Chapter 4

Dissipative systems

According to the definitions given in Section 1.5, *purely dissipative* systems are obtained by adding damping forces to the systems hitherto considered. Apart from nonworking reactions, noncirculatory and possibly gyroscopic loads, they contain dissipative reactions or loads. Systems of this type are important since dissipative reactions or loads. Systems of this type are important sine dissipative forces appear everywhere and can hardly be completely eliminated. Theoretically, it is possible that the dissipative forces manifest themselves merely in certain particular motions of the system. This happens, for example, in a double pendulum in which one of the two hinges is free of friction. Practically, however, they affect any type of motion, as in a double pendulum with friction in both hinges. The following section will be confined to the latter case, i.e., to the so-called *completely dissipative* systems.

4.1. General Aspect

If the system is linear, its differential equations are given by (1.113),

$$\sum_{k=1}^{n}(m_{ik}\ddot{q}_k + g_{ik}\dot{q}_k + c_{ik}q_k) = 0 \qquad (4.1)$$

Here, the matrices (m_{ik}), (g_{ik}), and (c_{ik}) are constant; (m_{ik})and (c_{ik})are symmetric, and (m_{ik})is positive definite. The matrix (g_{ik}) generally is asymmetric. Its symmetric part represents the dissipative forces and is positive definite in a completely dissipative system. The antimetric part stems from the gyroscopic forces and may be absent. Proceeding as in Section 3.1, we obtain the characteristic equation

$$\det(m_{ik}\lambda^2 + g_{ik}\lambda + c_{ik}) = 0 \qquad (4.2)$$

Here, however, the roots, in general, will not occur in equal and opposite pairs.

It is convenient to compare the given system with the nongyroscopic conservative system obtained from it by dropping the dissipative and gyroscopic terms. According to Lagrange's theorem(Section 1.6), both systems are stable (even in the nonlinear case) so long as the potential energy (assumed to be a continuous function) is positive definite. If the system is linear and free from dissipative and gyroscopic forces, it is unstable, according to Theorem 1 (Section 2.1), when its potential energy is not positive definite, i.e., when at least one of the root λ_i of (4.2) is zero or positive for (g_{ik}) (Figure 4.1).

Refer to Figure 4.1

The first case correpsonds to a static instability and is characterized by the fact that the constant term det c_{ik} in the characteristic equation (4.2) is zero. An instability of this type is obviously independent of the dissipative and gyroscopic forces and hence also occurs in the given system. In the second case it is easy to see that the given system also has at least one root with a positive real part. If this were not the case, the originally positive root would cross or at least touch the imaginary axis when the velocity-dependent forces were reintroduced, since the roots of (4.2) are continuous functions of the coefficients. This reintroduction can be realized by a proportional increase of the elements of (g_{ik}). Now, a purely imaginary root λ_i would correspond, together with $-\lambda_i$, to a periodic motion. However, periodic motions are not possible in a completely dissipative system, since the total energy decreases as long as the system moves. The only alternative is a vanishing root. However, a root $\lambda_i = 0$ is independent of (g_{ik}) and hence cannot appear as a consequence of the reintroduction of the velocity-dependent forces. Thus, we have proved the following theorem:

THEOREM 7. *A purely and, at the same time, completely dissipative* *linear system behaves exactly as the corresponding system with dissipative and gyroscopic forces.*

In other words, Theorem 1 (Section 2.1) also applies for purely and completely dissipative linear systems. As a consequence, the considerations contained in Section 2.1 concerning simple systems remain valid. In particular, the stability of the simple system is still illustrated by Figure 2.2, and we have :

THEOREM 8. *Theorems 1 and 2 remain valid for purely and, at the same time, completely dissipative systems.*

In order to generalize the foregoing results, let us consider a nonlinear system which can be linearized. So long as the quadratic approximation (2.10) of its potential energy is positive definite, so is the exact expressions, and it follows from Lagrange's theorem (Section

1.6) that the equilibrium is stable. When the quadratic approximation admits negative values in an arbitrarily small vicinity of the equilibrium configuration, the characteristic equation (4.2) has at least one root with a positive real part, provided that gyroscopic forces and damping are absent. It has been shown above that this remains true while the velocity-dependent forces are reintroduced, and it follows from Theorem D of Section 1.8 that the nonlinear system is unstable. We thus have :

