Lecture 7
Fluid Systems I

Brake systems

Brake Pedal → Vacuum Booster → Master Cylinder → Proportionnig Valve → Font Wheel / Rear Wheel
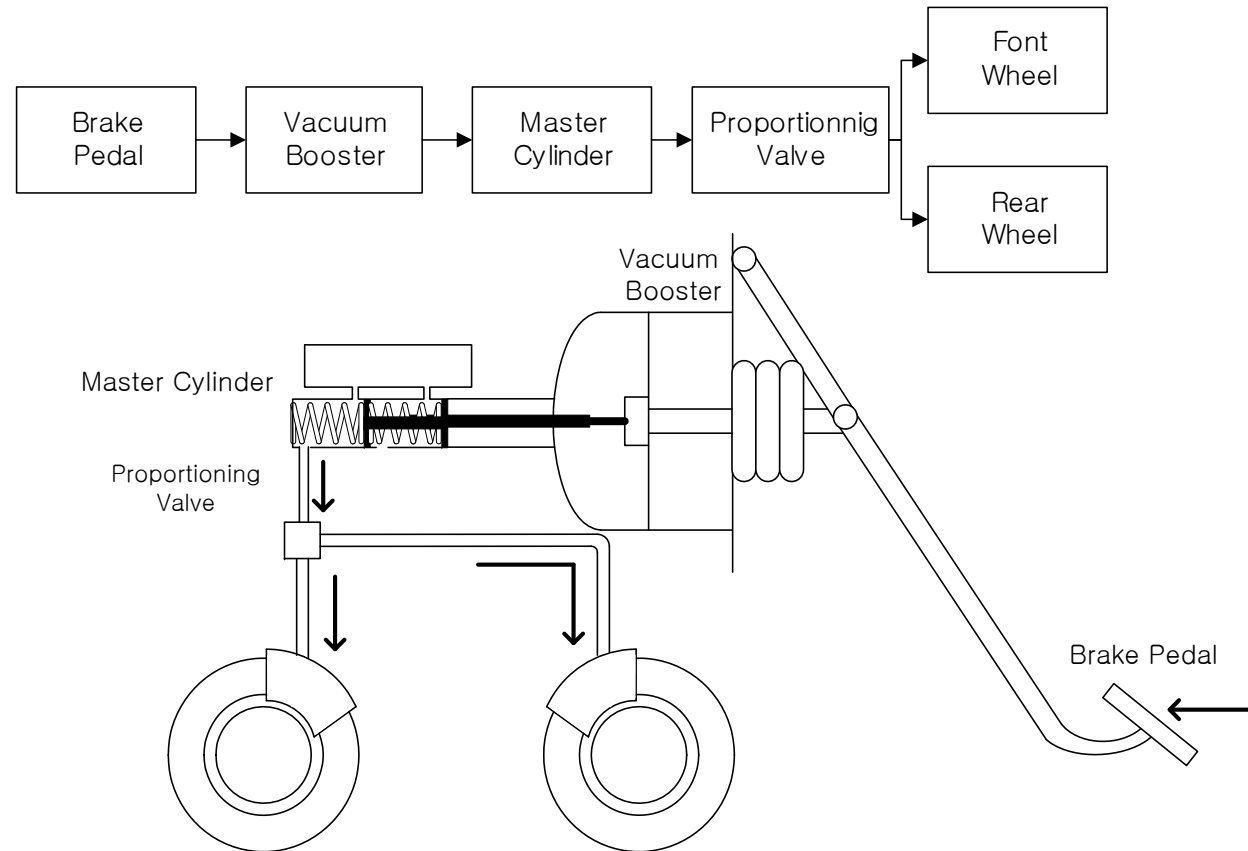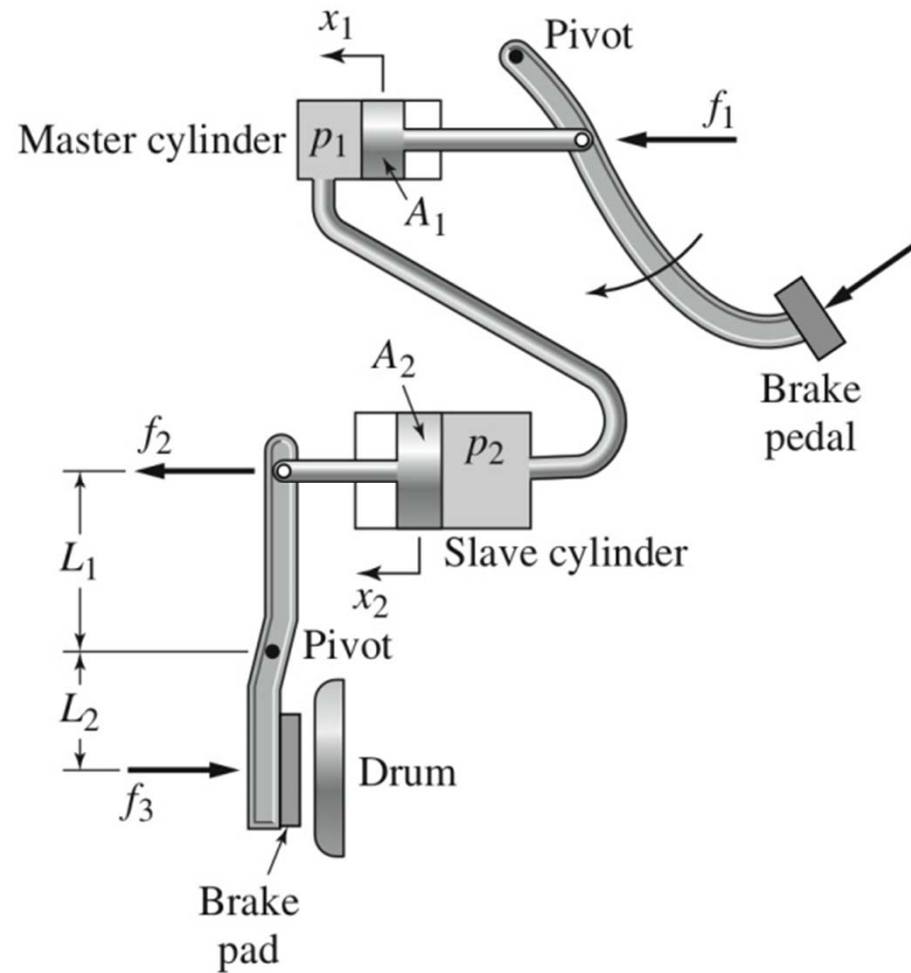
Vacuum Booster

