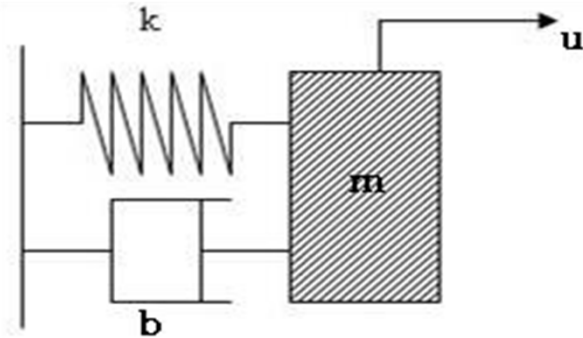


Lecture 5-1

Mathematical Modeling

of Dynamic Systems in State Space I

A Simple Problem



Consider the system equation :

$$m\ddot{y} + b\dot{y} + ky = u$$

Laplace transform : $(ms^2 + bs + k)Y(s) = U(s)$

$$\text{Transfer function: } \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

Consider the system equation again : $\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$

Let's change to multiple single order equations.

Choose variables : $x_1 = y, \quad x_2 = \dot{y}$

Then we get : $\dot{y} = \dot{x}_1 = x_2 \quad \ddot{y} = \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad \text{Simplify the matrix : } \dot{x} = Ax + Bu$$

Why State Space Modeling?

Transfer function Approach

Uses single, higher-order equation

Classical, Frequency Domain Approach.

Replace differential equations with algebraic equations → simplifies representation of individual subsystems, and modeling of interconnected subsystems.

Disadvantage: **Limited applicability**. Only can be applied to linear, time-invariant systems.

Advantage: Rapidly provide stability and transient response information. Can immediately see the effects of varying system parameters until an acceptable design is met.

State Space Approach

Consist of coupled first-order differential equations.
Uses vector and matrix notation.

Modern, time domain approach.

Unified approach for modeling, analyzing and designing a wide range of systems.

Can be used to represent **nonlinear systems** that have **backlash, saturation and dead zone**.

Can handle systems with **non-zero initial conditions** and **time-varying systems**.

Multi input and multi output systems can be compactly represented with a complexity similar to that of SISO system.

Numerous state space **software packages** available.

System analysis can be done using linear algebra. (Matrix form)

Disadvantage: It is **not as intuitive** as the classical approach.

State Space Modeling

If the system is linear time-invariant system, the system can be presented as n state variables, r input variables, and m output variables.

State equation :

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1r}u_r \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2r}u_r \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nr}u_r \end{aligned}$$

Output equation :

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$
$$\begin{aligned} y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1r}u_r \\ y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2r}u_r \\ &\vdots \\ y_m &= c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \cdots + d_{mr}u_r \end{aligned}$$

State Space Modeling

State of a dynamic system : The smallest set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$. And the variables are called *state variables*.

State vector : n state variables which is needed to completely describe the behavior of a given system can be considered the n components of a vector x . Such a vector is called a *state vector*.

State space : The n -dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis is called a *state space*.

Minimum number of state variables
= number of energy storage elements
= order of differential equation

State Space Modeling

State space equation : $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ → State Equation
 $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ → Output Equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix}$$

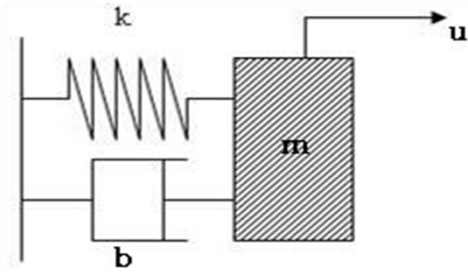
A : State matrix (System Matrix,
Transition matrix)

B : Input matrix

C : State output matrix

D : Control output matrix

A Simple Problem



$$m\ddot{y} + b\dot{y} + ky = u$$

$$\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

State vector: Choice should result in a set of 1st order eqns.

$$x_1 = y(\text{displacement}), \quad x_2 = \dot{y}(\text{speed}) \quad \dot{y} = \dot{x}_1 = x_2$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$\ddot{y} = \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u$$

Output Vector: What are you interested in?

$$\text{Total Force: } y_1 = u - kx - b\dot{x} \quad \text{Momentum: } y_2 = m\dot{x}$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -k & -b \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -k & -b \\ 0 & m \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Transformation of System Models

Step Response : $\text{sys} = \text{sys}(A,B,C,D)$

we use 'step(sys)' or 'step(A,B,C,D)' in MATLAB

Transfer Matrix : r inputs, u_1, u_2, \dots, u_r and m outputs, y_1, y_2, \dots, y_m

We define those vectors.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Then, the transfer matrix $G(s)$ expresses the relationship between $Y(s)$ and $U(s)$

$$Y(s) = G(s)U(s)$$

Transformation of System Models

$$\begin{aligned}\text{System equation : } \quad \dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

$$\begin{aligned}\text{Laplace Transformation : } \quad sX(s) - x(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

$$\begin{aligned}\text{Assume, } x(0)=0 : \quad X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= [C(sI - A)^{-1}B + D]U(s) \rightarrow \therefore G(s) = C(sI - A)^{-1}B + D\end{aligned}$$

ex)

