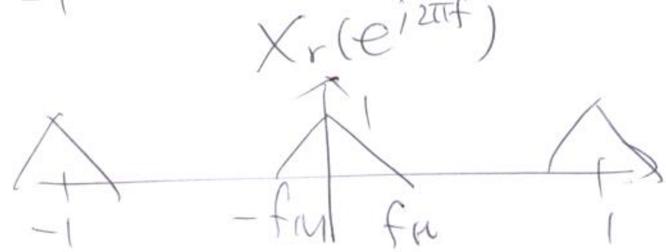
or











In time domain

$$H(e^{j2\pi f}) = N\,\mathrm{rect}\left(\frac{f}{f_s}\right) = N\,\mathrm{rect}\left(\frac{f}{2f_c}\right)$$

$$h(n) = 2Nf_c\,\mathrm{sinc}(2f_c n) = 2Nf_c\frac{\sin(2\pi f_c n)}{2\pi f_c n}$$

$$x_r(n) = x_p(n) * h(n)$$

$$x_r(n) = \sum_{k=-\infty}^{\infty} x(kN)\,2Nf_c\frac{\sin(2\pi f_c(n-kN))}{2\pi f_c(n-kN)}$$

7.5.2. Decimation & Interpolation

Decimation ; $x_b(n) = x(nN)$

Ex) data suppression

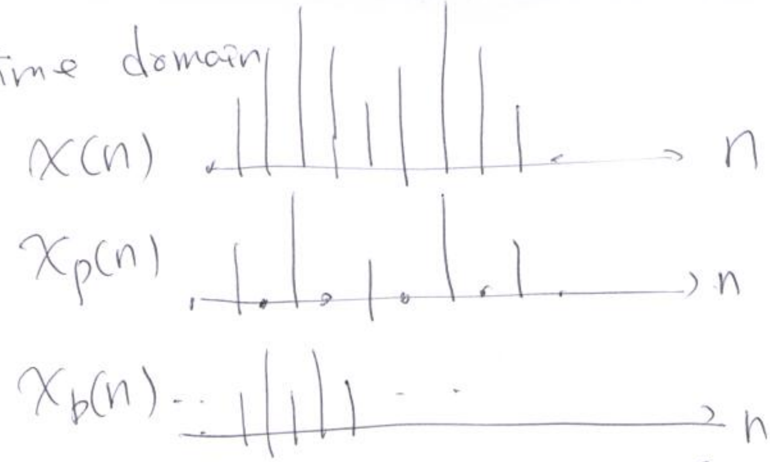
$$X_b(e^{j2\pi f}) = \sum_{k=-\infty}^{\infty} x_b(k)e^{-j2\pi fk}$$

$$= \sum_{k=-\infty}^{\infty} x_p(\underset{n}{kN})e^{-j2\pi fk}$$

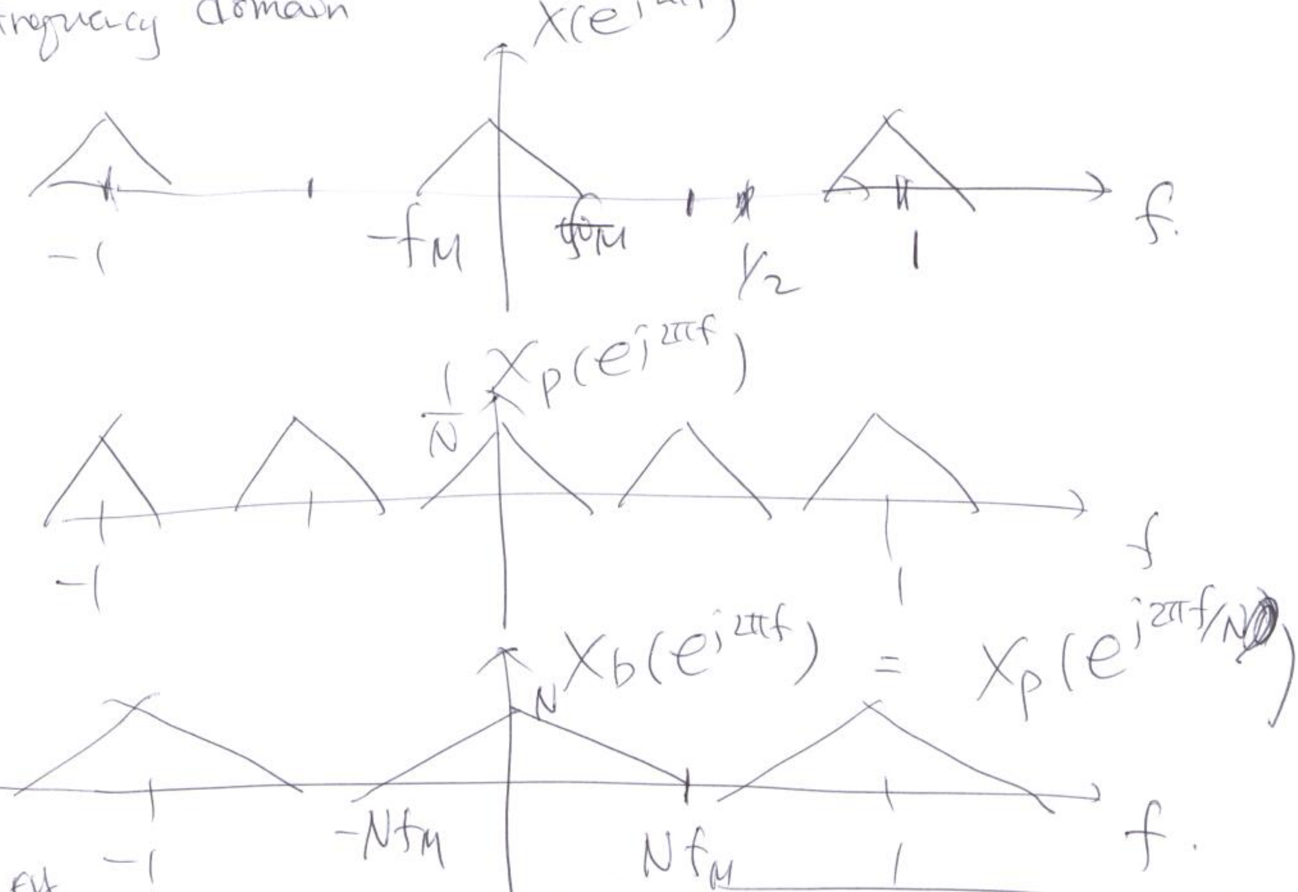
$$= \sum_{n=-\infty}^{\infty} x_p(n)e^{-j2\pi fn/N}$$