Master Cylinder

Proportioning Valve

Brake Pedal

Fundamental structure of a hydraulic brake

Conservation of Mass, Force and Pressure

Conservation of Mass, Force and Pressure

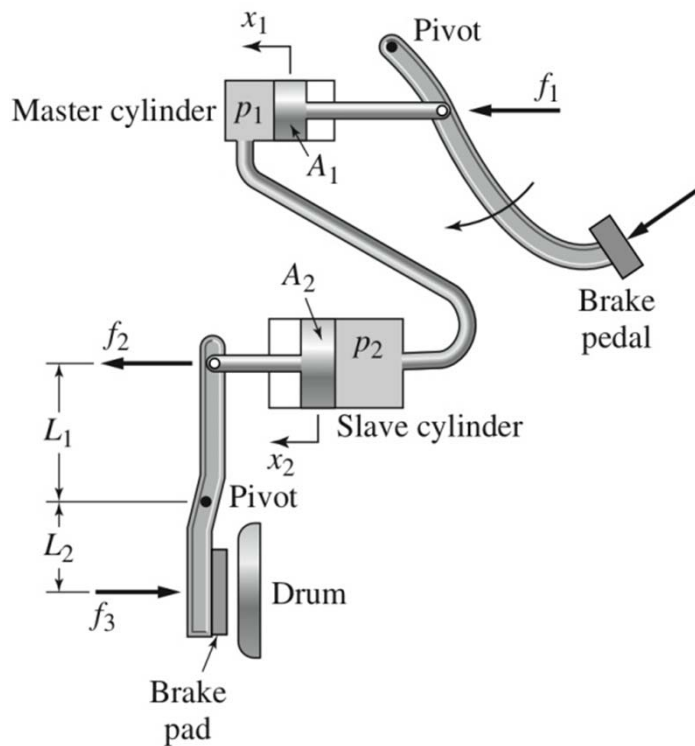$$\dot{m} = q_{mi} - q_{mo}$$

i) $Volume \quad A_1 dx_1 = A_2 dx_2$

$$\rightarrow dx_2 = \frac{A_1}{A_2} dx_1$$

ii) $\text{Pr}essue \quad p_2 = p_1 + \rho g h$

iii) **Force** $\quad f_1 = p_1 A_1$

$$f_2 = p_2 A_2$$

$$f_2 = \frac{A_2}{A_1} f_1 \quad (P_1 \approx P_2)$$

$$f_3 = \frac{L_1}{L_2} f_2 = \frac{L_1}{L_2} \frac{A_2}{A_1} f_1$$

x_1

Pivot

Master cylinder p_1

f_1

A_1

A_2

f_2

p_2

Brake pedal

L_1

Slave cylinder

x_2

Pivot

L_2

Drum

f_3

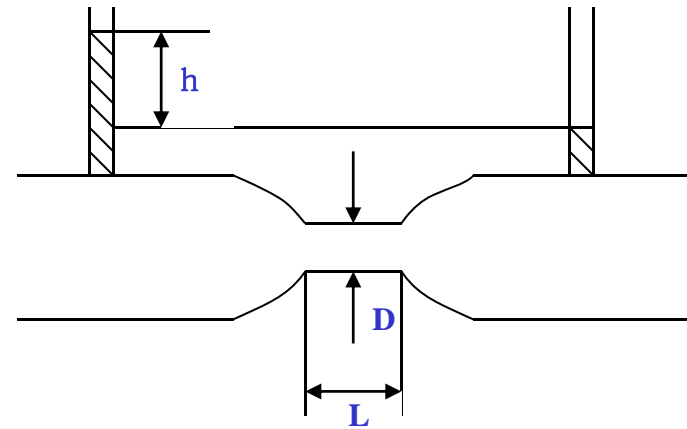Brake pad

Fluid Resistance

Electrical systems, resistance is defined as

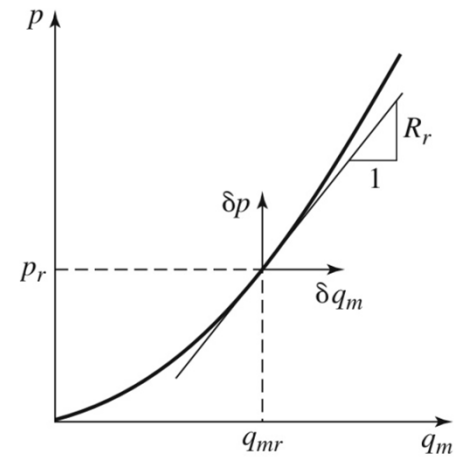$$i = \frac{v}{R}, \quad R = \frac{v}{i}$$

For fluid systems, R is defined as

$$q_m = \frac{P}{R}, \quad R = \frac{P}{q_m}$$

Mass flow rate,
pressure difference

Fluid Resistance

$$R = \frac{change\ in\ pressure}{change\ in\ flow\ rate} = \frac{dp}{dq_m}\bigg|_{q=q_{mr}}$$