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\end{aligned}$$
$$\begin{aligned}G(s) = C(sI - A)^{-1}B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix}\end{aligned}$$

Impulse Response

$$Y(s) = G(s)U(s)$$

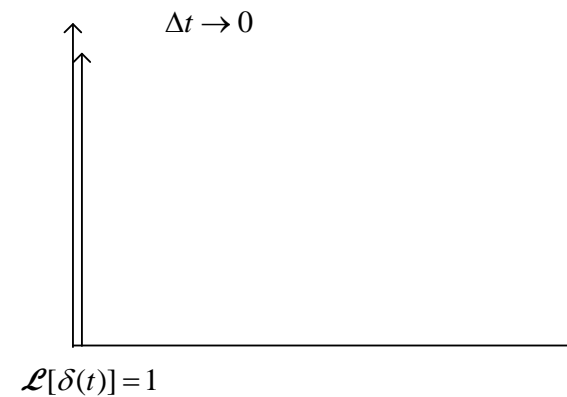
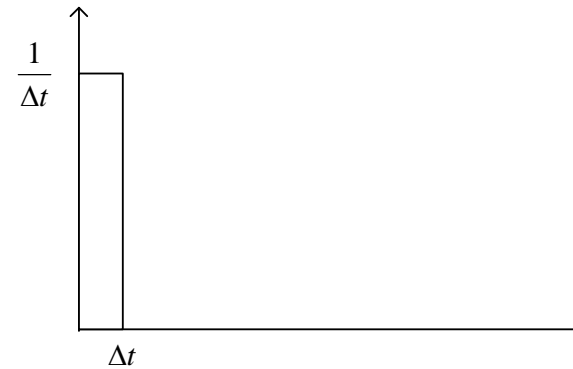
$$Y(s) = G(s)1 = G(s)$$

i.e. ; $\mathcal{L}[y(t)] = G(s)$
when $u(t) = \delta(t)$

$$u(t) = \delta(t)$$

$$\delta(t) = 0, t \neq 0$$

$$\int_{0^-}^{0^+} \delta(t) dt = 1$$



Impulse Response

Ex) Obtain the unit-impulse response

$$x(0) = 0, \dot{x}(0) = 0$$

What are the values of $x(0+)$ and $\dot{x}(0+)$?

$$\frac{Y(s)}{U(s)} = \frac{1}{2s^2 + 14s + 20} \quad u(t) = \delta(t)$$

Sol 1 : $u(s) = 1$

$$Y(s) = \frac{1}{2s^2 + 14s + 20} = \frac{1}{6} \frac{1}{s + 2} - \frac{1}{6} \frac{1}{s + 5}$$

$$y(t) = \frac{1}{6} (e^{-2t} - e^{-5t})$$

$$y(0+) = x(0+) = \lim_{t \rightarrow 0+} x(t) = \lim_{t \rightarrow 0+} \frac{1}{6} (e^{-2t} - e^{-5t}) = 0$$

$$\dot{y}(0+) = \dot{x}(0+) = \lim_{t \rightarrow 0+} \dot{x}(t) = \lim_{t \rightarrow 0+} \frac{-2e^{-2t} + 5e^{-5t}}{6} = \frac{1}{2}$$

Impulse Response

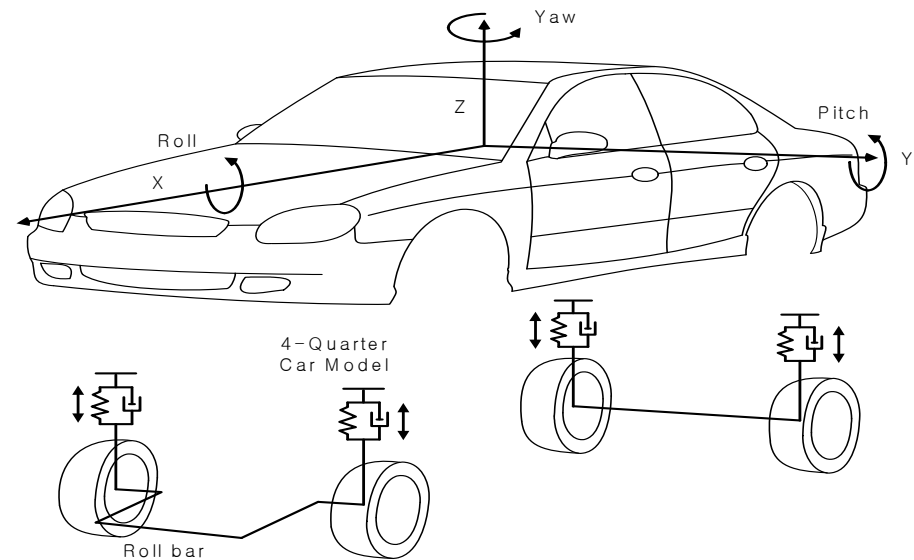
Sol 2 : Initial Value Theorem

$$y(0+) = \lim_{s \rightarrow \infty} sY(s) = \lim_{s \rightarrow \infty} s \frac{1}{2s^2 + 14s + 20} = 0$$

$$\dot{y}(0+) = \lim_{s \rightarrow \infty} s[sY(s) - y(0)] = \lim_{s \rightarrow \infty} s^2 \frac{1}{2s^2 + 14s + 20}$$