THEOREM 9. *Theorem 3 remains valid for purely and, at the same time, completely dissipative systems.*

4.2. Destabilization by Damping Forces

The results of Section 4.1 show that damping forces, provided that the dissipation is complete, do not affect the stability of a nongyroscopic conservative system. With respect to gyroscopic conservative systems, however, the situation is different. Comparing the results of Sections 3.1 and 4.1, we arrive at the unexpected conclusion that such systems, provided that they are stabilized by the gyroscopic terms, are again destabilized by the addition of dissipative forces. We thus have :

THEOREM 10. *Dissipative forces, applied to other than nongyroscopic conservative systems, may have a destabilizing effect. If they are added to a gyroscopic conservative system and if the dissipation is complete, they cancel the stabilizing effect of the gyroscopic forces.*

In view of this result, gyroscopic stabilization loses some of its importance. However, there are cases where the destabilizing effect of the dissipative forces manifests itself rather slowly or is again suspended by other influences. The sleeping top, e.g., is stabilized by the gyroscopic moment so long as the spin is sufficiently large. It may take a long time until the spin is sufficiently decreased by friction to make the top unstable.

It is interesting to discuss the various effects (Figure 4.1) in terms of the roots λ_i of (4.2). So long as velocity-dependent forces are absent, these roots are either purely imaginary, as in Figure 2.1, or real. Instability sets in when, in the course of an increase of the loading, a pair of conjugate imaginary roots meet at the origin and leave it as a pair of equal and opposite real roots. When at least two roots are positive, the appearance and subsequent increase of velocity-dependent terms may cause two positive roots to merge and to part again as a conjugate complex pair. In the case of complete dissipation, the pair stays in the open half-plane of positive real parts; in the absence of dissipative forces it may reach the imaginary axis.

As an example, let us reconsider a problem of critical speed. In Section 1.3 the particle of Figure 1.11 was considered as the model of a disk mounted on a shaft rotating with angular velocity ω. The case $c_1 = c_2$ corresponds to a shaft with a single flexural rigidity and is characterized by a single critical angular velocity ω_1. In order to study the influence of internal damping, it is convenient to treat the problem in a coordinate system rotating with the shaft. In a first approximation, we will represent the damping effect (Figure 4.2) by the force $-2mb \cdot (\dot{x}, \dot{y})$, where *b* is the damping constant. The differential equations of motion then are

$$\begin{aligned} &\ddot{x} + 2b\dot{x} - 2\omega\dot{y} + (\frac{c}{m} - \omega^2)x = 0 \\ &\ddot{y} + 2\omega\dot{x} - 2b\dot{y} + (\frac{c}{m} - \omega^2)y = 0 \end{aligned} \qquad (4.3)$$

Figure 4.2

They are straightforward modifications of (1.56). By means of (1.57) we readily obtain the characteristic equation

$$(\lambda^2 + 2b\lambda + \frac{c}{m} - \omega^2)^2 + 4\omega^2\lambda^2 = 0 \qquad (4.4)$$

or

$$\lambda^2 + 2(b \mp i\omega)\lambda + \frac{c}{m} - \omega^2 = 0 \qquad (4.5)$$

Let $\lambda_{1,2}$ be the two roots corresponding to the minus sign in (4.5), and let $\lambda_{3,4}$ correspond to the plus sign.

If there is no friction, (4.5) reduces to

$$\lambda^2 \mp 2i\omega\lambda + \frac{c}{m} - \omega^2 = 0 \qquad (4.6)$$

Hence,

$$\lambda_{1,2} = i(w \pm \sqrt{c/m}), \quad \lambda_{3,4} = -i(w \pm \sqrt{c/m}), \qquad (4.7)$$

These roots are purely imaginary; they also are distinct, provided $\omega^2 \neq c/m$. The result confirms the stability of the shaft for $\omega^2 \neq c/m$. Figure 4.3 shows

Refer to Figure 4.3

the distribution of the roots $\lambda_{1,2}$ for $\omega^2 >< c/m$; the roots $\lambda_{3,4}$ are obtained by reflection as the real axis.

