Vibrations are Everywhere

Modeling and Degrees of Freedom

The examples on the previous slide many degrees of freedom and many parts, we will start with one degree of freedom and work towards many.

- Recall from your study of statics and physics that a degree of freedom is the independent parameter needed to describe the configuration of a physical system

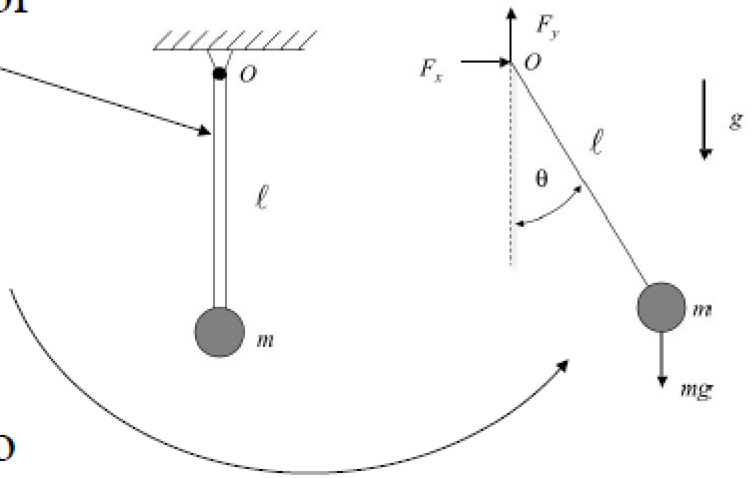
- So single degree of freedom system, which is where we start, is a system whose position in time and space can be defined by one coordinate, here a displacement or position.

Degrees of Freedom: The Minimum Number of coordinates to specify a configuration

- For a single particle confined to a line, one coordinate suffices so it has one degree of freedom
- For a single particle in a plane two coordinates define its location so it has two degrees of freedom
- A single particle in space requires three coordinates so it has three degrees of freedom

Example 1.1.1 The Pendulum

- Sketch the structure or part of interest
- Write down all the forces and make a "free body diagram"
- Use Newton's Law and/or Euler's Law to find the equations of motion



$$\sum \mathbf{M}_0 = J_0\alpha, \quad J_0 = m\ell^2$$

The problem is one dimensional, hence a scalar equation results

$$J_0\alpha(t) = -mgl\sin\theta(t) \Rightarrow m\ell^2\ddot{\theta}(t) + \underbrace{mg\ell\sin\theta(t)}_{\substack{restoring\\ force}} = 0$$

Here the over dots denote differentiation with respect to time *t*

This is a second order, nonlinear ordinary differential equation

We can *linearize* the equation by using the approximation $\sin\theta \approx \theta$

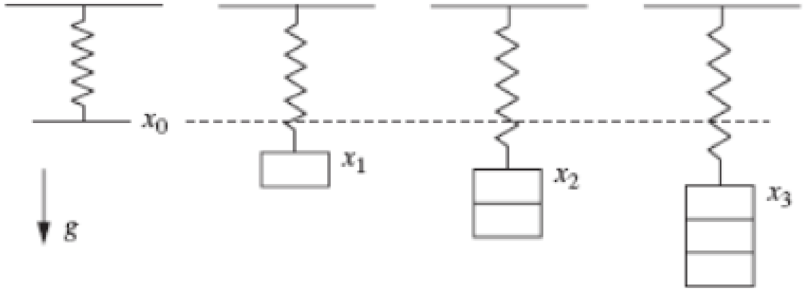
$$\Rightarrow m\ell^2\ddot{\theta}(t) + mg\ell\theta(t) = 0 \Rightarrow \ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0$$

Requires knowledge of $\theta(0)$ and $\dot{\theta}(0)$

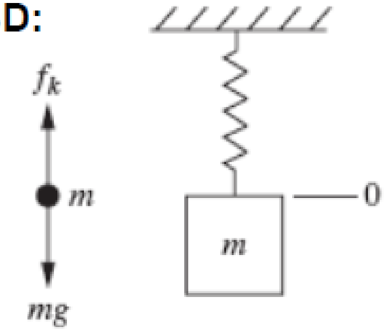
the initial position and velocity.

Next consider a spring mass system and perform a static experiment:

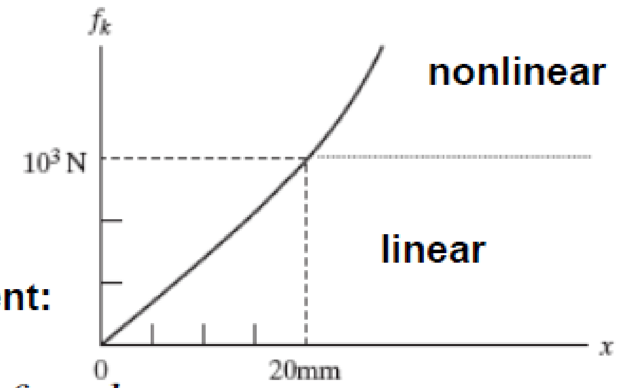
- From strength of materials recall:



FBD:

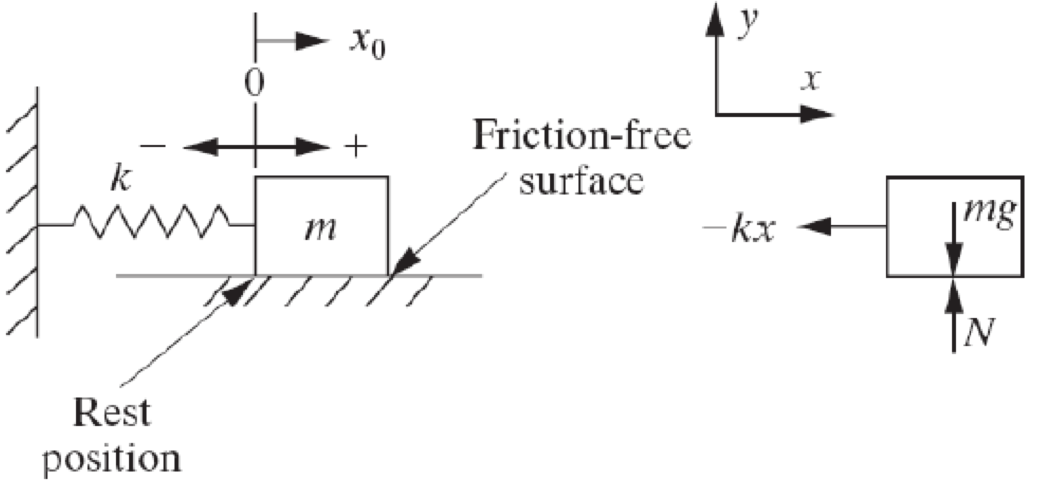


A plot of force versus displacement:



$$\text{experiment} \Rightarrow f_k = kx$$

Free-body diagram and equation of motion



•Newton's Law:

$$m\ddot{x}(t) = -kx(t) \Rightarrow m\ddot{x}(t) + kx(t) = 0$$

$$x(0) = x_0, \dot{x}(0) = v_0 \qquad \textbf{(1.2)}$$

Again a 2nd order ordinary differential equation

Stiffness and Mass

Vibration is cause by the interaction of two different forces one related to position (stiffness) and one related to acceleration (mass).

