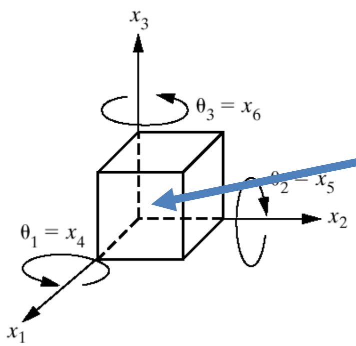
Chapter 4 Multiple Degree of Freedom Systems

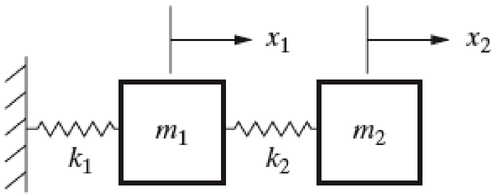
The Millennium bridge required many degrees of freedom to model and design with.

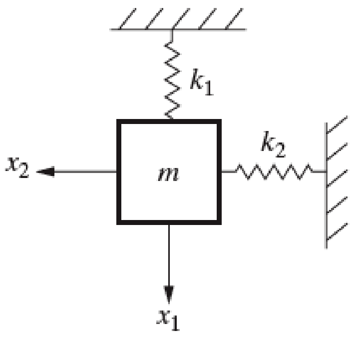


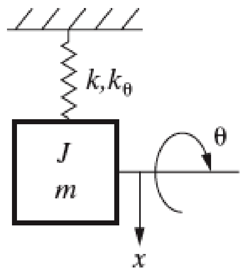
Extending the first 3 chapters to more then one degree of freedom

The first step in analyzing multiple degrees of freedom (DOF) is to look at 2 DOF

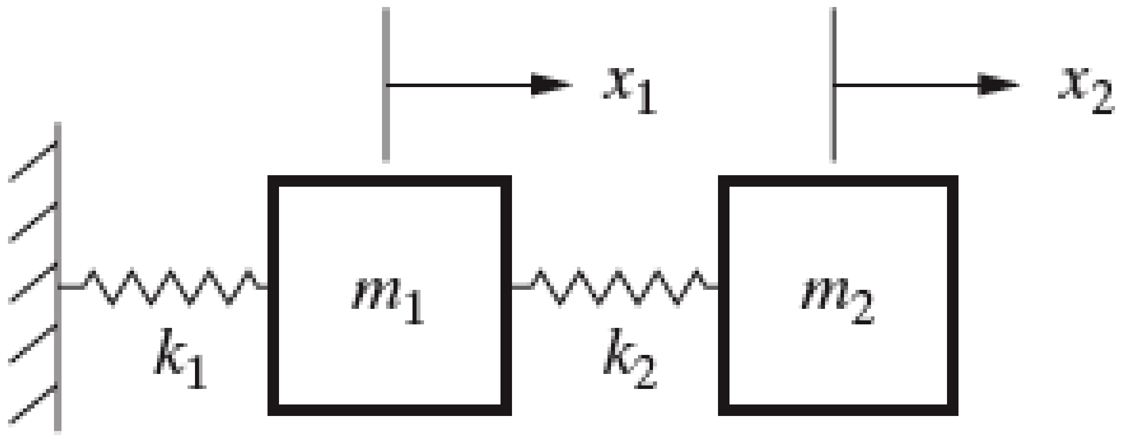
- DOF: Minimum number of coordinates to specify the position of a system
- Many systems have more than 1 DOF
- Examples of 2 DOF systems
  - Car with sprung and unsprung mass (both heave)
  - Elastic pendulum (radial and angular)
  - Motions of a ship (roll and pitch)
  - Airplane roll, pitch and yaw







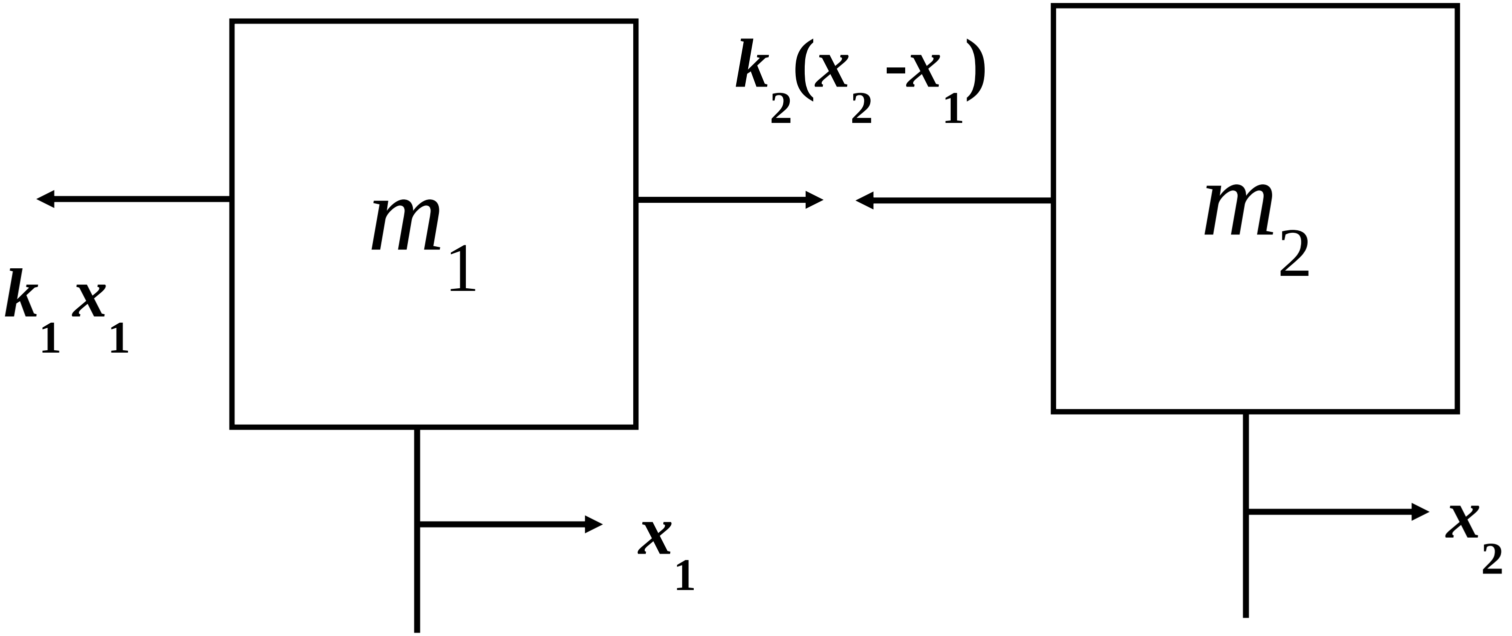
4.1 Two-Degree-of-Freedom Model (Undamped)



A 2 degree of freedom system used to base much of the analysis and conceptual development of MDOF systems on.

Free-Body Diagram of each mass

Figure 4.2



Summing forces yields the equations of motion:

$$
\begin{aligned}
m_1\ddot{x}_1(t) &= -k_1x_1(t) + k_2\left(x_2(t) - x_1(t)\right) \\
m_2\ddot{x}_2(t) &= -k_2\left(x_2(t) - x_1(t)\right)
\end{aligned}
\qquad (4.1)
$$

Rearranging terms:

$$
\begin{aligned}
m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) &= 0 \\
m_2\ddot{x}_2(t) - k_2x_1(t) + k_2x_2(t) &= 0
\end{aligned}
\qquad (4.2)
$$

Note that it is always the case that

- A 2 Degree-of-Freedom system has
  - Two equations of motion!
  - Two natural frequencies (as we shall see)!
- Thus some new phenomena arise in going from one to two degrees of freedom
  - Look for these as you proceed through the material
    - Two instead of one natural frequency
      - Leading to two possible resonance conditions
    - The concept of a mode shape arises

The dynamics of a 2 DOF system consists of 2 homogeneous and coupled equations

- **Free vibrations, so homogeneous eqs.**
- **Equations are coupled:**
  - **Both have x_1 and x_2.**
  - **If only one mass moves, the other follows**
  - **Example: pitch and heave of a car model**
- **In this case the coupling is due to k_2.**
  - **Mathematically and Physically**
  - **If $k_2 = 0$, no coupling occurs and can be solved as two independent SDOF systems**

Initial Conditions

- Two coupled, second -order, ordinary differential equations with constant coefficients
- Needs 4 constants of integration to solve
- Thus 4 initial conditions on positions and velocities

$$x_1(0) = x_{10},\ \dot{x}_1(0) = \dot{x}_{10},\ x_2(0) = x_{20},\ \dot{x}_2(0) = \dot{x}_{20}$$

Solution by Matrix Methods

The two equations can be written in the form of a single matrix equation (see pages 272-275 if matrices and vectors are a struggle for you) :

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \quad \ddot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} \qquad (4.4), (4.5)$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \qquad (4.6), (4.9)$$

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{aligned} m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2 x_2(t) &= 0 \\ m_2\ddot{x}_2(t) - k_2 x_1(t) + k_2 x_2(t) &= 0 \end{aligned}$$

Initial Conditions (two sets needed one for each equation of motion)

IC's can also be written in vector form

$$\mathbf{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{bmatrix}$$

The approach to a Solution:

For 1DOF we assumed the scalar solution $ae^{\lambda t}$ Similarly, now we assume the *vector* form:

$$\text{Let } \mathbf{x}(t) = \mathbf{u}e^{j\omega t} \tag{4.15}$$

$$j = \sqrt{-1}, \quad \mathbf{u} \neq \mathbf{0}, \quad \omega, \mathbf{u} \text{ unknown}$$

$$\Rightarrow \left(-\omega^2 M + K\right)\mathbf{u}e^{j\omega t} = \mathbf{0} \tag{4.16}$$

$$\Rightarrow \left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0} \tag{4.17}$$

This changes the differential equation of motion into algebraic vector equation:

$$\left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0} \qquad (4.17)$$

This is two algebraic equation in 3 uknowns

(1 vector of two elements and 1 scalar):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \text{ and } \omega$$

The condition for solution of this matrix equation requires that the the matrix inverse does not exist:

If the $\text{inv}\left(-\omega^2 M + K\right)$ exists $\Rightarrow \mathbf{u} = \mathbf{0}$: which is the static equilibrium position. For motion to occur

$$\mathbf{u} \neq \mathbf{0} \quad \Rightarrow \left(-\omega^2 M + K\right)^{-1} \text{ does not exist}$$

$$\text{or} \quad \det\left(-\omega^2 M + K\right) = \mathbf{0} \qquad (4.19)$$

The determinant results in 1 equation in one unknown ω

(called the *characteristic* equation)

Back to our specific system: the characteristic equation is defined as

$$\det\left(-\omega^2 M + K\right) = 0 \Rightarrow$$

$$\det\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} = 0 \Rightarrow$$

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0$$

Eq. (4.21) is quadratic in ω^2 so four solutions result:

$$\omega_1^2 \ \text{ and } \ \omega_2^2 \Rightarrow \pm\omega_1 \ \text{ and } \ \pm\omega_2$$

Once ω is known, use equation (4.17) again to calculate the corresponding vectors $\mathbf{u}_1$ and $\mathbf{u}_2$

This yields vector equation for each squared frequency:

$$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0} \qquad (4.22)$$

and

$$(-\omega_2^2 M + K)\mathbf{u}_2 = \mathbf{0} \qquad (4.23)$$

Each of these matrix equations represents 2 equations in the 2 unknowns components of the vector, but the coefficient matrix is singular so each matrix equation results in only 1 independent equation. The following examples clarify this.

Examples 4.1.5 & 4.1.6: calculating u and ω

- m_1=9 kg, m_2=1kg, k_1=24 N/m and k_2=3 N/m
- The characteristic equation becomes

$$\omega^4-6\omega^2+8=(\omega^2-2)(\omega^2-4)=0$$

$$\omega^2 = 2 \text{ and } \omega^2 = 4 \text{ or}$$

$$\omega_{1,3} = \pm\sqrt{2} \text{ rad/s}, \quad \omega_{2,4} = \pm 2 \text{ rad/s}$$

Each value of ω^2 yields an expression for **u:**

Computing the vectors **u**

For ω_1^2 =2, denote $\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$ then we have

$$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 27-9(2) & -3 \\ -3 & 3-(2) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$9u_{11} - 3u_{12} = 0 \text{ and } -3u_{11} + u_{12} = 0$$

2 equations, 2 unknowns but DEPENDENT!

(the 2nd equation is -3 times the first)

Only the direction of vectors u can be determined, not the magnitude as it remains arbitrary

$\frac{u_{11}}{u_{12}} = \frac{1}{3} \Rightarrow u_{11} = \frac{1}{3} u_{12}$ results from both equations:

only the direction, not the magnitude can be determined!

This is because: $\det(-\omega_1^2 M + K) = 0.$

The magnitude of the vector is arbitrary. To see this suppose that $\mathbf{u}_1$ satisfies

$(-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$, so does $a\mathbf{u}_1$, a arbitrary. So

$$(-\omega_1^2 M + K)a\mathbf{u}_1 = \mathbf{0} \Leftrightarrow (-\omega_1^2 M + K)\mathbf{u}_1 = \mathbf{0}$$

Likewise for the second value of ω^2

For $\omega_2^2 = 4$, let $\mathbf{u}_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$ then we have

$$(-\omega_1^2 M + K)\mathbf{u} = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 27 - 9(4) & -3 \\ -3 & 3 - (4) \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$-9u_{21} - 3u_{22} = 0 \quad \text{or} \quad u_{21} = -\frac{1}{3}u_{22}$$

Note that the other equation is the same

What to do about the magnitude!

Several possibilities, here we just fix one element:

Choose:

$$u_{12} = 1 \quad \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

Choose:

$$u_{22} = 1 \quad \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Thus the solution to the algebraic matrix equation is:

$$\omega_{1,3} = \pm\sqrt{2}, \text{ has mode shape } \mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\omega_{2,4} = \pm\ 2, \text{ has mode shape } \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Here we have introduce the name

mode shape **to describe the vectors**

$\mathbf{u}_1$ and $\mathbf{u}_2$. The origin of this name comes later

Return now to the time response:

We have computed four solutions:

$$\mathbf{x}(t) = \mathbf{u}_1 e^{-j\omega_1 t},\ \mathbf{u}_1 e^{j\omega_1 t},\ \mathbf{u}_2 e^{-j\omega_2 t}, \mathbf{u}_2 e^{j\omega_2 t} \Rightarrow \qquad (4.24)$$

Since linear, we can combine as:

$$\mathbf{x}(t) = a\mathbf{u}_1 e^{-j\omega_1 t} + b\mathbf{u}_1 e^{j\omega_1 t} + c\mathbf{u}_2 e^{-j\omega_2 t} + d\mathbf{u}_2 e^{j\omega_2 t}$$

$$\Rightarrow \mathbf{x}(t) = \left(ae^{-j\omega_1 t} + be^{j\omega_1 t}\right)\mathbf{u}_1 + \left(ce^{-j\omega_2 t} + de^{j\omega_2 t}\right)\mathbf{u}_2$$

$$= \underline{A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2} \qquad (4.26)$$

where $A_1, A_2, \phi_1,$ and ϕ_2 are constants of integration

determined by initial conditions.

Physical interpretation of all that math!

- Each of the TWO masses is oscillating at TWO *natural frequencies* ω_1 and ω_2
- The relative magnitude of each sine term, and hence of the magnitude of oscillation of m_1 and m_2 is determined by the value of $A_1 u_1$ and $A_2 u_2$
- The vectors u_1 and u_2 are called *mode shapes* because the describe the relative magnitude of oscillation between the two masses

What is a mode shape?

- **First note that A_1, A_2, Φ_1 and Φ_2 are determined by the initial conditions**

- **Choose them so that $A_2 = \Phi_1 = \Phi_2 = 0$**

- **Then:**

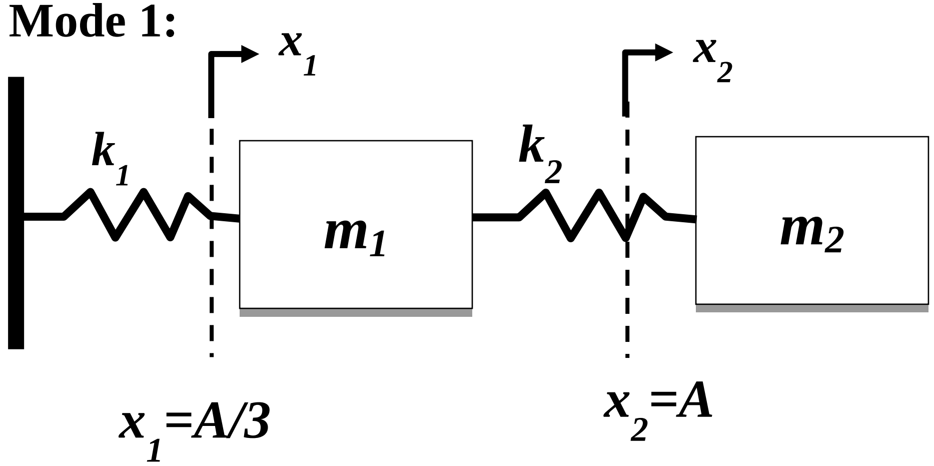
$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \sin \omega_1 t = A_1 \mathbf{u}_1 \sin \omega_1 t$$

- **Thus each mass oscillates at (one) frequency w_1 with magnitudes proportional to u_1 the 1st *mode shape***

A graphic look at mode shapes:

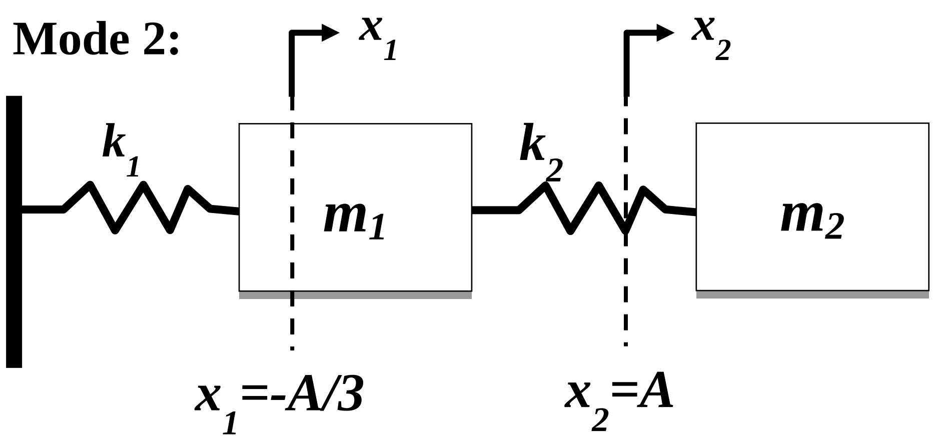
If IC's correspond to mode 1 or 2, then the response is purely in mode 1 or mode 2.