If friction occurs, *b* is positive and the position of the roots in Figure 4.3 is modified. It follows from (4.5) that

$$\lambda_1\lambda_2 = \frac{c}{m} - \omega^2, \qquad \frac{1}{2}(\lambda_1 + \lambda_2) = -b + i\omega \tag{4.8}$$

and, therefore, that neither of the two roots now lies on the imaginary axis. Because of the second equation (4.8), the center of the section connecting the points λ_1 and λ_2 is the point $-b+i\omega$ marked by a triangle; thus, at least one of the two points lies in the open half-plane of negative real parts. Because of the first equation, the sum of the arguments of λ_1 and λ_2 is 2π or π according as $\omega^2 >< c/m$. It is immediately obvious that, for $\omega^2 < c/m$, both points λ_1 and λ_2 lie on the left-hand side of the imaginary axis, since otherwise the sum of the arguments would be less or more than 2π. On the other hand, for $\omega^2 > c/m$, one of the two points must lie on the right-hand side of the imaginary axis, since otherwise the sum of the arguments would exceed π.
We have thus confirmed that the shaft is stable for $\omega^2 < c/m$ and unstable for $\omega^2 > c/m$. Without velocity-dependent forces the region $\omega^2 > c/m$ would

be unstable. It has been pointed out in Section 1.3 and confirmed in Section 3.1 that it is stabilized by the Coriolis forces. Now it turns out that it is again destabilized by damping.

In contrast to the results obtained here, experiments show that, at least within the domain of practically occurring angular velocities, $\omega_1 = \sqrt{(c/m)}$ is the only critical value. We conclude that there exist additional influences which again suspend the destabilizing effect of friction. According to Dimentberg [14], a more realistic representation of internal friction, taking account of hysteresis effects, is apt to remove the discrepancy between theory and experiment. Another possibility will be discussed in Section 4.3; it will be shown that external friction has a stabilizing tendency which may prevail over the destabilization by internal friction, at least in a certain domain of angular velocities.

At any rate, this simple example clearly shows that stability is very sensitive to secondary effects and that one should be very careful in the interpretation of results obtained by means of a simplified analysis.

Problem

1. In the problem of Figure 1.11 the centrifugal and Coriolis forces can be distinguished by writing ω_1 and ω_2, respectively, for the angular velocity ω. The characteristic equation (1.59) then takes the form

$$\lambda^4 + \left(\frac{c_1}{m} + \frac{c_2}{m} - 2\omega_1^2 + 4\omega_2^2\right)\lambda^2 + \left(\frac{c_1}{m} - \omega_1^2\right)\left(\frac{c_2}{m} - \omega_1^2\right) = 0 \qquad (4.9)$$

and the stabilizing effect of the Coriolis force can be studied by increasing ω_2 from 0 to ω_1.

Analyze the corresponding motion of the roots of (4.9) in the plane λ for $c_1 = c_2 = c$ and $\omega^2 = 2c/m$.

4.3. The Routh-Hurwitz Criteria

The characteristic equations encountered in the preceding sections have been quadratic or biquadratic. Accordingly, the discussion of their roots has been straightforward. In cases where the characteristic equation is more complicated, the discussion of the roots becomes more involved and the need for general criteria concerning the nature of the roots arises. Such criteria have been provided independently by Routh [57] and by Hurwitz [27]. They are not restricted to dissipative systems. However, since we will need them in the following sections, it is convenient to introduce them here. For the proofs, we refer to the original literature.

So far, we have been dealing with systems of differential equations of the type

$$\sum_{k=1}^{n}(m_{ik}\ddot{q}_k + g_{ik}\dot{q}_k + c_{ik}q_k) = 0 \qquad (4.10)$$

where n was the degree of freedom of the mechanical system and the matrices (m_{ik}), (g_{ik}), and (c_{ik}) had certain properties depending on the type of system. In this section we disregard the particular properties of these matrices, assuming merely that they are real and constant. This implies that we admit a rather general class of mechanical systems, including even nonholonomic cases, but excluding systems of the instationary type. In other words, we admit all kinds of linear autonomous systems. The differential equations considered here also occur in control mechanisms and in the theory of electrical networks. Setting

$$q_k = A_k e^{\lambda t} \qquad (k = 1, 2, \ldots, n) \tag{4.11}$$

we obtain, from (4.10),

$$\sum_{k=1}^{n} (m_{ik}\lambda^2 + g_{ik}\lambda + c_{ik}) A_k = 0 \qquad (i = 1, 2, \ldots, n) \tag{4.12}$$

and by exclusion of the trivial solution we finally obtain the characteristic equation

$$p_0\lambda^m + p_1\lambda^{m-1} + \cdots + p_{m-1}\lambda + p_m = 0 \tag{4.13}$$