$$= X_p(e^{j2\pi f/N})$$

Time domain



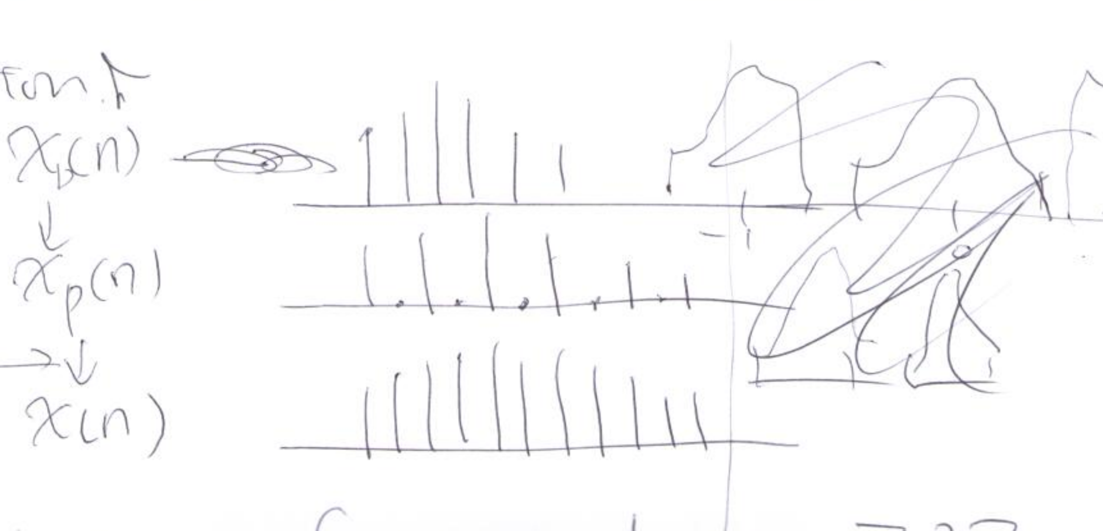
Frequency domain



Decimation: Down sampling ⟷ interpolation: Up sampling

Need redundancy in the original samples to avoid aliasing after decimation

Interpolation



Show in frequency domain 7.37

→ can be applied for signal recovery: Audio zips. downs.

Quiz

Write down Nyquist Sampling Theorem.

Announcements.

"FT & Sampling"

Exam. 5/14 9:15 – 10:45

301-118室.

Closed book/Note etc. phone

Same rule. bring your ID

① HW

② HW modified

③
④ advance problems.

Makeup class 5/16 (Sat)

10:00 – 11:15.

Will be video taped.

Chapter 6

Time & Frequency.

Chapter 6. Time & freq. Characterization of signals & systems (1)

6.1 Mag & Phas. in FT

$$X(f) = |X(f)| e^{j\angle X(f)}$$

decompose time-domain signal into frequency spectrum.