Fluid Capacitance

Electrical systems,
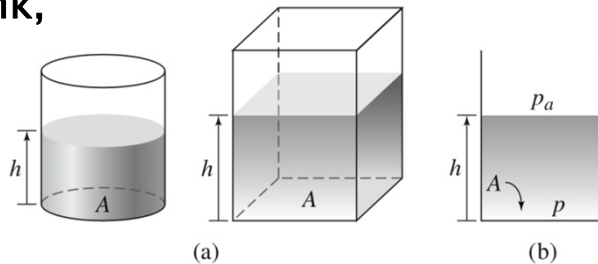
$$\frac{dv}{dt} = \frac{1}{C}i$$

Fluid systems,

$$\frac{dp}{dt} = \frac{1}{C}q_m, \qquad dp = \frac{1}{C}q_m dt, \qquad C = \frac{q_m dt}{dp}$$

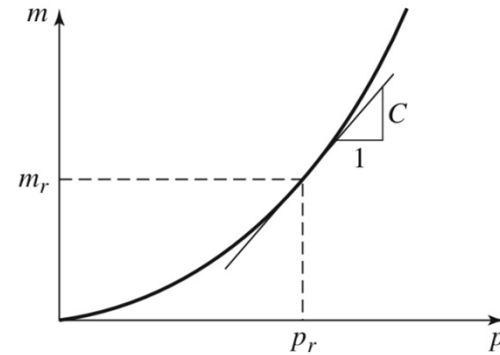$$C = \frac{change\ in\ stored\ mass}{change\ pressure} = \frac{dm}{dp}$$

Fluid Capacitance

$$C = \frac{change\ in\ stored\ mass}{change\ pressure} = \frac{dm}{dp}$$

For a tank,

$$C = \rho A \frac{dh}{dp} = \rho A \frac{1}{\rho g} = \frac{A}{g}$$

General fluid capacitance

$$\frac{dm}{dt} = C \frac{dp}{dt} = q_{mi} - q_{mo}$$

$$C \frac{dp}{dt} = \rho A \frac{dh}{dt} = q_{mi} - q_{mo}$$

For liquid level system, C is defined as

$$C = \frac{change\ in\ liquid\ stored}{change\ in\ head} \frac{m^3}{m} = A$$

use h(pressure head) , instead of p.

Fluid Systems vs Electrical Systems

Electrical systems,

Fluid systems,

Resistance

Capacitance

Inductance

Fluid Systems vs Electrical Systems

Electrical systems,

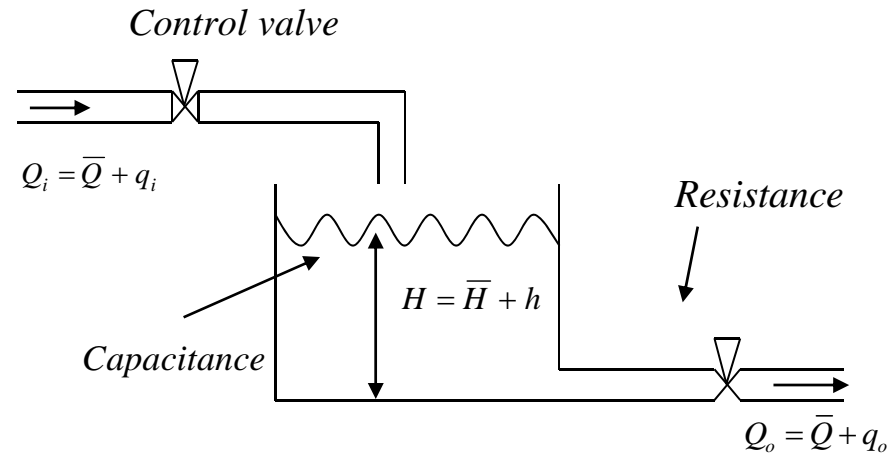Fluid systems,

Voltage   V

Pressure P

Current I (= dQ/dt)