Proportional to displacement

<u>Stiffness (*k*)</u>

$$f_k = -kx(t)$$

statics

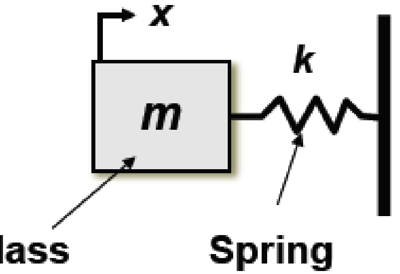
<u>Mass (*m*)</u>

$$f_m = ma(t) = m\ddot{x}(t)$$

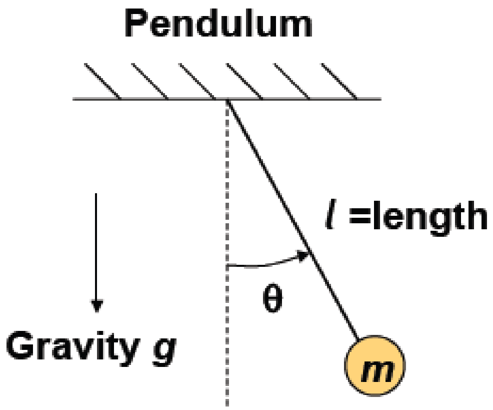
dynamics

Proportional to acceleration

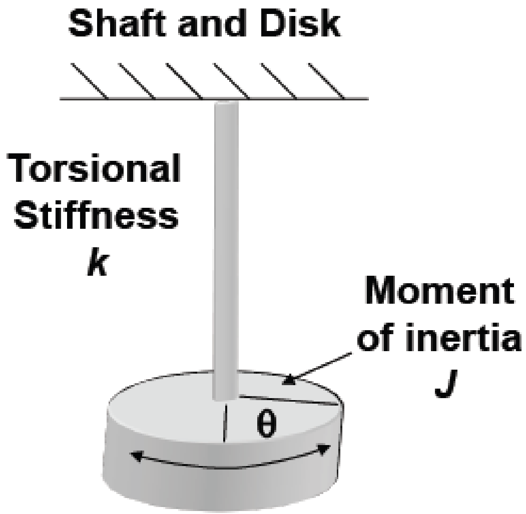
Displacement



Examples of Single-Degree-of-Freedom Systems



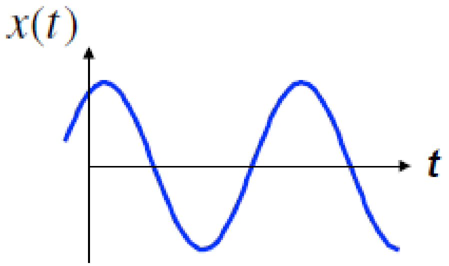
$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = 0$$



$$J\ddot{\theta}(t) + k\theta(t) = 0$$

Solution of 2nd order DEs

Lets assume a solution:



$$x(t) = A\sin(\omega_n t + \phi) \quad (1.3)$$

Differentiating twice gives:

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi) \quad (1.4)$$

$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi) = -\omega_n^2 x(t) \quad (1.5)$$

Substituting back into the equations of motion gives:

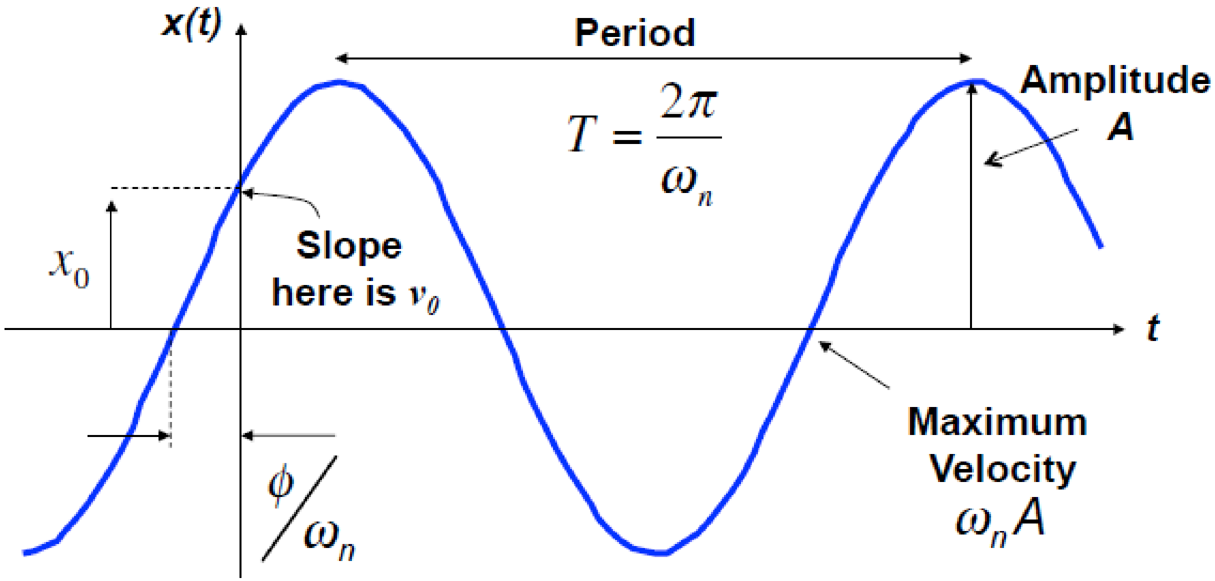
$$-m\omega_n^2 A\sin(\omega_n t + \phi) + kA\sin(\omega_n t + \phi) = 0$$

$$-m\omega_n^2 + k = 0 \quad \text{or} \quad \omega_n = \sqrt{\frac{k}{m}} \longleftarrow$$

Natural frequency

rad/s

Summary of simple harmonic motion



$$f_n = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi}\text{Hz}$$

Initial Conditions

If a system is vibrating then we must assume that something must have (in the past) transferred energy into to the system and caused it to move. For example the mass could have been:

•moved a distance x_0 and then released at $t = 0$ (i.e. given Potential energy) or

•given an initial velocity v_0 (i.e. given some kinetic energy) or

•Some combination of the two above cases

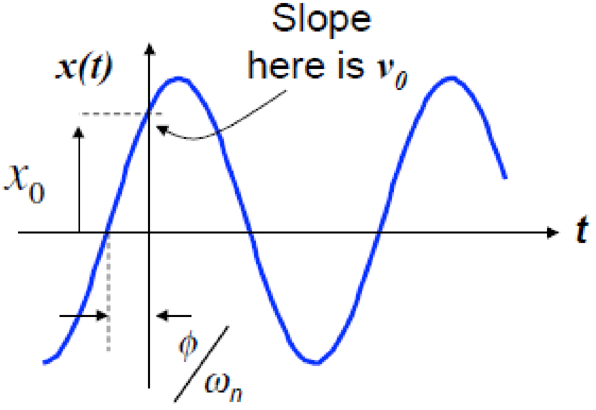
From our earlier solution we know that:

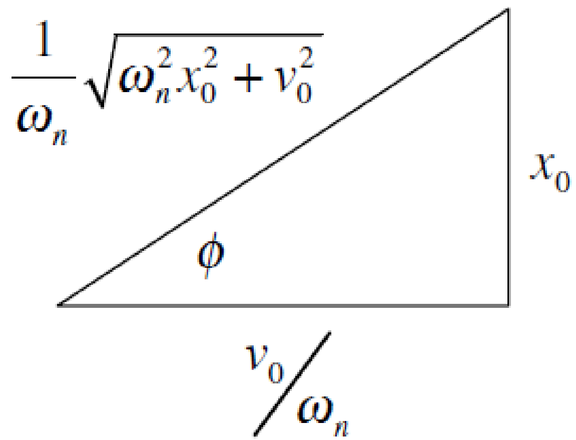
$$x_0 = x(0) = A\sin(\omega_n 0 + \phi) \;= A\sin(\phi)$$
$$v_0 = \dot{x}(0) = \omega_n A\cos(\omega_n 0 + \phi) \;= \omega_n A\cos(\phi)$$

Initial Conditions Determine the Constants of Integration

Solving these two simultaneous equations for *A* and ϕ gives:

$$\underbrace{A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}}_{\text{Amplitude}} \;,\quad \underbrace{\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)}_{\text{Phase}}$$





Thus the total solution for the spring mass system becomes:

$$x(t)=\frac{\sqrt{\omega_n^2x_0^2+v_0^2}}{\omega_n}\sin\left(\omega_n t+\tan^{-1}\frac{\omega_n x_0}{v_0}\right) \qquad \textbf{(1.10)}$$

Called the solution to a simple harmonic oscillator and describes oscillatory motion, or *simple harmonic motion*.