- magnetude
  $|X(f)|^2 \rightarrow$ energy density fn.
  $|X(f_0)|^2 \Delta f \rightarrow$ energy @ f_0 to $f+\Delta f$
- phase $\angle X(f) \rightarrow$ relative phase. (Fig 6.1)
  show the example of phase!
  $\cos(2\pi t+\phi_1) + \cos(4\pi t+\phi_2)$

is phase important?
YES. $x(-t) \xrightarrow{F} X(f) e^{-j\angle X(f)}$
reverse play of audio
Another example Fig. 6.2.

6.2. Mag & Phase of Freq. Response of LTI systems

$$Y(f) = H(f) X(f)$$

$$|Y(f)| = |H(f)||X(f)|$$ → gain

$$\angle Y(f) = \angle H(f) + \angle X(f)$$ → phase shift

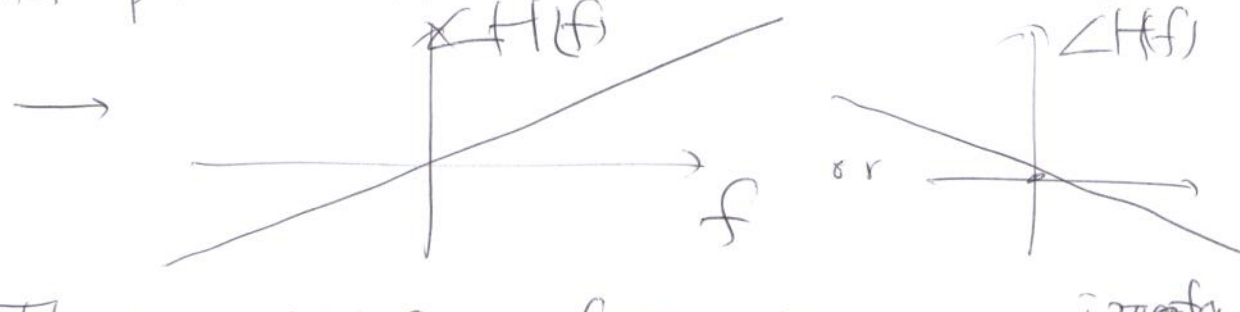
both of them are a func fn of "f"

Show equalizer!

If something in $H(f)$ results in undesirable results, ②
It is called as "distortion"

6.2.1. Linear & Nonlinear phase

If phase shift is linear over frequency

→ 

Then $\angle H(f) = -af2\pi$ $H(f) = e^{-j2\pi fa}$

If we assume $|H(f)| = 1$

$$Y(f) = e^{-j2\pi fa} X(f)$$

$$y(t) = x(t-a) \ !$$

⇒ linear phase means time delay

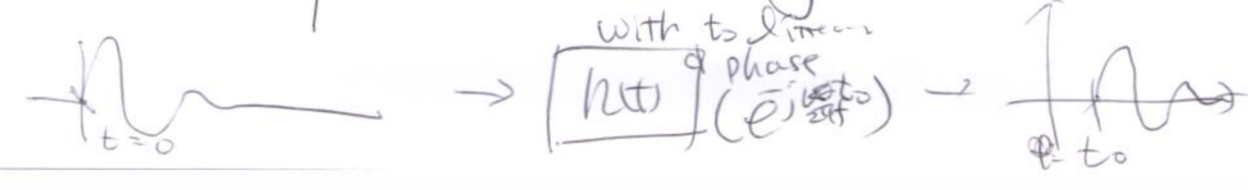
⇒ much benign (in most cases) than nonlinear cases. (Fig 6.3)

※ discrete case things are more complex. (why?)

Note that a filter with $|H(f)| = 1$ is called all-pass filter.
This filter can have significant effects on signals. (phase change)

6.2.2 Group delay

linear phase ⇒ delay in time



For a narrowband input (x(t) has FT zero except a small freq. band: example TDM) (3)

→ approximate the phase to be linear

$$\angle H(f) \simeq -\phi - 2\pi f \tau$$

$$Y(f) = X(f)\,|H(f)|\,e^{-j\phi}\,e^{-j2\pi f \tau}$$

(|H(f)|: magnitude; $e^{-j\phi}$: constant phase; $e^{-j2\pi f\tau}$: linear phase)

group delay: $\tau(f) = \frac{1}{2\pi}\frac{d}{df}\{\angle H(f)\}$

Example 6.1 ← very good one
Do it during the class if time permits

6.2.5 Log Mag / Bode plot

$$\log|Y(f)| = \log|H(f)| + \log|X(f)|$$

⟶ Including phase plot
now things are additive.

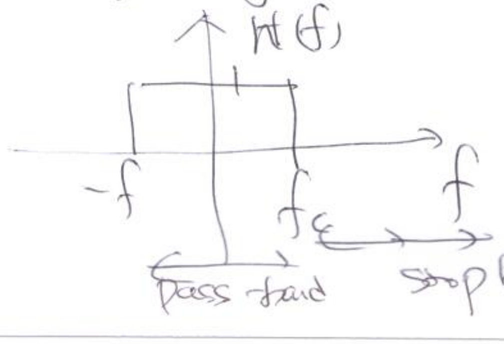
Amplitude: $20\log_{10}$ (dB)

0 dB : ×1
−20 dB : ×1/10
20 dB : ×10
6 dB : ×2

Power $10\log_{10}$ 3dB : ×2.