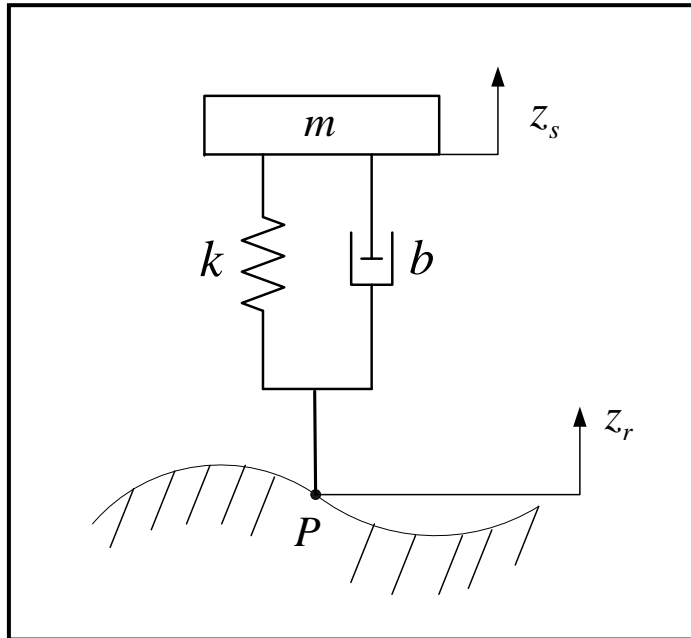
$$\mathcal{L}[\dot{y}(t)] = sY(s) - y(0) = \frac{1}{2}$$