Mass flow rate q

Charge Q

Fluid mass

Liquid Level Systems

Control valve

$Q_i = \overline{Q} + q_i$

Resistance

$H = \overline{H} + h$

Capacitance

$Q_o = \overline{Q} + q_o$

$$R = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate} \frac{m}{m^3 / s} \qquad C = \frac{change\ in\ liquid\ stored}{change\ in\ head} \frac{m^3}{m}$$

$$Steady\ state\ :\ \ Q_i = \overline{Q} = Q_0 = \frac{\overline{H}}{R}$$

$$consider,\ \ Q_o = \frac{H}{R}, \qquad Q_i - Q_o = C \frac{dH}{dt}$$

Basic Concepts

Laminar flow : When the viscous forces are dominant (slow flow, low Re) they are
sufficient enough to keep all the fluid particles in line,
then the flow is laminar.

Turbulent flow : It can be interpreted that when the inertial forces dominate over
the viscous forces (when the fluid is flowing faster and Re is larger)
then the flow is turbulent.

Reynolds number (Re) : The ratio of inertial forces to viscous forces.
It is used to determine whether a flow will be laminar or turbulent.

$$Re = \frac{\rho v D}{\mu}$$

μ : the dynamic viscousity of the fluid

ρ : density of the fluid

D : diameter

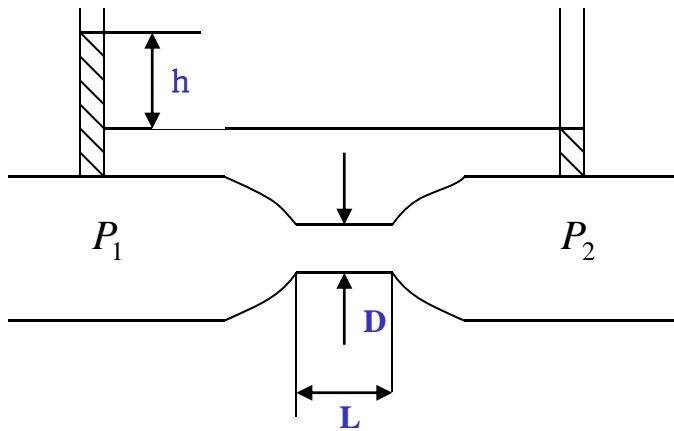$Re < 2000$: always laminar

$Re > 4000$: always turbulent

Laminar Flow

Cylindrical pipe

$$P_1 - P_2 = \rho g h, \quad Q \frac{128 \nu L}{g \pi D^4} = h$$

$\nu : viscosity,$

$L : length\ of\ pipe$

$D : diameter\ of\ pipe$

$$Q = \frac{h}{R} = K_l h, \qquad R = \frac{128 \nu L}{g \pi D^4} \quad [s/m^2]$$

$Q\ : steady - state\ flow\ rate$

$K_l : constant$

$h\ : steady - state\ head$
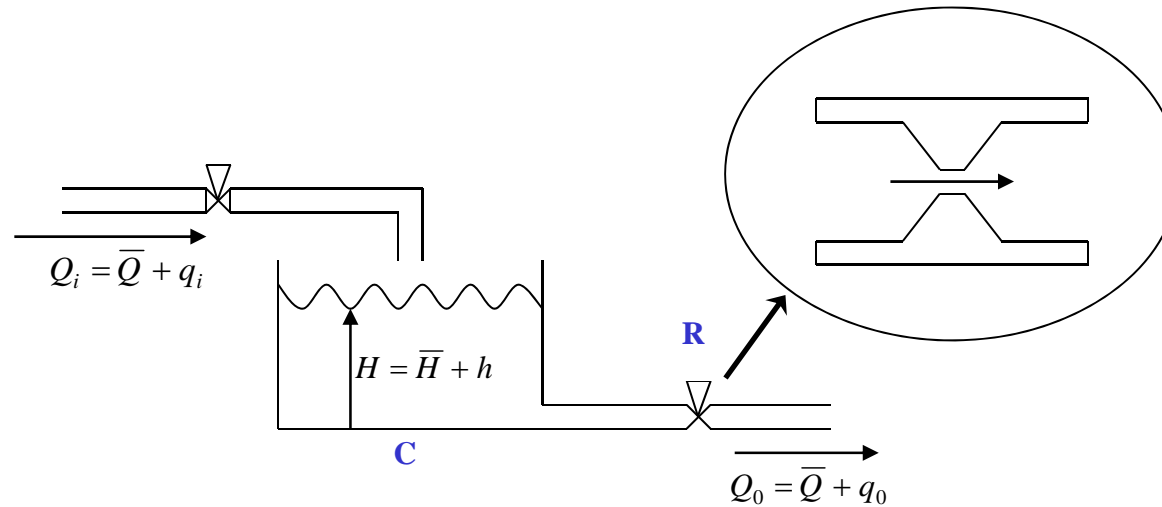
Laminar Flow