The coefficients are real, and the degree m does not exceed $2n$. It may be less, since (m_{ik}) is not assumed to be positive definite. Moreover, since nonholonomic systems have not been excluded, m is not necessarily even. In Section 4.4 we will deal with a case where $m=3$ since one of the equations (4.10) represents a nonholonomic constraint and hence does not contain an acceleration.

Inserting an arbitrary root of the characteristic equation in (4.12), we obtain a set $A_1, A_2, \ldots, A_n$ of amplitudes of which at least one is free. It represents a fundamental solution. So long as all the roots are different, superposition of these fundamental solutions yields the general integral, containing *m* free constants which can be chosen so as to satisfy the initial conditions. It has been shown in connection with (1.117) and (1.121) that this general solution is limited exactly as long as none of the roots has a positive real part. It follows that in the λ -plane (Figure 4.4) the imaginary axis divides the stable domain from the unstable one, and it would seem that

Refer to Figure 4.4

the imaginary axis itself belongs to the stable region. However, if $p_m = 0$ and hence $\lambda = 0$ is a root of the characteristic equation, the system is statically unstable. Thus the origin in Figure 4.4 belongs to the unstable domain. Moreover, it has been noted in Section 1.7 that multiple roots are apt to change the picture, insofar as a purely imaginary multiple root may render the system unstable. Therefore, the imaginary axis, with the exception of the origin, cannot be attributed unambiguously to the stable or the unstable domain.

The situation is exactly the same with respect to nonlinear systems, provided that they can be linearized. According to Theorems C and D (Section 1.8), the half-planes divided by the imaginary axis are stable and unstable, respectively. The origin marks a static instability, and the remainder of the imaginary axis contains the critical cases where the system may be stable or unstable. We will content ourselves, therefore, with sufficient stability conditions, ensuring that (Figure 4.4) all the roots of the characteristic equation have negative real parts. From a practical point of view this procedure is perfectly acceptable so long as we keep in mind that it supplies the whole interior of the stable domain and that the only stable cases not obtained in this way correspond to points on the boundary.

The roots λ_i of the characteristic equation (4.13) are completely determined by the coefficients $p_0, p_1, \dots, p_m$. Our problem is therefore reduced to the question : What are the restrictions on these coefficients which ensure that all the λ_i have negative real parts? The answer, supplied by Routh and Hurwitz may be formulated as follows:

Write the characteristic equation (4.13) so that $p_0 > 0$. Form the *m*-row determinant

$$D_4 = \begin{vmatrix} p_1 & p_0 & 0 & \cdots & & & & & 0 \\ p_3 & p_2 & p_1 & p_0 & 0 & \cdots & & & 0 \\ p_5 & p_4 & p_3 & p_2 & p_1 & p_0 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ p_{2m-1} & p_{2m-2} & \cdots & & & & & & p_m \end{vmatrix} \qquad (4.14)$$

starting with the element p_1 in the upper left-hand corner, completing the rows by successively decreasing the subscripts by 1, completing the columns by raising the subscripts by 2, and replacing all elements with subscripts $k<0$ or $k>m$ by zeros. Consider the test functions $D_1, D_2, \dots, D_m$ forming the so-called chain of principal minors of D_m, that is, the successive minors having the upper left-hand corner in common. The sufficient stability conditions we have been asking for then are

$$D_1 > 0, \quad D_2 > 0, \quad \dots, \quad D_m > 0 \tag{4.15}$$

In the case of a characteristic equation of degree $m = 4$, for example, the test functions are the determinant

$$D_4 = \begin{vmatrix} p_1 & p_0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & 0 & p_4 \end{vmatrix} \tag{4.16}$$

and the remainder of its chain of principal minors, i.e., the functions

$$\begin{aligned} D_1 &= p_1, \\ D_2 &= p_1 p_2 - p_0 p_3, \\ D_3 &= p_1(p_2 p_3 - p_1 p_4) - p_0 p_3^2, \\ D_4 &= p_1(p_2 p_3 p_4 - p_1 p_4^2) - p_0 p_3^2 p_4 \end{aligned} \tag{4.17}$$