6.3 Time-domain properties of ideal freq-selective filters.

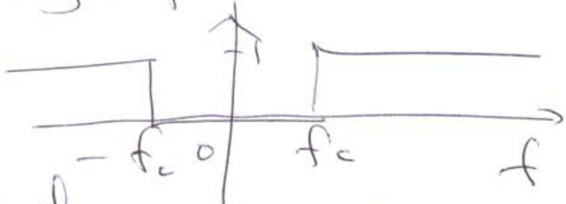
• Frequency-Selective filter.



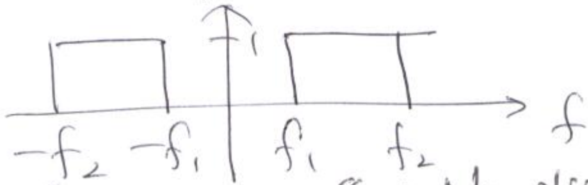
$$H(f) = \begin{cases} 1 & |f| \le f_c \\ 0 & \text{otherwise} \end{cases}$$

"ideal"
⇒ low pass filter

Ideal
· High-pass filter



Ideal
· Band pass filter



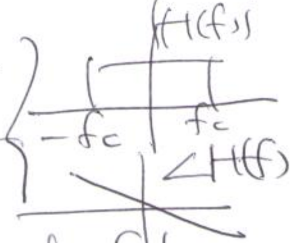
think about discrete case where high frq is around "π" and low frq is around "0"

→ "zero phase" : No distortion!

● What about "Phase"?

If nonlinear → output will be severely dependent on phase

If linear

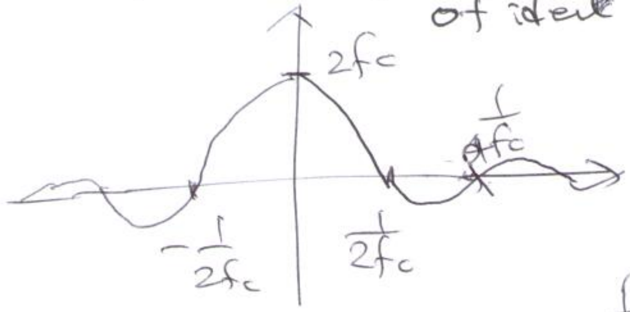


$\angle H(f) = -\alpha f$ → only delay in time.

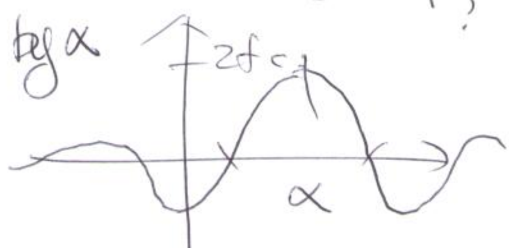
For this ideal filter

● Impulse response of ideal filter is $rec\left(\frac{f}{2f_c}\right) \to 2f_c \, sinc(2f_c t)$

but remember group delay?

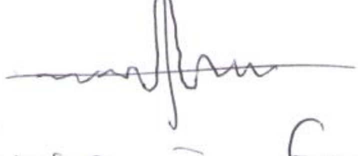


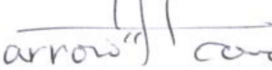
delay by α →



f_c : large → "wide" coverage in frequency

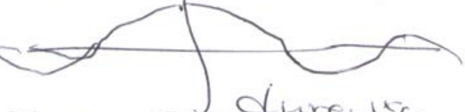
→ in time domain





f_c : small → "narrow" coverage in freq

→ in time domain



Remember sinc has infinite ~~coverage~~ duration

→ Ideal selective filter has infinite duration in time domain

→ Bad for implementation

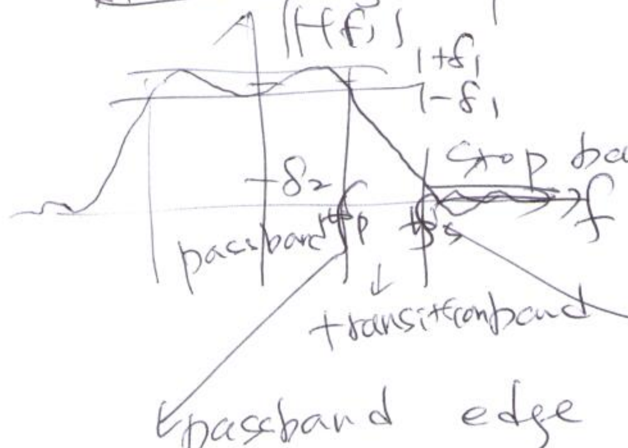
- Let's consider the step response of ideal low pass filter. (5)

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Q: What is the problem of noncausal ideal filter?
→ noncausal ideal filter
→ not applicable for real-time processing
→ more expensive to approximate this filter
(what does this mean?)