Liquid level dynamics :

$$C \frac{dH}{dt} = Q_i - Q_o$$

$$= \bar{Q} + q_i - (\bar{Q} + q_o) = q_i - q_o$$

$$\frac{dH}{dt} = \frac{d}{dt}(\bar{H} + h) = \frac{dh}{dt}$$

$$Q_o = \bar{Q} + q_o = \frac{\bar{H} + h}{R}, \qquad q_o = \frac{h}{R}$$

$$\rightarrow \quad \frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

Turbulent Flow

$Q_i = \overline{Q} + q_i$

$H = \overline{H} + h$

R

C

$Q_0 = \overline{Q} + q_0$

$$Q = C_d \cdot a \cdot \sqrt{\frac{2}{\rho}(P_1 - P_2)}$$

$\rho : density, \quad a : area, \quad C_d : discharge\ coefficient$

$steady\ state : \ Q_i = \overline{Q} = Q_0 = K_t \sqrt{\overline{H}}$

Turbulent Flow

Liquid level dynamics

$$C\frac{dH}{dt} = Q_i - Q_0 = \overline{Q} + q_i - K\sqrt{H}$$

$$\frac{dH}{dt} = \frac{1}{C}Q_i - \frac{1}{C}K\sqrt{H} = f(Q_i, H)$$

$$f(Q_i, H) = f(\overline{Q}_i, \overline{H}) + \frac{\partial f}{\partial Q_i}\bigg|_{\overline{Q}, \overline{H}}(Q_i - \overline{Q}) + \frac{1}{2!}\frac{\partial^2 f}{\partial Q_i^2}(Q_i - \overline{Q})^2 + \cdots$$

$$\cdots + \frac{\partial f}{\partial H}\bigg|_{\overline{Q}, \overline{H}}(H - \overline{H}) + \frac{1}{2!}\frac{\partial^2 f}{\partial H^2}(H - \overline{H})^2 + \cdots$$

$$= \frac{1}{C}\overline{Q} - \frac{1}{C}K\sqrt{\overline{H}} + \left(\frac{1}{C}q_i - \frac{K}{C \cdot 2\sqrt{\overline{H}}}h\right) + \text{high order term}$$

Turbulent Flow

$$\frac{dH}{dt} = \frac{d\overline{H}}{dt} + \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{C}q_i - \frac{K}{C \cdot 2\sqrt{\overline{H}}}h$$