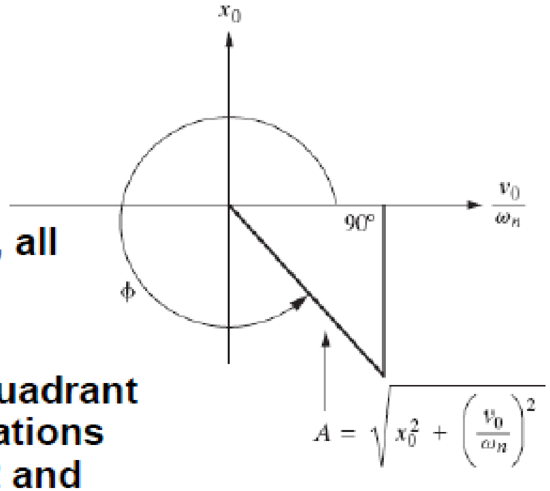
Note (Example 1.1.2)

$$x(0)=\frac{\sqrt{\omega_n^2x_0^2+v_0^2}}{\omega_n}\frac{\omega_n x_0}{\sqrt{\omega_n^2x_0^2+v_0^2}}=x_0$$

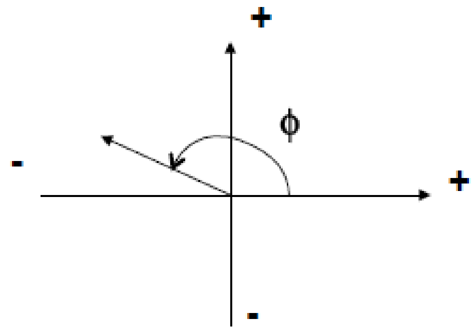
as it should

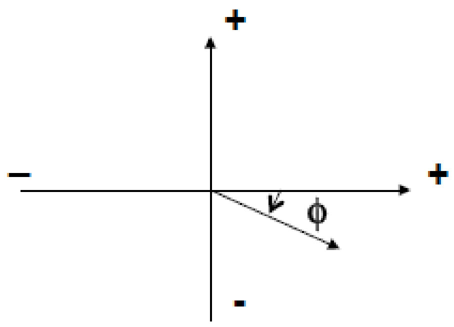
A note on arctangents

- Note that calculating arctangent from a calculator requires some attention. First, all machines work in *radians*.



- The argument atan(-/+) is in a different quadrant then atan(+/-), and usual machine calculations will return an arctangent in between -π/2 and +π/2, reading only the atan(-) for these two d*ifferent* cases.





In MATLAB, use the atan2(x,y) function to get the correct phase.

Example 1.1.3 wheel, tire suspension
m = 30 kg, f_n = 10 hz, what is *k*?

$$k = m\omega_n^2 = (30\text{ kg})\left(10\frac{\text{cylce}}{\text{sec}}\bullet\frac{2\pi\text{ rad}}{\text{cylce}}\right) = 1.184\times10^5\text{ N/m}$$

There are of course more complex models of suspension systems and these appear latter in the course as our tools develope

Section 1.2 Harmonic Motion

The period is the time elapsed to complete one complete cylce

$$T = \frac{2\pi \text{ rad}}{\omega_n \text{ rad/s}} = \frac{2\pi}{\omega_n} \text{s} \qquad (1.11)$$

The natural frequency in the commonly used units of hertz:

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi} \text{ Hz} \qquad (1.12)$$

For the pendulum:

$$\omega_n = \sqrt{\frac{g}{\ell}} \text{ rad/s}, \quad T = 2\pi\sqrt{\frac{\ell}{g}} \text{ s}$$

For the disk and shaft:

$$\omega_n = \sqrt{\frac{k}{J}} \text{ rad/s}, \quad T = 2\pi\sqrt{\frac{J}{k}} \text{ s}$$

Relationship between Displacement, Velocity and Acceleration

Displacement

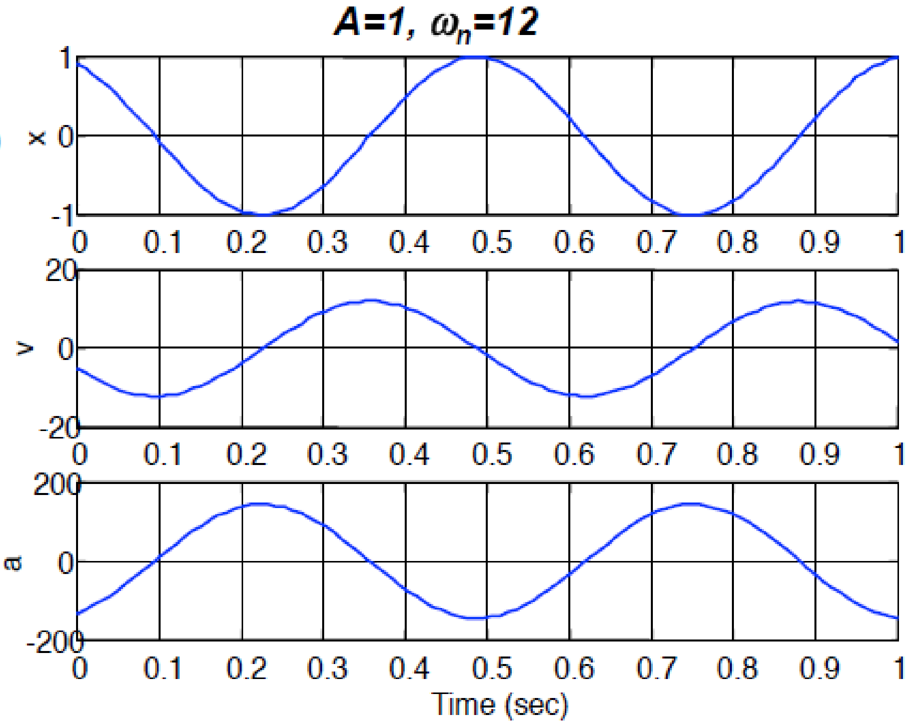
$$x(t) = A\sin(\omega_n t + \phi)$$

Velocity

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$$

Acceleration

$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi)$$



Note how the relative magnitude of each increases for $\omega_n > 1$

Example 1.2.1 Hardware store spring, bolt: m= 49.2x10^{-3} kg, k=857.8 N/m and x_0 =10 mm. Compute ω_n and the max amplitude of vibration.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8 \text{ N/m}}{49.2\times 10^{-3} \text{ kg}}} = 132 \text{ rad/s}$$

Note: common Units are Hertz

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz}$$

To avoid Costly errors use f_n when working in Hertz and ω_n when in rad/s

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = \frac{1}{21 \, {}^{cyles}\!/_{\text{sec}}} 0.0476 \text{ s}$$

$$x(t)_{\max} = A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + \cancelto{0}{v_0^2}} = x_0 = 10 \text{ mm}$$

Units depend on system

Compute the solution and max velocity and acceleration:

$$v(t)_{\max} = \omega_n A = 1320 \text{ mm/s} = 1.32 \text{ m/s}$$ 2.92 mph

$$a(t)_{\max} = \omega_n^2 A = 174.24 \times 10^3 \text{ mm/s}^2$$

$$= 174.24 \text{ m/s}^2 \approx 17.8g!$$

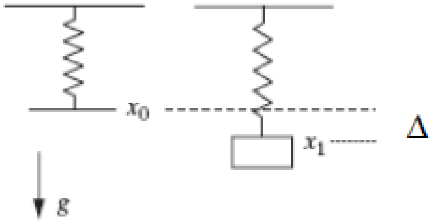
g = 9.8 m/s^2

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{0}\right) = \frac{\pi}{2} \text{ rad}$$ 90°

$$x(t) = 10\sin(132t + \pi/2) = 10\cos(132t) \text{ mm}$$

~0.4 in max

Does gravity matter in spring problems?



Let Δ be the deflection caused hanging a mass on a spring (Δ = x_1-x_0 in the figure)

Then from static equilibrium: $mg = k\Delta$

Next sum the forces in the vertical for some point $x > x_1$ measured from Δ

$$m\ddot{x} = -k\left(x+\Delta\right) + mg = -kx + \underbrace{mg - k\Delta}_{=0}$$

$$\Rightarrow m\ddot{x}(t) + kx(t) = 0$$

So no, gravity does not have an effect on the vibration

(note that this is not the case if the spring is nonlinear)

Example 1.2.2 Pendulums and measuring *g*

- A 2 m pendulum swings with a period of 2.893 s
- What is the acceleration due to gravity at that location?