Vehicle Suspension Problem

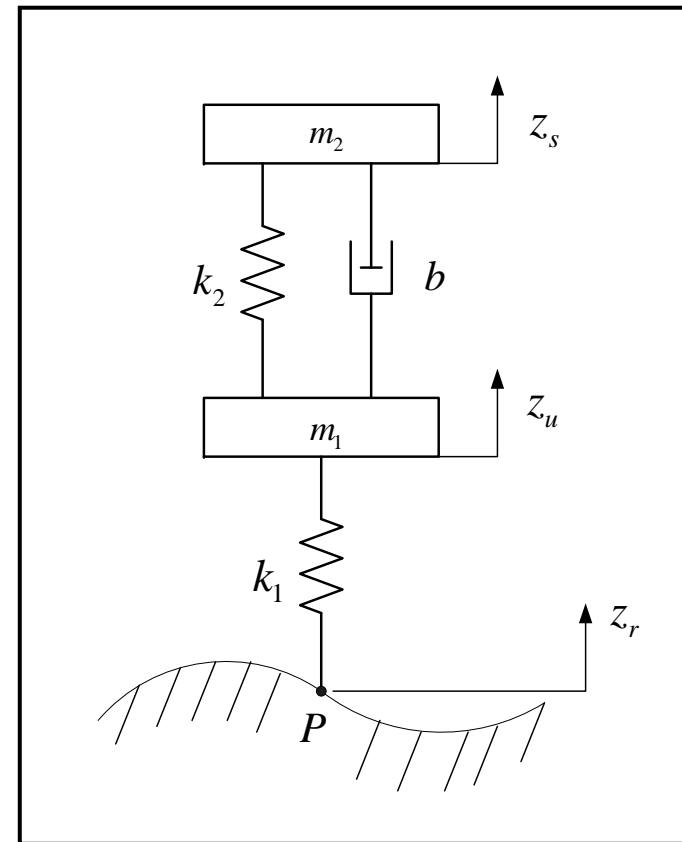


Vehicle Suspension Problem

ex1) Spring, damper, mass system

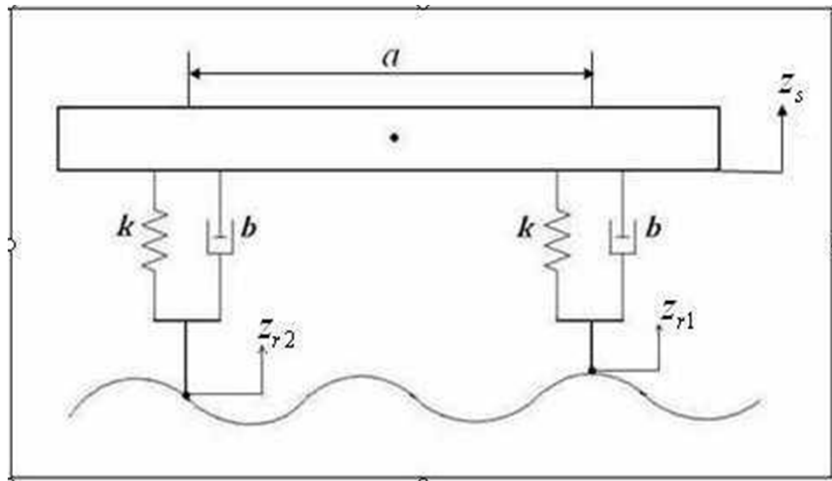


ex2) Body and tire model

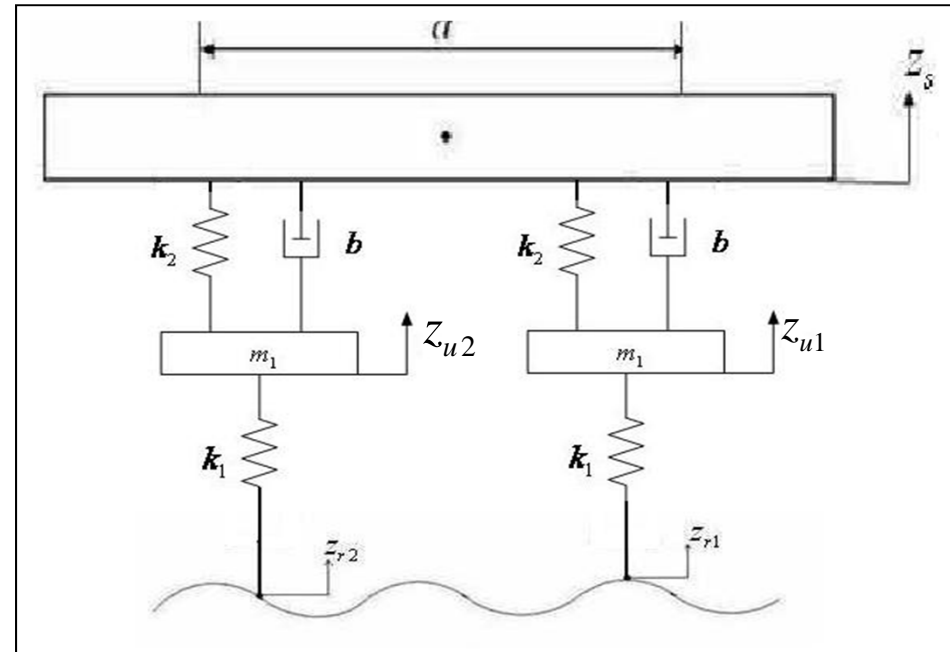


Vehicle Suspension Problem

ex3) Two inputs



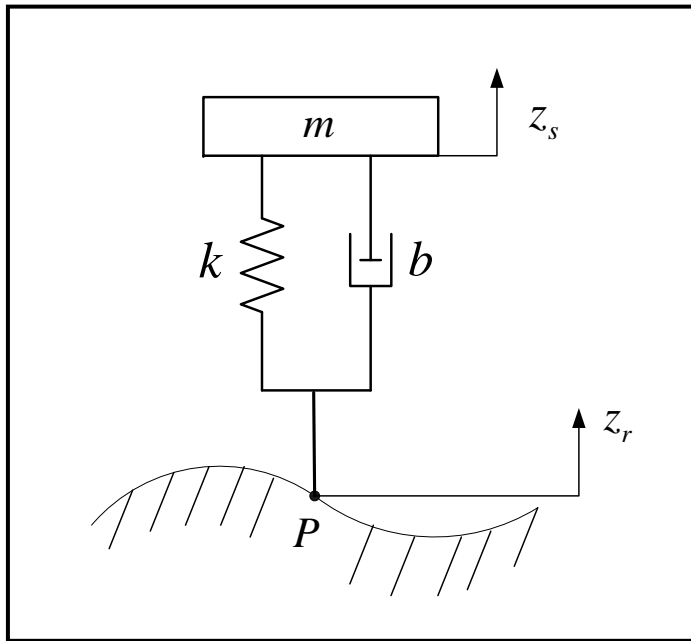
ex4) Two inputs, Body and Tire Model



Two inputs : z_{r1}, z_{r2} $z_{r2}(t) = z_{r1}(t - \tau)$

Vehicle Suspension Problem

ex1) Spring, damper, mass system



▪ Design Considerations

1. Ride Quality

→ Sprung mass acceleration : \ddot{z}_s

2. Rattle space

→ Suspension Deflection : z_s

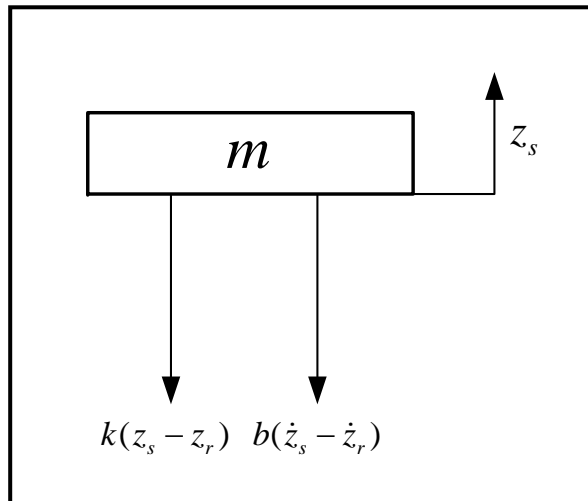
▪ Suspension Design Parameters

→ Spring Stiffness : k

→ Damping Ratio : b

Vehicle Suspension Problem

- Free Body Diagram



- Dynamic Equations

$$m\ddot{z}_s + b(\dot{z}_s - \dot{z}_r) + k(z_s - z_r) = 0$$