The calculation of the successive test functions and the discussion of the corresponding stability conditions are simplified if each one of the determinants is obtained by expansion with respect to its last row. In this way the previously calculated test functions can be used and appear in the result. Moreover, the last test function D_m can be replaced by the last coefficient p_m, since $D_m = p_m D_{m-1}$. For m=4, for example, the stability conditions turn out to be

$$\begin{aligned} p_1 &> 0, \\ p_1 p_2 - p_0 p_3 &> 0, \\ (p_1 p_2 - p_0 p_3) p_3 - p_1^2 p_4 &> 0, \\ p_4 &> 0. \end{aligned} \tag{4.18}$$

As an application, let us reconsider the problem of Figure 1.11 which, with $c_1 = c_2 = c$, corresponds to a shaft with a single flexural rigidity, rotating with the angular velocity ω. It was found in Section 4.2 that, because of internal damping, the entire domain $\omega^2 > c/m$ ought to be critical beside the experimentally verified critical value $\omega^2 = c/m$. It has been conjectured that external damping might rectify this result, at least to a certain extent.

If we treat the problem in a coordinate system at rest, the elastic force is given by $-c\cdot(x,y)$, and the external damping (air drag, friction in the bearings, etc.) may be represented by $-2mb_1\cdot(\dot{x},\dot{y})$. Since the transport velocity of *m* is $\omega\cdot(-y,x)$, the velocity relative to a coordinate system rotating with ω is given by $(\dot{x}+\omega y,\dot{y}-\omega x)$, and the internal damping can be represented by $-2mb_2\cdot(\dot{x}+\omega y,\dot{y}-\omega x)$. Thus, the differential equations of motion become

$$\begin{aligned}\ddot{x}+2(b_1+b_2)\dot{x}+\frac{c}{m}x+2b_2\omega y=0,\\ \ddot{y}+2(b_1+b_2)\dot{y}-2b_2\omega x+\frac{c}{m}y=0,\end{aligned}\qquad (4.19)$$

The coefficients of *x* and *y* show that the problem now is not only dissipative but also circulatory. By means of (1.57) we obtain the characteristic equation

$$p_0\lambda^4+p_1\lambda^3+p_2\lambda^2+p_3\lambda+p_4=0\qquad (4.20)$$

with the coefficients

$$\begin{aligned} p_0 &= 1, \\ p_1 &= 4(b_1 + b_2), \\ p_1 &= 4(b_1 + b_2)^2 + 2\frac{c}{m}, \\ p_1 &= 4(b_1 + b_2)\frac{c}{m}, \\ p_1 &= \frac{c^2}{m^2} + 4b_2^2\omega^2. \end{aligned} \tag{4.21}$$

The stability conditions (4.18) are

$$\begin{aligned} & b_1 + b_2 > 0, \\ & (b_1 + b_2)[4(b_1 + b_2)^2 + \frac{c}{m}] > 0, \\ & (b_1 + b_2)[(b_1 + b_2)^2\frac{c}{m} - b_2^2\omega^2] > 0, \\ & \frac{c^2}{m^2} + 4b_2^2\omega^2 > 0 \end{aligned} \tag{4.22}$$

The first two and the last one are satisfied as long as c, b_1, and b_2 are positive.

The remaining one may be written

$$\omega^2 < (1 + \frac{b_1}{b_2})^2\frac{c}{m} \tag{4.23}$$

If internal damping is small compared with external friction, the term in parentheses is large compared with 1. This may explain why the shaft is stable for $\omega^2 > c/m$, at any rate within the domain of practically occurring angular velocities. That (4.23) does not yield the critical value $\omega_1^2 = c/m$ is not surprising. It was shown in Section 3.2 that in a coordinate system at rest this value is due to resonance under the influence of imperfections which have been excluded here.