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\ell}{g}}$$

$$g = \frac{4\pi^2}{T^2}\ell = \frac{4\pi^2}{2.893^2\,\mathrm{s}^2}2\ \mathrm{m}$$

$$\Rightarrow g = 9.796\ \mathrm{m/s^2}$$

This is *g* in Denver, CO USA, at 1638m and a latitude of 40°

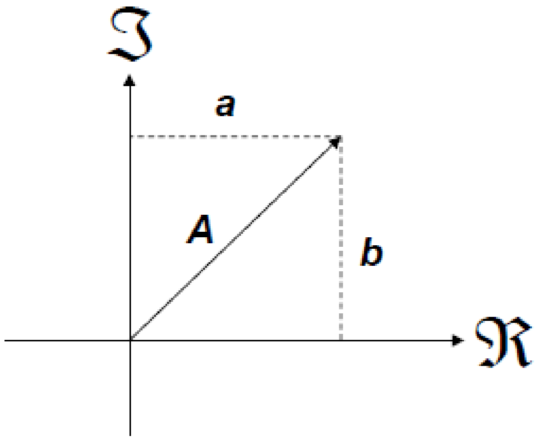
Review of Complex Numbers and Complex Exponential (See Appendix A)

A complex number can be written with a real and imaginary part or as a complex exponential

$$c = a + jb = Ae^{j\theta}$$

Where

$$a = A\cos\theta, b = A\sin\theta$$



Multiplying two complex numbers:

$$c_1 c_2 = A_1 A_2 e^{j(\theta_1 + \theta_2)}$$

Dividing two complex numbers:

$$\frac{c_1}{c_2} = \frac{A_1}{A_2} e^{j(\theta_1 - \theta_2)}$$

Equivalent Solutions to 2nd order Differential Equations (see Window 1.4)

All of the following solutions are equivalent:

$$x(t) = A\sin(\omega_n t + \phi)$$ Called the magnitude and phase form

$$x(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t$$ Sometimes called the Cartesian form

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$$ Called the polar form

The relationships between *A* and ϕ, *A_1 and A_2, and a_1 and a_2* can be found in Window 1.4 of the text, page 19..

- Each is useful in different situations
- Each represents the same information
- Each solves the equation of motion

Derivation of the solution

$$\text{Substitute} \quad x(t) = ae^{\lambda t} \quad \text{into} \quad m\ddot{x} + kx = 0 \Rightarrow$$

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + k = 0 \Rightarrow$$

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \Rightarrow$$

$$x(t) = a_1 e^{\omega_n jt} \quad \text{and} \quad x(t) = a_2 e^{-\omega_n jt} \Rightarrow$$

$$x(t) = a_1 e^{\omega_n jt} + a_2 e^{-\omega_n jt} \qquad \textbf{(1.18)}$$

This approach will be used again for more complicated problems

Is frequency always positive?

From the preceding analysis, $\lambda = \pm\,\omega_n$ then

$$x(t) = a_1 e^{\omega_n jt} + a_2 e^{-\omega_n jt}$$

Using the Euler relations* for trigonometric functions, the above solution can be written as (recall Window 1.4)

$$x(t) = A\sin\left(\omega_n t + \phi\right) \qquad (1.19)$$

It is in this form that we identify as the natural frequency ω_n and this is positive, because the $\pm$ sign being used up in the transformation from exponentials to the sine function.

* http://en.wikipedia.org/wiki/Euler's_formula

$$e^{ix} = \cos x + i\sin x$$

Calculating root mean square (*RMS*) values

May need to be limited due to physical constraints

$$A = \text{ peak value}$$

Not very useful since for a sine function the average value is zero

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt = \text{average value}$$

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt = \text{mean-square value} \qquad \textbf{(1.21)}$$

Proportional to energy

$$x_{rms} = \sqrt{\bar{x}^2} = \text{root mean square value}$$

Also useful when the vibration is random

The Decibel or dB scale

It is often useful to use a logarithmic scale to plot vibration levels (or noise levels). One such scale is called the *decibel* or dB scale. The dB scale is always relative to some reference value x_0. It is define as:

$$dB = 10\log_{10}\left(\frac{x}{x_0}\right)^2 = 20\log_{10}\left(\frac{x}{x_0}\right) \quad (1.22)$$

For example: if an acceleration value was 19.6m/s^2 then relative to 1g (or 9.8m/s^2) the level would be 6dB,

$$10\log_{10}\left(\frac{19.6}{9.8}\right)^2 = 20\log_{10}(2) = 6dB$$

Or for Example 1.2.1: The Acceleration Magnitude is 20log$_{10}$(17.8)=25*dB* relative to 1g.

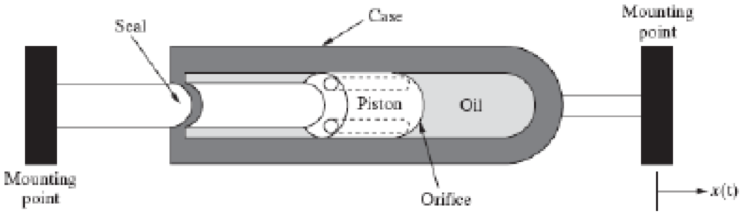
1.3 Viscous Damping

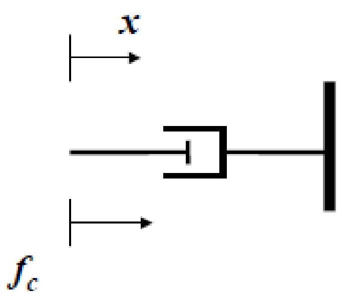
All real systems dissipate energy when they vibrate. To account for this we must consider damping. The most simple form of damping (from a mathematical point of view) is called viscous damping. A viscous damper (or dashpot) produces a force that is proportional to velocity.

Mostly a mathematically motivated form, allowing a solution to the resulting equations of motion that predicts reasonable (observed) amounts of energy dissipation.

Damper (c)

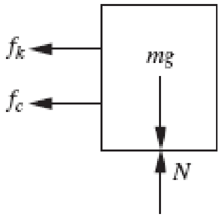
$$\boxed{f_c = -cv(t) = -c\dot{x}(t)}$$





Differential Equation Including Damping

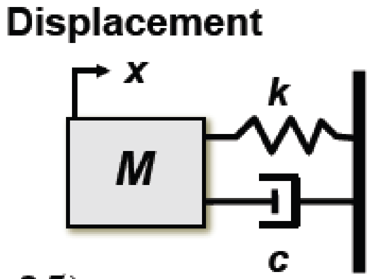
For this damped single degree of freedom system the force acting on the mass is due to the spring and the dashpot i.e. $f_m = -f_k - f_c$.



$$m\ddot{x}(t) = -kx(t) - c\dot{x}(t)$$

or

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad (1.25)$$



To solve this for of the equation it is useful to assume a solution of the form (again):

$$x(t) = ae^{\lambda t}$$

Solution to DE with damping included (dates to 1743 by Euler)

The velocity and acceleration can then be calculated as:

$$\dot{x}(t) = \lambda a e^{\lambda t}$$

$$\ddot{x}(t) = \lambda^2 a e^{\lambda t}$$

If this is substituted into the equation of motion we get:

$$a e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0 \qquad (1.26)$$

Divide equation by *m*, substitute for natural frequency and assume a non-trivial solution

$$a e^{\lambda t} \neq 0 \quad \Rightarrow \quad (\lambda^2 + \frac{c}{m}\lambda + \omega_n^2) = 0$$

Solution to our differential equation with damping included:

For convenience we will define a term known as the *damping ratio* as:

$$\zeta = \frac{c}{2\sqrt{km}} \qquad (1.30)$$

Lower case Greek zeta
NOT ξ as some like to use

The equation of motion then becomes:

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0$$

Solving for λ then gives,

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad (1.31)$$

Possibility 1. Critically damped motion

Critical damping occurs when ζ =1. The damping coefficient *c* in this case is given by:

$$\zeta = 1 \Rightarrow c = \underbrace{c_{cr} = 2\sqrt{km}}_{\text{definition of critical damping coefficient}} = 2m\omega_n$$