$$m\ddot{z}_s + b\dot{z}_s + kz_s = b\dot{z}_r + kz_r$$

- Laplace Transform

$$(ms^2 + bs + k)Z_s(s) = (bs + k)Z_r(s)$$

$$\text{Transfer function: } \frac{Z_s(s)}{Z_r(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Vehicle Suspension Problem

- General Form of State Equation $\dot{x} = Ax + Bu$

- The State variables $(u = \dot{z}_r)$

$$x_1 = z_s - z_r \quad : \text{Suspension Deflection}$$

$$x_2 = \dot{z}_s \quad : \text{absolute velocity of body}$$

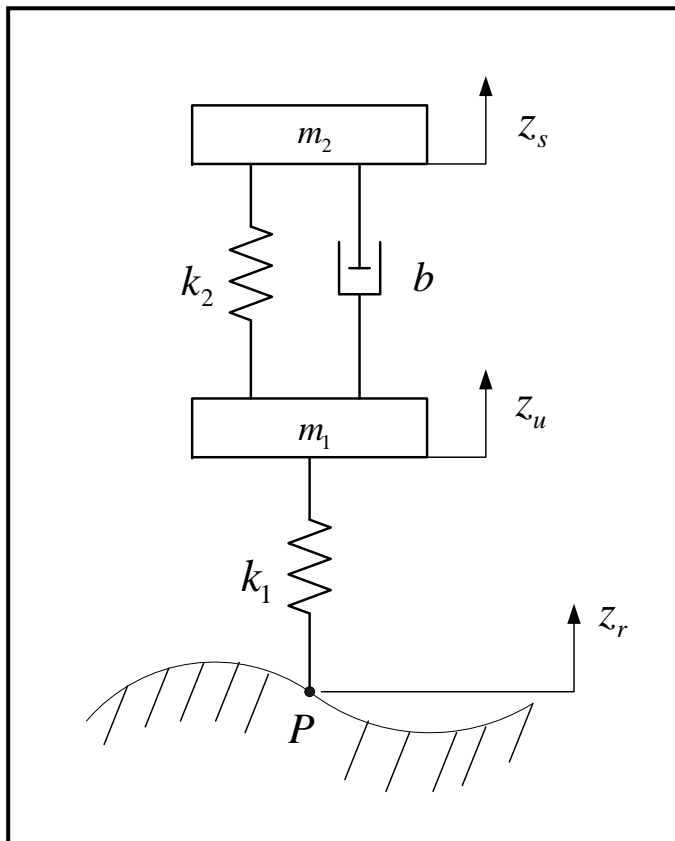
$$\dot{x}_1 = \dot{z}_s - \dot{z}_r = x_2 - \dot{z}_r,$$

$$\dot{x}_2 = \ddot{z}_s = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{z}_r \quad : \text{acceleration of body}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{b}{m} \end{bmatrix} u, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \ddot{z}_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix} u$$