Problems

1. Formulate sufficient stability criteria for a characteristic equation of degree 5.
2. The characteristic equation (4.4) of a rotating shaft subjected to linear internal damping has been discussed in Section 4.2 by reducing it to a quadratic equation with complex coefficients. Treat it as a fourth-degree equation with real coefficients and show that the stability conditions (4.18) confirm the result obtained in Section 4.2.

4.4 Shimmy of Trailers

It has been mentioned that the problem treated at the end of Section 4.3 is not purely dissipative. In this section we will treat another problem which is an instructive application of the criteria developed in Section 4.3 but differs considerably from the problems to which most of this book is devoted.

It sometimes happens that a trailer, being towed by a vehicle running smoothly on a perfectly straight and horizontal road, shows instability and starts to carry out dangerous lateral oscillations [77,78]. Similar phenomena have been observed in airplanes rolling on a runaway [13]. The problem, besides being not purely dissipative, is nonholonomic. The system exhibits negative dissipation since the constraints supply energy and thus give rise to self-excited oscillations.

Figure 4.5 shows a simplified version of a trailer with a single axle.

The two wheels are replaced by a single one, and the trailer by a rigid body hinged

Refer to Figure 4.5

at O and with centroid C. The compliance of the suspension is represented by a spring of stiffness c. The friction at O and at the hub H are neglected ; dry friction is assumed to act between the wheel and the ground. Initially, the wheel does not glide on the road; this is what makes the problem nonholonomic. Ina reference frame moving with the towing vehicle in a uniform rectilinear translation of velocity v, the coordinates of the trailer are x and φ, provided the mass of the wheel is neglected. The mass of the trailer will be denoted by m, its radii of gyration with respect to C and O by i_C and i_O, respectively. The external forces acting on the trailer are the traction Z, the spring force cx, and the friction F.

If we linearize for small values of *x* and φ, the momentum of the trailer, referred to the frame described above, has only a lateral component, given by $m(\dot{x}+r\dot{\varphi})$. The angular momentum, referred to the equilibrium position of the hinge *O*, is $mi_C^2\dot{\varphi}+mr(\dot{x}+r\dot{\varphi})$. The theorems of linear and angular momentum yield Z=0 and

$$\begin{aligned} m(\ddot{x}+r\ddot{\varphi}) &= -cx-F \\ m(r\ddot{x}+i_o^2\ddot{\varphi}) &= -Fl \end{aligned} \qquad (4.24)$$

Eliminating *F* from (4.24), we obtain a first differential equation of motion,

$$m[(1-\frac{r}{l})\ddot{x}+(r-\frac{i_o^2}{l})\ddot{\varphi}]+cx=0 \qquad (4.25)$$

In order to obtain a second differential equation, we have to express the fact that the wheel does not glide on the road. The absolute velocity of the hub *H*, referred to the ground and obtained by adding *v* and the contributions of $\dot{x}$ and $\dot{\varphi}$, has the direction *HO*. Therefore, in our approximation,

$$\dot{x}+l\dot{\varphi}+v\varphi=0 \qquad (4.26)$$

This second relation represents the nonholonomic constraint and is of the first order. The characteristic equation of the system (4.25)-(4.26) is

$$[(1-\frac{r}{l})\lambda^2+\kappa^2](l\lambda+\nu)-(r-\frac{i_o^2}{l})\lambda^3=0, \qquad (4.27)$$

where

$$\kappa^2=\frac{c}{m}>0. \qquad (4.28)$$

It is of the third degree and may be written

$$p_0\lambda^3+p_1\lambda^2+p_2\lambda+p_3=0 \qquad (4.29)$$

where

$$p_0=l-2r+\frac{i_o^2}{l}, \quad p_1=(1-\frac{r}{l})\nu, \qquad (4.30)$$
$$p_2=l\kappa^2, \quad p_3=\nu\kappa^2.$$