Solving for λ then gives,

$$\lambda_{1,2} = -1\omega_n \pm \omega_n\sqrt{1^2 - 1} = -\omega_n$$

A repeated, real root

The solution then takes the form

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t}$$

Needs two independent solutions, hence the *t* in the second term

Critically damped motion

***a_1* and *a_2* can be calculated from initial conditions *(t=0)*,**

$$x = (a_1 + a_2 t)e^{-\omega_n t}$$

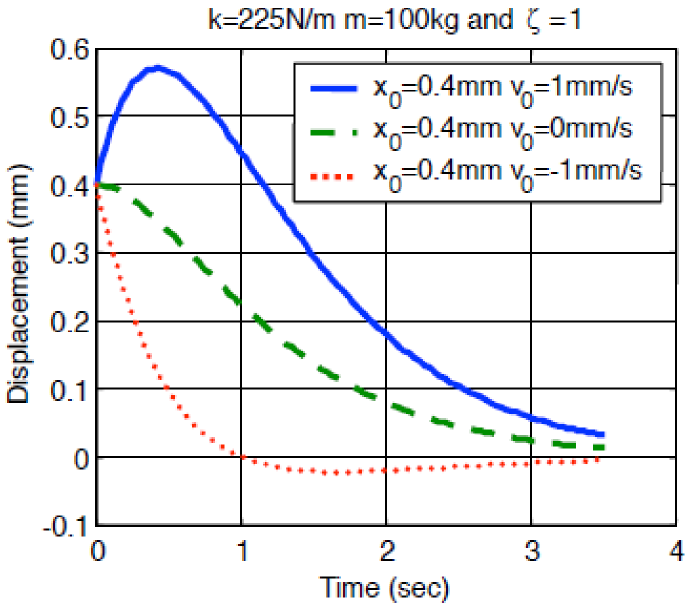
$$\Rightarrow a_1 = x_0$$

$$v = (-\omega_n a_1 - \omega_n a_2 t + a_2)e^{-\omega_n t}$$

$$v_0 = -\omega_n a_1 + a_2$$

$$\Rightarrow a_2 = v_0 + \omega_n x_0$$

- No oscillation occurs
- Useful in door mechanisms, analog gauges



Possibility 2: Overdamped motion

An overdamped case occurs when ζ >1. Both of the roots of the equation are again real.

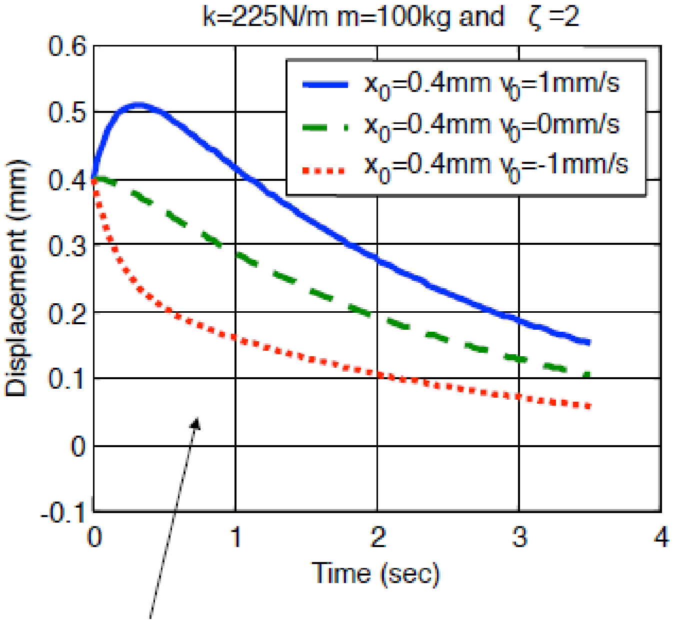
$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n t\sqrt{\zeta^2-1}} + a_2 e^{\omega_n t\sqrt{\zeta^2-1}}\right)$$

***a_1* and *a_2* can again be calculated from initial conditions *(t=0)*,**

$$a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$



Slower to respond than critically damped case

Possibility 3: Underdamped motion

An underdamped case occurs when ζ<1. The roots of the equation are complex conjugate pairs. This is the most common case and the only one that yields oscillation.

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$$

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{j\omega_n t\sqrt{1-\zeta^2}} + a_2 e^{-j\omega_n t\sqrt{1-\zeta^2}})$$

$$= Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

The frequency of oscillation ω_d is called the *damped natural frequency* is given by.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad (1.37)$$

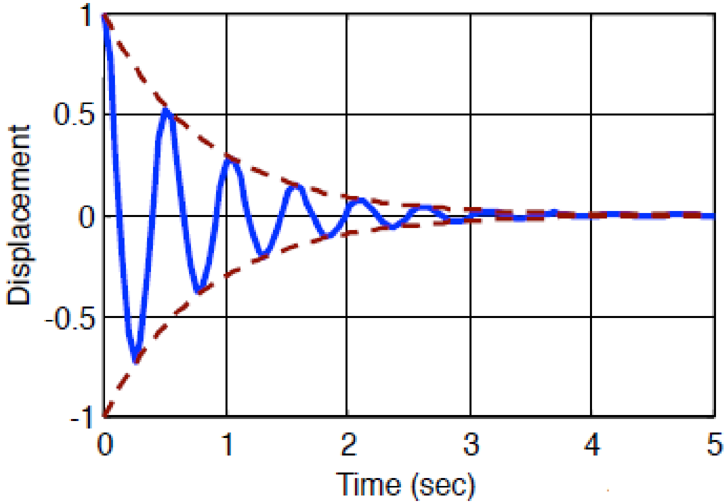
Constants of integration for the underdamped motion case

As before *A* and ϕ can be calculated from initial conditions (*t=0*),

$$A = \frac{1}{\omega_d}\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

- Gives an oscillating response with exponential decay
- Most natural systems vibrate with and underdamped response
- See Window 1.5 for details and other representations



Example 1.3.1: consider the spring of 1.2.1, if c = 0.11 kg/s, determine the damping ratio of the spring-bolt system.

$$m = 49.2 \times 10^{-3} \text{ kg}, \quad k = 857.8 \text{ N/m}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{49.2 \times 10^{-3} \times 857.8}$$

$$= 12.993 \text{ kg/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{0.11 \text{ kg/s}}{12.993 \text{ kg/s}} = 0.0085 \Rightarrow$$

the motion is *underdamped*

and the bolt will oscillate

Example 1.3.2

The human leg has a *measured* natural frequency of around 20 Hz when in its rigid (knee locked) position, in the longitudinal direction (i.e., along the length of the bone) with a damping ratio of $\zeta = 0.224$. Calculate the response of the tip if the leg bone to an initial velocity of $v_0 = 0.6$ m/s and zero initial displacement (this would correspond to the vibration induced while landing on your feet, with your knees locked form a height of 18 mm) and plot the response. What is the maximum acceleration experienced by the leg assuming no damping?

Solution:

$$\omega_n = \frac{20}{1}\frac{\text{cycles}}{s}\frac{2\pi\,\text{rad}}{\text{cycles}} = 125.66\ \text{rad/s}$$

$$\omega_d = 125.66\sqrt{1-(.224)^2} = 122.467\ \ \text{rad/s}$$

$$A = \frac{\sqrt{(0.6+(0.224)(125.66)(0))^2 + (0)(122.467)^2}}{122.467} = 0.005\ \text{m}$$

$$\phi = \tan^{-1}\left(\frac{(0)(\omega_d)}{v_0+\zeta\omega_n(0)}\right) = 0$$

$$\Rightarrow \underline{x(t) = 0.005e^{-28.148t}\sin(122.467t)}$$

Use the *undamped* formula to get maximum acceleration:

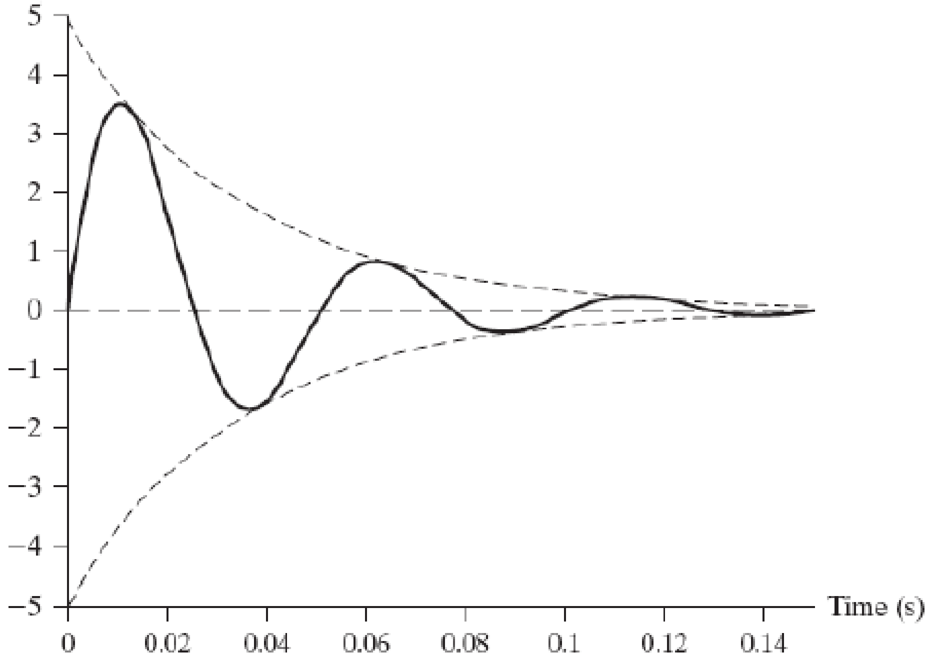
$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2},\ \omega_n = 125.66,\ v_0 = 0.6,\ x_0 = 0$$

$$A = \frac{v_0}{\omega_n}\text{m} = \frac{0.6}{\omega_n}\text{m}$$

$$\max(\ddot{x}) = \left|-\omega_n^2 A\right| = \left|-\omega_n^2\left(\frac{0.6}{\omega_n}\right)\right| = (0.6)\left(125.66\ \text{m/s}^2\right) = \underline{75.396\ \text{m/s}^2}$$

$$\text{maximum acceleration} = \frac{75.396\ \text{m/s}^2}{9.81\ \text{m/s}^2}\text{g} = 7.68\,\text{g's}$$

Here is a plot of the displacement response versus time



Example 1.3.3 **Compute the form of the response of an underdamped system using the Cartesian form of the solution given in Window 1.5.**

$$\sin(x+y) = \sin x \sin y + \cos x \cos y \Rightarrow$$

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t)$$

$$x(0) = x_0 = e^0(A_1 \sin(0) + A_2 \cos(0)) \Rightarrow \underline{A_2 = x_0}$$

$$\begin{aligned}\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t) \\ + \omega_d e^{-\zeta\omega_n t}(A_1 \cos\omega_d t - A_2 \sin\omega_d t)\end{aligned}$$

$$v_0 = -\zeta\omega_n(A_1 \sin 0 + x_0 \cos 0) + \omega_d(A_1 \cos 0 - x_0 \sin 0)$$

$$\Rightarrow \quad \underline{A_1 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}} \Rightarrow$$

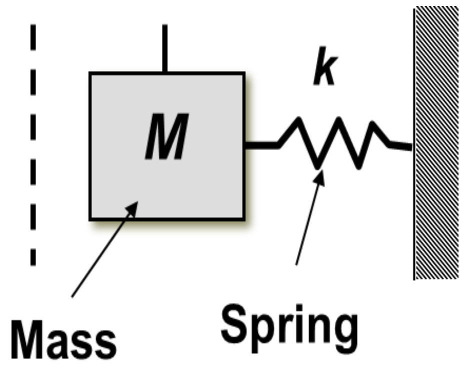
$$x(t) = e^{-\zeta\omega_n t}\left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t + x_0 \cos\omega_d t\right)$$

Section 1.4 Modeling and Energy Methods

- **Provides an alternative way to determine the equation of motion, and an alternative way to calculate the natural frequency of a system**
- **Useful if the forces or torques acting on the object or mechanical part are difficult to determine**
- **Very useful for more complicated systems later (MDOF and distributed mass systems)**

Potential and Kinetic Energy

The potential energy of mechanical systems *U* is often stored in "springs" (remember that for a spring *F* = *kx*)



$$U_{spring} = \int_0^{x_0} F\,dx = \int_0^{x_0} kx\,dx = \frac{1}{2}kx_0^2$$

The kinetic energy of mechanical systems *T* is due to the motion of the "mass" in the system

$$T_{trans} = \frac{1}{2}m\dot{x}^2, \quad T_{rot} = \frac{1}{2}J\dot{\theta}^2$$

Conservation of Energy

For a simple, conservative (i.e. no damper), mass spring system the energy must be conserved:

$$T + U = \text{constant}$$

or $$\frac{d}{dt}(T + U) = 0$$

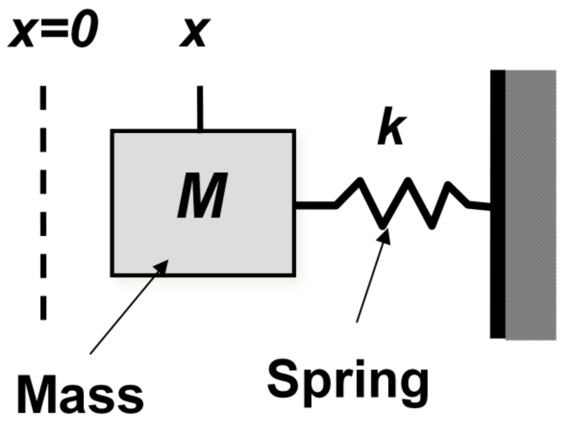
At two different times t_1 and t_2 the increase in potential energy must be equal to a decrease in kinetic energy (or visa-versa).

$$U_1 - U_2 = T_2 - T_1$$

and

$$U_{\max} = T_{\max}$$

Deriving the equation of motion from the energy approach



$$\frac{d}{dt}(T+U)=\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2\right)=0$$

$$\Rightarrow \dot{x}\left(m\ddot{x}+kx\right)=0$$

Since $\dot{x}$ cannot be zero for all time, then

$$m\ddot{x}+kx=0$$

Determining the Natural frequency directly from the energy

If the solution is given by $x(t) = A\sin(\omega t+\phi)$ then the maximum potential and kinetic energies can be used to calculate the natural frequency of the system

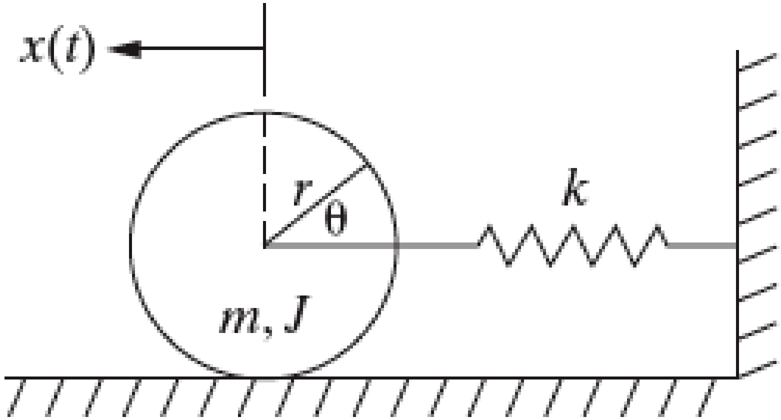
$$U_{\max} = \frac{1}{2}kA^2 \quad T_{\max} = \frac{1}{2}m(\omega_n A)^2$$

Since these two values must be equal

$$\frac{1}{2}kA^2 = \frac{1}{2}m(\omega_n A)^2$$

$$\Rightarrow k = m\omega_n^2 \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

Example 1.4.1



Compute the natural frequency of this roller fixed in place by a spring. Assume it is a conservative system (i.e. no losses) and rolls without slipping.

$$T_{\text{rot}} = \frac{1}{2} J\dot{\theta}^2 \quad \text{and} \quad T_{\text{trans}} = \frac{1}{2} m\dot{x}^2$$