Mode 1:



$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Mode 2:



$$\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Example 4.1.7 **given the initial conditions compute the time response**

$$\text{consider } \mathbf{x}(0)=\begin{bmatrix}1\\0\end{bmatrix}\text{ mm, } \dot{\mathbf{x}}(0)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}\frac{A_1}{3}\sin\left(\sqrt{2}t+\varphi_1\right)-\frac{A_2}{3}\sin\left(2t+\varphi_2\right)\\A_1\sin\left(\sqrt{2}t+\varphi_1\right)+A_2\sin\left(2t+\varphi_2\right)\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}\frac{A_1}{3}\sqrt{2}\cos\left(\sqrt{2}t+\varphi_1\right)-\frac{A_2}{3}2\cos\left(2t+\varphi_2\right)\\A_1\sqrt{2}\cos\left(\sqrt{2}t+\varphi_1\right)+A_2 2\cos\left(2t+\varphi_2\right)\end{bmatrix}$$

At *t* = 0 we have

$$\begin{bmatrix} 1\,\text{mm} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sin(\phi_1) - \frac{A_2}{3}\sin(\phi_2) \\ A_1\sin(\phi_1) + A_2\sin(\phi_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{A_1}{3}\sqrt{2}\cos(\phi_1) - 2\frac{A_2}{3}\cos(\phi_2) \\ A_1\sqrt{2}\cos(\phi_1) + 2A_2\cos(\phi_2) \end{bmatrix}$$

4 equations in 4 unknowns:

$$3 = A_1 \sin(\phi_1) - A_2 \sin(\phi_2)$$

$$0 = A_1 \sin(\phi_1) + A_2 \sin(\phi_2)$$

$$0 = A_1 \sqrt{2} \cos(\phi_1) - A_2 2 \cos(\phi_2)$$

$$0 = A_1 \sqrt{2} \cos(\phi_1) + A_2 2 \cos(\phi_2)$$

Yields:

$$A_1 = 1.5 \text{ mm}, A_2 = -1.5 \text{ mm}, \phi_1 = \phi_2 = \frac{\pi}{2} \text{ rad}$$

The final solution is:

$$x_1(t) = 0.5\cos\sqrt{2}t + 0.5\cos 2t \text{ mm}$$
$$x_2(t) = 1.5\cos\sqrt{2}t - 1.5\cos 2t \text{ mm} \qquad (4.34)$$

These initial conditions gives a response that is a combination of modes. Both harmonic, but their summation is not.

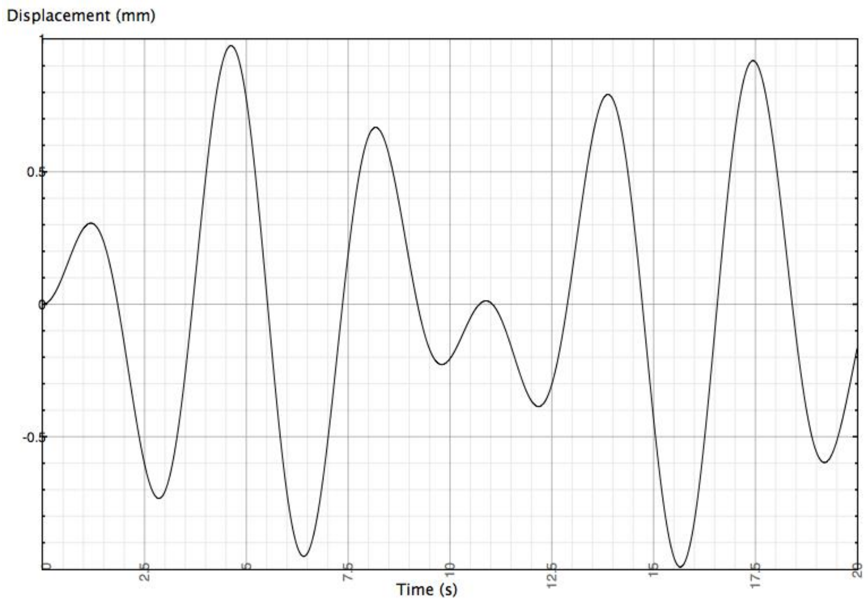


Figure 4.3a $x_1(t)$

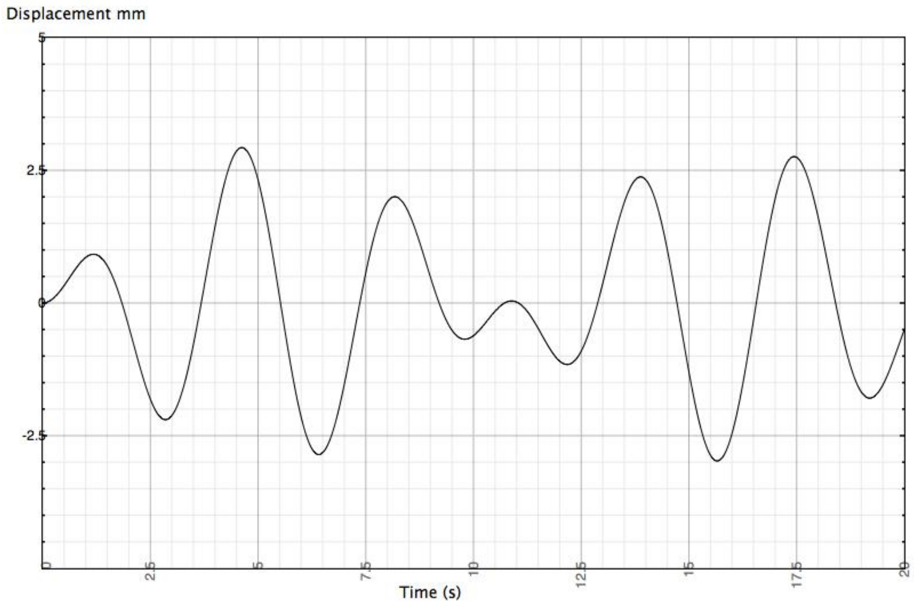


Figure 4.3b $x_2(t)$

Solution as a sum of modes

$$\mathbf{x}(t) = a_1\mathbf{u}_1 \cos\omega_1 t + a_2\mathbf{u}_2 \cos\omega_2 t$$

Determines how the first frequency contributes to the response

Determines how the second frequency contributes to the response

Things to note

- **Two degrees of freedom implies two natural frequencies**
- **Each mass oscillates at with these two frequencies present in the response and beats could result**
- **Frequencies are not those of two component systems**

$$\omega_1 = \sqrt{2} \neq \sqrt{\frac{k_1}{m_1}} = 1.63, \omega_2 = 2 \neq \sqrt{\frac{k_2}{m_2}} = 1.732$$

- **The above is not the most efficient way to calculate frequencies as the following describes**

Some matrix and vector reminders

$$A = \begin{bmatrix} a & b \\ c & c \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathbf{x}^T \mathbf{x} = x_1^2 + x_2^2$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \Rightarrow \mathbf{x}^T M \mathbf{x} = m_1 x_1^2 + m_2 x_2^2$$

$$M > 0 \Rightarrow \mathbf{x}^T M \mathbf{x} > 0 \quad \text{for every value of } \mathbf{x} \text{ except } 0$$

Then M is said to be *positive definite*

4.2 Eigenvalues and Natural Frequencies

- **Can connect the vibration problem with the algebraic eigenvalue problem developed in math**
- **This will give us some powerful computational skills**
- **And some powerful theory**
- **All the codes have eigen-solvers so these painful calculations can be automated**

Some matrix results to help us use available computational tools:

A matrix *M* is defined to be *symmetric* if

$$M = M^T$$

A symmetric matrix *M* is *positive definite* if

$$\mathbf{x}^T M\mathbf{x} > 0 \quad \text{for all nonzero vectors } \mathbf{x}$$

A symmetric positive definite matrix *M* can be factored

$$M = LL^T$$

Here L is upper triangular, called a Cholesky matrix

If the matrix *L* is diagonal, it defines the *matrix square root*

The matrix square root is the matrix $M^{1/2}$ such that

$$M^{1/2}M^{1/2} = M$$

If M is diagonal, then the matrix square root is just the root of the diagonal elements:

$$L = M^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix} \qquad (4.35)$$

A change of coordinates is introduced to capitalize on existing mathematics

For a diagonal, positive definite matrix M:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}, \quad M^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{bmatrix}$$

Let $\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$ and multiply by $M^{-1/2}$:

$$\underbrace{M^{-1/2}MM^{-1/2}}_{I \text{ identity}}\ddot{\mathbf{q}}(t) + \underbrace{M^{-1/2}KM^{-1/2}}_{\tilde{K} \text{ symmetric}}\mathbf{q}(t) = \mathbf{0} \qquad (4.38)$$

or $\quad \ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0}$ where $\tilde{K} = M^{-1/2}KM^{-1/2}$

$\tilde{K}$ is called the mass normalized stiffness and is similar to the scalar $\dfrac{k}{m}$ used extensively in single degree of freedom analysis. The key here is that $\tilde{K}$ is a SYMMETRIC matrix allowing the use of many nice properties and computational tools

How the vibration problem relates to the real symmetric eigenvalue problem

$$\text{Assume } \mathbf{q}(t) = \mathbf{v}e^{j\omega t} \text{ in } \ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0}$$

$$-\omega^2 \mathbf{v}e^{j\omega t} + \tilde{K}\mathbf{v}e^{j\omega t} = \mathbf{0}, \qquad \mathbf{v} \neq \mathbf{0} \text{ or}$$

$$\underbrace{\tilde{K}\mathbf{v} = \omega^2\mathbf{v}}_{\text{vibration problem}} \quad \Leftrightarrow \quad \underbrace{\tilde{K}\mathbf{v} = \lambda\mathbf{v}}_{\substack{\text{real symmetric} \\ \text{eigenvalue problem}}} \qquad \mathbf{v} \neq \mathbf{0}$$

(4.40) (4.41)

Note that the martrix $\tilde{K}$ contains the same type of information as does ω_n^2 in the single degree of freedom case.

Properties of the *n* x *n* Real Symmetric Matrix

- **There are *n* eigenvalues and they are all real valued**
- **There are *n* eigenvectors and they are all real valued**
- **The eigenvalues are all positive if and only if the matrix is positive definite**
- **The set of eigenvectors can be chosen to be orthogonal**
- **The set of eigenvectors are linearly independent**
- **The matrix is similar to a diagonal matrix**
- **Numerical schemes to compute the eigenvalues and eigenvectors of symmetric matrix are faster and more efficient**

Square n x n Matrix Review

- **Let a_{ik} be the ik^{th} element of A then A is symmetric if $a_{ik} = a_{ki}$ denoted $A^T = A$**
- **A is positive definite if $x^T A x > 0$ for all nonzero x (also implies each $\lambda_i > 0$)**
- **The stiffness matrix is usually symmetric and positive semi definite (could have a zero eigenvalue)**
- **The mass matrix is positive definite and symmetric (and so far, its diagonal)**

Normal and orthogonal vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \text{ inner product is } \quad \mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

$$\left.\begin{array}{l} \mathbf{x} \text{ orthogonal to } \mathbf{y} \text{ if } \mathbf{x}^T\mathbf{y} = 0 \\ \mathbf{x} \text{ is normal if } \mathbf{x}^T\mathbf{x} = 1 \end{array}\right\}$$

if a the set of vectores is is both orthogonal and normal it is called an *orthonormal* set

The norm of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$ (4.43)

Normalizing any vector can be done by dividing it by its norm:

$$\frac{\mathbf{x}}{\sqrt{\mathbf{x}^T\mathbf{x}}} \quad \text{has norm of 1} \qquad (4.44)$$

To see this compute

$$\left\| \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T\mathbf{x}}} \right\| = \sqrt{\frac{\mathbf{x}^T}{\sqrt{\mathbf{x}^T\mathbf{x}}} \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T\mathbf{x}}}} = \sqrt{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}}} = 1$$

Examples 4.2.2 through 4.2.4

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ which is symmetric.

$$\det(\tilde{K} - \lambda I) = \det \begin{bmatrix} 3\text{-}\lambda & \text{-}1 \\ \text{-}1 & 3\text{-}\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 8 = 0$$

which has roots: $\lambda_1 = 2 = \omega_1^2$ and $\lambda_2 = 4 = \omega_2^2$

$$(\tilde{K} - \lambda_1 I)\mathbf{v}_1 = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$v_{11} - v_{12} = 0 \Rightarrow \mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\|\mathbf{v}_1\| = \sqrt{\alpha^2(1+1)} = 1 \Rightarrow \alpha = 1/\sqrt{2}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The first normalized eigenvector

Likewise the second normalized eigenvector is computed and shown to be orthogonal to the first, so that the set is orthonormal

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1^T \mathbf{v}_2 = \frac{1}{2}(1-1) = 0$$

$$\mathbf{v}_1^T \mathbf{v}_1 = \frac{1}{2}(1+1) = 1$$

$$\mathbf{v}_2^T \mathbf{v}_2 = \frac{1}{2}(1+(-1)(-1)) = 1$$

$$\Rightarrow \mathbf{v}_i \text{ are orthonormal}$$

Modes u and Eigenvectors v are different but related:

$$\mathbf{u}_1 \neq \mathbf{v}_1 \text{ and } \mathbf{u}_2 \neq \mathbf{v}_2$$

$$\mathbf{x} = M^{-1/2}\mathbf{q} \Rightarrow \mathbf{u} = M^{-1/2}\mathbf{v} \qquad (4.37)$$

Note

$$M^{1/2}\mathbf{u}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{v}_1$$

This orthonormal set of vectors is used to form an *Orthogonal Matrix*

$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ called a matrix of eigenvectors (normalized)

$$P^T P = \begin{bmatrix} \mathbf{v}_1^T \mathbf{v}_1 & \mathbf{v}_1^T \mathbf{v}_2 \\ \mathbf{v}_2^T \mathbf{v}_1 & \mathbf{v}_2^T \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

***P* is called an orthogonal matrix**

$$P^T \tilde{K} P = P^T \begin{bmatrix} \tilde{K}\mathbf{v}_1 & \tilde{K}\mathbf{v}_2 \end{bmatrix} = P^T \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathbf{v}_1^T \mathbf{v}_1 & \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \\ \lambda_1 \mathbf{v}_2^T \mathbf{v}_1 & \lambda_2 \mathbf{v}_2^T \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \text{diag}(\omega_1^2, \omega_2^2) = \Lambda$$

***P* is also called a modal matrix**

Example 4.2.4 compute *P* and show that it is an orthogonal matrix

From the previous example:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow$$

$$P^T P = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$$

Example 4.2.5 Compute the square of the frequencies by matrix manipulation

$$P^T\tilde{K}P=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}2 & 4\\ 2 & -4\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}4 & 0\\ 0 & 8\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}=\Lambda=\begin{bmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{bmatrix}$$

$$\Rightarrow \omega_1=\sqrt{2}\ \text{rad/s}\ \ \text{and}\ \ \omega_2=2\ \text{rad/s}$$

In general:

$$\Lambda=P^T\tilde{K}P=\text{diag}\left(\lambda_i\right)=\text{diag}(\omega_i^2) \qquad (4.48)$$

Example 4.2.6

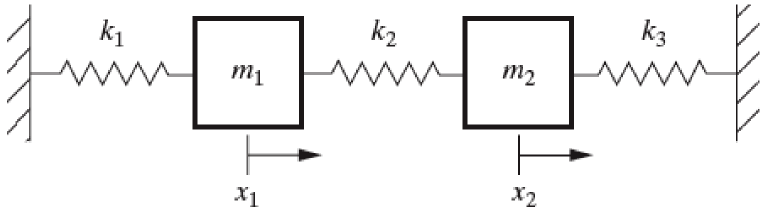


Figure 4.4

The equations of motion:

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 &= 0 \end{aligned} \qquad (4.49)$$

In matrix form these become:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\mathbf{x} = 0 \qquad (4.50)$$

Next substitute numerical values and compute *P* and Λ

$m_1 = 1$ kg, $m_2 = 4$ kg, $k_1 = k_3 = 10$ N/m and k_2=2 N/m

$$\Rightarrow M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad K = \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix}$$

$$\Rightarrow \tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 12 & -1 \\ -1 & 12 \end{bmatrix}$$

$$\Rightarrow \det\left(\tilde{K} - \lambda I\right) = \det\begin{bmatrix} 12-\lambda & -1 \\ -1 & 12-\lambda \end{bmatrix} = \lambda^2 - 15\lambda + 35 = 0$$

$$\Rightarrow \lambda_1 = 2.8902 \text{ and } \lambda_2 = 12.1098$$

$$\Rightarrow \omega_1 = 1.7 \text{ rad/s and } \omega_2 = 12.1098 \text{ rad/s}$$

Next compute the eigenvectors

For λ_1 equation (4.41) becomes:

$$\begin{bmatrix} 12 - 2.8902 & -1 \\ -1 & 3 - 2.8902 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0$$

$$\Rightarrow 9.1089 v_{11} = v_{21}$$

Normalizing $\mathbf{v}_1$ yields

$$1 = \|\mathbf{v}_1\| = \sqrt{v_{11}^2 + v_{21}^2} = \sqrt{v_{11}^2 + (9.1089)^2 v_{11}^2}$$

$$\Rightarrow v_{11} = 0.1091, \text{ and } v_{21} = 0.9940$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1091 \\ 0.9940 \end{bmatrix}, \quad \text{likewise } \mathbf{v}_2 = \begin{bmatrix} -0.9940 \\ 0.1091 \end{bmatrix}$$

Next check the value of *P* to see if it behaves as its suppose to:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.1091 & -0.9940 \\ 0.9940 & 0.1091 \end{bmatrix}$$

$$P^T \tilde{K} P = \begin{bmatrix} 0.1091 & 0.9940 \\ -0.9940 & 0.1091 \end{bmatrix} \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0.1091 & -0.9940 \\ 0.9940 & 0.1091 \end{bmatrix} = \begin{bmatrix} 2.8402 & 0 \\ 0 & 12.1098 \end{bmatrix}$$

$$P^T P = \begin{bmatrix} 0.1091 & 0.9940 \\ -0.9940 & 0.1091 \end{bmatrix} \begin{bmatrix} 0.1091 & -0.9940 \\ 0.9940 & 0.1091 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Yes!