- 6.4 Time / frequency domain aspects of nonideal filters

- Relax the constraints in ideal filter to ~~make~~ design practical filter

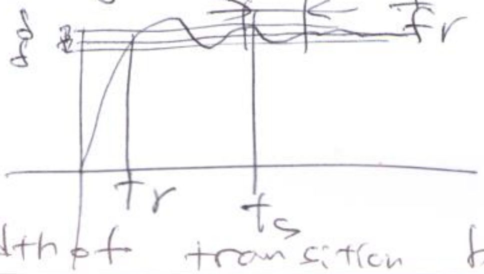


δ_1 : passband ripple

δ_2 : stopband ripple

- phase characteristics can be linear or nearly linear over passband.

- Time-domain behavior can be constrained by the step response.



t_r : rise time.
f_r : ringing freq.
t_s : settling time.

width of transition band $\propto$ 1/settling time of step fn

~~6.4~~, 6.5, 6.6, 6.7 will not be covered during the lecture.

Read them through!

Chapter 9.

Laplace Transform.

9.1 Laplace Transform. ①

~~CT-FT:~~ For LTI system, complex exponentials are eigen functions.

i.e $y(t) = H(s) e^{st}$
$H(s)$: system function.

CT-FT is great as it represent a signal as a linear combination of e^{st} where $s = j2\pi f$.

But there is a convergence issue. (Not all signals have FT)

Laplace Transform is a generalization of CT-FT by allowing $s = \sigma + j2\pi f$: complex value.
σ: non-zero.

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

So FT is a special case of LT when $s = j2\pi f$.

i.e $X(s)\Big|_{s=j2\pi f} = \mathcal{F}\{x(t)\}$.

In other words, LT can be viewed as

$$X(\sigma + j2\pi f) = \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t}\right] e^{-j2\pi f t} dt$$

i.e FT of $x(t)e^{-\sigma t}$
$e^{-\sigma t}$: decaying or growing.

Example 9.1

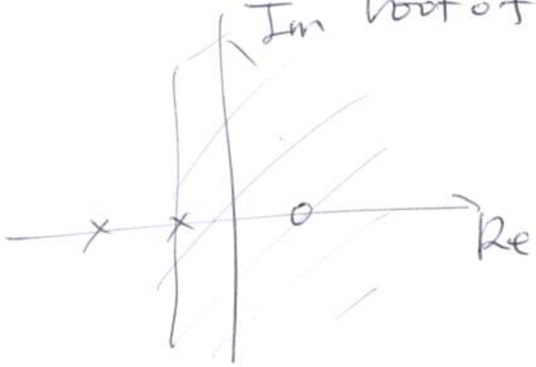
→ a [illegible] < 0 Laplace transform still exist but not FT.

Example 9.2 .

- Region of convergence : when does LT converges
  solutions of 9.1 & 9.2 the same
  but ROC are different .
  show Figure 9.1 .

  Example 9.3 .

- If LT ~~has~~ is a rational fn
  i.e $X(s) = \frac{N(s)}{D(s)}$
  (e.g. linear constant-coefficient differential Eq
  We can plot poles & zeros ~~to determine ROC~~
  poles ↓ root of $D(s)$; zeros → root of $N(s)$



← called "pole-zero plots" of $X(s)$"

- A rational LT is completely specified , to within a scale factor, by the pole-zero plot + ROC .

- If the order of the rational LT function is not even, we assume poles/zeros @ infinity

4.2 Region of Convergence

LT ← Requires ROC to be specified.

Property 1: ROC of $X(s)$ consists of strips parallel to $j\omega$-axis in the s-plane.

⇒ In LT, we check convergence of $x(t)e^{-\sigma t}$.
i.e. $\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty$.

Property 2: For rational LT, ROC excludes poles.

⇒ poles make $LT \to \infty$.

Property 3: $x(t)$ is of finite duration & absolutely integrable → ROC is entire s-plane

⇒ If $\int_{T_1}^{T_2} |x(t)| dt < \infty$, then $\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$
$\left(< e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)| dt \right)$

Property 4&5: If $x(t)$ is right sided (or left sided) & if $Re\{s\} = \sigma_0$ is in ROC then all values of s for $Re\{s\} > \sigma_0$ (or $Re\{s\} < \sigma_0$) will be in ROC

⇒ $x(t)$: right-sided → ROC is right-half plane
$x(t)$: left-sided → ROC is left-half plane

Property 6: $x(t)$ is two sided & $Re\{s\}=\sigma_0$ is in ROC → ROC will consist of a strip that includes $Re\{s\}=\sigma_0$

Solve Example 9.7.