Solution continued

$$x = r\theta \Rightarrow \dot{x} = r\dot{\theta} \Rightarrow T_{\text{Rot}} = \frac{1}{2} J \frac{\dot{x}^2}{r^2}$$

The max value of T happens at $v_{\max} = \omega_n A$

$$\Rightarrow T_{\max} = \frac{1}{2} J \frac{(\omega_n A)^2}{r^2} + \frac{1}{2} m(\omega_n A)^2 = \frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 A^2$$

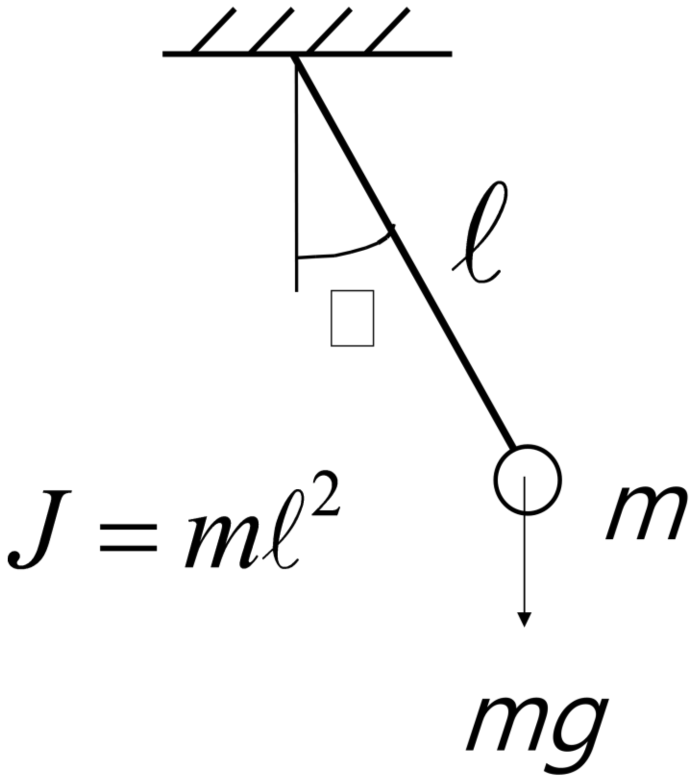
The max value of U happens at $x_{\max} = A$

$$\Rightarrow U_{\max} = \frac{1}{2} kA^2 \quad \text{Thus } T_{\max} = U_{\max} \Rightarrow$$

$$\frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 A^2 = \frac{1}{2} kA^2 \Rightarrow \omega_n = \sqrt{\frac{k}{\left(m + \frac{J}{r^2}\right)}}$$

Effective mass

Example 1.4.2 Determine the equation of motion of the pendulum using energy



Now write down the energy

$$T = \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m\ell^2 \dot{\theta}^2$$

$U = mg\ell(1-\cos\theta)$, the change in elevation is $\ell(1-\cos\theta)$

$$\frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(1-\cos\theta)\right) = 0$$

$$m\ell^2\dot{\theta}\ddot{\theta} + mg\ell(\sin\theta)\dot{\theta} = 0$$

$$\Rightarrow \dot{\theta}\left(m\ell^2\ddot{\theta} + mg\ell(\sin\theta)\right) = 0$$

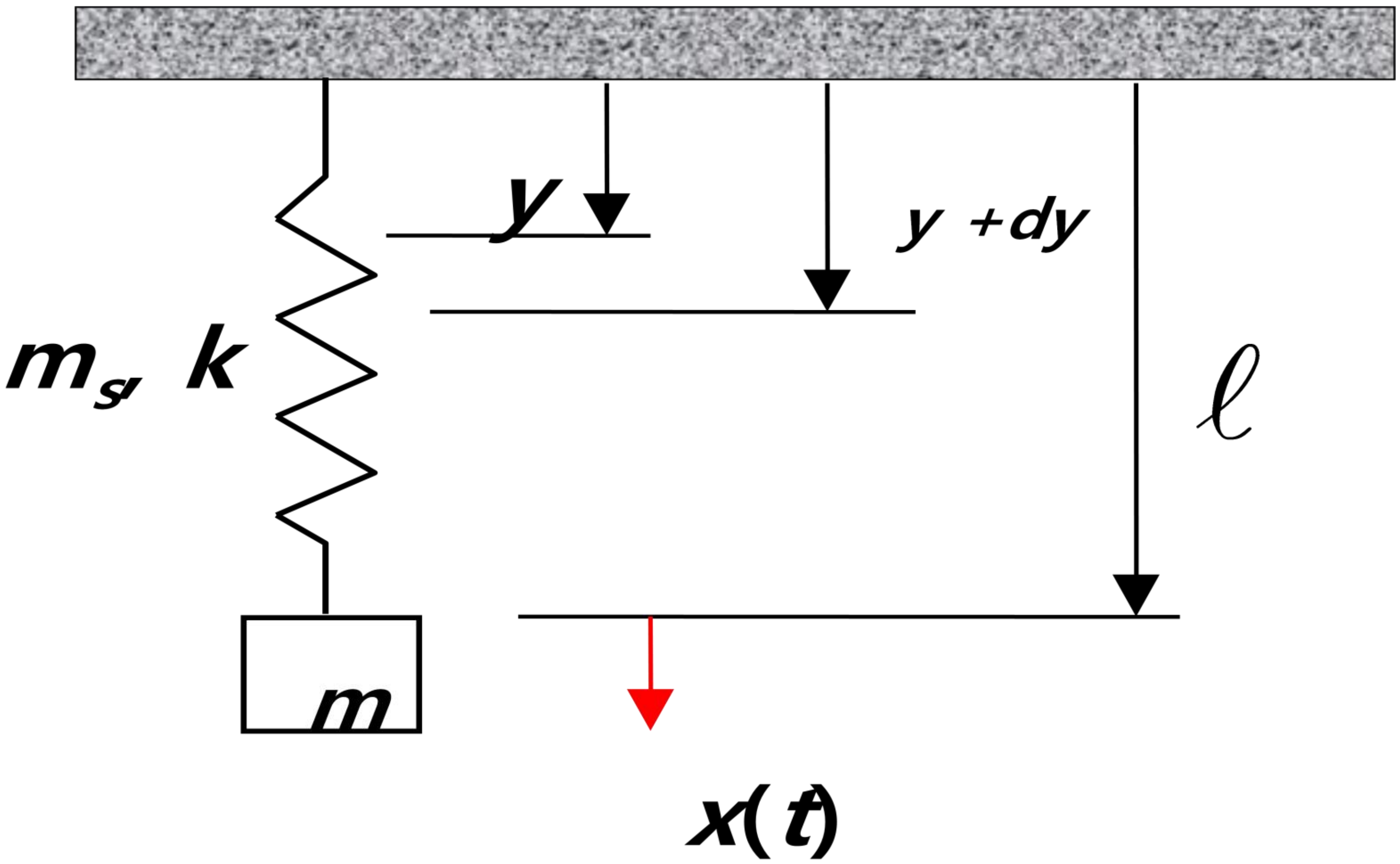
$$\Rightarrow m\ell^2\ddot{\theta} + mg\ell(\sin\theta) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{g}{\ell}\sin\theta(t) = 0$$

Using the small angle approximation for sine:

$$\Rightarrow \ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0 \qquad \Rightarrow \omega_n = \sqrt{\frac{g}{\ell}}$$

Example 1.4.4 The effect of including the mass of the spring on the value of the frequency.