A note on eigenvectors

In the previous section, we could have chosed $\mathbf{v}_2$ to be

$$\mathbf{v}_2 = \begin{bmatrix} 0.9940 \\ -0.1091 \end{bmatrix} \quad \text{instead of } \mathbf{v}_2 = \begin{bmatrix} -0.9940 \\ 0.1091 \end{bmatrix}$$

because one can always multiple an eigenvector by a constant and if the constant is -1 the result is still a normalized vector.

Does this make any difference?

No! Try it in the previous example

All of the previous examples can and should be solved by "hand" to learn the methods

However, they can also be solved on calculators with matrix functions and with the codes listed in the last section

In fact, for more then two DOF one must use a code to solve for the natural frequencies and mode shapes.

Next we examine 3 other formulations for solving for modal data

Matlab commands

- **To compute the inverse of the square matrix A: inv(A) or use A\eye(n) where n is the size of the matrix**
- **[P,D]=eig(A) computes the eigenvalues and normalized eigenvectors (watch the order). Stores them in the eigenvector matrix P and the diagonal matrix D (D=L)**

More commands

- To compute the matrix square root use sqrtm(A)
- To compute the Cholesky factor: L= chol(M)
- To compute the norm: norm(x)
- To compute the determinant det(A)
- To enter a matrix:

  K=[27 -3;-3 3]; M=[9 0;0 1];
- To multiply: K*inv(chol(M))

An alternate approach to normalizing mode shapes

From equation (4.17) $\left(-M\omega^2 + K\right)\mathbf{u} = 0, \qquad \mathbf{u} \neq 0$

Now scale the mode shapes by computing α such that

$$\left(\alpha_i \mathbf{u}_i\right)^T M \left(\alpha_i \mathbf{u}_i\right) = 1 \Rightarrow \alpha_i = \frac{1}{\sqrt{\mathbf{u}_i^T \mathbf{u}_i}}$$

$\mathbf{w}_i = \alpha_i \mathbf{u}_i$ is called *mass normalized* and it satisfies:

$$-\omega_i^2 M \mathbf{w}_i + K \mathbf{w}_i = 0 \Rightarrow \omega_i^2 = \mathbf{w}_i^T K \mathbf{w}_i, \quad i = 1, 2 \tag{4.53}$$

There are 3 approaches to computing mode shapes and frequencies

(i) $\omega^2 M\mathbf{u} = K\mathbf{u}$ (ii) $\omega^2\mathbf{u} = M^{-1}K\mathbf{u}$ (iii) $\omega^2\mathbf{v} = M^{-1/2}KM^{-1/2}\mathbf{v}$

(i) Is the Generalized Symmetric Eigenvalue Problem

easy for hand computations, inefficient for computers

(ii) Is the Asymmetric Eigenvalue Problem

very expensive computationally

(iii) Is the Symmetric Eigenvalue Problem

the cheapest computationally

Some Review: Window 4.3

Orthonormal Vectors

similar to the unit vectors of statics and dynamics

$\mathbf{x}_1$ and $\mathbf{x}_2$ are both *normal* if $\mathbf{x}_1^T\mathbf{x}_1 = 1$ and $\mathbf{x}_2^T\mathbf{x}_2 = 1$

and are *orthogonal* if $x_1^T x_2 = 0$

This is abbreviated by

$$\mathbf{x}_i^T\mathbf{x}_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

A set of *n* vectors $\mathbf{x}_i$ are set to be *orthonormal* if

$$\mathbf{x}_i^T\mathbf{x}_j = \delta_{ij}$$

for all values of *i* and *j*.

4.3 - Modal Analysis

- **Physical coordinates are not always the easiest to work in**
- **Eigenvectors provide a convenient transformation to modal coordinates**
  - **Modal coordinates are linear combination of physical coordinates**
  - **Say we have physical coordinates *x* and want to transform to some other coordinates *u***

$$\begin{aligned} u_1 &= x_1 + 3x_2 \\ u_2 &= x_1 - 3x_2 \end{aligned} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Review of the Eigenvalue Problem

Start with $M\ddot{\mathbf{x}}(t) + K\mathbf{x} = \mathbf{0}$ where $\mathbf{x}$ is a vector and M and K are matrices. Assume initial conditions $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$. Rewrite as

$$M^{\frac{1}{2}}\underbrace{M^{\frac{1}{2}}\ddot{\mathbf{x}}}_{\ddot{\mathbf{q}}} + K\mathbf{x} = \mathbf{0} \text{ and let}$$

$$M^{\frac{1}{2}}\mathbf{x} = \mathbf{q} \Rightarrow \mathbf{x} = M^{-\frac{1}{2}}\mathbf{q} \text{ (coord. trans. \#1)}$$

Eigenproblem (cont)

Premultiply by $M^{-\frac{1}{2}}$ to get

$$\underbrace{M^{-\frac{1}{2}}M^{\frac{1}{2}}}_{I}\ddot{\mathbf{q}} + \underbrace{M^{-\frac{1}{2}}KM^{-\frac{1}{2}}}_{\widetilde{K} = \widetilde{K}^T}\mathbf{q} = \ddot{\mathbf{q}} + \widetilde{K}\mathbf{q} = 0 \qquad (4.55)$$

- **Now we have a symmetric, real matrix**
- ***Guarantees* real eigenvalues and distinct, mutually orthogonal eigenvectors**

Eigenvectors = Mode Shapes?

Mode shapes are solutions to $M\omega^2\mathbf{u} = K\mathbf{u}$ in physical coordinates. Eigenvetors are characteristics of matrices. The two are related by a simple transformation, but they are not synonymous.

Eigenvectors vs. Mode Shapes

The eigenvectors of the symmetric PD matrix $\widetilde{K}$ are orthonormal, i.e., $P^T P = I$. Are the mode shapes orthonormal? Using the transformation $\mathbf{x} = M^{-\frac{1}{2}}\mathbf{q}$, the modes shapes $U = M^{-\frac{1}{2}}P \Rightarrow P = M^{\frac{1}{2}}U$. Now, $P^T P = U^T M^{\frac{1}{2}} M^{\frac{1}{2}} U = U^T M U = I$. Thus, the mode shapes are orthogonal only w.r.t. the mass matrix.

Similarly, $U^T K U = P^T \underbrace{M^{-\frac{1}{2}} K M^{-\frac{1}{2}}}_{\widetilde{K}} P = \Lambda$

The Matrix of eigenvectors can be used to decouple the equations of motion

If P orthonormal (unitary), $P^T P = I \Rightarrow P^T = P^{-1}$

Thus, $P^T \tilde{K} P = \Lambda =$ diagonal matrix of eigenvalues.

Back to $\ddot{q} + \tilde{K} q = 0$. Make the additional coordinate transformation $q = P_r$ and premultiply by P^T

$$P^T P\ddot{r} + P^T \tilde{K} P r = I\ddot{r} + \Lambda r = 0 \quad (4.59)$$

- **Now we have decoupled the EOM i.e., we have *n* independent 2nd-order systems in <u>modal coordinates r(*t*)</u>**

Writing out equation (4.59) yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{r}_1(t) \\ \ddot{r}_2(t) \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4.60)$$

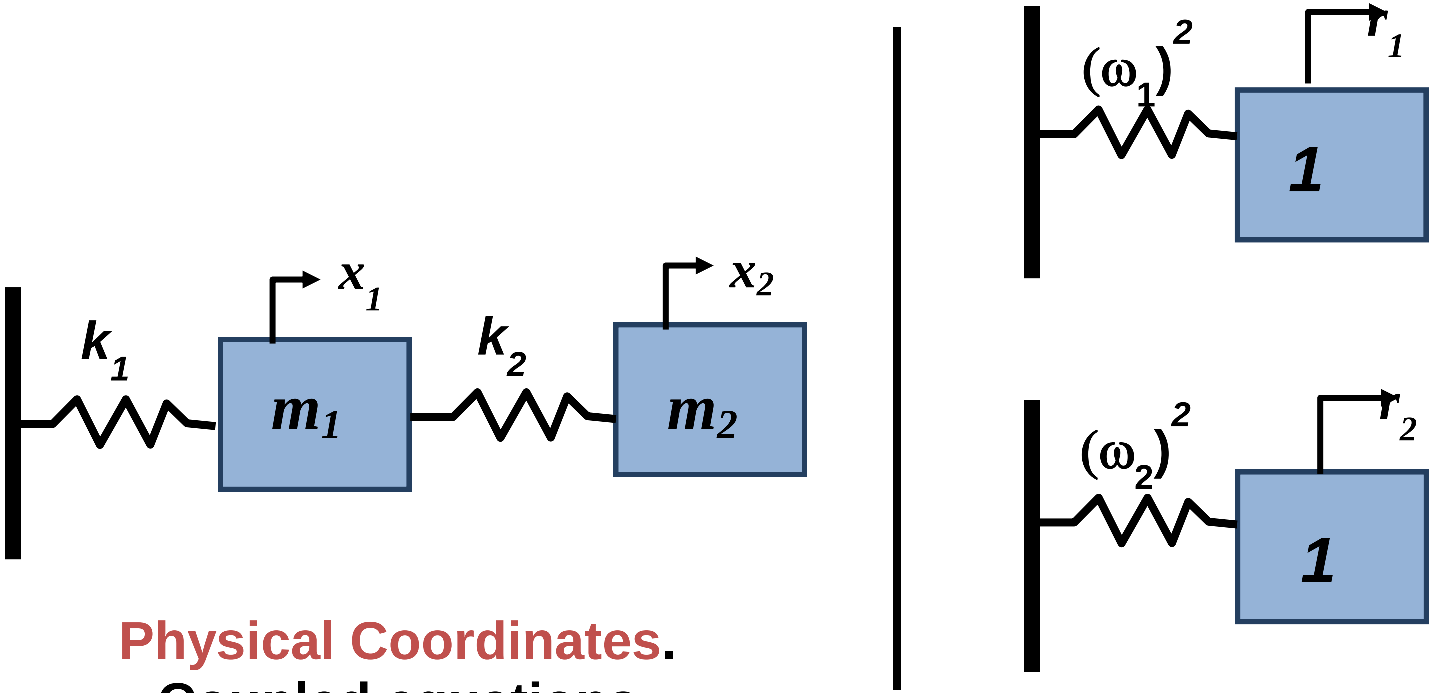
$$\Rightarrow \qquad \begin{array}{ll} \ddot{r}_1(t) + \omega_1^2 r_1(t) = 0 & (4.62) \\ \ddot{r}_2(t) + \omega_2^2 r_2(t) = 0 & (4.63) \end{array}$$

We must also transform the initial conditions

$$\mathbf{r}_0 = \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} = \begin{bmatrix} r_{10} \\ r_{20} \end{bmatrix} = P^T \mathbf{q}(0) = P^T M^{1/2} \mathbf{x}(0) \qquad (4.64)$$

$$\dot{\mathbf{r}}_0 = \begin{bmatrix} \dot{r}_1(0) \\ \dot{r}_2(0) \end{bmatrix} = \begin{bmatrix} \dot{r}_{10} \\ \dot{r}_{20} \end{bmatrix} = P^T \dot{\mathbf{q}}(0) = P^T M^{1/2} \dot{\mathbf{x}}(0) \qquad (4.65)$$

This transformation takes the problem from couple equations in the physical coordinate system in to decoupled equations in the *modal coordinates*



Physical Coordinates.
Coupled equations

Modal Coordinates.
Uncoupled equations

$$x = M^{\frac{-1}{2}} P r$$

Modal Transforms to SDOF

- **The modal transformation** $P^T M^{1/2}$ **transforms our 2 DOF into 2 SDOF systems**
- **This allows us to solve the two decoupled SDOF systems independently using the methods of chapter 1**
- **Then we can recombine using the inverse transformation to obtain the solution in terms of the physical coordinates.**

The free response is calculated for each mode independently using the formulas from chapter 1:

$$r_i(t) = \frac{\dot{r}_{i0}}{\omega_i} \sin \omega_i t + r_{i0} \cos \omega_i t, \qquad i = 1, 2$$

or (see Window 4.3 for a reminder)

$$r_i(t) = \sqrt{r_{i0}^2 + \frac{\dot{r}_{i0}^2}{\omega_i^2}} \sin(\omega_i t + \tan^{-1} \frac{\omega_i r_{i0}}{\dot{r}_{i0}}), \qquad i = 1, 2$$

Note, the above assumes neither frequency is zero

Once the solution in modal coordinates is determined (r_i) then the response in Physical Coordinates is computed:

- **With *n* DOFs these transformations are:**

$$\underset{n\times 1}{\mathbf{x}(t)} = \underset{n\times n}{S}\;\underset{n\times 1}{\mathbf{r}(t)}$$

$$\text{where} \quad \underset{n\times n}{S} = \underset{nxn}{M^{-1/2}}\;\underset{nxn}{P}$$

(where *n* = 2 in the previous slides)

Steps in solving via modal analysis (Window 4.5)

1. Calculate $M^{-1/2}$.
2. Calculate $\widetilde{K} = M^{-1/2}KM^{-1/2}$, the mass normalized stiffness matrix.
3. Calculate the symmetric eigenvalue problem for $\widetilde{K}$ to get ω_i^2 and $\mathbf{v}_i$.
4. Normalize $\mathbf{v}_i$ and form the matrix $P = [\mathbf{v}_1 \quad \mathbf{v}_2]$.
5. Calculate $S = M^{-1/2}P$ and $S^{-1} = P^T M^{1/2}$.
6. Calculate the modal initial conditions: $\mathbf{r}(0) = S^{-1}\mathbf{x}_0$, $\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}_0$.
7. Substitute the components of $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$ into equations (4.66) and (4.67) to get the solution in modal coordinate $\mathbf{r}(t)$.
8. Multiply $\mathbf{r}(t)$ by S to get the solution $\mathbf{x}(t) = S\mathbf{r}(t)$.

Note that S is the matrix of mode shapes and P is the matrix of eigenvectors.