Property 7 $X(s)$ is rational → ROC is bounded by poles or extends to ∞.

Property 8. $X(s)$ is rational & $x(t)$ is right sided (or left sided) → ROC is s-plane to right (left) of the rightmost (leftmost) pole.

Example 9.8.

- If ROC doesnot contain $s=j\omega$,

  F.T doesnot converge.

9.3. Inverse Laplace Transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

↑ Integration along a line parallel to $j\omega$ axis

→ This is difficult to evaluate.

So we will only solve the case for rational functions using partial fraction expansion

i.e. $X(s) = \sum_{i=1}^{n} \frac{A_i}{s + a_i}$

Then use ROC to determine $A_i e^{-a_i t} u(t)$ or $-A_i e^{-a_i t} u(-t)$

9.4. Geometric evaluation ~~Skip (but read)~~ → see next page

9.5 Property → Skip.

→ careful for ROC.

- $x(t) = 0$ for $t < 0$.

  & $x(t)$ has no singularity @ origin.

  $$x(0^+) = \lim_{s \to \infty} sX(s)$$

  "Initial-value theorem".

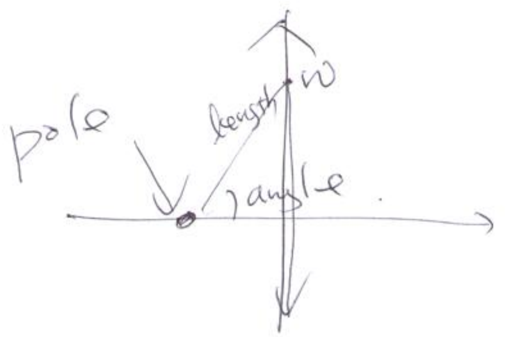
- $x(t) = 0$ for $t < 0$, $x(t)$ has a finite limit as $t \to \infty$

  $$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

  "Final-value theorem". ⇒ Will be your HW

9.4 Geometric evaluation of FT from pole-zero plot

- FT: evaluation of LT on $j\omega$-axis



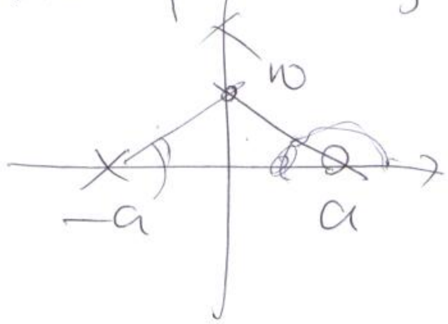
$$X(s) = \frac{1}{s+\frac{1}{2}} \quad Re\{s\} > -\frac{1}{2}$$

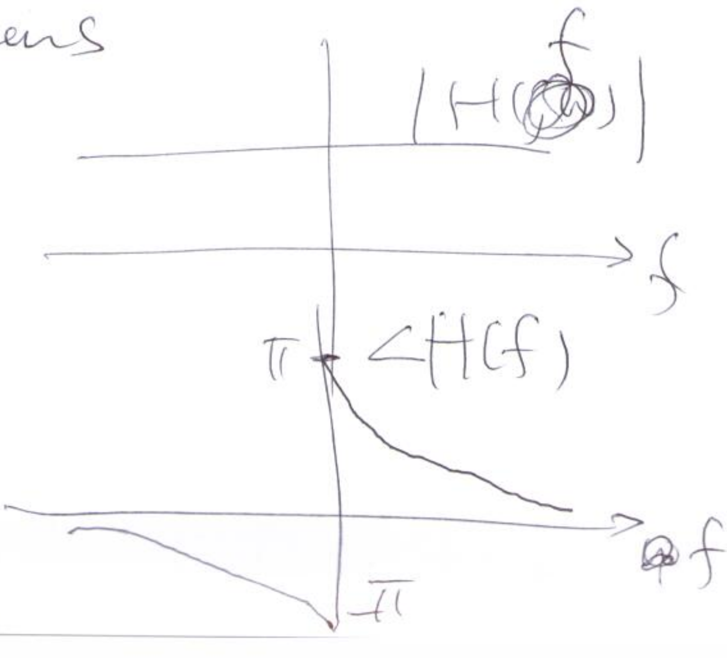
$$X(f) = \frac{1}{j2\pi f + 1/2}$$ from the graph

$$X(s) = M \frac{\prod_{i=1}^{R}(s-\beta_i)}{\prod_{j=1}^{P}(s-\alpha_j)} \Leftarrow \frac{\text{from } S \text{ to } \beta_i \text{ distance.}}{\text{from } S \text{ to } \alpha_j \text{ distance.}}$$

See 9.41 for first order system,
9.42 " Second order system.