Vehicle Suspension Problem

ex2) Body and tire model



▪ Design Considerations

1. Ride Quality

→ Sprung mass acceleration : \ddot{z}_s

2. Rattle space

→ Suspension Deflection : $z_s - z_u$

3. Tire Force Vibration

→ Tire Deflection : $z_u - z_r$

▪ Suspension Design Parameters

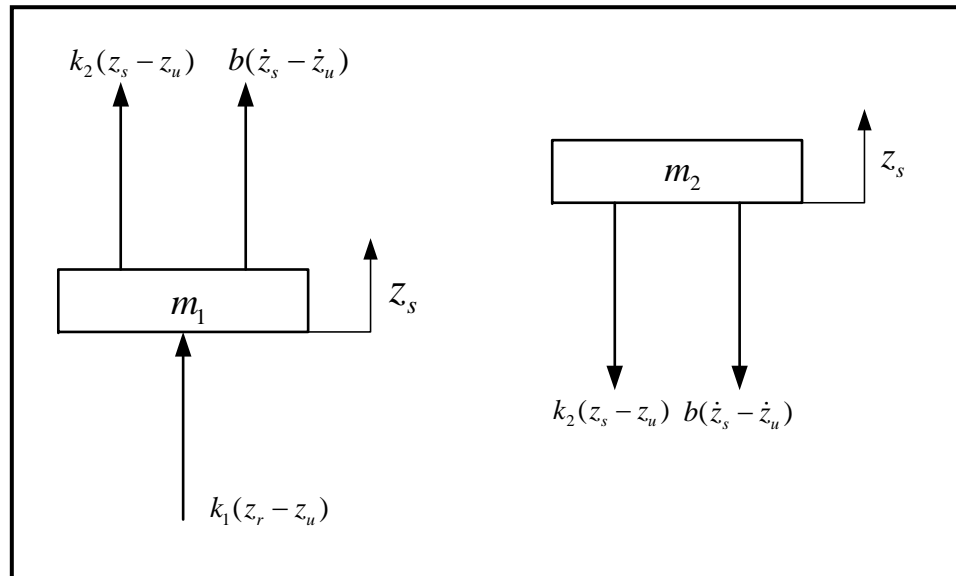
→ Spring Stiffness : k_2

→ Damping Ratio : b

→ Tire Stiffness : k_1

Vehicle Suspension Problem

- Free Body Diagram



- Dynamic Equations

$$m_1 \ddot{z}_u = k_2(z_s - z_u) + b(\dot{z}_s - \dot{z}_u) + k_1(z_r - z_u)$$

$$m_2 \ddot{z}_s = -k_2(z_s - z_u) - b(\dot{z}_s - \dot{z}_u)$$

- Laplace Transform

$$[m_1 s^2 + bs + (k_1 + k_2)]Z_u(s)$$

$$= (bs + k_2)Z_s(s) + k_1 Z_r(s)$$

$$[m_2 s^2 + bs + k_2]Z_s(s) = (bs + k_2)Z_u(s)$$

Vehicle Suspension Problem

- General Form of State Equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- Dynamic Equations

$$m_1 \ddot{z}_u = k_2(z_s - z_u) + b(\dot{z}_s - \dot{z}_u) + k_1(u - z_u)$$

$$m_2 \ddot{z}_s = -k_2(z_s - z_u) - b(\dot{z}_s - \dot{z}_u)$$

- The State variables $(x = z_u, \quad y = z_s)$

$$x_1 = z_s - z_u \quad : \text{Suspension Deflection}$$

$$x_2 = \dot{z}_s \quad : \text{absolute velocity of sprung mass}$$

$$x_3 = z_u - z_r \quad : \text{Tire Deflection}$$

$$x_4 = \dot{z}_u \quad : \text{absolute velocity of unsprung mass}$$

Vehicle Suspension Problem

- 1st order State equations

$$\dot{x}_1 = \dot{z}_s - \dot{z}_u = x_2 - x_4$$

$$\dot{x}_2 = -\frac{k_2}{m_2}(z_s - z_u) - \frac{b}{m_2}(\dot{z}_s - \dot{z}_u) = -\frac{k_2}{m_2}x_1 - \frac{b}{m_2}x_2 + \frac{b}{m_2}x_4$$

$$\dot{x}_3 = \dot{z}_u - \dot{u} = x_4 - \dot{z}_r$$

$$\dot{x}_4 = \frac{k_2}{m_1}(z_s - z_u) + \frac{b}{m_1}(\dot{z}_s - \dot{z}_u) - \frac{k_1}{m_1}(u - z_u) = \frac{k_2}{m_1}x_1 + \frac{b}{m_1}x_2 - \frac{k_1}{m_1}x_3 - \frac{b}{m_1}x_4$$

$$u = \dot{z}_r$$

- Matrix Form of State equations (system matrix & output matrix)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_2}{m_2} & -\frac{b}{m_2} & 0 & \frac{b}{m_2} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & \frac{b}{m_1} & -\frac{k_1}{m_1} & -\frac{b}{m_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{k_2}{m_2} & -\frac{b}{m_2} & 0 & \frac{b}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$y_1 = \ddot{x}_2 = -\frac{k_2}{m_2}x_1 - \frac{b}{m_2}x_2 + \frac{b}{m_2}x_4 \quad : \text{Sprung mass acceleration}$$

$$y_2 = z_s - z_u = x_1 \quad : \text{Suspension Deflection}, \quad y_3 = z_u - z_r = x_3 \quad : \text{Tire Deflection}$$