Example 4.3.1 via MATLAB (see text for hand calculations)

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- **Follow steps in Window 4.5 (page 337)**

1) Calculate $M^{-1/2}$ 2) Calculate $\tilde{K} = M^{-1/2} K M^{-1/2}$

```
» Minv2 = inv(sqrt(M))
Minv2 =
     0.3333          0
          0     1.0000
```

```
»Kt =Minv2*K*Minv2
Kt =
     3     -1
    -1      3
```

Example 4.3.1 solved using MATLAB as a calcuator

```
% 3) Calculate the symmetric eigenvalue problem for K tilde [P,D] = eig(Kt);
[lambda,I]=sort(diag(D)); % just sorts smallest to largest
P=P(:,I);  % reorder eigenvectors to match eigenvalues
»lambda =
     2
     4
P =
   -0.7071   -0.7071
   -0.7071    0.7071
```

Example 4.3.1 (cont)

```
% 4) Calculate S = M^(-1/2) * P and Sinv = P^T * M^(1/2)
S = Minv2 * P;
Sinv = inv(S);
% 5) Calculate the modal initial conditions
r0 = Sinv * x0;
rdot0 = Sinv * v0;
```

Example 4.3.1 (cont)

```
% 6) Find the free response in modal coordinates
tmax = 10;
numt = 1000;
t = linspace(0,tmax,numt);
[T,W]=meshgrid(t,lambda.^(1/2));
% Use Tony's trick
R0 = r0(:,ones(numt,1));
RDOT0 = rdot0(:,ones(numt,1));
r = RDOT0./W.*sin(W.*T) + R0.*cos(W.*T);
% 7) Transform back to physical space
x = S*r;
```

Example 4.3.1 (cont)

```
% Plot results
figure
subplot(2,1,1)
plot(t,r(1,:),'-',t,r(2,:),'--')
title('free response in modal coordinates')
xlabel('time (sec)')
legend('r_1','r_2')
subplot(2,1,2)
plot(t,x(1,:),'-',t,x(2,:),'--')
title('free response in physical coordinates')
xlabel('time (sec)')
legend('x_1','x_2')
```

Modal and Physical Responses

Modal Coordinates:
Independent oscillators

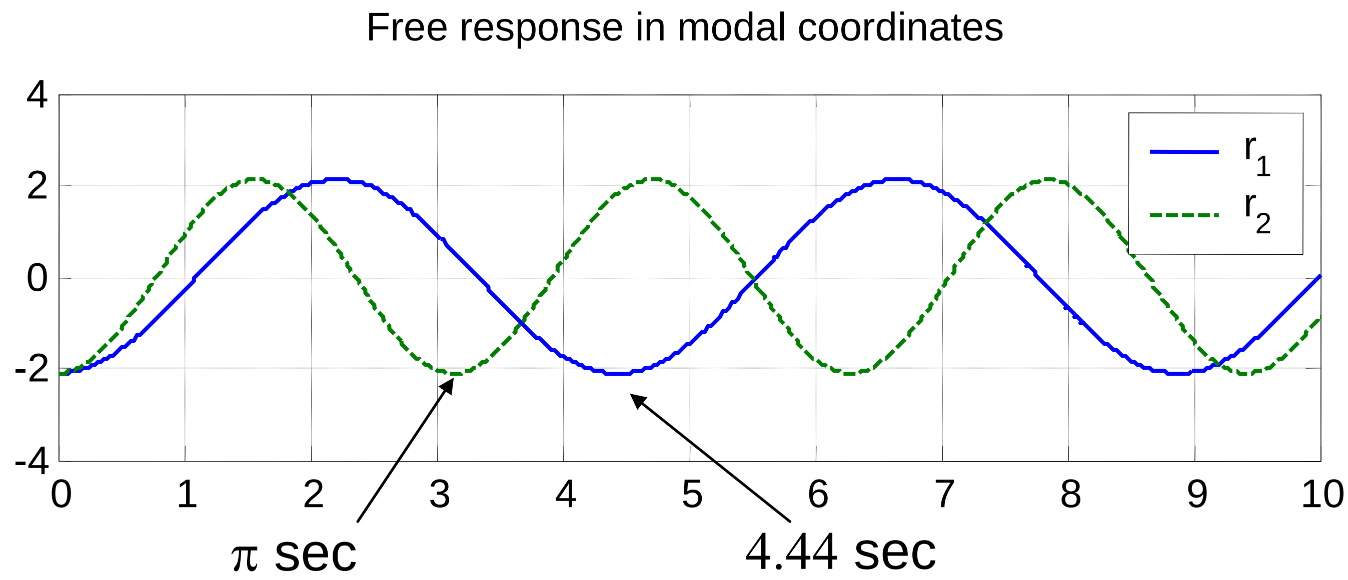
$$\lambda_1 = 2 \Rightarrow \omega_1 = \sqrt{2}$$

$$\Rightarrow T_1 = \frac{2\pi}{\omega_1}$$

$$= \sqrt{2}\pi = 4.44\,\text{sec},$$

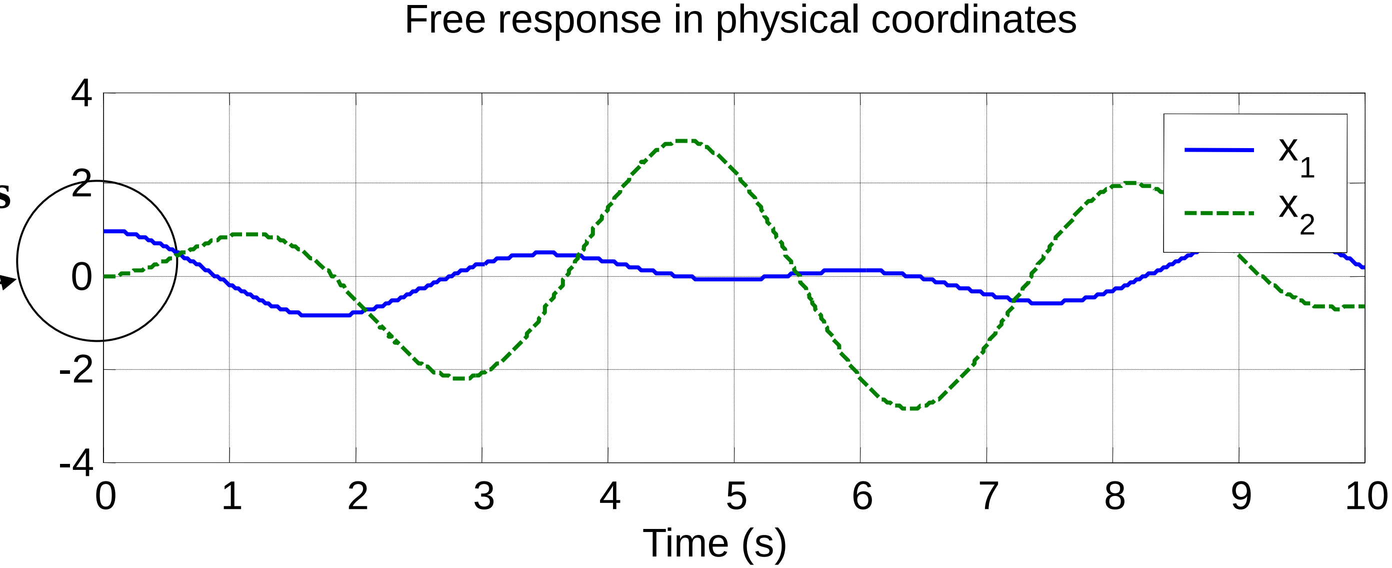
$$\lambda_2 = 4 \Rightarrow \omega_1 = 2$$

$$\Rightarrow T_2 = \pi\,\text{sec}$$



Physical Coordinates:
Coupled oscillators

Note IC



Section 4.4 More then 2 Degrees of Freedom

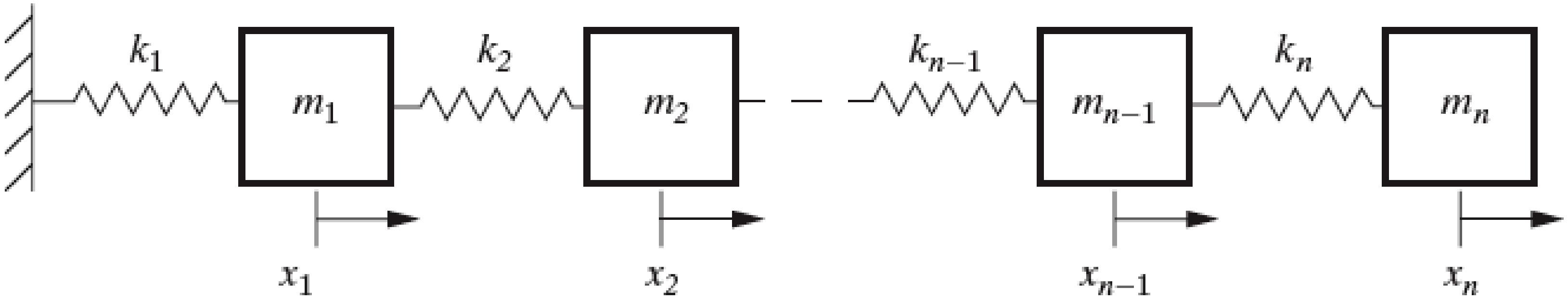


Fig 4.8

Extending previous section to any number of degrees of freedom

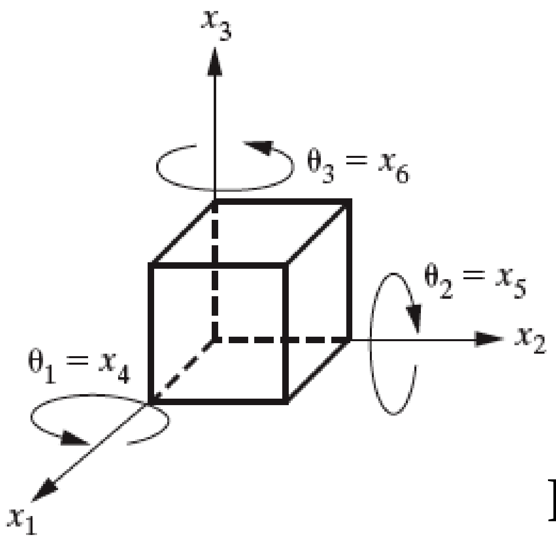


Fig 4.7

A FBD of the system of figure 4.8 yields the *n* equations of motion o the form:

$$m_i\ddot{x}_i + k_i\left(x_i - x_{i-1}\right) - k_{i+1}\left(x_{i-1} - x_i\right) = 0, \quad i = 1, 2, 3 \cdots n \qquad (4.83)$$

Writing all n of these equations and casting them in matrix form yields:

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}, \qquad (4.80)$$

where:

the relevant matrices and vectors are:

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & & 0 \\ 0 & -k_3 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & k_{n-1} + k_n & -k_n \\ 0 & 0 & \cdots & -k_n & k_n \end{bmatrix} \quad (4.83)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \ddot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \vdots \\ \ddot{x}_n(t) \end{bmatrix}$$

For such systems as figure 4.7 and 4.8 the process stays the same…just more modal equations result:

Process stays the same as section 4.3

$$\ddot{r}_1(t) + \omega_1^2 r_1(t) = 0$$
$$\ddot{r}_2(t) + \omega_2^2 r_2(t) = 0$$
$$\ddot{r}_3(t) + \omega_3^2 r_3(t) = 0$$
$$\vdots$$
$$\ddot{r}_n(t) + \omega_n^2 r_n(t) = 0$$

Just get more modal equations, one for each degree of freedom (*n* is the number of dof)

See example 4.4.2 for details

The Mode Summation Approach

- **Based on the idea that any possible time response is just a linear combination of the eigenvectors**

Starting with $\ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0}$ (4.88)

let $\mathbf{q}(t) = \sum_{i=1}^{n} \mathbf{q}_i(t) = \sum_{i=1}^{n} \left(a_i e^{-j\sqrt{\lambda_i}\, t} + b_i e^{j\sqrt{\lambda_i}\, t} \right) \mathbf{v}_i$

$\Rightarrow$ two linearly independent solutions for each term.

can also write this as $\mathbf{q}(t) = \sum_{i=1}^{n} d_i \sin\left(\omega_i t + \phi_i\right) \mathbf{v}_i$ (4.92)

Mode Summation Approach (cont)

Find the constants d_i and ϕ_i from the I.C.

$$\mathbf{q}(0) = \sum_{i=1}^{n} d_i \sin\phi_i \mathbf{v}_i \text{ and } \dot{\mathbf{q}}(0) = \sum_{i=1}^{n} d_i \quad \omega_i \cos\phi_i \mathbf{v}_i$$

Assuming eigenvectors normalized such that $\mathbf{v}_j^T \mathbf{v}_i = \delta_{ij}$

$$\mathbf{v}_j^T \mathbf{q}(0) = \mathbf{v}_j^T \sum_{i=1}^{n} d_i \quad \sin\phi_i \mathbf{v}_i = \sum_{i=1}^{n} d_i \quad \sin\phi_i \left(\mathbf{v}_j^T \mathbf{v}_i\right) = d_j \quad \sin\phi_j$$

Similarly for the initial velocities, $\mathbf{v}_j^T \dot{\mathbf{q}}(0) = d_j \quad \omega_j \cos\phi_j$

Mode Summation Approach (cont)

Solve for d_i and ϕ_i from the two IC equations

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin \phi_i} \quad \text{and} \quad \phi_i = \tan^{-1} \frac{\omega_i \, \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}$$

IMPORTANT NOTE about $\mathbf{q}(0)=\mathbf{0}$

if you just crank it through the above expressions

you might conclude that $d_i = 0,$ i.e., the trivial soln.

Be careful with $\dot{\mathbf{q}}(0) = 0$ as well.

Mode Summation Approach for zero initial displacement

If $\mathbf{q}(0) = 0$, the return to

$$\mathbf{q}(0) = \sum_{i=1}^{n} d_i \sin\phi_i \, \mathbf{v}_i$$

and realize that $\phi_i = 0$ instead of $d_i = 0$.

The compute d_i from the velocity expression

$$\mathbf{v}_i^T \dot{\mathbf{q}}(0) = \omega_i \, d_i \cos\phi_i$$

Mode Summation Approach with rigid body modes (ω_1 = 0)

if $\lambda_1 = 0$,

$$q_1(t) = (a_1 e^{-j\sqrt{0}t} + b_1 e^{j\sqrt{0}t})\mathrm{v}_i = (a_1 + b_1)\mathrm{v}_i$$

does not give two linearly independent solutions.

Now we must use the expansion

$$q(t) = \underline{(a_1 + b_1 t)}\mathrm{v}_1 + \sum_{i=2}^{n}(a_i e^{-j\sqrt{\lambda_i}t} + b_i e^{j\sqrt{\lambda_i}t})\mathrm{v}_i$$

and adjust calculation of the constants from the initial conditions accordingly.

Note that the underline term is a translational motion

Example 4.3.1 solved by the mode summation method

From before, we have $M^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

Appropriate IC are $\mathbf{q}_0 = M^{1/2}\mathbf{x}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\dot{\mathbf{q}}_0 = M^{1/2}\mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\phi_i = \tan^{-1}\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)} = \tan^{-1}\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{0} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -\frac{\pi}{2} \\ -\frac{\pi}{2} \end{bmatrix}$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{bmatrix}$$

Example 4.3.1 constructing the summation of modes

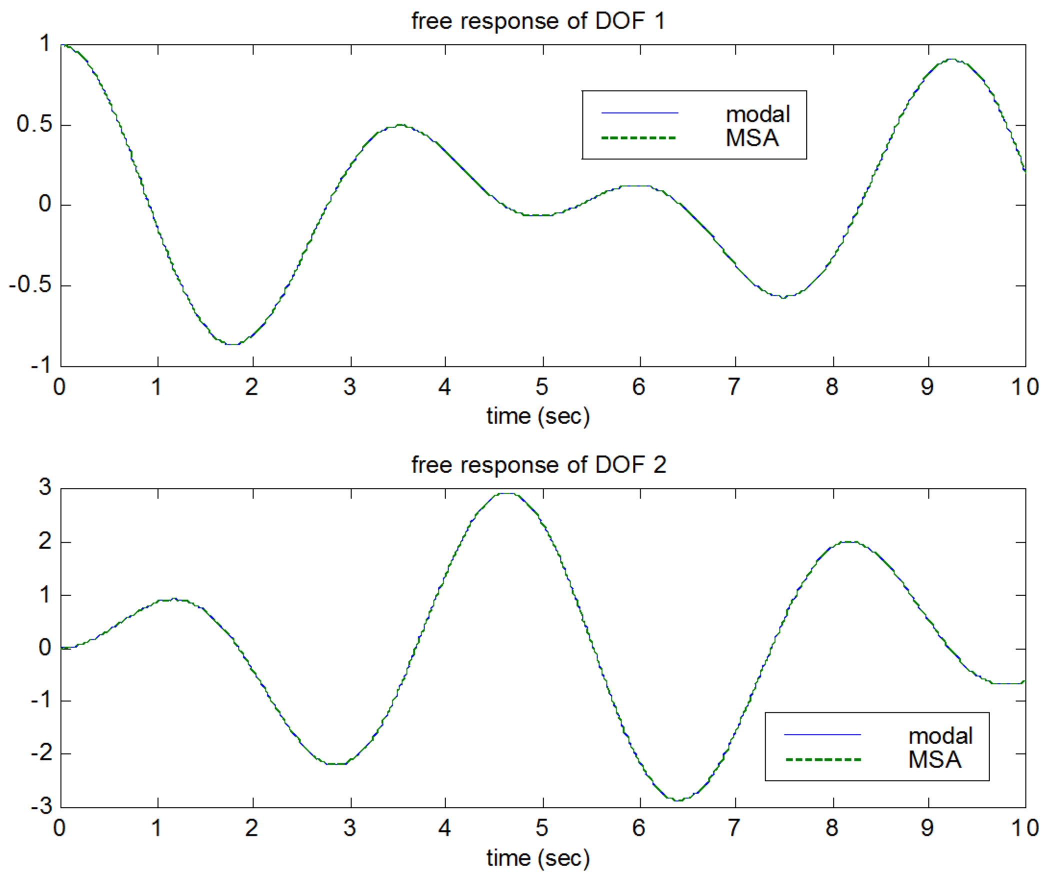
$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \underbrace{\frac{3\sqrt{2}}{2}\sin\left(\sqrt{2}t - \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{\text{the first mode}} + \underbrace{\frac{3\sqrt{2}}{2}\sin\left(2t - \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{the second mode}}$$

Transforming back to the physical coordinates yields:

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q} = \frac{3\sqrt{2}}{2}\sin\left(\sqrt{2}t - \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{3\sqrt{2}}{2}\sin\left(2t - \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{3\sqrt{2}}{2}\sin\left(\sqrt{2}t - \frac{\pi}{2}\right)\frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{3\sqrt{2}}{2}\sin\left(2t - \frac{\pi}{2}\right)\frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Example 4.3.1 a comparison of the two solution methods shows they yield identical results



Steps for Computing the Response By Mode Summation

1. **Write the equations of motion in matrix form, identify *M* and *K***
2. **Calculate $M^{-1/2}$ (or *L*)**
3. **Calculate** $\tilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$
4. **Compute the eigenvalue problem for the matrix** $\tilde{K}$ and get ω_i^2 and $\mathbf{v}_i$
5. **Transform the initial conditions to q(*t*)**

$$\mathbf{q}(0) = M^{\frac{1}{2}}\mathbf{x}(0) \text{ and } \dot{\mathbf{q}}(0) = M^{\frac{1}{2}}\dot{\mathbf{x}}(0)$$

Summary of Mode Summation Continued

6. **Calculate the modal expansion coefficients and phase constants**

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right), \quad d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i}$$