All-pass Systems





9.7. Analyses and Characterization of LTI system using L.T.

$$Y(s) = H(s)\, X(s)$$

- $H(s)$: LT of impulse response fn of the system — System function or Transfer function.
- $Y(s)$: LT of output.
- $X(s)$: LT of input

- Causality : initial rest.
  - ROC associated with the system function for a causal system is a right-half plane.
  - For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

- Stability : absolutely Integrable.
  - if & only if ROC of the system fn include the entire $j\omega$-axis.

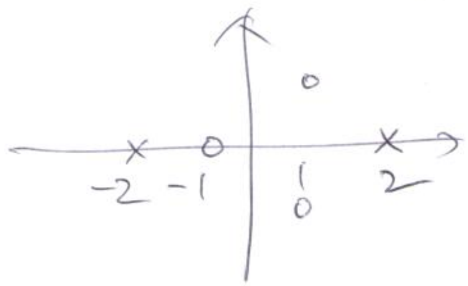
⟹ causal w rational system function is stable ⇄ all the poles are left-half of s-plane.
→ negative real parts.

Example 9.20

Ex 9.24 – 9.27

9.8 System fn algebra & Block diagram representation (Skip)

Quiz: draw potential ROC for the following pole-zero plot



Quiz: L.T (forward)

Chapter 10

z- transform

10.1 z-transform

- generalization of DT-FT
- In LTI system.

$y(n) = H(z)\, z^{n}$

Where $H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$

If $z = e^{+j2\pi f}$, ↓ DT-FT.

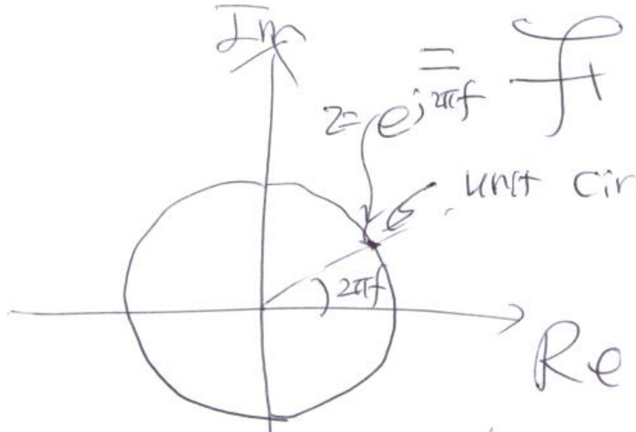
unit circle.

i.e. DT-FT is an evaluation of z-transform for a unit circle in z-plane.

- In general $z = r e^{j2\pi f}$

$$X(re^{j2\pi f}) = \sum_{n=-\infty}^{\infty} \left(x(n)\, r^{-n}\right) e^{-j2\pi f n}$$

$$= FT\{x(n)\, r^{-n}\}$$



Complex z-plane.

Convergen: $x(n)r^{-n}$ converges or not depending on r.

Example 1 & 2

10.2 ROC.

Property 1. ROC consists of a ring centered about the orgin.

→ $x(n)r^{-n}$ is absolutely summable.

Property 2. ROC does not contain any poles

Property 3. $x(n)$ finite duration → ROC entire z plane except $z=0$ or ∞.

Property 4. $x(n)$ right sided & $|z|=r_0$ in ROC

→ $|z|>r_0$ in ROC.

Property 5. $x(n)$ left-sided & $|z|=r_0$ in ROC

→ $0<|z|<r_0$ in ROC.

Property 6. $x(n)$ two sided & $|z|=r_0$ in ROC

→ ROC is a ring including $|z|=r_0$.

Property 7. $X(z)$ rational, ROC is bounded by poles or extends to infinity.

Property 8. $X(z)$ rational & right sided.

ROC: outside the outermost pole

& If causal ROC includes $z=\infty$.

Property 9. $X(z)$ rational & left sided.

ROC: inside the innermost nonzero pole

& If anticausal → ROC includes $z=0$.

10.3 Inverse z-transform.

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz .$$

Counter clockwise closed circular contour

Let's explain this:

$$X(re^{j2\pi f}) = \mathcal{F}\{x(n) r^{-n}\}$$

$$x(n) r^{-n} = \mathcal{F}^{-1}\{X(re^{j2\pi f})\}$$

$$x(n) = r^n \mathcal{F}^{-1}\{X(re^{j2\pi f})\}$$

$$= r^n \int X(re^{j2\pi f}) e^{j2\pi f n} df$$

$$= \int X(re^{j2\pi f}) (re^{j2\pi f})^n df$$

$$z = re^{j2\pi f}, \quad dz = j2\pi r e^{j2\pi f} df$$

$$\therefore x(n) = \frac{1}{j2\pi} \oint X(z) z^{n-1} dz$$

For rational z-transforms, (i.e. $\sum_{i=1}^{M} \frac{A_i}{1-a_i z^{-1}}$) use partial-fraction expansion

Solutions

$A_i a_i^n u(n)$ → ROC outside poles

$-A_i a_i^n u(-n-1)$ inside pole.