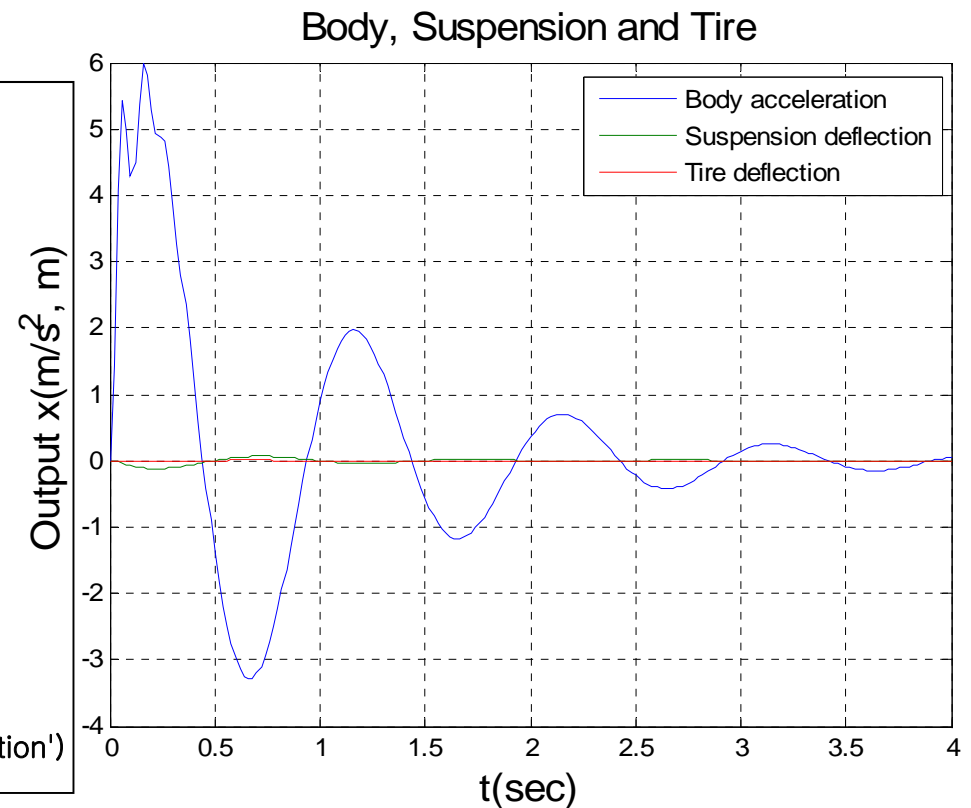
Vehicle Suspension Problem

Parameter : $M_1(\text{tire})=55\text{kg}$, $M_2(\text{body})=400\text{kg}$, $b=1000\text{Ns/m}$

$k_1(\text{tire})=180000\text{N/m}$, $k_2(\text{suspension})=18000\text{N/m}$

Input : Step input

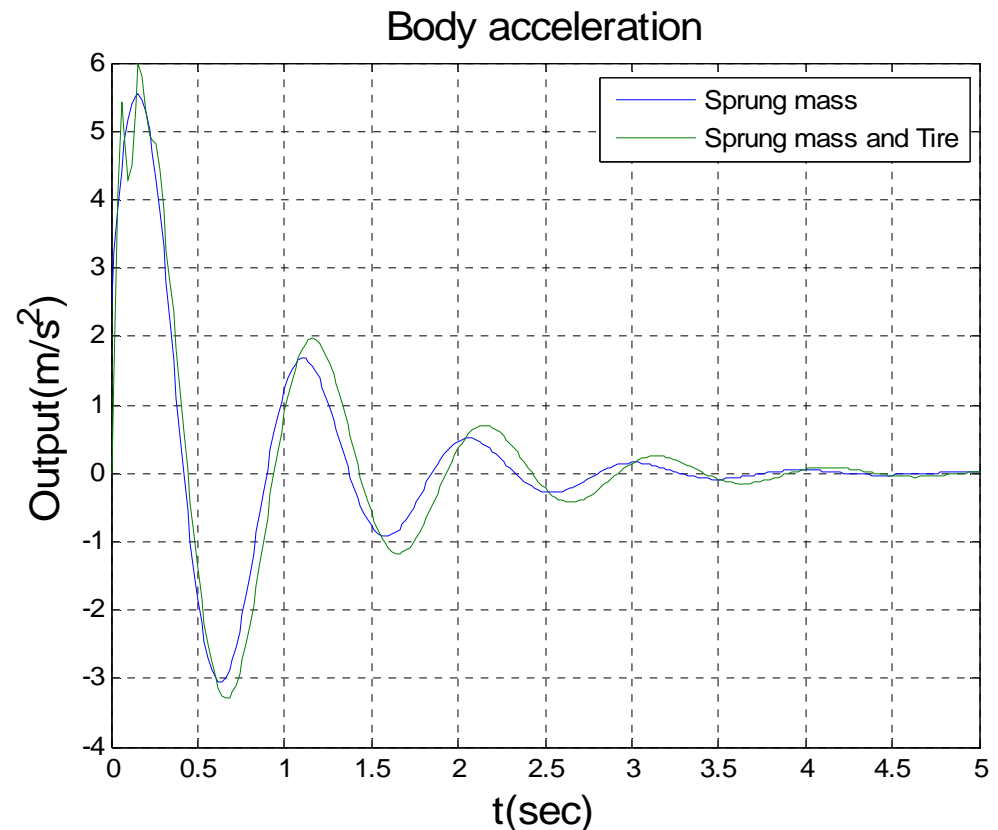
```
>>t=0:0.02:4;
>>A=[0 1 0 -1;-45 -2.5 0 2.5;0 0 0 1
      327.3 18.18 -3273 -18.18];
>>B=[0;0;-1;0];
>>C=[-45 -2.5 0 2.5;1 0 0 0;0 0 1 0];
>>D=0;
>>sys=ss(A,B,C,D);
>>[y,t]=step(sys,t);
>>plot(t,y)
>>grid
>>title('Body, Suspension and Tire','FontSize',15)
>>xlabel('t(sec)','FontSize',15)
>>ylabel('Output x(m/s^2, m)','FontSize',15)
>>legend('Body acceleration','Suspension deflection','Tire deflection')
```