7. **Assemble the time response for q**

$$\mathbf{q}(t) = \sum_{i=1}^{n} d_i \sin(\omega_i t + \phi_i)\mathbf{v}_i$$

8. **Transform the solution to physical coordinates**

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \sum_{i=1}^{n} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

Nodes of a Mode Shape

- **Examination of the mode shapes in Example 4.4.3 shows that the third entry of the second mode shape is zero!**
- **Zero elements in a mode shape are called *nodes*.**
- **A node of a mode means there is no motion of the mass or (coordinate) corresponding to that entry at the frequency associated with that mode.**

The second mode shape of Example 4.4.3 has a node

- **Note that for more then 2 DOF, a mode shape may have a zero valued entry**
- **This is called a *node* of a mode.**

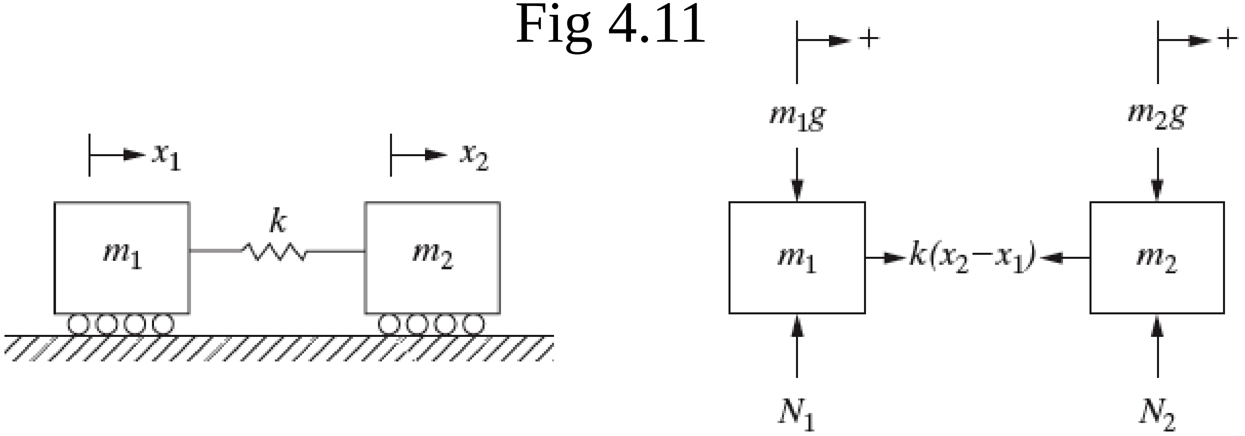
$$\mathbf{u}_2 = \begin{bmatrix} 0.2887 \\ 0.2887 \\ 0 \\ -0.2887 \end{bmatrix}$$

node

They make great mounting points in machines

A rigid body mode is the mode associated with a zero frequency

Fig 4.11



- **Note that the system in Fig 4.12 is not constrained and can move as a rigid body**
- **Physically if this system is displaced we would expect it to move off the page whilst the two masses oscillate back and forth**

Example 4.4.4 Rigid body motion

The free body diagram of figure 4.11 yields

$$m_1\ddot{x}_1 = k(x_2 - x_1) \quad \text{and} \quad m_2\ddot{x}_2 = -k(x_2 - x_1)$$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve for the free response given:

$m_1 = 1$ kg, $m_2 = 4$ kg, $k = 400$ N subject to

$$\mathbf{x}_0 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \text{ m and } \mathbf{v}_0 = 0$$

Following the steps of Window 4.5

1. $M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

2. $\tilde{K} = M^{-1/2} K M^{-1/2} = 400 \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 400 & -200 \\ -200 & 100 \end{bmatrix}$

3. $\det\left(\tilde{K} - \lambda I\right) = 100 \det\left(\begin{bmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 100\left(\lambda^2 - 5\lambda\right) = 0$

$\Rightarrow \lambda_1 = 0 \text{ and } \lambda_2 = 5 \Rightarrow \quad \omega_1 = 0, \quad \omega_2 = 2.236 \text{ rad/s}$

Indicates a rigid body motion

Now calculate the eigenvectors and note in particular that they cannot be zero even if the eigenvalue is zero

$$\lambda = 0 \Rightarrow 100\begin{bmatrix} 4-0 & -2 \\ -2 & 1-0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4v_{11} - 2v_{21} = 0$$

$$\Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ or after normalizing } \mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix}$$

$$\text{Likewise: } \mathbf{v}_2 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

As a check note that

$$P^T P = I \quad \text{and} \quad P^T \tilde{K} P = \text{diag}\begin{bmatrix} 0 & 5 \end{bmatrix}$$

5. Calculate the matrix of mode shapes

$$S = M^{-1/2}P = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix}$$

7. Calculate the modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}_0 = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.004472 \\ -0.008944 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}_0 = 0$$

7. Now compute the solution in modal coordinates and note what happens to the first mode.

Since $\omega_1 = 0$ the first modal equation is

$$\ddot{r}_1 + (0)r_1 = 0$$

$$\Rightarrow r_1(t) = a + bt$$

Rigid body translation

And the second modal equation is

$$\ddot{r}_2(t) + 5r_2(t) = 0$$

$$\Rightarrow r_2(t) = a_2 \cos\sqrt{5}t$$

Oscillation

Applying the modal initial conditions to these two solution forms yields:

$$r_1(0) = a = 0.004472$$

$$\dot{r}_1(0) = b = 0.0$$

$$\Rightarrow r_1(t) = 0.0042$$

as in the past problems the initial conditions for r_2 yield

$$r_2(t) = -0.0089\cos\sqrt{5}t$$

$$\Rightarrow \mathbf{r}(t) = \begin{bmatrix} 0.0042 \\ -0.0089\cos\sqrt{5}t \end{bmatrix}$$

8. Transform the modal solution to the physical coordinate system

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix} \begin{bmatrix} 0.0045 \\ -0.0089\cos\sqrt{5}t \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2.012 + 7.60\cos\sqrt{5}t \\ 2.012 - 1.990\cos\sqrt{5}t \end{bmatrix} \times 10^{-3} \text{ m}$$

Each mass is moved a constant distance and then oscillates at a single frequency.

Order the frequencies

- **It is convention to call the lowest frequency ω_1 so that $\omega_1 \leq \omega_2 \leq \omega_3 < \ldots$**
- **Order the modes (or eigenvectors) accordingly**
- **It really does not make a difference in computing the time response**
- **However:**
  - **When we measuring frequencies, they appear lowest to highest**
  - **Physically the frequencies respond with the highest energy in the lowest mode (important in flutter calculations, run up in rotating machines, etc.)**

The system of Example 4.1.5 solved by Mode Summation

From Example 4.1.6 we have:

$$\omega_1 = \sqrt{2},\ \mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \quad \omega_2 = 2,\ \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Use the following initial conditions and note that only one mode should be excited (why?)

$$\mathbf{x}(0) = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Transform coordinates

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow M^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus the initial conditions become

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Transform Mode Shapes to Eigenvectors

$$\mathbf{v}_1 = M^{1/2}\mathbf{u}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = M^{1/2}\mathbf{u}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

eigenvectors

Note that unlike the mode shapes, the eigenvectors are orthogonal:

Note that $\mathbf{v}_1^T\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0$, but $\mathbf{u}_1^T\mathbf{u}_2 = \begin{bmatrix} 1/3 & 1 \end{bmatrix} \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = \frac{2}{3} \neq 0$

Normalizing yields: $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

From Equation (4.92):

$$\mathbf{q}(t) = \sum_{i=1}^{2} d_i \sin(\omega_i t + \phi_i)\mathbf{v}_i \Rightarrow \dot{\mathbf{q}}(t) = \sum_{i=1}^{2} d_i \omega_i \cos(\omega_i t + \phi_i)\mathbf{v}_i$$

Set t=0 and multiply by $\mathbf{v}_1$:

$$\dot{\mathbf{q}}(0) = \sum_{i=1}^{2} d_i \omega_i \cos\phi_i \mathbf{v}_i$$

$$\Rightarrow \mathbf{v}_1^T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = d_1 \sqrt{2} \cos\phi_1 \mathbf{v}_1^T \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_2 2 \cos\phi_2 \mathbf{v}_1^T \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow 0 = d_1 \cos\phi_1 \Rightarrow \phi_1 = \pi / 2$$

Or directly from Eq. (4.97)

From the initial displacement:

$$d_1 = \frac{\mathbf{v}_1^T \mathbf{q}(0)}{\sin(\pi/2)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{2}} \qquad (4.98)$$

$$d_2 = \frac{\mathbf{v}_2^T \mathbf{q}(0)}{\sin(\pi/2)} = \begin{bmatrix} -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

Eigenvector 1

Eigenvector 2

thus

$$\mathbf{q}(t) = \sum_{i=1}^{2} d_i \sin(\omega_i t + \phi_i) \mathbf{v}_i$$

$$= \sqrt{2} \cos(\sqrt{2} t) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \cos(\sqrt{2} t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Transforming Back to Physical Coordinates:

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \cos(\sqrt{2}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}\cos\sqrt{2}t \\ \cos\sqrt{2}t \end{bmatrix}$$

$$\Rightarrow x_1(t) = \frac{1}{3}\cos\sqrt{2}t \quad \text{and} \quad x_2(t) = \cos\sqrt{2}t$$

So, the initial conditions generated motion only in the first mode (as expected)

Alternate Path to Symmetric Single-Matrix Eigenproblem

- **Square root of matrix conceptually easy, but computationally expensive**

$$M^{-\frac{1}{2}}M^{\frac{1}{2}}\ddot{\mathbf{q}} + M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\mathbf{q} = \ddot{\mathbf{q}} + \widetilde{K}\mathbf{q} = 0$$

- **More efficient to decompose *M* into product of upper and lower triangular matrices (Cholesky decomposition)**

Cholesky Decomposition

Let $M = U^T U$ where U is upper triangular

Introduce the coordinate transformation

$$U\mathbf{x} = \mathbf{q} \Rightarrow \mathbf{x} = U^{-1}\mathbf{q} \;\Rightarrow U^T \underset{\ddot{\mathbf{q}}}{U\ddot{\mathbf{x}}} + K \underset{U^{-1}\mathbf{q}}{\mathbf{x}} = 0$$

premultiply by U^{-T} to get

$$I\,\ddot{\mathbf{q}} + U^{-T} K U^{-1}\mathbf{q} = \ddot{\mathbf{q}} + \tilde{K}\,\mathbf{q} = \mathbf{0}$$

note that: $\left[U^{-T} K U^{-1}\right]^T = \left[U^{-1}\right]^T K^T \left[U^{-T}\right]^T = U^{-T} K U^{-1}$

Cholesky (cont)

- Is this really faster? Let's ask MATLAB

$$M = M^{\frac{1}{2}} M^{\frac{1}{2}} \qquad\qquad M = U^T U$$

```
»M = [9   0 ; 0   1];
»flops(0); sqrtm(M); flops
ans = 65
```

```
»M = [9   0 ; 0   1];
»flops(0); chol(M); flops
ans = 5
```

- **sqrtm requires a singular value decomposition (SVD), whereas Cholesky requires only simple operations**

Note that $M^{\frac{1}{2}} = U$ for diagonal M

Section 4.5 Systems with Viscous Damping

The solution of $m\ddot{x} + c\dot{x} + kx = 0$, $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, or $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ is (for the underdamped case $0 < \zeta < 1$)

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \theta)$$

where $\omega_n = \sqrt{k/m}$, $\zeta = c/(2m\omega_n)$, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, and

$$A = \left[\frac{(\dot{x}_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}\right]^{1/2} \qquad \theta = \tan^{-1}\frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}$$

from equations (1.36), (1.37), and (1.38).

Extending the first 4 sections to included the effects of viscous damping (dashpots)

Viscous Damping in MDOF Systems

- **Two basic choices for including damping**
  - **Modal Damping**
    - **Attribute some amount to each mode based on experience, i.e., an artful guess or**
    - **Estimate damping due to viscoelasticity using some approximation method**
  - **Model the damping mechanism directly (hard and still an area of research-good for physicists but engineers need models that are correct enough).**

Modal Damping Method

Solve the undamped vibration problem following Window 4.5

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = 0 \Rightarrow I\ddot{\mathbf{r}}(t) + \Lambda\mathbf{r}(t) = 0$$

Here the mode shapes and eigenvectors are real valued and form orthonormal sets, even for repeated natural frequencies

(known because $\tilde{K} = M^{-1/2} K M^{-1/2}$ **is symmetric)**

Modal Damping (cont)

- **Decouple system based on *M* and *K*, i.e., use the "undamped" modes**
- **Attribute some z_i (zeta) to each mode of the decoupled system (a guess. Not known beforehand. Can be tested with gross data like *x*):**

here $\omega_{di} = \omega_i \sqrt{1-\zeta_i^2}$

$$\ddot{r}_i + 2\zeta_i \omega_i \dot{r}_i + \omega_i^2 r_i = 0 \quad (4.106)$$

$$\Rightarrow r_i(t) = A_i e^{-\zeta_i \omega_i t} \sin\left(\omega_{di} t + \phi_i\right) \quad (4.107)$$

Alternately: $r_i(t) = e^{-\zeta_i \omega_i t}(A_i \sin \omega_{di} t + B_i \cos \omega_{di} t)$

Transform Back to Get Physical Solution

- **Use modal transform to obtain modal initial conditions and compute A_i and F$_i$:**

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = P^T M^{1/2}\mathbf{x}(0) = P^T M^{1/2}\mathbf{x}_0$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = P^T M^{1/2}\dot{\mathbf{x}}(0) = P^T M^{1/2}\dot{\mathbf{x}}_0$$

- **With r(t) known, use the inverse transform to recover the physical solution:**

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = M^{-1/2}P\mathbf{r}(t) = S\mathbf{r}(t)$$

Modal Damping by Mode Summation

- **Can also use mode summation approach**
- **Again, modes are from undamped system**
- **The higher the frequency, the smaller the effect (because of the exponential term). So just few first modes are enough.**

$$\mathbf{q}(t) = \sum_{i=1}^{n} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{v}_i \quad \text{where}$$

$$M^{-1/2} K M^{-1/2} \mathbf{v}_i = \omega_i^2 \mathbf{v}_i, \text{ and } \omega_{di} = \omega_i \sqrt{1-\zeta_i^2}$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \quad \text{and} \quad \phi_i = \tan^{-1} \frac{\omega_{di} \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0) + \zeta_i \omega_i \mathbf{v}_i^T \mathbf{q}(0)}$$

Compute q(*t*), Transform back

- **To get the proper initial conditions use:**

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0), \text{ and } \dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0)$$

- **Use the above to compute q(*t*) and then:**

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

the response in physical coordinates.

Example

Consider:

$$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Subject to initial conditions: $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dot{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Experiments do not give *C*. They provide zeta (in modal coordinates) by the half power method.
Compute the solution assuming modal damping of:

$$\zeta_1 = 0.01 \text{ and } \zeta_2 = 0.1$$

Compute the modal decomposition

L =sqrt(M)

$$L = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \;\; \widetilde{K} = L^{-1}KL^{-1} = \begin{bmatrix} 0.667 & -0.333 \\ -0.333 & 0.500 \end{bmatrix}$$

$$\widetilde{K}\mathbf{v} = \lambda\mathbf{v} \Rightarrow \lambda_1 = 0.240, \mathbf{v}_1 = \begin{bmatrix} 0.615 \\ 0.788 \end{bmatrix}, \text{and } \lambda_2 = 0.947, \mathbf{v}_2 = \begin{bmatrix} -0.788 \\ 0.615 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.615 & -0.788 \\ 0.788 & 0.615 \end{bmatrix}$$

Compute the modal initial conditions:

$$S = L^{-1}P = \begin{bmatrix} 0.205 & -0.263 \\ 0.394 & 0.308 \end{bmatrix} \Rightarrow \mathbf{r}_0 = S^{-1}\mathbf{x}_0 = \begin{bmatrix} 1.846 \\ -2.365 \end{bmatrix}$$

$$\dot{\mathbf{r}}_0 = 0$$

Compute the modal solutions:

$$\zeta_1 = 0.01, \zeta_2 = 0.1,$$

$$\omega_1 = 0.49, \omega_{d1} = 0.49, \omega_2 = 0.963, \omega_{d2} = 0.958$$

Using eq (4.108) and (4.109) yields

$$r_1(t) = 4.208e^{-0.004896t}\sin(0.49t + 1.561)$$

$$r_2(t) = 3.346e^{-0.096t}\sin(0.958t + 1.471)$$

Then use $\mathbf{x}(t) = S\mathbf{r}(t)$

$$x(t) = Sr(t) = \begin{bmatrix} 0.205 & -0.263 \\ 0.394 & 0.308 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

$$x_1(t) = 0.863e^{-0.004896t} \sin(0.49t + 1.561) - 0.88e^{-0.096t} \sin(0.958t + 1.471)$$
$$x_2(t) = 1.658e^{-0.004896t} \sin(0.49t + 1.561) + 1.029e^{-0.096t} \sin(0.958t + 1.471)$$

So, first separate solutions in the modal coordinates were found and then the modes were assembled by the use of S.

The response in the physical coordinates is therefore a combination of the modal responses just as in the undamped case. See page 357 for an additional example.