$$q_o = \frac{h}{R}, \qquad R = \frac{2\sqrt{\overline{H}}}{K} = \frac{2\overline{H}}{\overline{Q}}, \quad \left(\overline{Q} = K\sqrt{\overline{H}}\right)$$
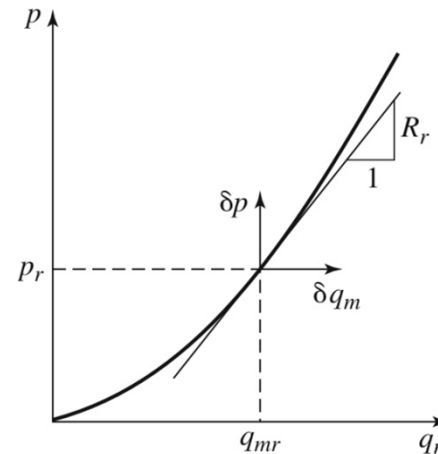
$$\therefore \quad \begin{cases} \dfrac{dh}{dt} = -\dfrac{1}{CR}h + \dfrac{1}{C}q_i \\[3mm] q_o = \dfrac{h}{R} \end{cases} \qquad for \;\; small \;\; q_i$$

Fluid Resistance(laminar flow vs turbulent flow)

$$R = \frac{change\ in\ pressure}{change\ in\ flow\ rate} = \frac{dp}{dq_m}\bigg|_{q=q_{mr}}$$
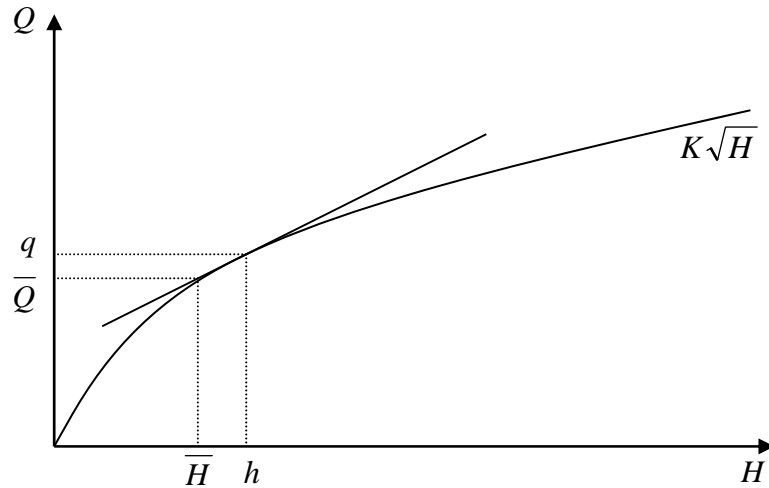
$$q_m = \frac{p}{R} \qquad \text{For laminar flow}$$

$$q_m = \sqrt{\frac{p}{R_l}} \qquad \text{For turbulent flow}$$

For liquid level system, R is defined as

$$R = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate}\ \frac{m}{m^3/s} = \frac{dh}{dq_m}$$

Linearization

$$Q = \overline{Q} + q = K\sqrt{\overline{H}} + (\quad) \cdot h$$

$$y = f(x), \qquad \overline{y} = f(\overline{x}), \qquad x = \overline{x} + \Delta x$$

$$y = f(\overline{x}) + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} \cdot \Delta x + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\overline{x}} \cdot \Delta x^2 + \cdots$$

$$\approx \overline{y} + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} \cdot \Delta x$$

$$y - \overline{y} = \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} (x - \overline{x}) = K(x - \overline{x})$$

$$y = f(x_1, x_2) \approx f(\overline{x_1}, \overline{x_2}) + \left.\frac{\partial f}{\partial x_1}\right|_{\overline{x_1}, \overline{x_2}} (x_1 - \overline{x_2}) + \left.\frac{\partial f}{\partial x_2}\right|_{\overline{x_1}, \overline{x_2}} (x_2 - \overline{x_2})$$

$$= \overline{y} + K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

$$\therefore \ y - \overline{y} = K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