We note that

$$p_0=l-2r+\frac{1}{l}(i_C^2+r^2)=\frac{1}{l}[(1-r)^2+i_C^2]>0. \qquad (4.31)$$

Thus, the stability criteria (4.15) can be applied. They require

$$p_1>0, \quad p_1p_2-p_0p_3>0, \quad p_3>0. \qquad (4.32)$$

Inserting (4.30) and (4.31) in (4.32) and observing that $\kappa^2 l > 0$, we obtain

$$(l-r)v > 0, \quad (l-r)v - [(l-r)^2 + i_C^2]\frac{v}{l} > 0, \qquad v > 0. \tag{4.33}$$

The third inequality (4.33) confirms the well-known fact that the trailer is unstable when moving backwards. When $v > 0$, that is, in forward motion, the first condition requires that the centroid *C* be situated ahead of the hub *H*.

Moreover, the second inequality may be written

$$lr - r^2 - i_C^2 > 0 \qquad \text{or} \qquad (l-r)r > i_C^2 \tag{4.34}$$

It includes the first condition and is satisfied by concentrating the mass of the trailer as much as possible in the vicinity of the center of the distance *OH*.

The stability conditions obtained by this simplified approach are independent of the magnitude of *v*. If damping in the hinge *O* and other effects, such as the lateral motion of the rear of the truck, are taken into account, one finds that there exists a critical speed v_1 beyond which instability is to be expected. The author [77,78] has considered the problem in detail, including an analysis of the trailer with two axles. He has proposed a number of rules which are to be observed in order to obtain a trailer that is stable up to sufficiently high velocities. Slibar and Paslay [60] have approached the same problem, taking the characteristic of the tires into account.

Problem

1. Treat the shimmy problem for a trailer with a single axle, including a damping moment of magnitude $b\dot{\varphi}$ at the hinge *O*. Show that the critical speed in forward motion is $v_1 = (bl/m)(i_O^2 - rl)$.

4.5. A Theorem Concerning the Constraints

The results obtained so far for simple linear systems have been represented by Figures 2.2 and 3.1. They are compounded in Figure

4.6 Diagonal lines indicate that the behavior of the system is uncertain; crosshatching indicates instability. Purely dissipative systems behave as those of the nongyroscopic

Figure 4.6

conservative type provided that the dissipation is complete. We observe that in the three cases represented in Figure 4.6 the system is stable for $P < P_1$ and at least statically unstable for P_1. In all these cases P_1 may be obtained by the energy method since it is the smallest load for which V is not positive definite.

Let us consider two systems A and B of the type just discussed. We assume that they differ merely in their constraints: in addition to all the constraints of A, system B is supposed to have certain additional constraints which are nonworking and consistent with the presence of the equilibrium configuration $q_1 = q_2 = \cdots = q_n = 0$. In configuration space

$q_1, q_2, \ldots, q_n$ (Figure 4.7)

Refer to Figure 4.7

the admissible configurations of the two systems are represented by the two domains *A* and *B*. Because of the assumptions made, either of them contains the point *O* corresponding to the common equilibrium configuration; moreover, *B* is contained in *A*. This means that any admissible configuration of the system *B* is also admissible for system *A* while, on account of the additional constraints of *B*, the reverse is not true. It immediately follows that *V*, if positive definite in *A*, is also positive definite in *B*. As a consequence, we have:

THEOREM 11. *If, in a simple linear system containing neither instationary nor circulatory forces, nonworking constraints are added which do not alter the equilibrium configuration, the smallest critical* *load does not decrease.*

In general, the smallest critical load is increased by the additional constraints. The theorem is particularly useful for nongyroscopic systems, since here P_1 marks the transition between the stable and unstable regions.
In the nonlinear case the information contained in Figure 4.6 remains correct if V is interpreted as the quadratic approximation and if we remember that static instabilities do not necessarily occur under the loads $P_1, P_2, \ldots$
For nongyroscopic conservative systems and for those of the purely and completely dissipative type, P_1 still marks the transition between the stable and the unstable domains. We thus have:

THEOREM 12. *Theorem 11 remains valid for kinetic instability of simple nonlinear systems belonging to the nongyroscopic conservative* *and to the purely and completely dissipative types.*

Problem

1. Show that the buckling loads (1.10) in Euler's cases (Table 1.1, Section 1.2) are consistent with Theorem 11.