$$\left.\begin{array}{l} \text{mass of element } dy: \dfrac{m_s}{\ell}\, dy \\[2ex] \text{velocity of element } dy\text{: } v_{dy} = \dfrac{y}{\ell}\,\dot{x}(t), \end{array}\right\} \text{ assumptions}$$

$$T_{spring} = \frac{1}{2}\int_0^{\ell} \frac{m_s}{\ell}\left[\frac{y}{\ell}\,\dot{x}\right]^2 dy \quad \text{(adds up the KE of each element)}$$

$$= \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$$

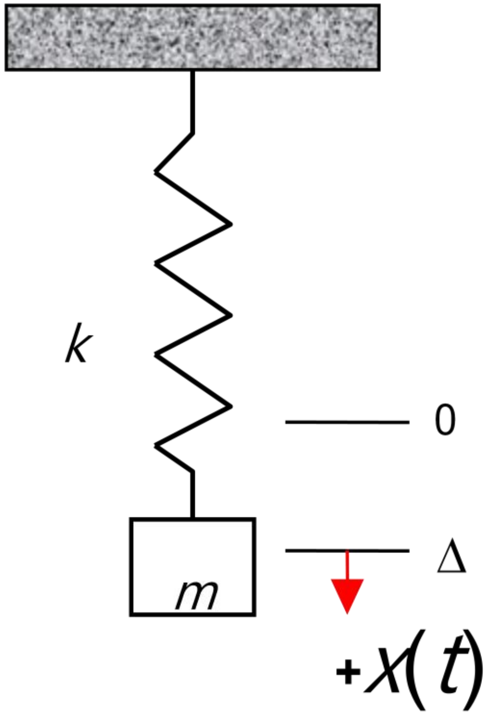
$$T_{mass} = \frac{1}{2} m\dot{x}^2 \Rightarrow T_{tot} = \left[\frac{1}{2}\left(\frac{m_s}{3}\right) + \frac{1}{2} m\right]\dot{x}^2 \Rightarrow T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\omega_n^2 A^2$$

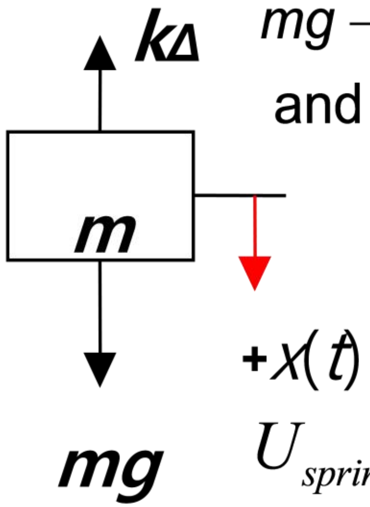
$$U_{\max} = \frac{1}{2} kA^2$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m + \frac{m_s}{3}}}}$$

- **This provides some simple design and modeling guides**

What about gravity?





$mg - k\Delta = 0$, from FBD, and static equilibrium

$$U_{spring} = \frac{1}{2}k(\Delta + x)^2$$

$$U_{grav} = -mgx$$

$$T = \frac{1}{2}m\dot{x}^2$$

Now use $\dfrac{d}{dt}(T+U)=0$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 - mgx + \frac{1}{2}k(\Delta + x)^2\right] = 0$$

$$\Rightarrow m\dot{x}\ddot{x} - mg\dot{x} + k(\Delta + x)\dot{x}$$

$$\Rightarrow \dot{x}(m\ddot{x} + kx) + \dot{x}(\underbrace{k\Delta - mg}_{\substack{0 \text{ from static} \\ \text{equilibiurm}}}) = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

- **Gravity does not effect the equation of motion or the natural frequency of the system for a linear system as shown previously with a force balance.**

Lagrange's Method for deriving equations of motion.

Again consider a conservative system and its energy.

It can be shown that if the Lagrangian *L* is defined as

$$L = T - U$$

Then the equations of motion can be calculated from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad (1.62)$$

Which becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = 0 \qquad (1.63)$$

Here *q* is a generalized coordinate

Example 1.4.7 Derive the equation of motion of a spring mass system via the Lagrangian

$$T = \frac{1}{2} m\dot{x}^2 \quad \text{and} \quad U = \frac{1}{2} kx^2$$

Here *q* = *x*, and and the Lagrangian becomes

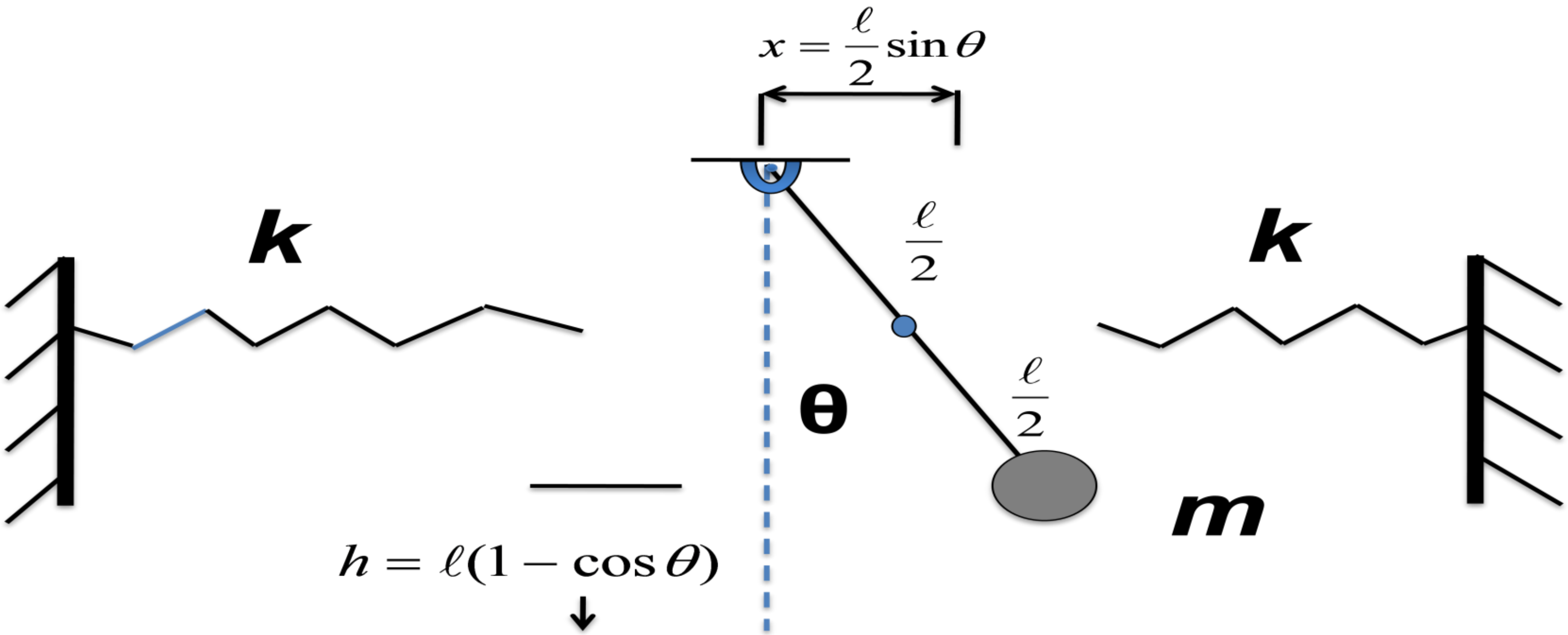
$$L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

Equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = \frac{d}{dt}(m\dot{x}) - 0 + kx = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

Example



$$U = \frac{1}{2}kx^2 + \frac{1}{2}kx^2 + mg\ell(1-\cos\theta)$$

$$= \frac{k\ell^2}{4}\sin^2\theta + mg\ell(1-\cos\theta)$$

The Kinetic energy term is : $T = \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m\ell^2 \dot{\theta}^2$

Compute the terms in Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m\ell^2 \dot{\theta}\right) = m\ell^2 \ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{k\ell^2}{4}\sin^2\theta + mg\ell(1-\cos\theta)\right) = \frac{k\ell^2}{2}\sin\theta\cos\theta + mg\ell\sin\theta$$

Lagrange's equation (1.64) yields:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = m\ell^2\ddot{\theta} + \frac{k\ell^2}{2}\sin\theta\cos\theta + mg\ell\sin\theta = 0$$

Does it make sense:

$$m\ell^2\ddot{\theta} + \underbrace{\frac{k\ell^2}{2}\sin\theta\cos\theta}_{0 \text{ if } k=0} + mg\ell\sin\theta = 0$$

Linearize to get small angle **case:**

$$m\ell^2\ddot{\theta} + \frac{k\ell^2}{2}\theta + mg\ell\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left(\frac{k\ell + 2mg}{2m\ell}\right)\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k\ell + 2mg}{2m\ell}}$$

What happens if you linearize first?