Lumped Damping models

- **In some cases (FEM, machine modeling), the damping matrix is determined directly from the equations of motion.**
- **Then our analysis must start with:**

$$M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0},$$

$$\text{subject to } \mathbf{x}_0 \text{ and } \dot{\mathbf{x}}_0$$

Generic Example:

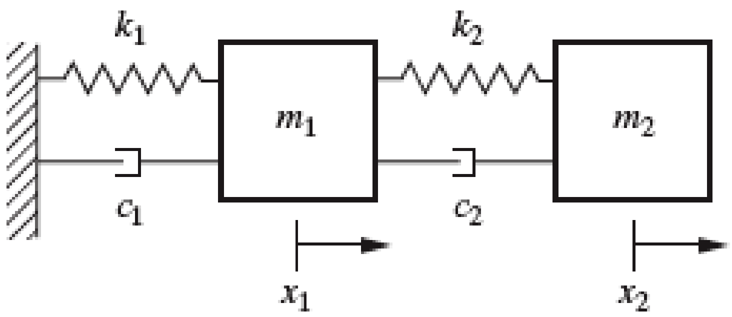
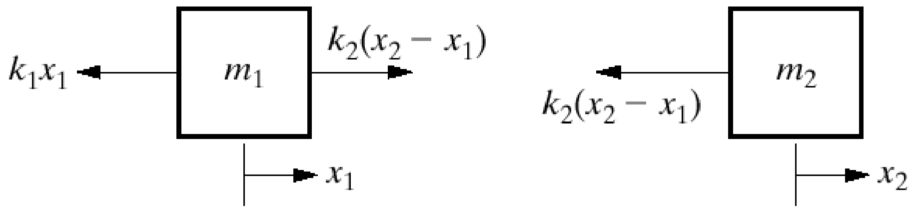


Fig 4.15

- If the damping mechanisms are known then
- Sum forces to find the equations of motion

Free Body Diagram:



$$m_1\ddot{x}_1 = -c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1)$$
$$- k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -c_2(\dot{x}_2 - \dot{x}_1)$$
$$- k_2(x_2 - x_1)$$

Matrix form of Equations of Motion:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The *C* and *K* matrices have the same form.

It follows from the system itself that consisted damping and stiffness elements in a similar manner.

A Question of matrix decoupling

- **Can we decouple the system with the same coordinate transformations as before?**

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = 0$$

$$I\ddot{\mathbf{r}} + \underbrace{P^T M^{-1/2} C M^{-1/2} P}_{\text{diagonal ?}}\dot{\mathbf{r}} + \Lambda\mathbf{r} = 0$$

- **In general, these can not be decoupled since *K* and *C* can not be diagonalized simultaneously**

A Little Matrix Theory

- Two symmetric matrices have the same eigenvectors

  if and only if the matrices commute

- Define $\tilde{C} = M^{-1/2} C M^{-1/2}$

- Transform the damped equations of motion into:

$$I\ddot{\mathbf{q}}(t) + \tilde{C}\dot{\mathbf{q}}(t) + \hat{K}\mathbf{q}(t) = 0$$

- Let P be the matrix of eigenvectors v_i of $\tilde{K}$ and $\Lambda = P^T \tilde{K} P$

Then $P^T \tilde{C} P$ will be diagonal if and only if transformed

K and C have same eigenvectors, i.e.

$$\tilde{C}\mathbf{v}_i = \beta_i \mathbf{v}_i \text{ for all } i, \text{ so } \tilde{C}\tilde{K} = \tilde{K}\tilde{C}$$

More Matrix Stuff and Normal Mode Systems

$$\tilde{C}\tilde{K} = \tilde{K}\tilde{C} \Rightarrow$$

$$M^{-1/2}CM^{-1/2}M^{-1/2}KM^{-1/2} = M^{-1/2}KM^{-1/2}M^{-1/2}CM^{-1/2}$$

$$\Rightarrow M^{-1/2}CM^{-1}KM^{-1/2} = M^{-1/2}KM^{-1}CM^{-1/2}$$

$$\Rightarrow CM^{-1}K = KM^{-1}C$$

Happens if and only if $CM^{-1}K$ is symmetric

- **This does not require a matrix square root to check**
- **This informs us explicitly whether or not the equations of motion can be decoupled**
- **If true, such systems are called "normal mode" systems or said to possess "classical normal modes"**

Proportional Damping

- **It turns out that $CM^{-1}K$ = symmetric is a necessary and sufficient condition for C to be diagonalizable by the eigenvectors of the "undamped" system, i.e., those based on M,K**

- **Best known example is "proportional"damping.**

- **The coefficients are obtained through experiments or just by guess.**

$$C = \alpha M + \beta K = \text{linear combination of } M \text{ and } K.$$

$$CM^{-1}K = \left(\alpha M + \beta K\right) M^{-1}K = \underbrace{\alpha K + \beta KM^{-1}K}_{\text{both symmetric}}$$

Proportional Damping (cont)

Write the system as $M\ddot{\mathbf{x}} + \left(\alpha M + \beta K\right)\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$

$$\mathbf{q} = M^{1/2}\mathbf{x} \Rightarrow I\ddot{\mathbf{q}} + \left(\alpha I + \beta \tilde{K}\right)\dot{\mathbf{q}} + \tilde{K}\mathbf{q} = \mathbf{0}$$

$$\mathbf{q} = P\mathbf{r} \Rightarrow I\ddot{\mathbf{r}} + \underbrace{\left(\alpha I + \beta \Lambda\right)}_{\text{diagonal!}}\dot{\mathbf{r}} + \Lambda\mathbf{r} = \mathbf{0}$$

Thus, the damping ratios in the decoupled system are

$$2\zeta_i\omega_i = \alpha + \beta\omega_i^2 \Rightarrow \zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \qquad (4.124)$$

Generalized Proportional Damping

For any value of *n* up to the number of degrees of freedom:

$$C = \sum_{i=1}^{n} \beta_{i-1} K^{i-1}$$

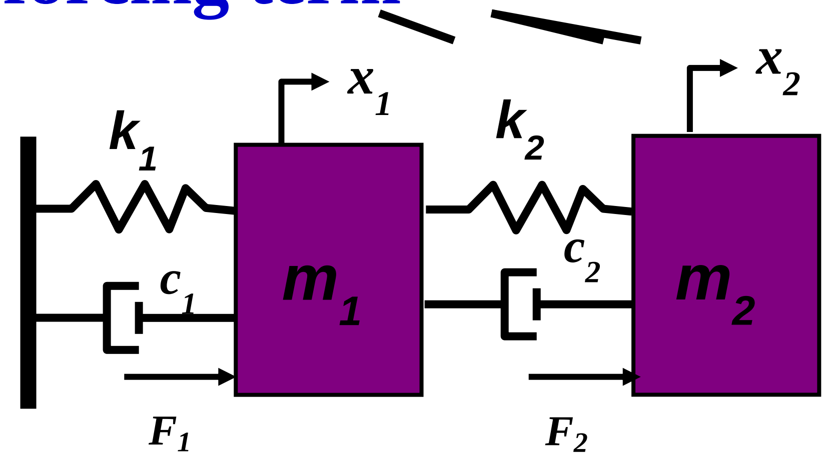
For example for *n* = 2 we get the previous proportional damping formulation:

$$C = \beta_0 K^0 + \beta_1 K = \alpha I + \beta K$$

Section 4.6 Modal Analysis of the Forced Response

Extending the chapters 2 and 3 to more then one degree of freedom

Forced Response: the response of an mdof system to a forcing term



$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = B\mathbf{F}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_4(t) \end{bmatrix} \qquad (4.126)$$

Assume C diagonalizable for now, i.e.,

$$\ddot{\mathbf{q}} + \tilde{C}\dot{\mathbf{q}} + \tilde{K}\mathbf{q} = M^{-1/2}\mathbf{F} \text{ where } \tilde{C} = M^{-1/2}CM^{-1/2}$$

If the system of equations decouple then the methods of Chapters 2 and 3 can be applied

Decouple the system with the eigenvalues of $\tilde{K}$

$$I\,\ddot{\mathbf{r}} + \begin{bmatrix} \ddots & & \\ & 2\zeta_i\omega_i & \\ & & \ddots \end{bmatrix}\dot{\mathbf{r}} + \begin{bmatrix} \ddots & & \\ & \lambda_i & \\ & & \ddots \end{bmatrix}\mathbf{r} = P^T M^{-1/2} B\mathbf{F} \tag{4.129}$$

so the i^{th} equation would be $\ddot{r}_i + 2\zeta_i\omega_i\dot{r}_i + \omega_i^2 r_i = f_i(t)$ (4.130)

- **Responses to harmonic, periodic, or general forces as in Chapters 2 and 3**
- **Note that the modal forcing function is a linear combination of many physical forces**

With the modal equation in hand the general solution is given

$$\ddot{r}_i(t) + 2\zeta_i\omega_i\dot{r}_i(t) + \omega_i^2 r_i(t) = f_i(t) \qquad (4.130)$$

$$\Rightarrow r_i(t) = d_i e^{-\zeta_i\omega_i t}\sin\left(\omega_{di}t + \phi_i\right)$$

$$+\frac{1}{\omega_{di}} e^{-\zeta_i\omega_i t}\int_0^t f_i(t) e^{\zeta_i\omega_i\tau}\sin\omega_{di}\left(t-\tau\right)d\tau \qquad (4.131)$$

The response of an underdamped system

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

(with zero initial conditions) is given by (for $0 < \zeta < 1$)

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t}\int_0^t F(\tau) e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\,d\tau$$

where $\omega_n = \sqrt{k/m}$, $\zeta = c/(2m\omega_n)$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. With nonzero initial conditions this becomes

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \frac{1}{\omega_d} e^{-\zeta\omega_n t}\int_0^t f(\tau) e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\,d\tau$$

where $f = F/m$ and A and ϕ are constants determined by the initial conditions.

The applied force is distributed across the all of the modes except in a special case.

$$\mathbf{f}(t) = P^T M^{-1/2} B\mathbf{F}(t) \text{ for the decoupled EOM.}$$

- **An excitation on a single physical DOF may "spread" to all modal DOFs (one F generates many f's)**
- **It is actually possible to drive a MDOF system at one of its natural frequencies and not experience resonant response (an unusual circumstance)**

Let $\mathbf{F}(t) = \mathbf{b}\, f(t),$ where $\mathbf{b}$ is some spatial vector and $f(t)$ is any fuction of time. What if $\mathbf{b}$ happens to be related to the i^{th} mode shape by $\mathbf{b} = M\,\mathrm{u}_i$?

Example 4.6.1
A 2-dof system

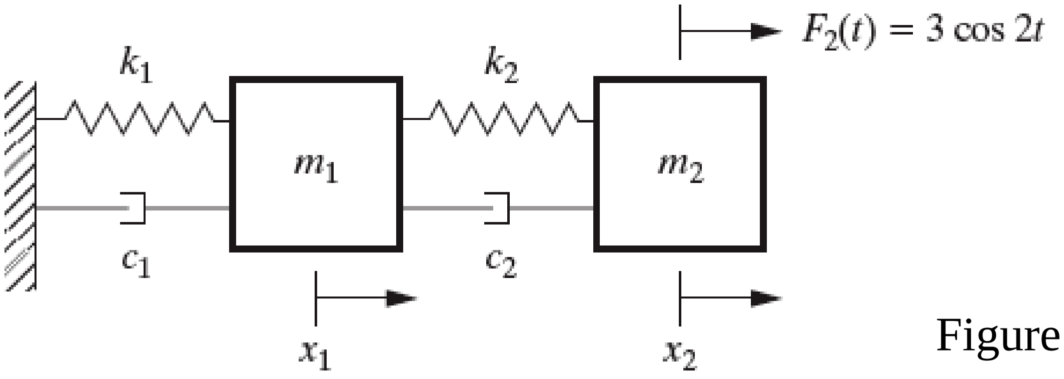


Figure 4.16

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ F_1(t) \end{bmatrix}$$

$$M^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, M^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{C} = M^{-1/2} C M^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix}$$

Compute the mass normalized stiffness matrix and its eigen solution

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

From before:

$$\tilde{K}\mathbf{v} = \lambda \mathbf{v} \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 4 \end{cases}, \quad P = 0.707 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Transform the damping matrix, the forcing function and write down the modal equations

$$P^T\tilde{C}P = 0.707\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix}0.707\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$P^T\tilde{K}P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{f}(t) = P^T M^{-1/2} B\mathbf{F}(t) = \begin{bmatrix} 0.2357 & 0.7071 \\ -0.2357 & 0.7071 \end{bmatrix}\begin{bmatrix} 0 \\ F_2(t) \end{bmatrix}$$

From the above coefficients the modal equations become (note that the force is distributed to each mode)

$$\ddot{r}_1(t) + 0.2\dot{r}_1(t) + 2r_1(t) = (0.7071)(3)\cos 2t = 2.1213\cos 2t$$

$$\ddot{r}_2(t) + 0.4\dot{r}_2(t) + 4r_2(t) = (0.7071)(3)\cos 2t = 2.1213\cos 2t$$

Compute the modal values using the single degree of freedom formulas

- **The modal damping ratios and damped natural frequencies are computed using the usual formulas and the coefficients from the terms in the modal equations:**

$$\zeta_1 = \frac{0.2}{2\sqrt{2}} = 0.0707$$

$$\zeta_2 = \frac{0.4}{2(2)} = 0.1000$$

$$\omega_{d1} = \omega_1\sqrt{1-\zeta_1^2} = 1.41$$

$$\omega_{d2} = \omega_2\sqrt{1-\zeta_2^2} = 1.99$$

Use SDOF formula for the particular solution given in equation (2.36)

$$r_{1p}(t) = 1.040\cos(2t + 0.1974)$$

$$r_{2p}(t) = 2.6516\sin(2t)$$

Now transform back to physical coordinates

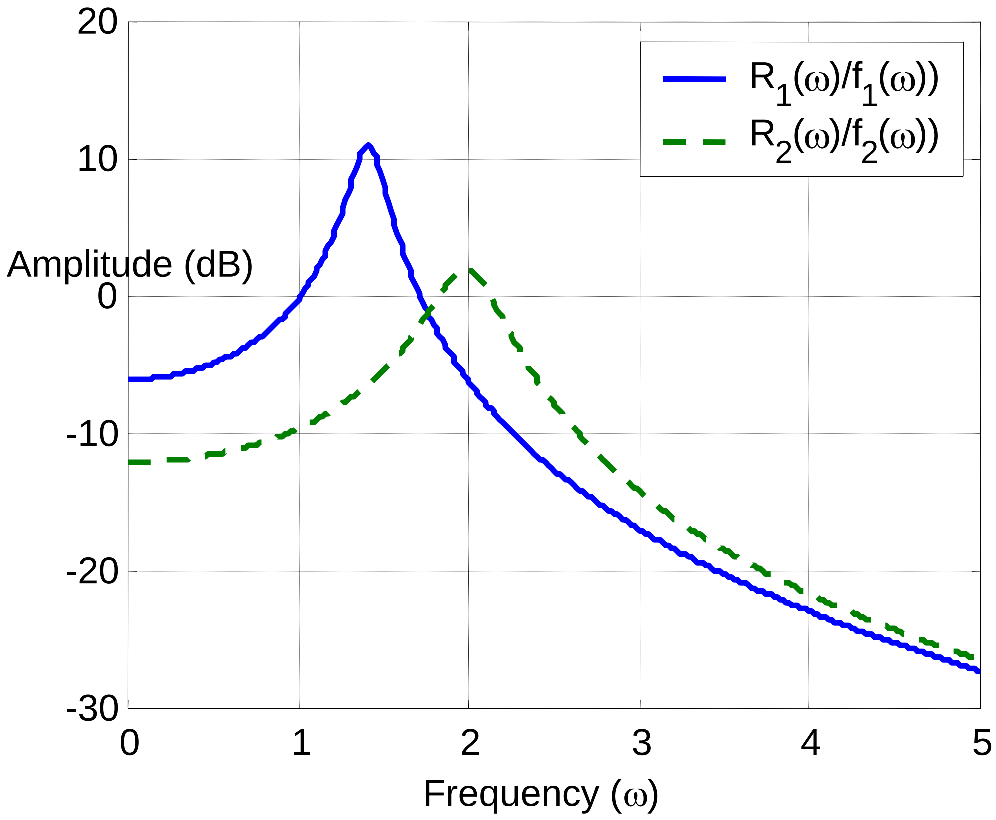
$$\mathbf{x}_{ss}(t) = M^{-1/2}P\begin{bmatrix} 1.040\cos(2t + 0.1974) \\ 2.6516\sin(2t) \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1(t) = 0.2451\cos(2t + 0.1974) - 0.6249\sin 2t \\ x_2(t) = 0.7354\cos(2t + 0.1974) + 1.8749\sin 2t \end{cases}$$

Note that the force effects both degrees of freedom even though it is applied to one.