Vehicle Suspension Problem

```
t=0:0.02:5;
A=[0 1;-45 -2.5];
B=[-1;2.5];
C=[-45 -2.5];
D=[2.5];
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
A2=[0 1 0 -1;-45 -2.5 0 2.5;0 0 0 1
     327.3 18.18 -3273 -18.18];
B2=[0;0;-1;0];
C2=[-45 -2.5 0 2.5];
D2=0;
sys2=ss(A2,B2,C2,D2);
[y2,t]=step(sys2,t);
plot(t,y,t,y2)
grid
title('Body acceleration','FontSize',15)
xlabel('t(sec)','FontSize',15)
ylabel('Output(m/s^2)','FontSize',15)
legend('Sprung mass','Sprung mass and Tire')
```

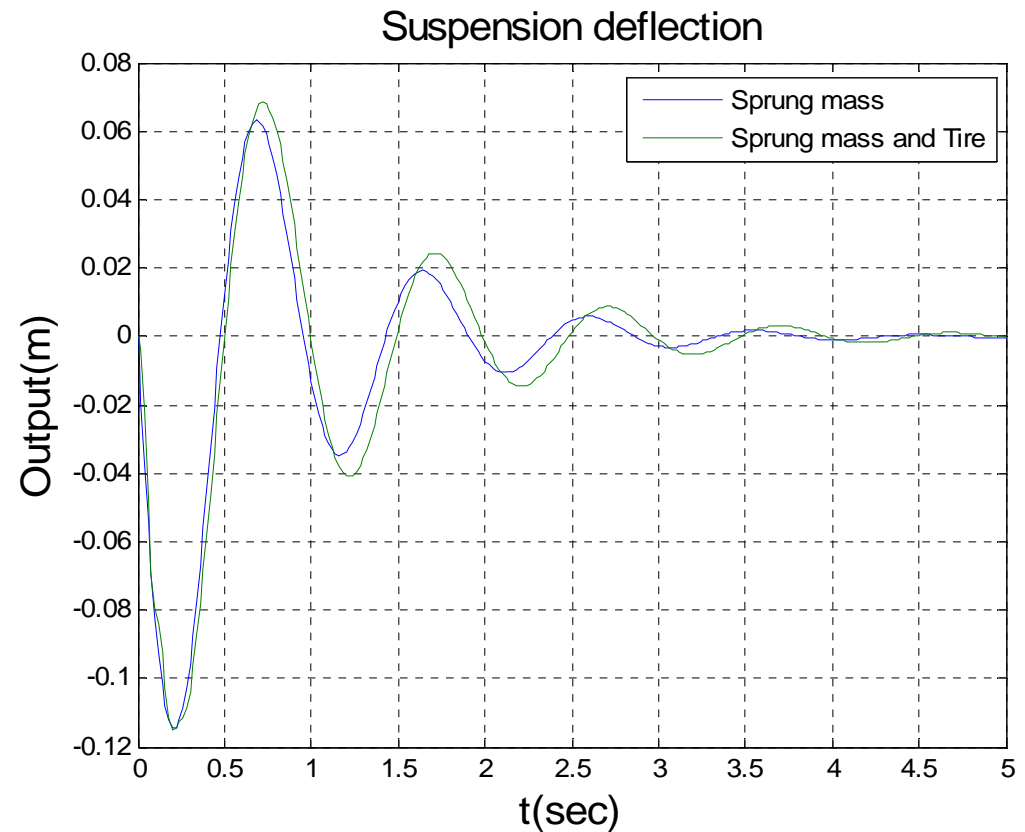
Comparison 1. Acceleration of sprung mass (body)



Vehicle Suspension Problem

```
t=0:0.02:5;
A=[0 1;-45 -2.5];
B=[-1;2.5];
C=[1 0];
D=0;
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
A2=[0 1 0 -1;-45 -2.5 0 2.5;0 0 0 1
     327.3 18.18 -3273 -18.18];
B2=[0;0;-1;0];
C2=[1 0 0 0];
D2=0;
sys2=ss(A2,B2,C2,D2);
[y2,t]=step(sys2,t);
plot(t,y,t,y2)
grid
title('Suspension deflection','FontSize',15)
xlabel('t(sec)','FontSize',15)
ylabel('Output(m)','FontSize',15)
legend('Sprung mass','Sprung mass and Tire')
```

Comparison 2. Suspension deflection



End of lecture 5(I)