Summary of Liquid Level Systems

Laminar flow

$$\frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

$$\bar{Q} = K \cdot \bar{H} \qquad R = \frac{\bar{H}}{\bar{Q}}$$

$$R = \frac{128\nu L}{g\pi D^4}$$
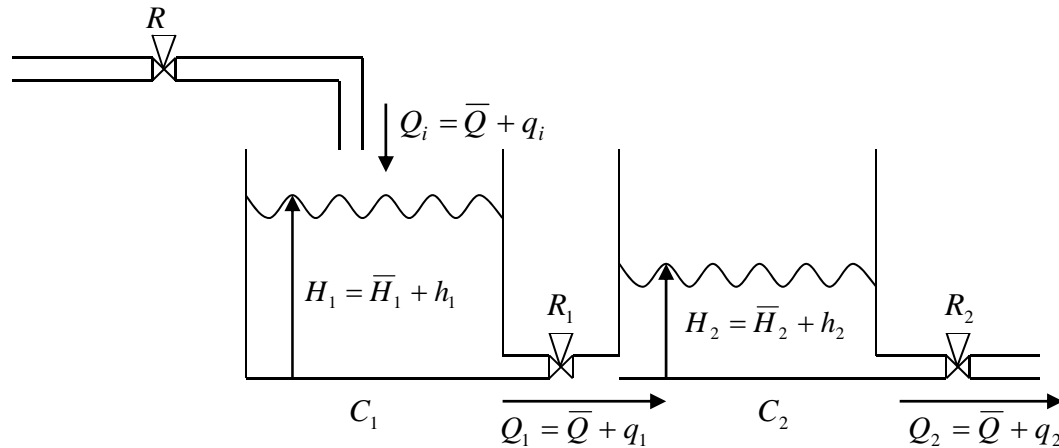
$$q_o = \frac{h}{R}$$

Turbulent flow

$$\frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

$$q_o = \frac{h}{R}$$

$$R = \frac{2\bar{H}}{\bar{Q}} = \frac{2\sqrt{\bar{H}}}{K} = \frac{2\sqrt{\bar{H}}}{C_d a\sqrt{2g}}$$

$$\left(\because \bar{Q} = K\sqrt{\bar{H}} = C_d a\sqrt{\frac{2}{\rho}\rho g \bar{H}} = C_d a\sqrt{2g}\sqrt{\bar{H}} \right)$$

Liquid Level Systems with Interaction

Steady state :

$$Q_i = \bar{Q} = Q_1 = Q_2$$

$$= \frac{\bar{H}_1 - \bar{H}_2}{R_1} = \frac{\bar{H}_2}{R_2}$$

Liquid level dynamics :

$$C_1 \frac{dH_1}{dt} = \bar{Q} + q_i - \left(\bar{Q} + q_1\right) \qquad \rightarrow \quad C_1 \frac{dh_1}{dt} = q_i - q_1$$

$$Q_1 = \bar{Q} + q_1 = \frac{1}{R_1}\left(\bar{H}_1 + h_1 - (\bar{H}_2 + h_2)\right) \quad \rightarrow \quad q_1 = \frac{h_1 - h_2}{R_1}$$

$$C_2 \frac{dH_2}{dt} = \bar{Q} + q_1 - \left(\bar{Q} + q_2\right) \qquad \rightarrow \quad C_2 \frac{dh_2}{dt} = q_1 - q_2$$

$$Q_2 = \bar{Q} + q_2 = \frac{\bar{H}_2 + h_2}{R_2} \qquad\qquad \rightarrow \quad q_2 = \frac{h_2}{R_2}$$

Liquid Level Systems with Interaction

$$
\therefore \begin{cases}
\dfrac{dh_1}{dt} = -\dfrac{1}{C_1}\left(\dfrac{h_1 - h_2}{R_1}\right) + \dfrac{1}{C_1}q_i \\[3em]
\dfrac{dh_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{h_1 - h_2}{R_1}\right) - \dfrac{1}{C_2}\dfrac{h_2}{R_2}
\end{cases}
$$

$$
Transfer\ Function = \frac{Q_2(S)}{Q_i(S)} = \frac{1}{R_1C_1R_2C_2S^2 + \left(R_1C_1 + R_2C_2 + R_2C_1\right)S + 1}
$$

End of Lecture 7