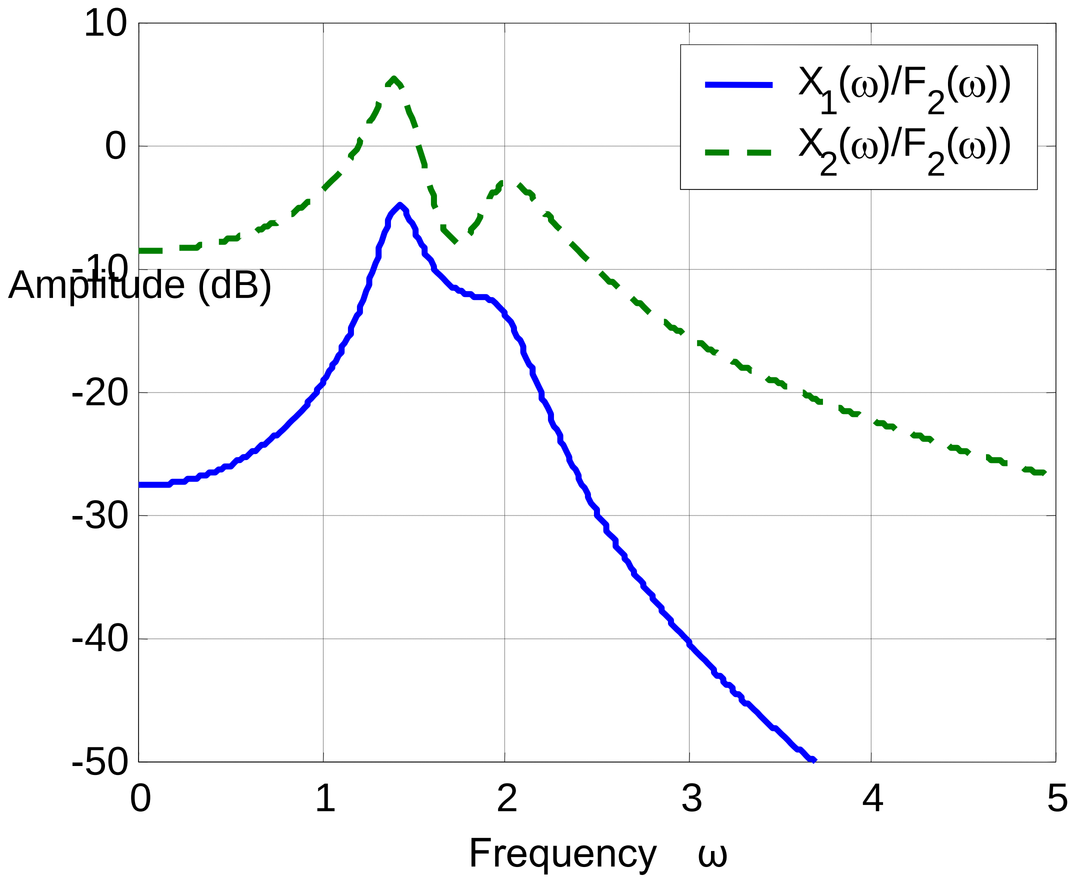
The Frequency Response of each mode is plotted:

- This graph shows the amplitude of each mode due to an input ***modal force*** f_1 and f_2.
- A force applied to mass # 2 F_2 will contribute to both modal forces!



The frequency response of each degree of freedom is plotted

- **This graph shows the amplitude of each mass due to an input force on mass #2.**
- **Each mass is excited by the force on mass #2**
- **Both masses are effected by both modes**



Resonance for multiple degree of freedom systems can occur at each of the systems natural frequencies

- **Note that the frequency response of the previous example shows *two* peaks**

Special cases:

- **If in the odd case that b is orthogonal to one of the mode shapes then resonance in that mode may not occur (see example 4.6.2)**

- **If the modes are strongly coupled the resonant peaks may combine (see X_1/F_2 in the previous slide) and be hard to notice**

Example: Illustrating the effect of the input force allocation

Consider: $$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\cos 2t$$

Compute the modal equations and discuss resonance.

Solution:

$$M^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{x} = M^{-1/2}\mathbf{q}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\mathbf{q} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\cos 2t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t$$

Calculating the natural frequencies and mode shapes yields:

$$\omega_1 = \sqrt{2} \text{ and } \omega_2 = 2 \text{ rad/s}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

The mass normalized eigenvectors are:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Transform and compute the modal equations:

$\Rightarrow \mathbf{q} = P\mathbf{r}$ yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t \Rightarrow$$

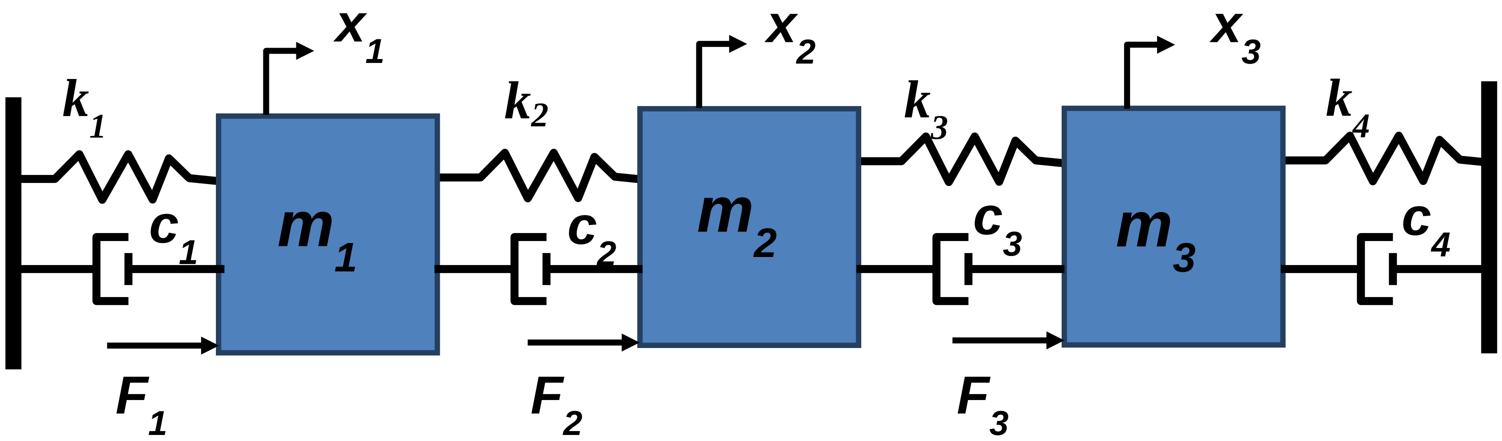
$$\ddot{r}_1 + 2r_1 = \sqrt{2}\cos 2t$$

$$\ddot{r}_2 + 4r_2 = 0$$

No resonance even though

$\omega_2 = 2 = \omega$, the driving frequency

An example with three masses



$m_1=m_2=m_3=2Kg$ $\quad$ $k_1=k_2=k_3= k_1=3N/m$ $\quad$ $C=0.02K$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

Solving a system with 3 masses is best done using a code.

Using Matlab we can calculate the eigenvectors and eigenvalues and hence the mode shapes and natural frequencies.

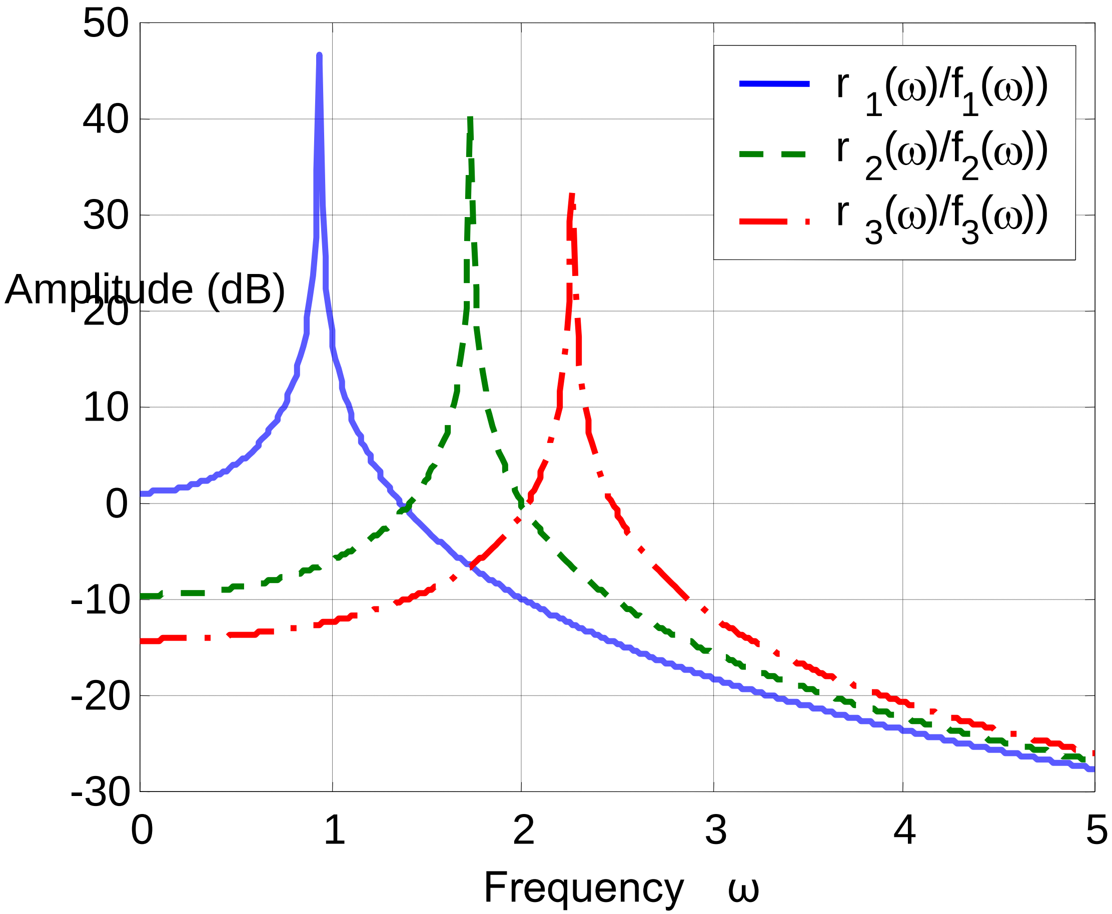
$$M^{-1/2} = \begin{bmatrix} 0.707 & 0 & 0 \\ 0 & 0.707 & 0 \\ 0 & 0 & 0.707 \end{bmatrix} \qquad P = \begin{bmatrix} 0.5 & 0.707 & 0.5 \\ 0.707 & 0 & -0.707 \\ 0.5 & -0.707 & 0.5 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.354 & 0.5 & 0.354 \\ 0.5 & 0 & -0.5 \\ 0.354 & -0.5 & 0.354 \end{bmatrix}$$

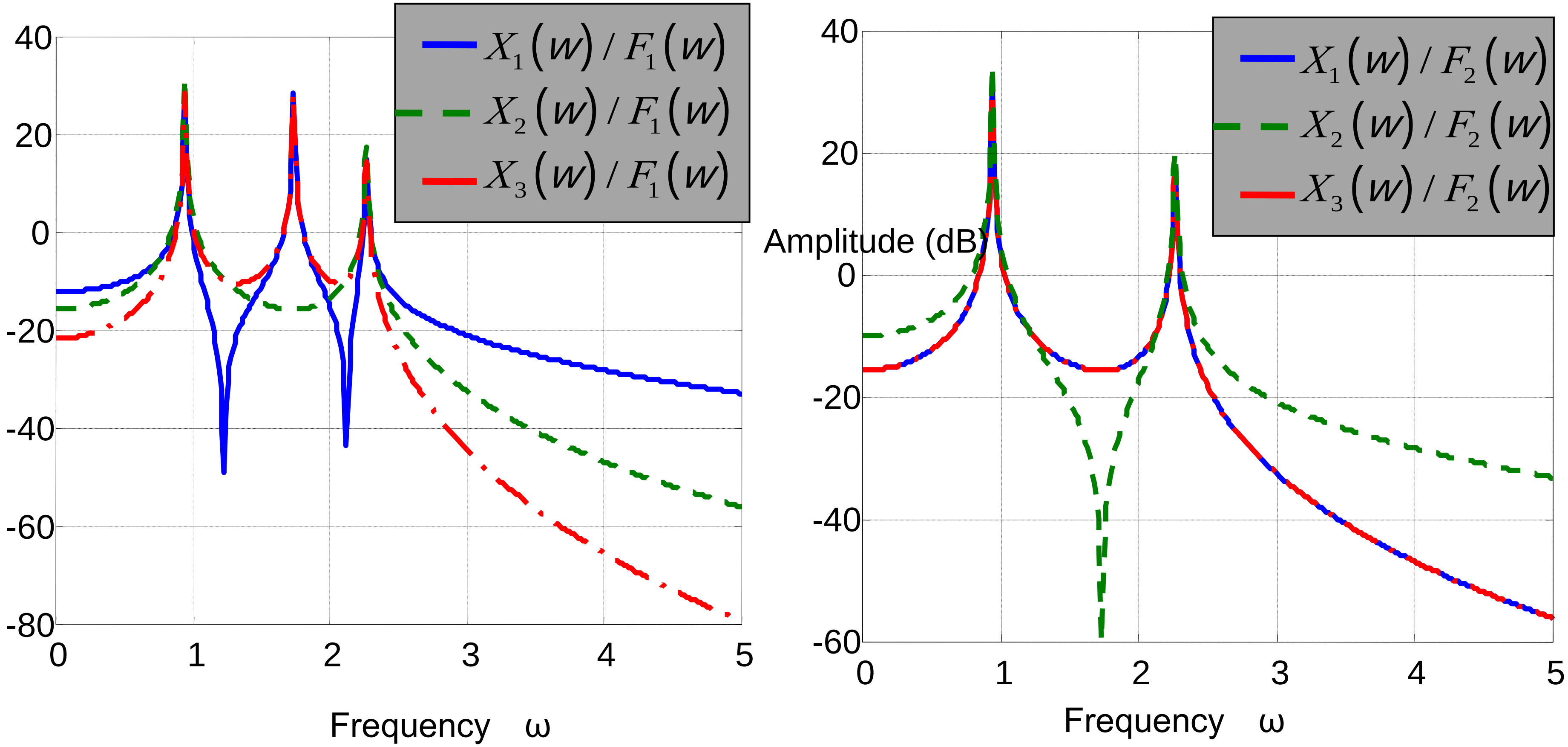
$$\omega_1 = 0.94 \quad \omega_2 = 1.73 \quad \omega_3 = 2.26$$

$$\zeta_1 = 0.0094 \quad \zeta_2 = 0.017 \quad \zeta_3 = 0.0226$$

The frequency response of each mode computed separately:



A comparison of the Frequency response between driving mass #1 and driving mass #2



Computing the forced response via the mode summation technique

Consider $$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t)$$

Transform: $$\ddot{\mathbf{q}}(t) + \hat{K}\mathbf{q}(t) = M^{-\frac{1}{2}}\mathbf{F}(t)$$

From eq. (4.92) the homogeneous solution in mode summation form is

$$\mathbf{q}_H(t) = \sum_{i=1}^{n} d_i \sin\left(\omega_i t + \phi_i\right)\mathbf{v}_i \qquad (4.134)$$

The total solution in mode summation form is:

$$\mathbf{q}(t) = \underbrace{\sum_{i=1}^{n} [b_i \sin \omega_i t + c_i \cos \omega_i t] \mathbf{v}_i}_{\text{homogenous}} + \underbrace{\mathbf{q}_p(t)}_{\text{particular}} \qquad (4.135)$$

But

$$\mathbf{q}_p(t) = M^{1/2} \mathbf{x}_p(t)$$

$$\mathbf{q}(t) = \sum_{i=1}^{n} \left(b_i \sin \omega_i t + c_i \cos \omega_i t \right) \mathbf{v}_i + M^{1/2} \mathbf{x}_p(t) \qquad (4.136)$$

Next use the initial conditions and orthogonality to evaluate the constants

$$\mathbf{v}_i^T\mathbf{q}_0 = c_i + \mathbf{v}_i^T M^{1/2}\mathbf{x}_p(0) \qquad (4.138)$$

$$\mathbf{v}_i^T\dot{\mathbf{q}}_0 = \omega_i b_i + \mathbf{v}_i^T M^{1/2}\dot{\mathbf{x}}_p(0) \qquad (4.139)$$

$$\Rightarrow$$

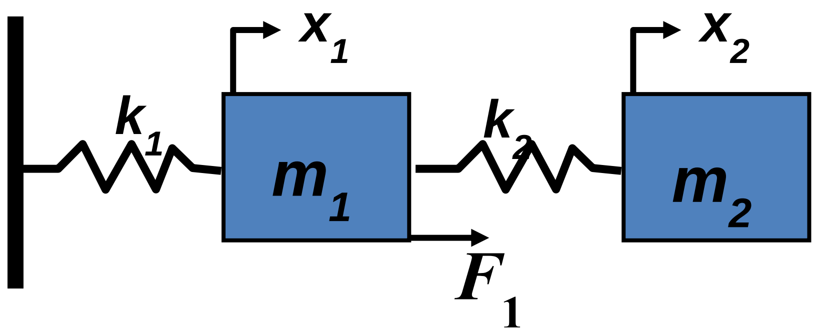
$$c_i = \mathbf{v}_i^T\mathbf{q}_0 - \mathbf{v}_i^T M^{1/2}\mathbf{x}_p(0)$$

$$b_i = \frac{1}{\omega_i}\left(\mathbf{v}_i^T\dot{\mathbf{q}}_0 - \mathbf{v}_i^T M^{1/2}\dot{\mathbf{x}}_p(0)\right)$$

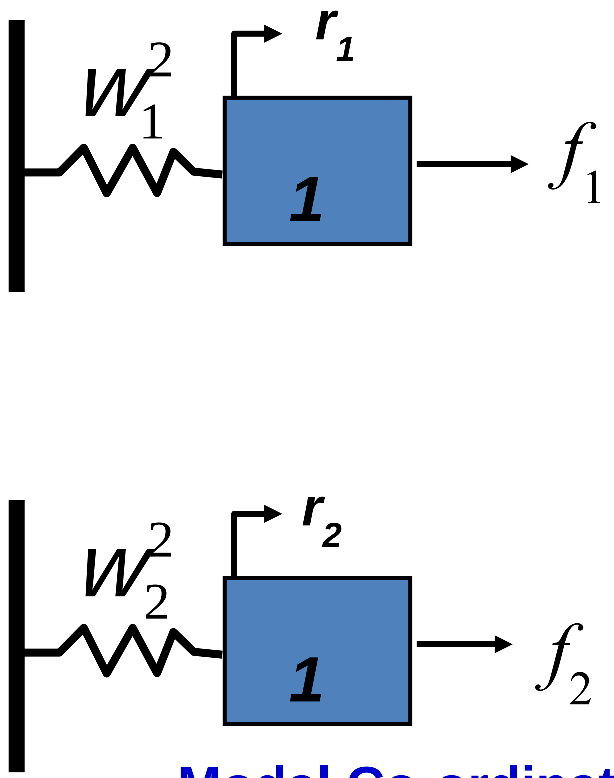
Substitution of the constants into Equation (4.136) and multiplying by $M^{-1/2}$ yields

$$\mathbf{x}(t) = \sum_{i=1}^{n} \left(d_i \sin \omega_i t + c_i \cos \omega_i t \right) \mathbf{u}_i + \mathbf{x}_p(t) \tag{4.141}$$

Decoupled Forced EOM



Physical Co-ordinates.
Coupled equations

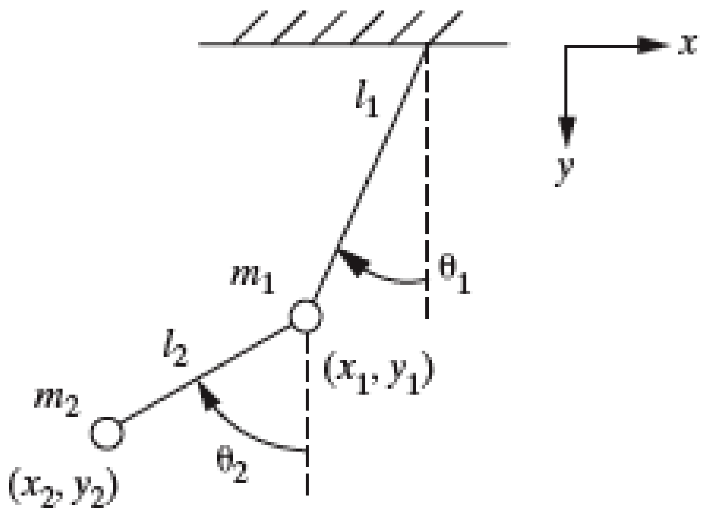


Modal Co-ordinates.
Uncoupled equations

$$\mathbf{x}(t) = M^{-1/2} P \, \mathbf{r}(t)$$

4.7 Lagrange's Equations

Defining work, energy and virtual displacements and work we will learn an *alternate* method of deriving equations of motion



Generalized coordinates: 2 not 4!