Ex 10.9 & 10, 11

10.4. Geometric evaluation..

FT → use pole-zero plot and evaluate on the contour $|z|=1$.

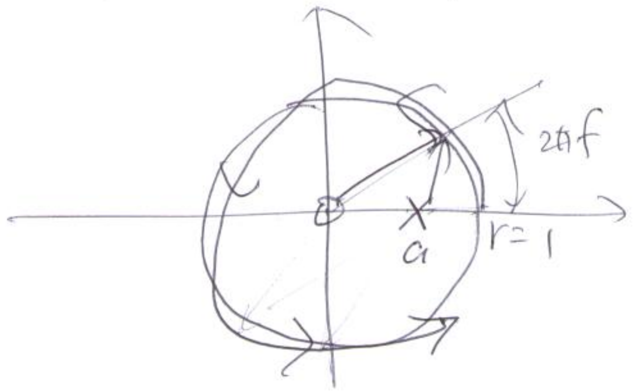
Example $h(n) = a^n u(n)$

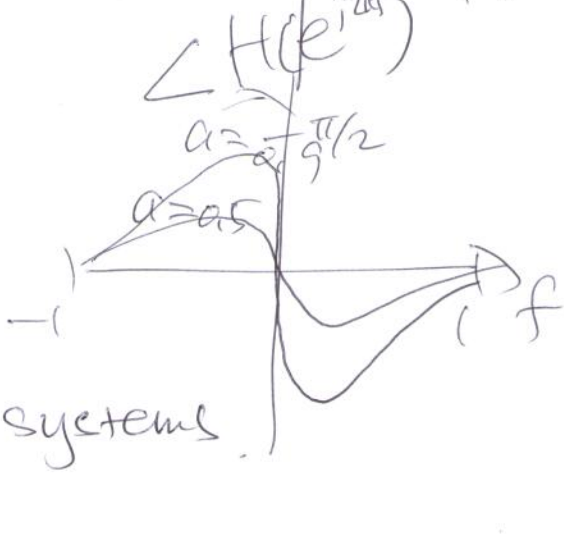
$$H(z) = \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

for $|a|<1$ → ROC include $|z|=1$ → FT exists

$$H(e^{j2\pi f}) = \frac{1}{1-ae^{-j2\pi f}}$$

From pole-zero plot





Check 10.4.2 for second order systems.

10.5 properties

- linearity. ROC: $R_1 \cap R_2$.
- Time Shifting $x(n-n_0) \longleftrightarrow z^{-n_0}X(z)$

  ROC: R except addition or deletion of ⓐ infinity or zero
- Scaling in z

  $z_0^n x(n) \longleftrightarrow X\left(\frac{z}{z_0}\right)$ ROC $= |z_0| R$
- Time Reversal

  $x(-n) \longleftrightarrow X\left(\frac{1}{z}\right)$ ROC $= \frac{1}{R}$
- Time expansion

  $$x_{(k)}(n) = \begin{cases} x(n/k) & \text{if } n \text{ is multiple of } k \\ 0 & \text{otherwise.} \end{cases}$$

  $x_{(k)}(n) \longleftrightarrow X(z^k)$ with ROC $= R^{1/k}$
- Conjugate

  $x^*(n) \longleftrightarrow X^*(z^*)$
- Convolution

  $x_1(n) * x_2(n) \longleftrightarrow X_1(z)X_2(z)$ $R_1 \cap R_2$
- Differentiation in z

  $n\,x(n) \longleftrightarrow -z\frac{dX(z)}{dz}$
- Initial-Value Theorem.

If $x(n) = 0$ for $n < 0$

$$x(0) = \lim_{z \to \infty} X(z)$$

10.7 LTI system.

In discrete-time LTI,

$$Y(z) = H(z)X(z)$$

$H(z)$: system function or transfer function

↓

becomes frequency response if evaluated for $z = e^{j2\pi f}$

- Causality
  $h(n) = 0$ for $n < 0$ ⟺ Right-sided!
  ⟺ ROC ~~extends~~ exterior of a circle including ∞

  Check book

- Stability
  - An LTI system is stable iff ROC of $H(z)$ includes $|z| = 1$.
  - A causal LTI with rational system is stable iff all poles of $H(z)$ lies inside the unit circle.

we are not covering 10.8 & 10.9.

Where do we go from now?

DSP.

통신의기초 → 통신시스템, DSP

→ 확률신호론 (대학원)

화상/음성신호처리, 정보이론

제어공학개론, DSP → 지능시스템개론.

· 로봇공학개론

→ 최신제어기법

최적화. 확률신호론(대학원), 신호시스템이론,

DSP, 생체계측 (2학기 대학원), Biophysics 3학년 (생물물리학) 2학기. "New"

Bioimaging (1학기, 대학원)

생체전자공학 특강 (1학기, 대학원)

신경보완기술.

Samplay → 신호처리특강 (TBA)