Recall equations (1.63) and (1.64)

Definitions (from Dynamics)

Kinetic Energy: $T = \frac{1}{2} m \dot{\mathbf{r}} \bullet \dot{\mathbf{r}} = \frac{1}{2} m \dot{\mathbf{r}}^T \dot{\mathbf{r}}$

Work Done by a force: $W_{1\to 2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$

$\mathbf{r}_0$ a reference position then the potential energy is

$$V(r) = \int_{\mathbf{r}_1}^{\mathbf{r}_0} \mathbf{F} \cdot d\mathbf{r}$$

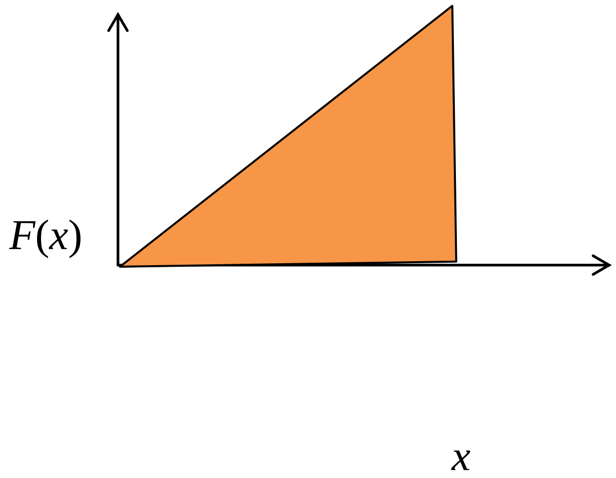
Strain Energy in a Spring

Strain energy (elastic potential energy)

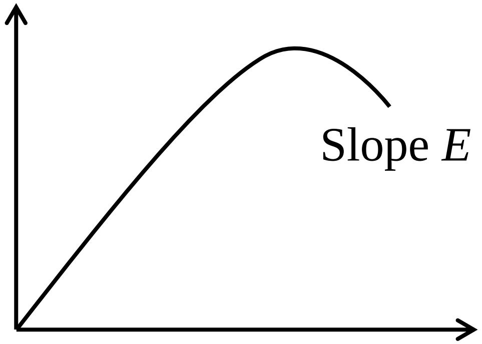
for a spring: $F = -kx$

$$V(x) = \int_x^0 F(\eta)d\eta = \int_x^0 -k\eta d\eta = \frac{1}{2}kx^2$$

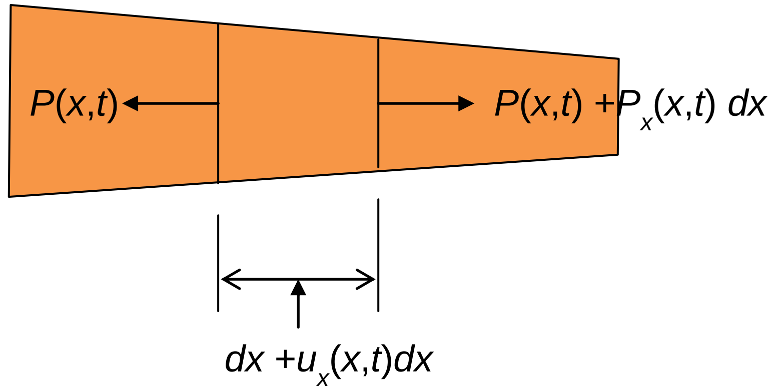
which is the area under the $F(x)$ vs x curve



Strain energy in an elastic material



Example of a bar of cross section *A*(*x*) elongated by force *P*(*x*,*t*)



The variation of dx, denoted $\delta(dx)$ is given by

$$\delta(dx) = \frac{\partial u(x,t)}{\partial x} dx = \varepsilon(x,t) dx$$

The axial stress is $\sigma(x,t) = \dfrac{P(x,t)}{A(x)} = E\varepsilon(x,t)$ so $P = EA\varepsilon$

Strain energy continued

$$dV = \frac{1}{2}P(x,t)\delta(dx) = \frac{1}{2}P(x,t)\varepsilon(x,t)dx$$

$$= \frac{1}{2}[EA(x)\varepsilon(x,t)]\varepsilon(x,t)dx$$

$$= \frac{1}{2}EA(x)\varepsilon^2(x,t)dx$$

Integrating yields the strain energy for axial tension in a bar element:

$$V = \frac{1}{2}E\int_0^{\ell} A(x)\varepsilon^2(x,t)dx$$

$$= \frac{1}{2}E\int_0^{\ell} A(x)\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx$$

Virtual Reality (actually: virtual displacement)

$$\delta r$$

↑

Variation or
Change in:

A virtual displacement
Based on variational math
Small or infinitesimal changes compatible with constraints
No time associated with change

Consequence of satisfying the constraint:

Constraint: $f(\mathbf{r}) = c$, a constant

$$\Rightarrow f(\mathbf{r} + \delta\mathbf{r}) = c$$

Taylor expansion:

$$f(\mathbf{r}) + \underbrace{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \delta x_i \right)}_{\frac{\partial f}{\partial \mathbf{r}} \cdot \delta\mathbf{r}} = c$$

$$\Rightarrow \frac{\partial f}{\partial \mathbf{r}} \cdot \delta\mathbf{r} = 0$$

Virtual work

Suppose the i^{th} mass is acted on by $\mathbf{f}_i$ with system in static equilibrium

$$\Rightarrow \delta W_i = \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0, \Rightarrow$$

the principle of virtual work:

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

which states that if a system is in equilibrium, the work done by externally applied forces through a virtual displacement is zero: $\Rightarrow \delta V = 0$

$\Rightarrow V$ has an critical value

Dynamic Equilibrium

D'Alembert's Principle $\Rightarrow$ move inertia force to left side and treat dynamics as statics. From Newton's law in terms of change in momentum:

$$\sum \mathbf{F}_i = \dot{\mathbf{p}} \Rightarrow \left(\sum \mathbf{F}_i - \dot{\mathbf{p}} \right) = 0$$

This allows us to use virtual work in the dynamic case:

$$\Rightarrow \left(\sum \mathbf{F}_i - \dot{\mathbf{p}} \right) \cdot \delta \mathbf{r} = 0$$

$$\left(\sum \mathbf{F}_i - m\ddot{\mathbf{r}} \right) \cdot \delta \mathbf{r} = 0$$

Hamilton's Principle

$$\frac{d}{dt}(\dot{\mathbf{r}} \cdot \delta \mathbf{r}) = \ddot{\mathbf{r}} \cdot \delta \mathbf{r} + \dot{\mathbf{r}} \delta \dot{\mathbf{r}}$$

$$= \ddot{\mathbf{r}} \cdot \delta \mathbf{r} + \delta(\frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})$$

$$\Rightarrow \sum \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = \sum \frac{d}{dt}(\dot{\mathbf{r}} \cdot \delta \mathbf{r}) - \sum \delta(\frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}), \quad \text{multiply by } m$$

$$\Rightarrow \delta W = \sum m \frac{d}{dt}(\dot{\mathbf{r}} \cdot \delta \mathbf{r}) - \delta T$$

$$\Rightarrow \delta T + \delta W = \sum m \frac{d}{dt}(\dot{\mathbf{r}} \cdot \delta \mathbf{r})$$

Integrate this last expression

$$\int_{t_1}^{t_2}\left(\delta T+\delta W\right)dt=\int_{t_1}^{t_2}\sum m\frac{d}{dt}(\dot{\mathbf{r}}\cdot\delta\mathbf{r})dt$$

$$\int_{t_1}^{t_2}\left(\delta T+\delta W\right)dt=\underbrace{\sum m\dot{\mathbf{r}}\cdot\delta\mathbf{r}\Big|_{t_1}^{t_2}}_{\text{path indepence}}=0\Rightarrow$$

$$\int_{t_1}^{t_2}\left(\delta T+\delta W\right)dt=0,\quad \text{for conservative forces } \delta W=-\delta V$$

$$\Rightarrow\delta\int_{t_1}^{t_2}\left(T-V\right)dt=0,\quad \text{Hamilton's principle}$$

Lagrange's Equation

Let $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3 \ldots q_n, t)$, q_i called generalized coordinates

Let $Q_i = \dfrac{\delta W}{\delta q_i}$ a generalized force (or moment) (4.143)

The Lagrange formulation, derived from Hamilton's principle for determining the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad (4.144)$$

The Lagrangian, *L*

Let *L* = (*T* - *U*), called the Lagrangian
Then (4.145) becomes:

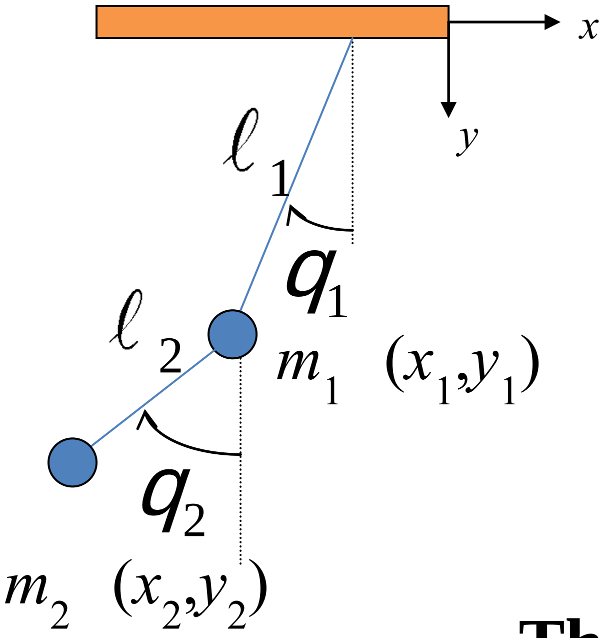
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \ldots n \qquad (1.146)$$

For the (common) case that the potential energy does not depend on the velocity: $\frac{\partial U}{\partial \dot{q}_i} = 0$

Advantages

- **Equations contain only scalar quantities**
- **One equation for each degree of freedom**
- **Independent of the choice of coordinate system since the energy does not depend on coordinates**
- **See examples in Section 4.7 pages 369-377**
- **Useful in situations where F = *m*a is not obvious**

Example of Generalize Coordinates



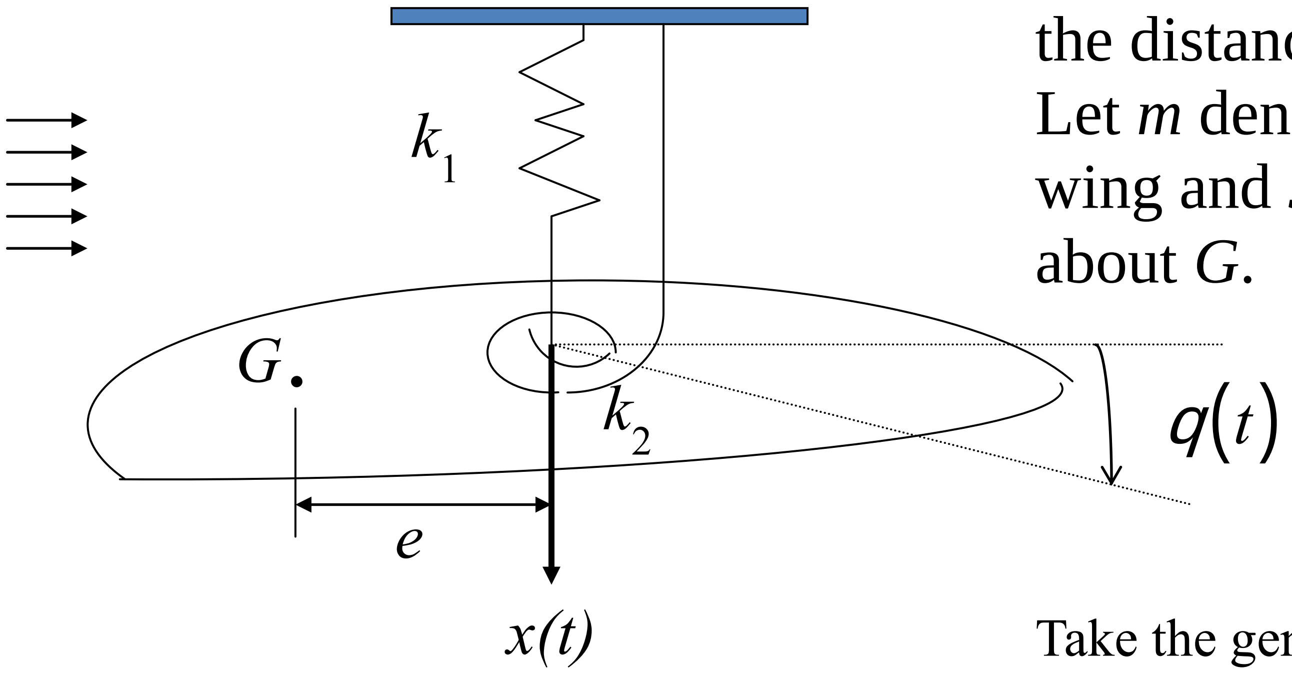
How many dof?
What are they?
Are there constraints?

$$x_1^2 + y_1^2 = \ell_1^2 \qquad (x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell_2^2$$

There are only 2 DOF and one choice is:

$$q_1 = \theta_1 \text{ and } q_2 = \theta_2$$

Example 4.7.3 (also illustrates linear approximation method)



Here G is mass center and e is the distance to the elastic axis. Let m denote the mass of the wing and J the rotational inertia about G.

Take the generalized coordinates to be:

$$q_1 = x(t),\ \ q_2 = \theta(t)$$

Called the pitch and plunge model

Computing the Energies

The Kinetic Energy is $$T = \frac{1}{2} m\dot{x}_G^2 + \frac{1}{2} J\dot{\theta}^2$$

The relationship between x_G and x is

$$x_G(t) = x(t) - e\sin\theta(t)$$

$$\Rightarrow \dot{x}_G(t) = \dot{x}(t) - e\cos\theta(t)\frac{d\theta}{dt} = \dot{x}(t) - e\dot{\theta}\cos\theta(t)$$

So the kinetic energy is

$$T = \frac{1}{2} m[\dot{x} - e\dot{\theta}\cos\theta]^2 + \frac{1}{2} J\dot{\theta}^2$$

Potential Energy and the Lagrangian

The potential energy is: $U = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2\theta^2$

The Lagrangian is:

$$L = T - U =$$

$$\frac{1}{2}m\left[\dot{x} - e\dot{\theta}\cos\theta\right]^2 + \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1 x^2 - \frac{1}{2}k_2\theta^2$$

Computing Derivatives for Equation (1.146)

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{x}} = m[\dot{x} - e\dot{\theta}c\,\text{os}\,\theta]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} - me\ddot{\theta} + me\dot{\theta}^2 \sin\theta$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial x} = -k_1 x$$

Now use the Lagrange equation to get:

$$m\ddot{x} - me\ddot{\theta}\cos\theta + em\dot{\theta}^2 \sin\theta + k_1 x = 0$$

Likewise differentiation with respect to $q_2 = \theta$ yields:

$$J\ddot{\theta} + me\cos\theta\ddot{x} + me^2\cos^2\theta\ddot{\theta} - me^2\dot{\theta}^2 \sin\theta\cos\theta + k_2\theta = 0$$

Next Linearize and write in matrix form

Using the small angle approximations:

$$\sin\theta \to \theta \qquad \cos\theta \to 1$$

In matrix form this becomes:

$$\begin{bmatrix} m & -me \\ -me & me^2 + J \end{bmatrix} \begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that this is a dynamically coupled system

Next consider the Single Spring-Mass System and compute the equation of motion using the Largranian approach

$$T = \frac{1}{2} m\dot{x}^2, \qquad U = \frac{1}{2} kx^2$$

$$L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \underline{m\ddot{x} + kx = 0}$$